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NOTES

A NOTE ON THE ANALYTICAL SOLUTION TO THE NEOCLASSICAL GROWTH MODEL WITH LEISURE

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In this note, we study the basic Ramsey models with labor–leisure choice. We first study the deterministic model and find that a closed-form solution exists and is represented by the Gauss hypergeometric functions. We next incorporate stochastic productivity shocks into the model. We prove that the analytical solution path still exists if the coefficient of relative risk aversion is equal to the capital share.

Keywords: Optimal Growth, Closed-Form Solution, Ramsey Model

1. INTRODUCTION

The Gauss hypergeometric functions are becoming popular in macroeconomics. Pioneering works of Boucekkine and Ruiz-Tamarit (2008) (henceforth BR) and Boucekkine et al. (2008) use them and obtain an explicit solution path to the two-sector endogenous growth model of Lucas (1988). Ruiz-Tamarit (2008) and Hiraguchi (2009) consider human capital externalities in the Lucas model and find that the closed-form solution path still exists.

The special functions have been applied to several growth models. Guerrini (2010) studies the AK model [in which the production function linearly depends on technology (A) and capital (K)] with logistic population growth. Pérez-Barahona (2011) studies the AK model with energy resources. Both find that solution paths are expressed in terms of the hypergeometric functions. A recent paper of Benchekroun and Withagen (2011), on the other hand, finds the closed form solution to the neoclassical growth model with exhaustible resources (Dasgupta–Heal–Solow–Stiglitz model) by using another type of the special function, the exponential integral.

In this note, we study the Ramsey models with leisure. First we study the deterministic model and show that the solution path is represented by the hypergeometric functions. We use the additively separable utility function, which is constant relative risk aversion (CRRA) in consumption, and the disutility from labor is linear. Such a function is commonly used in dynamic macroeconomics.

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This note is close to Smith (2006) and Nagata (2008), who obtain the explicit solution paths to the deterministic Ramsey models. The main difference between their models and ours is that they impose a parametric restriction so that the differential equation on capital becomes a simple Bernoulli type, whereas we do not.

Next we introduce stochastic productivity shocks as in Wälde (2011) into the model and obtain the solution path. The stochastic process is a combination of a Wiener process and Poisson processes. Here we assume that the coefficient of the relative risk aversion is equal to the capital share. Wälde (2011) studies the AK model and guesses that one may leave the AK framework by adding parameter restrictions. His guess is correct.

The note is organized as follows. Section 2 studies the deterministic model. Section 3 considers the stochastic model. The conclusions are in Section 4.

2. MODEL

In this section, we study the deterministic Ramsey model with leisure.

2.1. Mathematical Preparation

Here we describe some equalities we use in this note. First, a solution to the differential equation $\dot{x}_t = a_t x_t + b_t$ is

$$x_t = \exp\left(\int_0^t a_s ds\right)\left\{x_0 + \int_0^t \left[b_s \exp\left(-\int_0^s a_z dz\right)\right] ds\right\}. \quad (1)$$

In particular, if $a_t = a$ and $b_t = b$ are constant, then x_t is written as $x_t = -b/a + e^{at}(x_0 + b/a)$ and it satisfies $\dot{x}_t/x_t = a + b/x_t$. Thus $\exp(\int_0^t b/x_s ds) = (x_t/x_0)\exp(-at)$.

Second, the hypergeometric function satisfies

$$\int_x^\infty e^{-a_1 y}(b_1 + b_2 e^{-a_2 y})^{a_3} dy = \frac{e^{-a_1 x} b_1^{a_3}}{a_1} {}_2\mathrm{F}_1\left(-a_3, \frac{a_1}{a_2}, 1 + \frac{a_1}{a_2}; -\frac{b_2}{b_1} e^{-a_2 x}\right), \quad (2)$$

where ${}_2\mathrm{F}_1(a, b, c; z) = \sum_{n=0}^{\infty}[(a)_n(b)_n/(c)_n]z^n/n!$, $(a)_n = \Gamma(a+n)/\Gamma(a)$, $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, and $b_1 > 0$. BR also use a similar property. (The proof is in the Appendix.)

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2.2. Setup

We consider the following problem:

$$(\mathrm{P}_1)\ :\ \max_{\{c_t,k_t,n_t\}} \int_0^\infty e^{-\rho t}\left(\frac{c_t^{1-\theta}}{1-\theta}-\frac{n_t}{b}\right)dt,$$
$$\text{s.t. } \dot{k}_t = Ak_t^\alpha n_t^{1-\alpha} - \delta k_t - c_t, \tag{3}$$

where ρ is the discount factor, c_t is consumption, θ is the coefficient of relative risk aversion, n_t is labor supply, b and A are parameters, k_t is capital, δ is the depreciation rate and $\alpha \in (0, 1)$ is the capital share. The initial capital k_0 is given. In the following, we use four additional parameters, $\omega = (1/\alpha - 1)(\rho + \delta) > 0$, $\beta = 1/(1-\alpha)$, $\psi = A[(1-\alpha)Ab]^{1/\alpha-1} > 0$, and $\gamma = \alpha/[\theta(1-\alpha)] > 0$. We define a growth rate x_t as $\hat{x} = \dot{x}/x$.

