On some resistance properties of a certain network containing inductances and capacities and their analogies in a vibrating mechanical system. By Mr E. B. MOULLIN, Downing College. (Communicated by Dr G. F. C. SEARLE.)

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(1) *Introduction.*

In the network shown diagrammatically in Fig. 1, A_0A_3, A_3A_5, ... are resistances of values a_1, a_3, ... joined in series with one another, and A_3B_3, A_5B_5, ... are resistances of values $1/a_2$, $1/a_4$, ...: the points B_0B_3, ... are all on a cable of negligible resistance. The members A_0A_3, A_3A_5, ... will be called the series members of the network, and the members A_3B_3, A_5B_5, ... will be called the shunt members of the network: the points A_0B_0, will be called the input terminals of the network. If a potential difference is maintained between the input terminals, currents will flow in the members of the network.

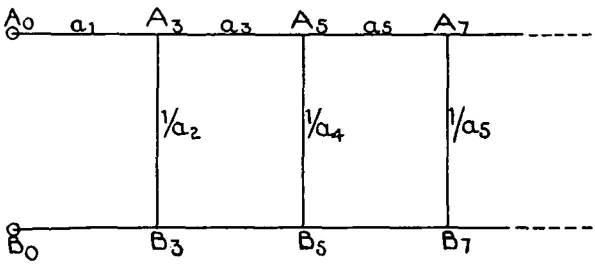

Fig. 1.

The more general network of which the system of Fig. 1 is a degenerate form is important in telegraphy and telephony; its properties depend on the character of the various members. In the degenerate case each member is a simple resistance, and then a_1, a_2, ... may represent portions of an overhead line, while $1/a_2$, $1/a_4$, ... represent leaks at the points of support. The special case when $a_1 = a_3$ ... and $1/a_2 = 1/a_4$ ... has been examined previously. (See Dr G. F. C. Searle, F.R.S., *Proc. Camb. Phil. Soc.* 1915, Vol. XVIII, p. 111, 'Calculations of the electrical resistance of a certain network of conductors.')

The resistance Z_0, measured between the input terminals of the network of Fig. 1 may be represented by the continued fraction

$$Z_0 = a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3 +}\,\frac{1}{a_4 +}\cdots\frac{1}{a_r +}\,\frac{1}{a_{r+1} +}\cdots\frac{1}{a_n} \quad \text{......(1).}$$

We shall call Z_0 the input resistance of the network.

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In an example of more general interest the branches of the network contain inductances and capacities, and an alternating potential difference $v = V \sin pt$ is maintained between the input terminals.

If a current $i = I \sin (pt + \alpha)$ then enters the network along A_0A_3 and leaves it along B_0B_3, we term the ratio V/I the impedance of the network; this quantity replaces the input resistance of the degenerate system. Important examples of the more general network are the "filter circuits" used in telephony. In these circuits the receiving apparatus is the final shunt member of the network and each series member combined with the succeeding shunt member is called a stage; thus a filter has as many stages as it has shunt members. The voltage between the input terminals is often not simple harmonic and it may be undesirable that currents of certain frequencies should flow through the receiver: a filter system is then used to reduce these undesired components of current to a negligible amount and the degree of elimination depends on the number of stages in the filter.

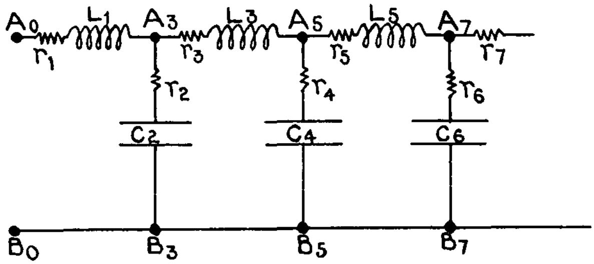

Fig. 2.

The circuit of Fig. 2 is called the "low pass" filter and is used to exclude from the receiver (for example A_7B_7) currents whose frequencies are greater than some predetermined value. If we transpose the inductances and the capacities, as in Fig. 3, we obtain a "high pass" filter which excludes from the receiver currents whose frequencies are less than a certain value. If the individual members of the network contain both inductance and capacity we may arrange to obtain a "band pass" filter or a "band stop" filter which permits or excludes respectively only currents whose frequencies lie within certain specified limits.

