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Bisections of Graphs Without Short Cycles

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Bollobás and Scott (*Random Struct. Alg.* **21** (2002) 414–430) asked for conditions that guarantee a bisection of a graph with m edges in which each class has at most $(1/4+o(1))m$ edges. We demonstrate that cycles of length 4 play an important role for this question. Let G be a graph with m edges, minimum degree δ, and containing no cycle of length 4. We show that if (i) G is 2-connected, or (ii) $\delta \geqslant 3$, or (iii) $\delta \geqslant 2$ and the girth of G is at least 5, then G admits a bisection in which each class has at most $(1/4+o(1))m$ edges. We show that each of these conditions are best possible. On the other hand, a construction by Alon, Bollobás, Krivelevich and Sudakov shows that for infinitely many m there exists a graph with m edges and girth at least 5 for which any bisection has at least $(1/4-o(1))m$ edges in one of the two classes.

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1. Introduction

It is well known that every graph G with m edges has a bipartition $V(G)=V_1\cup V_2$ with $e(V_1,V_2)\geqslant m/2$, where $e(V_1,V_2)$ is the number of edges of G with one end in V_1 and the other end in V_2. Edwards [8, 9] proved that G has a bipartition $V(G)=V_1\cup V_2$ such that

$$e(V_1,V_2)\geqslant \frac{m}{2}+\frac{1}{4}\left(\sqrt{2m+\frac{1}{4}}-\frac{1}{2}\right),$$

and this bound is best possible for K_{2k+1}. For special classes of graphs, such as subcubic graphs [21, 25], the main term in Edwards' bound can be improved. Moreover, for certain ranges of m, Alon [1] gave an additive improvement.

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Judicious partitioning problems for graphs ask for partitions of graphs that maximize or minimize several quantities simultaneously. Bollobás and Scott [4] initiated a systematic study of such problems by proving that each graph with m edges has a partition into k classes, each of which contains at most $m/k^2 + O(\sqrt{m})$ edges. This result was improved by Xu and Yu [22, 23], and extended to hypergraphs with edges of sizes at most 2 by Hou and Zeng [12]. For more results and problems we refer the reader to [11, 16, 18].

In this paper, we focus on bisections of graphs. First, we give some definitions and notation. Let G be a graph. We use $|G| = |V(G)|$ to denote the *order* of G, and $e(G) = |E(G)|$ to denote the *size* of G. For a vertex $v \in V(G)$, we use $N_G(v)$ to denote the neighbourhood of v in G, and $d_G(v) = |N_G(v)|$ to denote the *degree* of v. For subsets S and T of $V(G)$, $e_G(S,T)$ denotes the number of edges of G with one end in S and the other end in T, and $e_G(S)$ denotes the number of edges of G with both ends in S. In particular, if $S = \{v\}$, we simply write $e_G(v,T)$ for $e_G(\{v\},T)$. A *k-cycle* is a cycle of length k. The *girth* of G, denoted by $g(G)$, is the length of a shortest cycle in G. We will drop the reference to G when there is no danger of confusion. A *bisection* of G is a partition $V(G) = V_1 \cup V_2$ with $||V_1| - |V_2|| \leqslant 1$.

Recent works on graph bisections are partly motivated by a conjecture of Bollobás and Scott [5]. If G is a graph with minimum degree $\delta(G) \geqslant 2$, then G admits a bisection $V(G) = V_1 \cup V_2$ such that for $i = 1,2$, $e(V_i) \leqslant e(G)/3$. The star $K_{1,n}$ shows that the requirement of minimum degree is necessary, and the triangle shows that the bound is best possible. After a series of papers [13, 19, 20], the conjecture was finally established by Xu and Yu [24]. For large minimum degree, Lee, Loh and Sudakov [13] proved that every graph G with minimum degree $\delta \geqslant 2$, where δ is even, has a bisection $V(G) = V_1 \cup V_2$ such that, for $i = 1,2$,

$$e(V_i) \leqslant \left(\frac{\delta+2}{4(\delta+1)} + o(1)\right)e(G),$$

and a similar bound holds for odd δ by applying the above result for the even integer $\delta - 1$. The complete bipartite graph $K_{\delta+1,n-\delta-1}$ shows that the bound is asymptotically tight.

By arbitrary pairing of the vertices of a graph G and then separating each pair independently and uniformly at random, we obtain a bisection $V(G) = V_1 \cup V_2$ such that the expected number of edges in each V_i is $e(G)/4$. However, $e(V_1) \leqslant (1/4 + o(1))e(G)$ and $e(V_2) \leqslant (1/4 + o(1))e(G)$ do not necessarily hold at the same time. In [5], Bollobás and Scott posed the following problem.

Problem 1.1. *Under what conditions can we guarantee a bisection of a graph G in which each class contains at most* $(1/4 + o(1))e(G)$ *edges?*

We believe that 3-cycles and 4-cycles play a natural role in this problem. The graph $K_{3,n}$ shows that excluding 3-cycles alone is not sufficient for the bound $(1/4 + o(1))e(G)$. Let G denote the graph obtained from the disjoint union of n triangles by choosing one vertex from each triangle and identifying them to a single vertex, denoted by u. Then G has $2n+1$ vertices, $3n$ edges and no 4-cycles. For a bisection $V(G) = V_1 \cup V_2$ of G, the vertex class containing u, say V_1, contains at least $n-1$ neighbourhoods of u, which implies that

$$e(V_1) \geqslant n - 1 = e(G)/3 - 1.$$

Let G be a graph and let $v \in V(G)$. We use $t_G(v)$ to denote the number of triangles $vxyv$ with $d_G(x) = d_G(y) = 2$. The following result shows that $\max\{t_G(v) : v \in V(G)\}$ also plays a role in judicious bisections of graphs without 4-cycles.

Theorem 1.2. *Let G be a graph with minimum degree $\delta(G) \geqslant 2$ and without 4-cycles, and let $t = \max\{t_G(v) : v \in V(G)\}$. Then G has a bisection $V(G) = V_1 \cup V_2$ such that, for $i = 1, 2$,*

$$e(V_i) \leqslant \frac{1}{4}(e(G) + t) + o(e(G)).$$

Note that $t = 0$ when $\delta(G) \geqslant 3$, or $g(G) \geqslant 5$, or G is 2-connected and not a triangle. So we have the following consequence of Theorem 1.2.

Corollary 1.3. *Let G be a graph with minimum degree $\delta(G) \geqslant 2$ and without 4-cycles. Then G admits a bisection $V(G) = V_1 \cup V_2$ such that, for $i = 1, 2$,*

$$e(V_i) \leqslant \left(\frac{1}{4} + o(1)\right) e(G)$$

if one of the following conditions holds:

(1) *G is 2-connected, or*
(2) *$\delta(G) \geqslant 3$, or*
(3) *$g(G) \geqslant 5$.*

Now we consider the max-bisection problem. Given a graph G, find a bisection $V(G) = V_1 \cup V_2$ that maximizes $e(V_1, V_2)$. As noticed in [10, 17], Edwards' bound implicitly implies that a connected graph G admits a bipartition $V(G) = V_1 \cup V_2$ with $e(V_1, V_2) \geqslant e(G)/2 + (|G| - 1)/4$. However, every bisection of the star $K_{1,n-1}$ has size at most $\lceil n/2 \rceil$. In [13], Lee, Loh and Sudakov proved that every connected graph G has a bisection of size at least $e(G)/2 + (|G| + 1 - \Delta(G))/4$, and that the bound is tight. The following result gives a similar bound on bisections of graphs without short cycles.