The Hamiltonian is $H_1 = c_t^{1-\theta}/(1-\theta) - n_t/b + \lambda_t(Ak_t^\alpha n_t^{1-\alpha} - \delta k_t - c_t)$, where λ_t is the multiplier. The optimal path satisfies the first-order conditions (FOCs) and the transversality condition (TC)

$$\text{FOC[k]} : \frac{\dot{\lambda}_t}{\lambda_t} = \rho + \delta - \alpha A\left(\frac{k_t}{n_t}\right)^{\alpha-1}, \tag{4}$$

$$\text{FOC[c]} : c_t^{-\theta} = \lambda_t, \tag{5}$$

$$\text{FOC[n]} : 1 = (1-\alpha)\lambda_t Ab\left(\frac{k_t}{n_t}\right)^{\alpha}, \tag{6}$$

$$\text{TC} : \lim_{t\to\infty}\left[\lambda_t k_t e^{-\rho t}\right] = 0. \tag{7}$$

A path $\{c_t, k_t, n_t\}$ is optimal if it satisfies equations (3), (4), (5), (6), and (7) for some λ_t.

The steady state $(\bar{c}, \bar{k}, \bar{n})$ satisfies the FOCs $\rho + \delta = \alpha A(\bar{k}/\bar{n})^{\alpha-1}$ and $\bar{c}^\theta = (1-\alpha)Ab(\bar{k}/\bar{n})^\alpha$ and the resource constraint $\bar{c} = A\bar{k}^\alpha \bar{n}^{1-\alpha} - \delta\bar{k}$. Thus we have

$$(\bar{c}, \bar{k}, \bar{n}) = \frac{\{(1-\alpha)bA\}^{1/\theta}(A\alpha)^\gamma}{(\rho+\delta)^\gamma}\left[1, \frac{1}{\omega+\rho}, \frac{1}{\omega+\rho}\left(\frac{A\alpha}{\rho+\delta}\right)^\beta\right].$$

The multiplier in the steady state is $\bar{\lambda} = \bar{c}^{-\theta}$.

2.3. Analytical Solution

Equations (4) and (6) imply that the variable $\mu_t = \lambda_t^{1-1/\alpha}$ satisfies $\dot{\mu}_t = -\omega\mu_t + \psi(1-\alpha)$. Thus $\exp\int_0^t \psi/\mu_s ds = (\mu_t/\mu_0)^\beta e^{(\omega+\rho+\delta)t}$ (see Section 2.1). We have

$$\mu_t = \bar{\mu} + e^{-\omega t}(\mu_0 - \bar{\mu}), \tag{8}$$

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where $\bar{\mu} = \psi(1-\alpha)/\omega > 0$. Because $\omega > 0$, $\lim_{t\to\infty} \mu_t = \bar{\mu}$. Equations (3), (5), and (6) imply that

$$\dot{k}_t = [A(k_t/n_t)^{\alpha-1} - \delta]k_t - c_t = (\psi/\mu_t - \delta)k_t - \mu_t^{\gamma}. \quad \mathbf{(9)}$$

The solution to equation (9) is written as

$$k_t = \left(\frac{\mu_t}{\mu_0}\right)^{\beta} e^{(\omega+\rho)t}\left[k_0 - \mu_0^{\beta}\int_0^t e^{-(\omega+\rho)s}(\mu_s)^{\gamma-\beta}ds\right]. \quad \mathbf{(10)}$$

We show the following lemma.

LEMMA 1. *The transversality condition* (7) *holds if and only if* μ_0 *satisfies*

$$k_0 = \mu_0^{\beta}\int_0^{\infty} e^{-(\omega+\rho)s}(\mu_s)^{\gamma-\beta}ds. \quad \mathbf{(11)}$$

Proof. See the Appendix.

We can easily show that equation (11) uniquely determines μ_0. Using equation (11), we have

$$k_t = \mu_t^{\beta} e^{(\omega+\rho)t}\int_t^{\infty} \mu_s^{\gamma-\beta} e^{-(\omega+\rho)s}ds. \quad \mathbf{(12)}$$

Using equation (2), we can express the integral part of equation (12) as

$$\int_t^{\infty} \mu_s^{\gamma-\beta} e^{-(\omega+\rho)s}ds = \frac{e^{-(\omega+\rho)t}\bar{\mu}^{\gamma-\beta}}{\omega+\rho}\ {}_2\mathrm{F}_1\left(\beta-\gamma, 1+\frac{\rho}{\omega}, 2+\frac{\rho}{\omega}; -\mu_0' e^{-\omega t}\right), \quad \mathbf{(13)}$$

where $\mu_0' = \mu_0/\bar{\mu} - 1$. Now we get the following proposition.