In connexion with these circuits, certain impedance problems arise which deserve attention in general. Thus if Z_0 is the input impedance of the network when the alternator is connected between the input terminals we may wish to know what the impedance will be at the same frequency if the input terminals are short circuited and the alternator is inserted in series with some other shunt or series member. Again, since both inductance and capacity are present, resonance will be possible and we may enquire under what conditions resonance will occur. Further, if resonance occurs at a given frequency when the alternator is between A_0 and B_0, will resonance necessarily occur at the same frequency if $A_0 B_0$ are joined and the alternator is inserted in some other member of the network? Also if resonance does occur again when the alternator is transferred to some other particular member, would it necessarily have occurred if the alternator had been transferred to any one arbitrarily selected member?

In either of the networks of Figs. 2 and 3, two forms of resonance are possible. In one case the system behaves as an "acceptor" circuit and the current it receives is a maximum and the impedance a minimum. In the other case the system behaves as a "rejector" circuit and the current is a minimum and the impedance a maximum. In the case in which the system is a rejector circuit to a given alternator placed in one member of the network, it is important to know if it will also be a rejector circuit when the

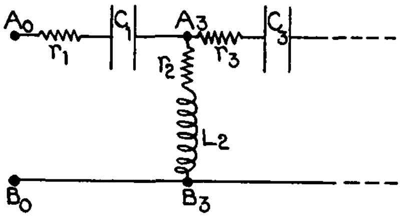

Fig. 3.

same alternator is placed in any one other member, or if it will then become an acceptor circuit or if it will fail to exhibit resonance.

Similar problems arise in mechanical vibrations of certain systems of interconnected masses, for the equations of the currents in the network of Fig. 2 are the same as the equations of motion for a system of fly-wheels fixed on a light elastic shaft turning freely in bearings, or of a suspended system of heavy particles connected together by a series of light vertical springs.

The torsional problem is of practical interest in the investigation of torsional oscillations of the crank shaft masses of a multi-cylinder engine, and here one of our problems is as follows:—If any one of the cranks is actuated by pressure behind a light piston and resonance of torsional oscillations is produced, will resonance be produced if the same piston actuates any other crank and what will be the result of actuating every crank simultaneously by a similar piston?

(2) *Expression for the impedance of the network.*

We may still use the continued fraction (1) to express the impedance of the general network, typified diagrammatically by Fig. 1 and exemplified in Figs. 2 and 3, if we specify the meanings to be attributed to the quantities a_1, a_2, ... The network is actuated by a periodic voltage $v = V\epsilon^{jpt}$ applied between A_0 and B_0, and in the final steady state, which we are here considering, the currents in the various branches will also be proportional to ϵ^{jpt}. We may therefore replace $\dfrac{d}{dt}$ by jp and $\dfrac{d^2}{dt^2}$ by $-p^2$: if this is done, we find that A_0A_3 in Fig. 3 is represented by the complex quantity $(r_1 + jpL_1)$. We may express the same fact more directly as follows:—Every current can be analysed into a Fourier series of simple harmonic terms and therefore every component may be represented instantaneously by the projection of a uniformly rotating line of constant length. Now the voltage across A_0A_3 consists of a component ri in phase with the current and a component pL_1i in phase quadrature with the current. So we may represent the voltage A_0A_3 by another uniformly rotating line which is related to the current line by the vector equation $V = I(r_1 + jpL_1)$. Hence, in accordance with the well-known methods of making alternating current calculations, we may replace the a's by complex quantities such that $a_1 = \{r_1 + jpL_1\}$ and $1/a_2 = \{r_2 - j/(pC_2)\}$, where $j \equiv \sqrt{-1}$.

We may now proceed to use the properties of continued fractions; the reader will note that all the quotients are in general complex quantities.

Let p_r/q_r be the rth convergent of fraction (1). Then

$$Z_0 = \frac{Y_r p_{r-1} + p_{r-2}}{Y_r q_{r-1} + q_{r-2}} = \frac{p_{r-1} + p_{r-2}/Y_r}{q_{r-1} + q_{r-2}/Y_r} \quad \text{.............(2)},$$

where $$Y_r = a_r + \frac{1}{a_{r+1}+}\,\frac{1}{a_{r+2}+}\cdots\frac{1}{a_n}.$$

Now let A_0 and B_0 be joined and let the network be opened at A_r and let an alternator be inserted between A_r and A_r' as indicated in Fig. 4. We may note that the quantity Y_r defined above is the impedance of the part of the network to the right of the points A_r' and B_r'. Let X_r be the impedance of the network to the left of A_rB_r. Then according to the arrangement of Fig. 4 we have

$$X_r = \frac{1}{a_{r-1}+}\,\frac{1}{a_{r-2}+}\,\frac{1}{a_{r-3}+}\cdots\frac{1}{a_1}.$$