Theorem 1.4. *Let G be a graph with minimum degree $\delta(G) \geqslant 2$ and without 4-cycles. If $\delta(G)$ is even, then G has a bisection of size at least*

$$\frac{e(G)}{2} + \frac{|G| - 1}{4} - \frac{|M|}{2\delta(G)},$$

where M is a maximum matching in G. Moreover, if the girth $g(G) \geqslant 5$, then G has a bisection of size at least $e(G)/2 + (|G| - 1)/4$.

Remark. For graphs G with odd $\delta(G) \geqslant 3$, one can deduce a similar bound by applying the above result for the even integer $\delta(G) - 1$. The bound in Theorem 1.4 is tight when $\delta(G) = 2$ by considering the union of vertex disjoint triangles. The complete bipartite graph $K_{\delta, n-\delta}$ shows that the condition of not having 4-cycles in Theorem 1.4 cannot be removed.

We organize the paper as follows. In Section 2, we collect some lemmas that will be used in this paper. We then consider graphs without 4-cycles in Section 3 and prove Theorem 1.2.

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In Section 4, we prove Theorem 1.4. In Section 5, we discuss the bounds in Theorems 1.2 and 1.4.

2. Lemmas

We begin with a result of Lee, Loh and Sudakov [13].

Lemma 2.1 (Lee, Loh and Sudakov [13]). *Let ε be a fixed positive constant and let G be a graph such that (i) $e(G) \geqslant \varepsilon^{-2}|G|$, or (ii) $\Delta(G) \leqslant \varepsilon^2|G|/2$. If $|G|$ is sufficiently large, then G admits a bisection $V(G) = V_1 \cup V_2$ such that for $i = 1,2$, $e(V_i) \leqslant (1/4+\varepsilon)e(G)$.*

In their analysis of bisections, Lee, Loh and Sudakov [13] introduced the notion of a tight component in a graph. A connected graph T is *tight* if

- for every vertex $v \in V(T)$, $T - v$ contains a perfect matching, and
- for every vertex $v \in V(T)$ and every perfect matching M of $T - v$, no edge in M has exactly one end adjacent to v.

Note that K_1 is tight, and we call it *trivial*. Recently, Lu, Wang and Yu [14] studied the tight graphs and proved the following.

Lemma 2.2 (Lu, Wang and Yu [14]). *A connected graph G is tight if and only if every block of G is an odd clique.*

Note that, if G is a graph without 4-cycles, T is a triangle of G and $v \in V(G)$ is not incident with T, then $e(v,V(T)) \leqslant 1$. Hence, we have the following easy consequence of Lemma 2.2.

Lemma 2.3. *Let G be a graph without 4-cycles and let $A \subseteq V(G)$. Let J be the union of some tight components of $G \setminus A$, let J_1 be the subgraph of J consisting of trivial components in $G \setminus A$, and let $J_2 = J \setminus V(J_1)$. Then for $v \in A$, $e(v,V(J)) \leqslant |V(J_1)| + |V(J_2)|/3$.*

The following result bounds the number of tight components in a graph.

Lemma 2.4 (Lee, Loh and Sudakov [13]). *Let H be an arbitrary graph. For each integer i, let d_i be the number of vertices in H with degree equal to i. Then the number of tight components τ in H satisfies*

$$\tau \leqslant \frac{d_0}{1} + \frac{d_2}{3} + \frac{d_4}{5} + \cdots .$$

A standard approach to finding a good judicious partition is to first partition certain large degree vertices, and then apply a randomized algorithm to distribute the remaining vertices. The main tool we use for the second step is the following lemma given in [13].

Lemma 2.5 (Lee, Loh and Sudakov [13]). *Given any real constants $C,\varepsilon > 0$, there exist $\gamma, n_0 > 0$ for which the following holds. Let G be a given graph with $n \geqslant n_0$ vertices and at* *most Cn edges, and let $A \subseteq V(G)$ be a set of $\leqslant \gamma n$ vertices which has already been partitioned into $A_1 \cup A_2$. Let $\overline{A} = V(G) \setminus A$, and suppose that every vertex in $\overline{A}$ has degree at most γn (with respect to the full G). Let τ be the number of tight components in $G[\overline{A}]$. Then, there is a bisection $V(G) = V_1 \cup V_2$ with $A_1 \subseteq V_1$ and $A_2 \subseteq V_2$, such that, for $i = 1,2$,*

$$e(V_i) \leqslant e(A_i) + \frac{e(A_i, \overline{A})}{2} + \frac{e(\overline{A})}{4} - \frac{n-\tau}{8} + \varepsilon n.$$

The following lemma, given by Bollobás and Scott [4], shows that one can find a bipartition of a graph which satisfies both Edwards' bound and a best possible upper bound on $\max\{e(V_1), e(V_2)\}$.

Lemma 2.6 (Bollobás and Scott [4]). *Every graph G admits a partition $V(G) = V_1 \cup V_2$ such that*

$$e(V_1, V_2) \geqslant \frac{e(G)}{2} + \frac{\sqrt{2e(G)+1/4} - 1/2}{4},$$

and for $i = 1,2$,

$$e(V_i) \leqslant \frac{e(G)}{4} + \frac{\sqrt{2e(G)+1/4} - 1/2}{8}.$$

We also need the following result of Bondy and Simonovits [7] on the maximum number of edges in graphs without cycles of a given even length.

Lemma 2.7 (Bondy and Simonovits [7]). *Let $l \geqslant 2$ be an integer and let G be a graph with n vertices. If G contains no cycle of length $2l$, then $e(G) \leqslant 100ln^{1+1/l}$.*

We end this section with a lemma concerning the degree of vertices in a graph without 4-cycles. For a graph G, let $d_1(G)$ denote the number of vertices with degree 1 in G.

Lemma 2.8. *Let G be a graph without 4-cycles, let $v \in A \subseteq V(G)$, and let $C \subseteq G \setminus A$ be connected. Then $e(v, V(C)) \leqslant |C|/2 + d_1(C)$ if $|C| \geqslant 2$, and $e(v, V(C)) \leqslant 2|C|/3$ if $|C| \geqslant 3$.*

Proof. Suppose $|C| \geqslant 3$. Let $S = N_G(v) \cap V(C)$ and $T = \left(\bigcup_{u \in S} N_C(u)\right) \setminus S$. Then $\Delta(C[S]) \leqslant 1$ since G has no 4-cycle. Hence, $T \neq \emptyset$ as $|C| \geqslant 3$ and C is connected. Since C is connected, if x is an isolated vertex of $C[S]$ then $e_C(x, T) \geqslant \delta(C) \geqslant 1$, and if xy is an edge of $C[S]$ then $e_C(x, T) \geqslant 1$ or $e_C(y, T) \geqslant 1$. Hence,

$$e_C(S, T) = \sum_{x \in S} e_C(x, T) \geqslant |S|/2.$$

Note that no two vertices in S have common neighbourhoods in T as G has no 4-cycle. Thus, $|T| = e_C(S, T) \geqslant |S|/2$. Since $|S| + |T| \leqslant |C|$, $e_G(v, V(C)) = |S| \leqslant 2|C|/3$.

Now assume $|C| \geqslant 2$. We show $e(v, V(C)) \leqslant |C|/2 + d_1(C)$ by inducting on $|C|$. First, suppose $|C| \leqslant 3$. If $d_1(C) = 2$ then $e(v, V(C)) \leqslant |C|/2 + d_1(C)$. So assume $d_1(C) \leqslant 1$. Then C is a triangle.