PROPOSITION 1. *The solution path* $\{c_t, k_t, n_t\}$ *is unique and is denoted as*

$$c_t = \bar{c}(1+\mu_0' e^{-\omega t})^{\alpha/[\theta(1-\alpha)]}, \quad \mathbf{(14)}$$

$$k_t = \bar{k}(1+\mu_0' e^{-\omega t})^{1/(1-\alpha)} \cdot {}_2\mathrm{F}_1\left[\frac{\theta-\alpha}{\theta(1-\alpha)}, 1+\frac{\rho}{\omega}, 2+\frac{\rho}{\omega}; -\mu_0' e^{-\omega t}\right], \quad \mathbf{(15)}$$

$$n_t = \bar{n} \cdot {}_2\mathrm{F}_1\left[\frac{\theta-\alpha}{\theta(1-\alpha)}, 1+\frac{\rho}{\omega}, 2+\frac{\rho}{\omega}; -\mu_0' e^{-\omega t}\right]. \quad \mathbf{(16)}$$

Equation (11) *determines* μ_0. *The path converges to the steady state* $\{\bar{c}, \bar{k}, \bar{n}\}$.

Proof. See the Appendix.

As BR, Smith (2006), and Nagata (2008) point out, the solutions are simple when the coefficient of relative risk aversion θ is equal to the capital share α, because the differential equation on capital becomes Bernoulli-type. When $\theta = \alpha$, equations (15) and (16) are in fact simplified because ${}_2\mathrm{F}_1(0, q, 1+q; z) = 1$ for all q and z. Note that ${}_2\mathrm{F}_1$ has the Euler integral representation ${}_2\mathrm{F}_1(p, q, 1+q; z) = q\int_0^1 t^{q-1}(1-tz)^{-p}dt$.

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3. STOCHASTIC TECHNOLOGY SHOCK

This section considers a stochastic technology shock. The setup is similar to that in Smith (2007), but we adopt the more general stochastic process of Wälde (2011). The productivity A_t is a combination of one Wiener and several Poisson processes:

$$\frac{dA_t}{A_t} = \mu dt + \sigma dz_t + \sum_{i=1}^{N} \beta_i dq_{i,t}. \qquad \mathbf{(17)}$$

The term z_t is a geometric Brownian motion, whereas the term $q_{i,t}$ follows independent Poisson processes with arrival rate λ_i. The model is described as follows:

$$(\mathrm{P}_2) : \max \mathbf{E}\left[\int_0^\infty e^{-\rho t}\left(\frac{c_t^{1-\alpha}}{1-\alpha} - \frac{n_t}{b}\right)dt\right], \qquad \text{s.t. } \dot{k}_t = A_t k_t^\alpha n_t^{1-\alpha} - \delta k_t - c_t.$$

Here we assume $\theta = \alpha$. The value function J satisfies the Bellman equation:

$$\rho J(k, A) = \max_{c,n}\left[\frac{c^{1-\alpha}}{1-\alpha} - \frac{n}{b} + J_k(Ak^\alpha n^{1-\alpha} - c)\right]$$
$$- J_k\delta k + J_A A\mu + J_{AA}A^2\frac{\sigma^2}{2} + \sum_{i=1}^{N}\lambda_i\{J[k, A(1+\beta_i)] - J(k, A)\}. \qquad \mathbf{(18)}$$

We guess that the value function has the form

$$J(k, A) = x^{-\alpha}\frac{k^{1-\alpha}}{1-\alpha} + zA^{1/\alpha}, \qquad \mathbf{(19)}$$

where $x > 0$ and $z > 0$ are unknown. The maximization problem in equation (18) is

$$\max_{c,n}\left[\frac{c^{1-\alpha}}{1-\alpha} - \frac{n}{b} + x^{-\alpha}(An^{1-\alpha} - k^{-\alpha}c)\right]$$
$$= (\alpha x - \delta)x^{-\alpha}\frac{k^{1-\alpha}}{1-\alpha} + \frac{\alpha}{x}[(1-\alpha)\, b]^{1/\alpha - 1}A^{1/\alpha}. \qquad \mathbf{(20)}$$