When the alternator is placed between the points A_r and A_r' the circuit consists of an impedance Y_r to the right of A_r' which

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is in series with the impedance X_r to the left of A_r. Accordingly the impedance Z_r, between the points A_r and A_r' is given by the expression

$$Z_r = (X_r + Y_r) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

We wish to relate Z_0 and Z_r. It follows readily from the theory of continued fractions that

$$\frac{p_{r-1}}{p_{r-2}} = a_{r-1} + \frac{1}{a_{r-2} +} \, \frac{1}{a_{r-3} +} \cdots \frac{1}{a_1} = \frac{1}{X_r} \quad \ldots\ldots\ldots(4),$$

$$\frac{q_{r-1}}{q_{r-2}} = a_{r-1} + \frac{1}{a_{r-2} +} \, \frac{1}{a_{r-3} +} \cdots \frac{1}{a_2} \ldots\ldots\ldots\ldots\ldots\ldots(5),$$

and

$$\frac{p_{r-1}}{p_{r-2}} - \frac{q_{r-1}}{q_{r-2}} = \frac{(-1)^{r-1}}{p_{r-2}q_{r-2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

In our notation r is an odd integer. Hence by (4) and (6) we have

$$\frac{q_{r-1}}{q_{r-2}} = \frac{1}{X_r} - \frac{1}{p_{r-2}q_{r-2}}.$$

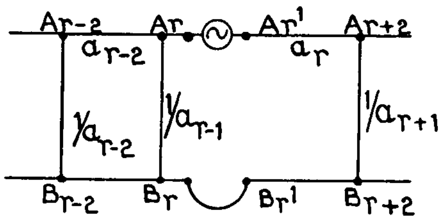

Fig. 4.

On substituting in (2), we have by (3) that

$$Z_0 = \frac{p_{r-2}\left(\frac{1}{X_r} + \frac{1}{Y_r}\right)}{q_{r-2}\left(\frac{1}{X_r} + \frac{1}{Y_r}\right) - \frac{1}{p_{r-2}}} = \frac{p_{r-2}^2 Z_r}{p_{r-2}q_{r-2}Z_r - X_r Y_r} \quad \ldots(7).$$

Equation (7) connects Z_0 with Z_r, X_r, Y_r, p_{r-2}, q_{r-2}.

(3) *Resonance conditions.*

We will now find the conditions for resonance in a network whose members are subject to the following restrictions.

(α) Every member is resistanceless, so that at resonance the impedance is zero or infinite according as the circuit is an acceptor or a rejector.

(β) No one of the impedances $a_1, a_3, a_5, \ldots$ is infinite and no one of the impedances $1/a_2, 1/a_4, 1/a_6, \ldots$ is zero. This means that no series member may contain inductance and capacity in parallel and no shunt member may contain inductance and capacity in series.

Since p_r and q_r etc. consist of various products of the a's no p or q can be infinite on account of restriction (β).

(i) Acceptor circuit from input terminals.

We will first suppose that neither X_r nor Y_r is infinite. Then, from (7) Z_0 will be zero if $p_{r-2}=0$, or if $Z_r=0$, or if both p_{r-2} and Z_r are zero, provided that the denominator of (7) is not zero.

If $Z_r=0$, the system is an acceptor circuit, with A_0B_0 joined, to an alternator placed between A_r and A_r' and by (7) it is then also an acceptor circuit to an alternator placed between A_0 and B_0. Thus we have proved that the network can be an acceptor circuit when it is energised by an alternator either situated in series with the rth series member or connected between A_0 and B_0.

Now we see from (6) that $p_{r-1}q_{r-2}-p_{r-2}q_{r-1}=1$, since r is odd. Hence, if p_{r-2} is zero, then p_{r-1} is not zero, since no p or q is infinite.

But, by (4), we have $p_{r-1}/p_{r-2}=1/X_r$ and hence, if $p_{r-2}=0$, then $X_r=0$ also.

But we have seen from (7) that Z_0 is zero if $p_{r-2}=0$ and we see from (2) that this entails that $Y_r=0$, since neither p_{r-1} nor q_{r-2} can be zero when $p_{r-2}=0$; accordingly $X_r=0$ and $Y_r=0$ and $Z_r=(X_r+Y_r)=0$. Hence the necessary and sufficient condition for the case where X_r and Y_r are finite is that Z_0 should be zero. Thus we have proved that if the network is an acceptor circuit to an alternator placed in series with a certain series member then it must be an acceptor circuit to an alternator placed between the input terminals.

If one and one only of X_r and Y_r is infinite then $Z_r=X_r+Y_r$ will certainly be infinite; is it then possible that Z_0 should be zero?