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Since G has no 4-cycle, $e(v,V(C)) \leqslant 1 \leqslant |C|/2$. Thus, we may assume that $|C| \geqslant 4$ and the result holds for connected subgraphs of $G \setminus A$ with fewer than $|C|$ vertices.

Suppose $\delta(C) \geqslant 2$. Then $e_C(x,T) \geqslant 2$ if x is an isolated vertex in $C[S]$, and $e_C(x,T) \geqslant 1$ and $e_C(y,T) \geqslant 1$ if xy is an edge of $C[S]$. Hence,

$$|T| = e_C(S,T) \geqslant |S|,$$

which implies $e(v,V(C)) = |S| \leqslant |C|/2$.

So we may assume $\delta(C) = 1$. Let $x_1x_2\cdots x_k$ be a maximal path in C such that $d_C(x_1) = 1$, $d_C(x_i) = 2$ for $i = 2,\dots,k-1$, and $d_C(x_k) \neq 2$. If $d_C(x_k) = 1$, then C is a path and $d_1(C) = 2$; hence, since G has no 4-cycle, $e(v,V(C)) \leqslant (|C|+1)/2 \leqslant |C|/2 + d_1(C)$. So assume that $d_C(x_k) \geqslant 3$. Consider the graph $C' = C - \{x_1,\dots,x_{k-1}\}$. Then, C' is connected, $|C'| \geqslant 3$, $|C'| = |C| - k + 1$, and $d_1(C') = d_1(C) - 1$. By the induction hypothesis,

$$e(v,V(C')) \leqslant \frac{1}{2}|C'| + d_1(C') = \frac{1}{2}|C| + d_1(C) - \frac{k+1}{2}.$$

On the other hand, it is easy to see that

$$e(v,\{x_1,\dots,x_{k-1}\}) \leqslant \frac{k+1}{2}.$$

Thus,

$$e(v,V(C)) = e(v,V(C')) + e(v,\{x_1,\dots,x_{k-1}\}) \leqslant \frac{1}{2}|C| + d_1(C),$$

which completes the proof. □

3. Graphs without 4-cycles

In this section, we prove Theorem 1.2. The idea of the proof is the same as in Lee, Loh and Sudakov [13], which was used first by Bollobás and Scott [3] and again by Ma, Yen and Yu [15]. First we partition a set of large degree vertices using an additional idea from [13]. Then we study the properties of the partition. Next we apply Lemma 2.5 to partition the remaining vertices.

Proof of Theorem 1.2. It suffices to show that for any small $\varepsilon > 0$, there exists integer $n_0 > 0$ such that if $n \geqslant n_0$, then any graph G with n vertices has a bisection $V(G) = V_1 \cup V_2$ such that for $i = 1,2$, $e(V_i) \leqslant (1/4 + \varepsilon)e(G) + t/4$.

Since $\delta(G) \geqslant 2$, $e(G) \geqslant n$. By Lemma 2.1, we may assume that

$$n \leqslant e(G) \leqslant \varepsilon^{-2} n \quad \text{and} \quad \Delta(G) \geqslant \varepsilon^2 n/2.$$

Let

$$A = \{v \in V(G) : d_G(v) \geqslant n^{3/4}\} \quad \text{and} \quad \overline{A} = V(G) \setminus A.$$

Note that

$$2e(G) = \sum_{v \in V(G)} d(v) \geqslant \sum_{v \in A} d(v) \geqslant |A| n^{3/4}.$$

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This together with $e(G) \leqslant \varepsilon^{-2} n$ yields

$$|A| \leqslant 2\varepsilon^{-2} n^{1/4} = O(n^{1/4}), \tag{3.1}$$

and hence

$$e(A) \leqslant \binom{|A|}{2} = O(n^{1/2}). \tag{3.2}$$

We now estimate the number of edges between A and $\overline{A}$. Let

$$S = \{x \in \overline{A} : e(x,A) \geqslant 2\}.$$

For $i \geqslant 0$, let d_i be the number of vertices in $G[\overline{A}]$ with degree equal to i. We need these parameters later to bound the number of tight components in $G[\overline{A}]$.

Claim 3.1. *$|S| \leqslant \binom{|A|}{2}$ and hence $d_0 + d_1 + \cdots + d_{\delta(G)-2} \leqslant |S| = O(n^{1/2})$.*

Since G does not have 4-cycles, any pair of vertices in A cannot be adjacent to more than one vertex in S. So we have

$$\binom{|A|}{2} \geqslant \sum_{w \in S} \binom{e(w,A)}{2} \geqslant |S|.$$

Hence by (3.1), $|S| = O(n^{1/2})$. By the definitions of S and d_i for $i = 0, \ldots, \delta(G) - 2$, it is easy to see that $d_0 + d_1 + \cdots + d_{\delta(G)-2} \leqslant |S|$.

Claim 3.2. *$e(A, \overline{A}) \leqslant n + O(n^{3/4})$.*

Let $S' = \{v \in \overline{A} : e(v,A) = 1\}$. Then

$$e(A,\overline{A}) = e(A,S) + e(A,S') = e(A,S) + |S'| \leqslant e(A,S) + n.$$

Since G contains no 4-cycle, we may apply Lemma 2.7 to $G[A \cup S]$; so

$$e(A,S) = O((|A| + |S|)^{3/2}) = O(n^{3/4})$$

(by Claim 3.1). Thus, $e(A,\overline{A}) \leqslant n + O(n^{3/4})$.

Claim 3.3. *For $v \in A$,*

$$e(v,\overline{A}) \leqslant \frac{n-1}{\delta(G)-1}.$$

This is obvious when $\delta(G) = 2$. So assume $\delta(G) \geqslant 3$. Let $B = N_G(v) \cap \overline{A}$. Then $\Delta(G[B]) \leqslant 1$ as G has no 4-cycle. Let $B' = (\cup_{u \in B} N_G(u)) \setminus (B \cup \{v\})$. Then, for $u \in B$,

$$e(u,B') = d_G(u) - e(u,B) - 1 \geqslant \delta(G) - 2.$$

Therefore, since G contains no 4-cycle,

$$|B'| = e(B,B') \geqslant (\delta(G) - 2)|B|.$$

Since $|B'|+|B|+1 \leqslant n$, we conclude that

$$e(v,\overline{A}) = |B| \leqslant \frac{n-1}{\delta(G)-1}.$$

This completes the proof of Claim 3.3.

Partition $A = A_1 \cup A_2$ in such a way that $e(A_1,\overline{A}) \geqslant e(A_2,\overline{A})$ and, subject to this,

$$\theta = e(A_1,\overline{A}) - e(A_2,\overline{A})$$

is minimized.