The optimal c and n are respectively $c = xk$ and $n = [(1-\alpha)\, Ab]^{1/\alpha}/x$. The other terms of equation (18) are written as $-\delta x^{-\alpha}k^{1-\alpha} + \eta z A^{1/\alpha}$, with $\eta = \mu/\alpha + (1-\alpha)\sigma^2/(2\alpha^2) + \sum_i \lambda_i\{(1+\beta_i)^{1/\alpha} - 1\} > 0$. Thus the functional form (19) solves equation (18) if and only if

$$\rho = \alpha x - (1-\alpha)\delta, \qquad \mathbf{(21)}$$

$$\rho z = \frac{\alpha}{x}[(1-\alpha)\, b]^{1/\alpha - 1} + \eta z \qquad \mathbf{(22)}$$

Equation (21) implies that $x = [(1-\alpha)\delta + \rho]/\alpha > 0$. Equation (22) then

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means that $z = \alpha\{(1-\alpha)b\}^{1/\alpha-1}/\{x(\rho-\eta)\}$. To ensure that $z > 0$, we must have

$$\rho > \frac{\mu}{\alpha} + \frac{1-\alpha}{2\alpha^2}\sigma^2 + \sum_{i=1}^{N}\lambda_i[(1+\beta_i)^{1/\alpha} - 1]. \qquad \mathbf{(23)}$$

PROPOSITION 2. *If equation* (23) *holds, the value function of the problem* (P_2) *is given by equation* (19). *The optimal path satisfies* $c_t = xk_t$ *and* $n_t = [(1-\alpha)Ab]^{1/\alpha}/x$.

4. CONCLUSION

In this note, we get a closed-form solution path to the Ramsey model with leisure by using the special function. As a future study, we hope to study the stochastic model without assuming that the capital share is equal to the coefficient of relative risk aversion. The parametric restriction is not empirically supported and we admit that it represents a major weakness of the paper. We also want to investigate a model with a utility function that is multiplicatively separable between consumption and labor, because such a utility function allows the existence of long-run balanced growth.

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APPENDIX

A.1. PROOF OF EQUATION (2)

For a variable $z = e^{-a_1 y}$, we get $dz = -a_1 e^{-a_1 y} dy$. Thus

$$\int_x^{\infty} e^{-a_1 y}(b_1 + b_2 e^{-a_2 y})^{a_3} dy = \frac{b_2^{a_3}}{a_1} \int_0^{e^{-a_1 x}} (b_1/b_2 + z^{(a_2/a_1)})^{a_3} dz. \qquad \textbf{(A.1)}$$

Because a variable $u = xt^{1/b}$ satisfies $du = xt^{1/b-1}/bdt$, we get

$${}_2\mathrm{F}_1\left(-c, \frac{1}{b}, 1 + \frac{1}{b}; -\frac{1}{a}x^b\right) xa^c = \frac{1}{b}\int_0^1 xt^{1/b-1}(a + x^b t)^c dt = \int_0^x (a + u^b)^c du.$$

Moreover, ${}_2\mathrm{F}_1\,(a, b, c; 0) = 1$ for any a, b, and c. Thus we get

$$\int_0^{e^{-a_1 x}} (b_1/b_2 + z^{(a_2/a_1)})^{a_3} dz = {}_2\mathrm{F}_1\left(-a_3, \frac{a_1}{a_2}, 1 + \frac{a_1}{a_2}; -\frac{b_2}{b_1}e^{-a_2 x}\right) e^{-a_1 x}\,(b_1/b_2)^{a_3} \quad \textbf{(A.2)}$$

Substitution of Eq. (A.2) into Eq. (A.1) yields the result. ■

A.2. PROOF OF LEMMA 1

If equation (11) does not hold, $\lim_{t\to\infty} \hat{k} = \omega + \rho$ and the growth rate of the term $\lambda_t k_t e^{-\rho t}$ converges to $\omega > 0$. Thus equation (7) does not hold. On the other hand, if equation (11) holds, one has $k_t = \mu_t^{1-\alpha} \int_0^{\infty} \mu_{t+i}^{\gamma} e^{-(\omega+\rho)i} di$. Thus $\lim_{t\to\infty} k_t < \infty$ and equation (7) holds. Therefore equations (11) and (7) are equivalent. ■

A.3. PROOF OF PROPOSITION 1

First, equation (14) holds, because $c_t = \mu_t^{\gamma}$. Second, substitution of equation (13) into equation (12) yields equation (15). Third, equation (6) implies $n_t = [(1-\alpha)Ab]^{1/\alpha} \mu_t^{-\beta} k_t$ and equation (16) holds. Finally, $\omega > 0$ and then $\lim_{t\to\infty} {}_2\mathrm{F}_1(a, b, c; e^{-\omega t}) = 1$. Thus $\lim_{t\to\infty}(c_t, k_t, n_t) = (\bar{c}, \bar{k}, \bar{n})$. ■

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