We see from (2) that if Y_r is infinite, then Z_0 is zero only if p_{r-1} is zero. But we see from (4) that if p_{r-1} is zero then X_r is infinite. Hence we find that Z_0 is zero if $X_r+Y_r=0$ or if both X_r and Y_r are infinite*. In the first case the system is an acceptor from both positions of the driving alternator and in the second condition it is an acceptor from one place and a rejector from the other.

* Though both X_r and Y_r become infinite at a certain frequency, it may be shown that their sum cannot be zero. Both X_r and Y_r consist of two parallel branches, one whose impedance is positive and increasing and the other whose impedance is negative and decreasing for a steady increase of frequency: hence the rate of change of each branch impedance is positive. From this it follows that X_r/Y_r cannot have the value -1.

We have now proved that if the network is an acceptor circuit from the input terminals, then the same alternator will produce resonance if it is inserted in any series member whatsoever provided the input terminals have been short-circuited. There will always be resonance but not necessarily acceptor resonance. There will be rejector resonance if, and only if, the alternator is placed in a member which previously carried no current: that is to say only if it is situated in a member which carried no current, and so may be called a current node, when the alternator was feeding the input terminals. Conversely we see that if an alternator at the input terminals causes a certain member to be a current node, then this member will remain a current node if the alternator is moved to it.

(ii) Rejector circuit from input terminals.

We will now examine the condition that the network should be a rejector circuit from the input terminals, in which condition Z_0 must be infinite. Remembering that no p or q is infinite we see by equation (2) that Z_0 is infinite only if

$$Y_r = -q_{r-2}/q_{r-1} \qquad \text{(8)}.$$

Now both Z_r and Z_0 can be infinite only if either X_r or Y_r is infinite: if both are infinite we have seen above that Z_0 is zero. Now if q_{r-1} is zero, neither p_{r-1} nor q_{r-2} can be zero. Therefore both Z_0 and Z_r will be infinite if Y_r is infinite and q_{r-1} is zero simultaneously: also they will both be infinite if X_r is infinite and Y_r satisfies condition (8). We may note that if X_r is infinite q_{r-2} cannot be zero and so $Y_r = 0$ cannot satisfy (8), but that if Y_r is infinite the condition that $q_{r-1} = 0$ does not preclude the possibility that X_r should be zero.

So we find that if the network is a rejector circuit from the input terminals it will not be a rejector circuit when energised from some series member unless certain special conditions are satisfied.

(4) *Network excited from a shunt member.*

We will now consider briefly the conditions obtaining when the alternator is placed in a shunt member as shown in Fig. 5. Let X_r' be the impedance of the part of the network to the left of A_rB_r (excluding the shunt member A_rB_r). Then the impedance Z_r' presented to the alternator, consists of the impedance X_r' in parallel with the impedance Y_r, and these two together are in series with the impedance $1/a_{r-1}$. Hence

$$Z_r' = \frac{X_r' Y_r}{X_r' + Y_r} + \frac{1}{a_{r-1}}.$$

Now
$$X_r' = a_{r-2} + \frac{1}{a_{r-3}+}\,\frac{1}{a_{r-4}+}\cdots\frac{1}{a_1},$$

and then
$$X_r = \frac{1}{a_{r-1} + 1/X_r'}.$$

Hence
$$X_r' = -\frac{X_r}{(X_r a_{r-1} - 1)} \quad\text{.....................(9).}$$

Now
$$Z_r' = \frac{X_r'(Y_r a_{r-1} + 1) + Y_r}{a_{r-1}(X_r' + Y_r)} = \frac{a_{r-1} + 1/Y_r + 1/X_r'}{a_{r-1}(1/Y_r + 1/X_r')} \quad\text{...(9)}a,$$

whence, using (9) we have
$$Z_r' = \frac{1/X_r + 1/Y_r}{a_{r-1}(1/Y_r + 1/X_r - a_{r-1})} \quad\text{..............(10).}$$

We see from (10) that $Z_r' = 0$ if $X_r + Y_r = 0$, or if $X_r + Y_r$ are both infinite. But we saw previously that $Z_0 = 0$ in both these circumstances. If therefore the circuit is an acceptor circuit from

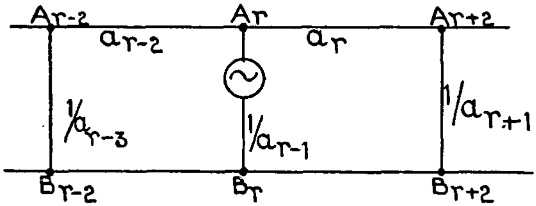

Fig. 5.

the input terminals it will always be an acceptor circuit to an alternator placed in any shunt member, provided only the impedance of that particular member is not infinite.