Then, for any $v \in A_1$, $e(v,\overline{A}) \geqslant \theta$; for otherwise, the partition $A_1 \setminus \{v\}, A_2 \cup \{v\}$ of A contradicts the choice of A_1, A_2. Moreover, we have $e(A_2,\overline{A}) \leqslant e(A_1,\overline{A}) = (e(A,\overline{A})+\theta)/2$. Since $|A| = O(n^{1/4})$ and n is sufficiently large (by choosing n_0 large), it follows from Lemma 2.5 that G has a bisection $V(G) = V_1 \cup V_2$ such that, for $i = 1,2$,

$$e(V_i) \leqslant e(A_i) + \frac{e(A,\overline{A})+\theta}{4} + \frac{e(\overline{A})}{4} - \frac{n-\tau}{8} + \frac{\varepsilon}{3}n,$$

where τ is the number of tight components in $G[\overline{A}]$. Since $e(A_i) = O(n^{1/2})$ and $e(G) \geqslant n$,

$$e(V_i) \leqslant \frac{1}{4}\left(\theta + \frac{\tau}{2} - \frac{n}{2}\right) + \left(\frac{1}{4} + \frac{\varepsilon}{2}\right)e(G). \tag{3.3}$$

We now bound θ from below. Combining Lemma 2.4 and Claim 3.1, we obtain

$$\tau \leqslant (|\overline{A}| - d_0)/3 + d_0 \leqslant n/3 + O(n^{1/2}).$$

If $\theta \leqslant n/3 + \varepsilon e(G) + t$, then

$$\theta + \frac{\tau}{2} \leqslant \frac{n}{3} + t + \varepsilon e(G) + \frac{1}{2} \times \left(\frac{n}{3} + O(n^{1/2})\right) \leqslant \frac{n}{2} + 2\varepsilon e(G) + t,$$

which together with (3.3) yields $e(V_i) \leqslant (1/4+\varepsilon)e(G) + t/4$ for $i = 1,2$; so V_1, V_2 give the desired bisection. Thus, we may assume that

$$\theta > n/3 + \varepsilon e(G) + t. \tag{3.4}$$

Let

$$\alpha = |\{v \in A : e(v,\overline{A}) \geqslant \theta\}|, \quad R = \{v \in A_2 : e(v,\overline{A}) < \theta\} \quad \text{and} \quad \rho = \sum_{v \in R} e(v,\overline{A}).$$

Then

$$e(A,\overline{A}) = \sum_{v \in A} e(v,\overline{A}) \geqslant \alpha\theta + \rho.$$

By Claim 3.2 and (3.4), we have $\alpha \leqslant 2$ (as $e(G) \geqslant n$ and n is large).

Claim 3.4. $\alpha = 1$.

Let $u \in A_1$. Recall that $e(u,\overline{A}) \geqslant \theta$. First, we show that $\rho \leqslant e(u,\overline{A}) - \theta$ as in Lee, Loh and Sudakov [13]. Indeed, suppose $\rho > e(u,\overline{A}) - \theta$. Note that

$$e(A_1 \setminus \{u\},\overline{A}) = (e(A,\overline{A})+\theta)/2 - e(u,\overline{A}) \leqslant (e(A,\overline{A})-\theta)/2.$$

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Let $C_1 = (A_1 \setminus \{u\}) \cup R$ and $C_2 = (A_2 \cup \{u\}) \setminus R$. Then

$$e(C_1, \overline{A}) = (e(A, \overline{A}) + \theta)/2 - e(u, \overline{A}) + \rho > (e(A, \overline{A}) - \theta)/2$$

and hence $e(C_2, \overline{A}) < (e(A, \overline{A}) + \theta)/2$. Thus, there exists $R' \subseteq R$ such that for $C_1' = (A_1 \setminus \{u\}) \cup R'$ and $C_2' = (A_2 \cup \{u\}) \setminus R'$,

$$(e(A, \overline{A}) - \theta)/2 < e(C_i', \overline{A}) < (e(A, \overline{A}) + \theta)/2$$

for $i = 1, 2$, which contradicts the choice of A_1, A_2 (minimizing θ).

Suppose $\alpha = 2$, and let $v_1, v_2 \in A$ be distinct with $e(v_1, \overline{A}) \geqslant e(v_2, \overline{A}) \geqslant \theta$. If $v_1, v_2 \in A_1$, then $A_1 = \{v_1, v_2\}$ and $\rho \leqslant e(v_2, \overline{A}) - \theta$ by the above arguments. Recall that $\theta = e(A_1, \overline{A}) - e(A_2, \overline{A})$. Thus,

$$\theta = e(v_1, \overline{A}) + e(v_2, \overline{A}) - \rho \geqslant e(v_1, \overline{A}) + e(v_2, \overline{A}) - (e(v_2, \overline{A}) - \theta) \geqslant 2\theta,$$

a contradiction. So we may assume $v_1 \in A_1$ and $v_2 \in A_2$. Note that

$$\theta = e(v_1, \overline{A}) - (e(v_2, \overline{A}) + \rho) \leqslant e(v_1, \overline{A}) - e(v_2, \overline{A}) \leqslant e(v_1, \overline{A}) - \theta,$$

which together with (3.4) yields

$$e(v_1, \overline{A}) \geqslant 2\theta > \frac{2}{3}n + 2\varepsilon e(G) + 2t. \tag{3.5}$$

Next, we derive a contradicting upper bound on $e(v_1, \overline{A})$ by bounding $e(v_1, V(C))$ for each connected component C of $G[\overline{A}]$. If $|C| = 1$ then $V(C) \subseteq S$. If $|C| \geqslant 3$, then $e(v_1, V(C)) \leqslant 2|C|/3$ by Lemma 2.8. If $|C| = 2$ and $V(C) \nsubseteq N_G(v_1)$, then $e(v_1, V(C)) \leqslant 1$. Now suppose $|V(C)| = 2$ and $V(C) \subseteq N_G(v_1)$. Let $V(C) = \{x, y\}$. If $d_G(x) = d_G(y) = 2$, then C is a connected component of $G \setminus \{v_1\}$ and the number of such C is at most t. If $d_G(x) \geqslant 3$ or $d_G(y) \geqslant 3$, $e(\{x, y\}, A_2) \geqslant 1$. This implies that $\{x, y\} \cap S \neq \emptyset$. Thus, by summing over all components C of $G[\overline{A}]$, we have

$$\begin{aligned} e(v_1, \overline{A}) &= \sum_C e(v_1, V(C)) \\ &\leqslant 2|S| + 2t + \frac{2}{3}(n - 2t - |S|) \\ &= \frac{2}{3}(n + t + 2|S|), \\ &\leqslant \frac{2}{3}(n + t) + O(n^{1/2}) \quad \text{(by Claim 3.1)}. \end{aligned}$$

This contradicts (3.5) (as $e(G) \geqslant n$) and completes the proof of Claim 3.4.

Since $e(v, \overline{A}) \geqslant \theta$ for $v \in A_1$, it follows from Claim 3.4 that $|A_1| = 1$. Let $A_1 = \{v\}$. Thus v is the unique vertex of G with degree $\Delta(G)$. Then $e(A, \overline{A}) = e(v, \overline{A}) + \rho$, and $\theta = e(v, \overline{A}) - \rho$.

Next, we give an upper bound on $e(v, \overline{A})$. Let J (respectively, J_1) be the union of all (respectively, all trivial) tight components of $G[\overline{A}]$, and let $J_2 = J \setminus V(J_1)$. By Claim 3.1, $|V(J_1)| = d_0 = O(n^{1/2})$. Hence, by Lemma 2.3,

$$e(v, V(J)) \leqslant |V(J_1)| + \frac{|V(J_2)|}{3} = d_0 + \frac{|V(J_2)|}{3}.$$

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Let L (respectively, $\mathcal{L}$) be the union (respectively, collection) of all non-tight components of $G[\overline{A}]$. Then, by Lemma 2.8,

$$e(v,V(L)) = \sum_{C\in\mathcal{L}} e(v,V(C)) \leqslant \sum_{C\in\mathcal{L}} \left(\frac{1}{2}|C| + d_1(C)\right) = \frac{|V(L)|}{2} + d_1.$$

Combining the above two inequalities, we have

$$e(v,\overline{A}) = e(v,V(J)) + e(v,V(L)) \leqslant \frac{|\overline{A}|}{2} - \frac{|V(J_2)|}{6} + d_0 + d_1. \tag{3.6}$$

We may assume that $\delta(G) = 2$. For, suppose $\delta(G) \geqslant 3$. Then by Claim 3.1, $d_0 + d_1 = O(n^{1/2})$. Recall that τ is the number of tight components of $G[\overline{A}]$. Thus, by Lemma 2.4,

$$\tau \leqslant |V(J_1)| + \frac{|V(J_2)|}{3} = d_0 + \frac{|V(J_2)|}{3}.$$

Hence, by (3.6)

$$\theta + \frac{\tau}{2} \leqslant e(v,\overline{A}) + \frac{\tau}{2} \leqslant \frac{1}{2}n + O(n^{1/2}).$$

Therefore, by (3.3), $e(V_i) \leqslant (1/4+\varepsilon)e(G)$ for $i = 1,2$.

Now that $\delta(G) = 2$, the bound $\theta \leqslant (n-1)/(\delta(G)-1)$ in Claim 3.3 is no longer sufficient, and the bound on $e(v,\overline{A})$ in (3.6) is not enough as we cannot bound d_1. Thus, we introduce a new idea by considering some special triangles containing v. A triangle $vxyv$ in G is of *type I* if $d_G(x) = d_G(y) = 2$, and of *type II* if $d_G(x) = 2$ and $d_G(y) = 3$. Let t_1 (respectively, t_2) denote the number of type I (respectively, type II) triangles containing v. Note that if $vxyv$ is a triangle of type I or type II then $x,y \in V(L)$.

We construct a new graph G' from G by deleting some vertices in L contained in triangles of type I or type II. If $t_1 \geqslant \lfloor \theta/2 \rfloor$ then let $\alpha = \lfloor \theta/2 \rfloor$ and $\beta = 0$; otherwise, let $\alpha = t_1$ and $\beta = \lfloor \theta/2 \rfloor - t_1$. When $\theta \leqslant 2(t_1+t_2)$, we choose α type I triangles and β type II triangles and delete the vertices in those triangles except v. When $\theta > 2(t_1+t_2)$, we delete the vertices except v in all triangles of type I or type II. Let G' denote the resulting graph, and let $B = V(G') \setminus A$.

Claim 3.5. *If G' has a bipartition $V(G') = V_1' \cup V_2'$ such that for $i = 1,2$, $e_{G'}(V_i') \leqslant e(G')/4 + \varepsilon e(G)$, then G has a bipartition $V(G) = V_1 \cup V_2$ such that $|V_1| - |V_2| = |V_1'| - |V_2'|$, and for $i = 1,2$, $V_i' \subseteq V_i$ and $e(V_i) \leqslant (e(G)+t)/4 + \varepsilon e(G)$.*

To prove this claim, let $V(G') = V_1' \cup V_2'$ be a bipartition of G' such that for $i = 1,2$, $e_{G'}(V_i') \leqslant e(G')/4 + \varepsilon e(G)$. We may assume that $v \in V_1'$. We now define a bipartition $V(G) = V_1 \cup V_2$ of G such that $V_i' \subseteq V_i$ for $i = 1,2$. Let $vxyv$ be a triangle with $x,y \notin V(G')$. If $vxyv$ is of type I, then let $x \in V_1$ and $y \in V_2$. If $vxyv$ is of type II we may let $d_G(y) = 3$ and $\{z\} = N_G(y) \setminus \{v,x\}$. If $z \in V_1'$ then let $x \in V_1$ and $y \in V_2$; and otherwise, let $x \in V_2$ and $y \in V_1$. Note that $|V_1| - |V_2| = |V_1'| - |V_2'|$, and each triangle $vxyv$ with $x,y \notin V(G')$ increases $e(V_1')$ by one but has no effect on $e(V_2')$.

Let $a = \alpha$ and $b = \beta$ if $\theta \leqslant 2(t_1+t_2)$, and let $a = t_1$ and $b = t_2$ if $\theta > 2(t_1+t_2)$. Then $e(G) = e(G') + 3a + 4b$. Hence,

$$e(V_1) = e(V_1') + a + b \leqslant \frac{1}{4}e(G') + a + b + \varepsilon e(G) \leqslant \frac{1}{4}(e(G)+a) + \varepsilon e(G),$$

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and

$$e(V_2) = e(V_2') \leqslant \frac{1}{4}e(G) + \varepsilon e(G).$$

Note that $a \leqslant t_1 \leqslant t$. We have $e(V_i) \leqslant (e(G)+t)/4 + \varepsilon e(G)$ for $i = 1,2$, which completes the proof of Claim 3.5.

Let J' (respectively, J_1') be the union of all (respectively, all trivial) tight components of $G'[B]$, let $J_2' = J' - V(J_1')$, and let L' be the union of all non-tight components of $G'[B]$. Note that a tight component of $G[\overline{A}]$ is also a tight component of $G'[B]$, as each vertex deleted from G has degree 1 in $G[L]$ or is adjacent to a degree 1 vertex in $G[L]$. Thus, $V(J_i) \subseteq V(J_i')$ for $i = 1,2$. Let τ' denote the number of tight components of $G'[B]$. Then $\tau' \geqslant \tau$.

If $|G'| < \varepsilon n/4$, then, by Lemma 2.6, G' admits a bipartition $V(G') = V_1' \cup V_2'$ such that for $i = 1,2$, $e_{G'}(V_i') \leqslant e(G')/4 + O(\sqrt{e(G')})$. Thus, by Claim 3.5, G has a bipartition $V(G) = V_1 \cup V_2$ such that $|V_1| - |V_2| = |V_1'| - |V_2'|$ and for $i = 1,2$, $V_i' \subseteq V_i$ and

$$e(V_i) \leqslant \frac{1}{4}(e(G)+t) + \varepsilon e(G)/4.$$

Since $||V_1| - |V_2|| = ||V_1'| - |V_2'|| < |G'| \leqslant \varepsilon n/4$, we may equalize $|V_1|$ and $|V_2|$ by moving at most $\varepsilon n/4$ vertices of degree at most 3. This affects each $e(V_i)$ by most $3\varepsilon n/4 \leqslant 3\varepsilon e(G)/4$, thereby producing the desired bisection.

Suppose that $|G'| \geqslant \varepsilon n/4$. We now show that if $|G'|$ is sufficiently large (by making n large since $|G'| \geqslant \varepsilon n/4$), then G' admits a bisection $V(G') = V_1' \cup V_2'$ such that for $i = 1,2$, $e_{G'}(V_i') \leqslant e(G')/4 + \varepsilon e(G)$; so the assertion of Theorem 1.2 follows from Claim 3.5.

By Lemma 2.1, we only consider the case that $e(G') \leqslant C|G'|$ for some constant C. Note that for any vertex $x \in B$, $d_{G'}(x) \leqslant d_G(x) \leqslant n^{3/4} = O(|G'|^{3/4})$, and $|A| = O(n^{1/4}) = O(|G'|^{1/4})$. Applying Lemma 2.5 to G', we produce a bisection $V(G') = V_1' \cup V_2'$ of G' such that for $i = 1,2$, $A_i \subseteq V_i'$ and

$$e(V_i') \leqslant e_{G'}(A_i) + \frac{e_{G'}(A_i,B)}{2} + \frac{e_{G'}(B)}{4} - \frac{|G'| - \tau'}{8} + \frac{\varepsilon}{3}|G'|. \tag{3.7}$$

Recall that $A_1 = \{v\}$ and $\rho = e(A_2, \overline{A})$. Note that for a triangle $vxyv$ with $x,y \notin V(G')$, if $e(\{x,y\},A_2) \geqslant 1$ then either $x \in S$ or $y \in S$. Hence, by Claim 3.1, we have $e_{G'}(A_2,B) = \rho - O(|G'|^{1/2})$.

We may assume $\theta > 2(t_1+t_2)$. For, suppose $\theta \leqslant 2(t_1+t_2)$. Then, by definition, G' is obtained from G by deleting the vertices x,y from $\lfloor \theta/2 \rfloor$ triangles $vxyv$ of type I or type II. Thus

$$e_{G'}(v,B) = e(v,B) = e(v,\overline{A}) - 2\lfloor \theta/2 \rfloor = (\theta + \rho) - 2\lfloor \theta/2 \rfloor.$$

Hence $e_{G'}(A_1,B) - e_{G'}(A_2,B) = O(|G'|^{1/2})$, which means $e_{G'}(A_i,B) = e_{G'}(A,B)/2 + O(|G'|^{1/2})$ for $i = 1,2$. By (3.7) and the fact $|G'| \geqslant \tau'$, we have

$$e_{G'}(V_i') \leqslant e_{G'}(A_i) + \frac{e_{G'}(A,B) + O(|G'|^{1/2})}{4} + \frac{e_{G'}(B)}{4} + \frac{\varepsilon}{3}|G'| = \frac{e(G')}{4} + \frac{\varepsilon}{3}|G'| + O(|G'|^{1/2}),$$

for $i = 1,2$. Since $|G'| \leqslant n \leqslant e(G)$, $e_{G'}(V_i') \leqslant e(G')/4 + \varepsilon e(G)$, as desired.

Therefore, G' is obtained from G by deleting the vertices x,y for every triangle $vxyv$ of type I or type II. So $e_{G'}(v,B) = e(v,\overline{A}) - 2(t_1+t_2) > \rho$ and G' contains neither type I nor type II triangles containing v as G has no 4-cycle. Since $e_{G'}(A_2,B) = \rho - O(|G'|^{1/2})$ and $|G'| \leqslant n \leqslant e(G)$, it follows from (3.7) that, to prove $e_{G'}(V_i') \leqslant e(G')/4+\varepsilon e(G)$, it suffices to show that

$$e_{G'}(v,B)+\frac{\tau'}{2} \leqslant \frac{|G'|}{2}+\rho+\varepsilon e(G). \tag{3.8}$$

So we need to bound $e_{G'}(v,B)$ and τ'.

First, we show $\tau' \leqslant d_0+\rho+|V(J')|/3$. Recall that $V(J_1)$ is the set of isolated vertices in $G[\overline{A}]$ and $V(J_1')$ is the set of isolated vertices in $G'[B]$. We have $|V(J_1)| = d_0$ and $V(J_1) \subseteq V(J_1')$ by the construction of G'. If $z \in V(J_1') \setminus V(J_1)$, then there exists some triangle $vxyv$ of type II with $d_G(y) = 3$ and $z \in N_G(y) \setminus \{v,x\}$. Thus, $d_{G'}(z) = d_G(z)-1$ and $vz \notin E(G)$ (as G has no 4-cycle). Since $\delta(G) = 2$, $e_{G'}(z,A_2) = e_G(z,A_2) \geqslant 1$. Therefore,

$$|V(J_1')| = |V(J_1)|+|V(J_1') \setminus V(J_1)| \leqslant d_0+e_{G'}(A_2,V(J_1')) = d_0+e(A_2,V(J_1')).$$

Hence, by Lemma 2.3,

$$\tau' \leqslant |V(J_1')|+\frac{|V(J_2')|}{3} \leqslant d_0+e(A_2,V(J_1'))+\frac{|V(J_2')|}{3} \leqslant d_0+\rho+\frac{|V(J')|}{3}. \tag{3.9}$$

Next we bound $e_{G'}(v,B)$ by bounding $e_{G'}(v,V(J'))$ and $e_{G'}(v,V(L'))$. By Lemma 2.2,

$$e_{G'}(v,V(J')) \leqslant |V(J_2')|/3+e_{G'}(v,V(J_1')).$$

Since for any $z \in V(J_1') \setminus V(J_1)$, $vz \notin E(G)$ (as G has no 4-cycle), we have

$$e_{G'}(v,V(J_1')) = e(v,V(J_1)) \leqslant d_0.$$

Hence,

$$e_{G'}(v,V(J')) \leqslant \frac{|V(J_2')|}{3}+d_0 \leqslant \frac{|V(J')|}{3}+d_0. \tag{3.10}$$

We may assume

$$e_{G'}(v,V(L')) > \frac{|V(L')|+|A_2|}{2}+|S|. \tag{3.11}$$

For otherwise, by (3.9) and (3.10), we have

$$\begin{aligned} e_{G'}(v,B)+\frac{\tau'}{2} &= e_{G'}(v,V(J'))+e_{G'}(v,V(L'))+\frac{\tau'}{2} \\ &\leqslant \frac{|V(J')|}{3}+d_0+\frac{|V(L')|+|A_2|}{2}+|S|+\frac{|V(J')|/3+\rho+d_0}{2} \\ &= \frac{|G'|+\rho-1}{2}+|S|+\frac{3}{2}d_0. \end{aligned}$$

By Claim 3.1 and the fact that $e(G) \geqslant n$, it is easy to see that (3.8) holds.

Let L_2' be the union of those components of $G'[B]$ with just two vertices. Then

$$e_{G'}(v,V(L_2')) \leqslant \frac{|V(L_2')|}{2}+|S|. \tag{3.12}$$

To see this, we consider a component C of $G'[B]$ of order two, say xy. Suppose $e_{G'}(v,C) = 2$. Then $vx, vy \in E(G)$, so $d_{G'}(x) = d_G(x)$ and $d_{G'}(y) = d_G(y)$, since G has no 4-cycle. Since $vxyv$ is not a type I or type II triangle in G, $e_{G'}(\{x,y\},A_2) \geqslant 1$, so we have $x \in S$ or $y \in S$. Thus, if s

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denotes the number of components C of $G'[B]$ with $|C| = 2$ and $e_{G'}(v,C) = 2$, then $s \leqslant |S|$ and

$$e_{G'}(v,V(L_2')) \leqslant \frac{|V(L_2')| - 2s}{2} + 2s \leqslant \frac{|V(L_2')|}{2} + |S|.$$

Now we consider components C of $G'[B]$ such that C is not tight and $|C| \geqslant 3$. Let

$$H_C = G'[V(C) \cup (N_{G'}(V(C)) \cap A_2)].$$

Then H_C is connected and $|H_C| \geqslant 3$. Let X_C be the neighbourhood of v in C and $Y_C = N_{H_C}(X_C) \setminus X_C$. We claim that

$$e_{G'}(v,V(C)) \leqslant (|X_C| + |Y_C|)/2. \tag{3.13}$$

Since G has no 4-cycle, $\Delta(H_C[X_C]) \leqslant 1$ and $|Y_C| = e_{H_C}(X_C,Y_C)$. Note that, since H_C is connected, $e_{H_C}(x,Y_C) \geqslant 1$ for any isolated vertex x of $H_C[X_C]$. Now let $x,y \in X_C$ with $xy \in E(H_C)$. Since G has no 4-cycle and $vx, vy \in E(G)$, neither x nor y is adjacent to a vertex in some triangle of type II containing v. Thus $d_G(x) = d_{H_C}(x)$ and $d_G(y) = d_{H_C}(y)$. Therefore, since G has no 4-cycle, $e_{H_C}(\{x,y\},Y_C) = d_G(x) + d_G(y) - 4$. Since $vxyv$ is not of type I or type II, $d_G(x) + d_G(y) \geqslant 6$. So $e_{H_C}(\{x,y\},Y_C) \geqslant 2$. Hence, $e_{G'}(v,V(C)) = |X_C| \leqslant e_{H_C}(X_C,Y_C) = |Y_C|$ (as G has no 4-cycle), and we have (3.13).

Let L'' (respectively, $\mathcal{L}''$) be the union (respectively, collection) of all non-tight components C of $G'[B]$ with $|C| \geqslant 3$. Since G has no 4-cycle, $Y_C \cap Y_{C'} = \emptyset$ for any distinct components C, C' of $G'[B]$ contained in $\mathcal{L}''$. Combining (3.12) and (3.13), we have

$$\begin{aligned} e_{G'}(v,V(L')) &= e_{G'}(v,V(L_2')) + e_{G'}(v,V(L'')) \\ &\leqslant \frac{|V(L_2')|}{2} + |S| + \sum_{C \in \mathcal{L}''} \frac{|X_C| + |Y_C|}{2} \\ &\leqslant \frac{|V(L')| + |A_2|}{2} + |S|. \end{aligned}$$

This contradicts (3.11) and completes the proof. □

4. Max-bisection

In this section, we give a proof of Theorem 1.4 using ideas from Lee, Loh and Sudakov [13] and Xu, Yan and Yu [19]: pair the vertices using a maximum matching, order the pairs appropriately, and separate the pairs to optimize the bisection.

Proof of Theorem 1.4. Let $M = \{v_1w_1, \ldots, v_sw_s\}$ be a maximum matching in G and $W = V(G) \setminus V(M)$. Since M is a maximum matching in G, W is an independent set in G and, for any $x, y \in W$ and $v_iw_i \in M$,

$$\{v_ix, w_iy\} \nsubseteq E(G) \quad \text{and} \quad \{v_iy, w_ix\} \nsubseteq E(G). \tag{4.1}$$

Divide the vertices in W into pairs $\{x_1,y_1\}, \ldots, \{x_r,y_r\}$ and let u be the unique vertex in $W \setminus \{x_1,y_1,\ldots,x_r,y_r\}$ if $|G|$ is odd. Note that $s + r = \lfloor |G|/2 \rfloor$.

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We order the pairs $\{v_i,w_i\}$ and $\{x_j,y_j\}$ into a sequence: arrange the pairs $\{v_1,w_1\},\dots,\{v_s,w_s\}$ in this order. For each j, since $\delta(G)\geqslant 2$ and W is independent in G,

$$e(\{x_j,y_j\},V(M)) = d_G(x_j)+d_G(y_j)\geqslant 4. \tag{4.2}$$

On the other hand, by (4.1), we have, for each v_iw_i, $e(\{v_i,w_i\},\{x_j,y_j\})\leqslant 2$. If there is some $i\in\{1,\dots,s\}$ such that $e(\{v_i,w_i\},\{x_j,y_j\})=1$, then choose the smallest such i, and place $\{x_j,y_j\}$ between $\{v_i,w_i\}$ and $\{v_{i+1},w_{i+1}\}$. Arbitrarily order those pairs $\{x_j,y_j\}$ which are placed between two consecutive pairs $\{v_i,w_i\}$ and $\{v_{i+1},w_{i+1}\}$. Let t denote the number of pairs $\{x_j,y_j\}$ with $e(\{v_i,w_i\},\{x_j,y_j\})=0$ or 2 for all $i\in\{1,\dots,s\}$. We append those t pairs at the end of the sequence in arbitrary order. Let $P_1,\dots,P_l$ denote the final sequence of pairs, where $l=s+r=\lfloor |G|/2\rfloor$.

We claim that $t=0$ if $g(G)\geqslant 5$. For a pair $\{x_j,y_j\}$, if $e(\{v_i,w_i\},\{x_j,y_j\})=2$ for some i, then at most one of $\{v_ix_j,w_ix_j\}$ (respectively, $\{v_iy_j,w_iy_j\}$) is in $E(G)$; hence by (4.1), $v_ix_j,v_iy_j\in E(G)$ or $w_ix_j,w_iy_j\in E(G)$. Therefore, if $t>0$ then there exist pairwise distinct i,i',j such that

$$e(\{v_i,w_i\},\{x_j,y_j\}) = e(\{v_{i'},w_{i'}\},\{x_j,y_j\}) = 2,$$

so G contains a 4-cycle. Hence $t=0$ if $g(G)\geqslant 5$.

We also claim that $t\leqslant s/\delta(G)$ if $\delta(G)$ is even. By (4.1), if $e(\{v_i,w_i\},\{x_j,y_j\})=2$ for some i, then either x_j,y_j have a common neighbourhood in $\{v_i,w_i\}$, or $x_jv_iw_ix_j$ or $y_jv_iw_iy_j$ is a triangle in G. Since G does not contain 4-cycles, x_j,y_j have at most one common neighbourhood. Since $\delta(G)$ is even, there are at least $\delta(G)/2$ pairs $\{v_i,w_i\}$ such that $x_jv_iw_ix_j$ (respectively, $y_jv_iw_iy_j$) is a triangle in G. Since each edge of G is contained in at most one triangle, $((\delta(G))/2)\cdot 2t\leqslant s$, which implies $t\leqslant s/\delta(G)$.

We now form a bisection of G by separating P_i in the order $i=1,\dots,l$ and placing u in the end if $|G|$ is odd. Let $V_0^1=V_0^2=\emptyset$. For each $i\in\{1,\dots,l\}$, we construct V_i^1 and V_i^2 such that

(a) $V_{i-1}^1\subseteq V_i^1$, $V_{i-1}^2\subseteq V_i^2$, $|V_i^1|=|V_{i-1}^1|+1=|V_{i-1}^2|+1=|V_i^2|$ and $|V_i^1\cap P_i|=|V_i^2\cap P_i|=1$, and
(b) subject to (a), $e(V_i^1,V_i^2)$ is maximum.

If $|G|$ is even, let $V_1=V_l^1$ and $V_2=V_l^2$. If $|G|$ is odd, let $V_1=V_l^1\cup\{u\}$ and $V_2=V_l^2$ when $e(u,V_l^1)\leqslant e(u,V_l^2)$; and let $V_1=V_l^1$ and $V_2=V_l^2\cup\{u\}$ when $e(u,V_l^1)>e(u,V_l^2)$. Clearly, $V(G)=V_1\cup V_2$ is a bisection of G.

We bound $e(V_1,V_2)$. Let $G_i=G[V_i^1\cup V_i^2]$, for $i=1,\dots,l$. By (b),

$$e(V_i^1,V_i^2)-e(V_{i-1}^1,V_{i-1}^2)\geqslant (e(G_i)-e(G_{i-1}))/2$$

for $i=1,\dots,l$. Moreover, if $|G|$ is odd, then

$$e(V_1,V_2)-e(V_l^1,V_l^2)\geqslant (e(G)-e(G_l))/2.$$

In fact, for $1\leqslant i\leqslant \lfloor |G|/2\rfloor - t$, either $P_i=\{v_j,w_j\}$ for some j or $P_i=\{x_{j'},y_{j'}\}$ for some j' with $d_{G_i}(x_{j'})+d_{G_i}(y_{j'})$ odd; hence

$$e(V_i^1,V_i^2)-e(V_{i-1}^1,V_{i-1}^2)\geqslant \frac{e(G_i)-e(G_{i-1})+1}{2}.$$

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Therefore, write $k = \lfloor |G|/2 \rfloor - t$,

$$\begin{aligned} e(V_1,V_2) &= \sum_{i=1}^{l}(e(V_i^1,V_i^2) - e(V_{i-1}^1,V_{i-1}^2)) + \frac{e(G)-e(G_l)}{2} \\ &\geqslant \frac{1}{2}\sum_{i=1}^{k}(e(G_i)-e(G_{i-1})+1) + \frac{1}{2}\sum_{i=k+1}^{l}(e(G_i)-e(G_{i-1})) + \frac{e(G)-e(G_l)}{2} \\ &= \frac{1}{2}(e(G)+\lfloor |G|/2 \rfloor - t). \end{aligned}$$

If $g(G) \geqslant 5$ then $t = 0$, so $e(V_1,V_2) \geqslant e(G)/2 + (|G|-1)/4$. If $\delta(G)$ is even then $t \leqslant s/\delta(G)$; so $e(V_1,V_2) \geqslant e(G)/2 + (|G|-1)/4 - s/(2\delta(G))$. Thus $V(G) = V_1 \cup V_2$ is the desired bisection of G. □

5. Concluding remarks

In [2], Alon, Bollobás, Krivelevich and Sudakov studied maximum bipartitions of graphs without short cycles and proved that there exists a constant $c > 0$ such that every graph G with m edges and $g(G) \geqslant 5$ has a bipartition of size $m/2 + cm^{5/6}$. They also showed that there exists a constant c' such that, for infinitely many m, there exists a graph G with m edges and $g(G) \geqslant 5$ whose Max-Cut has size at most $m/2 + c'm^{5/6}$. It would be interesting to see if similar results hold for bisections. Combining Theorem 1.4 and Lemma 2.7, it is easy to see that there exists a constant $c > 0$ such that every graph G with m edges, $\delta(G) \geqslant 2$ and $g(G) \geqslant 5$ has a bisection of size at least $m/2 + cm^{2/3}$.

Let G be a graph with $\delta(G) \geqslant 2$ and without 4-cycles. Our proof of Theorem 1.2 shows that G admits a bisection $V(G) = V_1 \cup V_2$ such that

$$e(V_1) \leqslant (1/4 + o(1))e(G) + t/4 \quad \text{and} \quad e(V_2) \leqslant (1/4 + o(1))e(G).$$

Certain triangles incident with a vertex of degree $\Delta(G)$ play a role. If we delete those triangles, the resulting graph has an optimal judicious bisection. On the other hand, the lower bound by Alon, Bollobás, Krivelevich and Sudakov shows that the bound in Theorem 1.2 is asymptotically tight for infinitely many m: there exists a graph G with m edges and $g(G) \geqslant 5$ such that for any bisection $V(G) = V_1 \cup V_2$ of G,

$$\max\{e(V_1), e(V_2)\} > m/4 - c'm^{5/6}.$$

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References

[1] Alon, N. (1996) Bipartite subgraphs. *Combinatorica* **16** 301–311.

[2] Alon, N., Bollobás, B., Krivelevich, M. and Sudakov, B. (2003) Maximum cuts and judicious partitions in graphs without short cycles. *J. Combin. Theory Ser. B* **88** 329–346.

[3] Bollobás, B. and Scott, A. D. (1997) Judicious partitions of hypergraphs. *J. Combin. Theory Ser. A* **78** 15–31.

[4] Bollobás, B. and Scott, A. D. (1999) Exact bounds for judicious partitions of graphs. *Combinatorica* **19** 473–486.
[5] Bollobás, B. and Scott, A. D. (2002) Problems and results on judicious partitions. *Random Struct. Alg.* **21** 414–430.
[6] Bollobás, B. and Scott, A. D. (2002) Better bounds for Max Cut. In *Contemporary Combinatorics*, Vol. 10 of Bolyai Society Mathematical Studies, pp. 185–246.
[7] Bondy, A. and Simonovits, M. (1974) Cycles of even length in graphs. *J. Combin. Theory Ser. B* **16** 97–105.
[8] Edwards, C. S. (1973) Some extremal properties of bipartite graphs. *Canad. J. Math.* **3** 475–485.
[9] Edwards, C. S. (1975) An improved lower bound for the number of edges in a largest bipartite subgraph. In *Proc. 2nd Czechoslovak Symposium on Graph Theory*, pp. 167–181.
[10] Erdős, P., Gyárfás, A. and Kohayakawa, Y. (1997) The size of the largest bipartite subgraphs. *Discrete Math.* **177** 267–271.
[11] Fan, G. and Hou, J. (2017) Bounds for pairs in judicious partitioning of graphs. *Random Struct. Alg.* **50** 59–70.
[12] Hou, J. and Zeng, Q. (2017) Judicious partitioning of hypergraphs with edges of size at most 2. *Combin. Probab. Comput.* **26** 267–284.
[13] Lee, C., Loh, P. and Sudakov, B. (2013) Bisections of graphs. *J. Combin. Theory Ser. B* **103** 590–629.
[14] Lu, C., Wang, K. and Yu, X. (2015) On tight components and anti-tight components. *Graphs Combin.* **31** 2293–2297.
[15] Ma, J., Yen, P. and Yu, X. (2010) On several partitioning problems of Bollobás and Scott. *J. Combin. Theory Ser. B* **100** 631–649.
[16] Ma, J. and Yu, X. (2016) On judicious bipartitions of graphs. *Combinatorica* **36** 537–556.
[17] Poljak, S. and Tuza, Z. (1994) Bipartite subgraphs of triangle-free graphs. *SIAM J. Discrete Math.* **7** 307–313.
[18] Scott, A. D. (2005) Judicious partitions and related problems. In *Surveys in Combinatorics*, Vol. 327 of London Mathematical Society Lecture Note Series, Cambridge University Press, pp. 95–117.
[19] Xu, B., Yan, J. and Yu, X. (2010) Balanced judicious partitions of graphs. *J. Graph Theory* **63** 210–225.
[20] Xu, B., Yan, J. and Yu, X. (2010) A note on balanced bipartitions. *Discrete Math.* **310** 2613–2617.
[21] Xu, B. and Yu, X. (2008) Triangle-free subcubic graphs with minimum bipartite density. *J. Combin. Theory Ser. B* **98** 516–537.
[22] Xu, B. and Yu, X. (2009) Judicious k-partition of graphs. *J. Combin. Theory Ser. B* **99** 324–337.
[23] Xu, B. and Yu, X. (2011) Better bounds for k-partitions of graphs. *Combin. Probab. Comput.* **20** 631–640.
[24] Xu, B. and Yu, X. (2014) On judicious bisection of graphs. *J. Combin. Theory Ser. B* **106** 30–69.
[25] Zhu, X. (2009) Bipartite density of triangle-free subcubic graphs. *Discrete Appl. Math.* **157** 710–714.