The conditions that the circuit of Fig. 5 should be a rejector circuit are readily followed from (9)a. In order that the impedances Z_0 and Z_r' should be infinite simultaneously, very special selection of the members of the network may be necessary.

If we remove restriction (β) we can examine the resonance conditions only when we have specified which of the a's have become infinite.

(5) *Illustration.*

To illustrate the various resonance conditions just developed let us consider the circuit depicted in Fig. 6, which represents a five-stage filter in which all the inductances are equal to L and all the capacities are equal to C and all the resistances are zero. If we write $1/LC = p_0^2$, and make suitable substitutions in (1) we can show that the system is an acceptor circuit to an alternator placed between A_0 and B_0 when p^2/p_0^2 is a root of the equation

$$y^5-10y^4+36y^3-56y^2+35y-6=0,$$

and the roots of this are

$$y=0{\cdot}27,\ 1,\ 2,\ 3,\ 3{\cdot}73.$$

Acceptor condition for $p^2/p_0^2=1$.

That $p=p_0$ is a possible condition is readily seen, for at this frequency the portion to the right of A_9B_9 has infinite impedance and its presence is not felt by C_4. But L_4C_4 is an acceptor circuit and therefore is a complete short circuit across C_3 which will therefore carry no current. Hence A_7 is in effect joined to B_7 and hence the portion to the right of A_3 has infinite impedance and in effect we are left with only L_1 and C_1 in series. The current will be large everywhere except in L_2, L_5 and C_3 where it is zero.

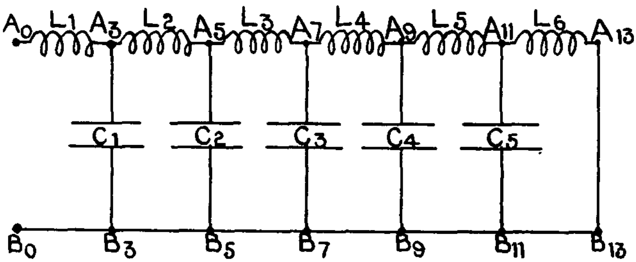

Fig. 6.

Let us now join A_0B_0 and place an alternator in series with L_4. The portion to the right of A_7 has zero impedance and thus $Y=0$. The portion to the left of C_3 reduces to a zero impedance shunting C_3 and hence $X=0$ and the condition $X+Y=0$ is satisfied.

If we now remove the alternator from L_4 and place it in series with L_5, we see readily that Y is infinite. But we have just seen that C_3 is short-circuited by the system to its left and so in effect A_7 is joined to B_7. Consequently X consists of L_4 in parallel with C_4 and this circuit has infinite impedance. Consequently the condition that both X and Y are infinite is satisfied and we have an acceptor circuit.

Acceptor condition for $p^2/p_0^2=2$.

Now consider the condition when $p^2=2/LC$, corresponding to the root $p^2/p_0^2=2$: then each inductance has an impedance $pL\sqrt{2}$ and each condenser has an impedance $1/(pC\sqrt{2})$. If we denote the inductive impedance by $2R$, then the condensive impedance will equal $-R$. Then C_5 and L_6 in parallel evidently have impedance $-2R$ and this in series with L_5, whose impedance is $2R$, will make a short circuit across C_4. Following the same argument, we see that C_2 is short-circuited and that the initial part of the system (L_1, C_1, L_2) has zero impedance and thus the current will be large everywhere except in C_2 and C_4 where it is zero.

Now let us insert an alternator at A_7 in series with L_4. We have just seen that C_4 is short-circuited by the acceptor circuit beyond it and so Y reduces in effect to L_4 alone, and this impedance we have called $2R$. We now consider the circuit to the left of A_7B_7, and see that C_2 is short-circuited by the acceptor circuit beyond it. Therefore in effect X consists of L_3 in parallel with C_3 and evidently the impedance of this is $-2R$. Therefore we find $X = -2R$, and $Y = 2R$, and so the condition $X + Y = 0$ is satisfied though neither X nor Y is zero: the system is an acceptor circuit whether the alternator is at A_0 or A_7.

Rejector conditions with $p^2/p_0^2 = 1$.

To consider the rejector circuit conditions we will suppose that in Fig. 6, $A_{11}B_{11}$ are joined by a thick conductor. We then have a system which is a rejector circuit to an alternator placed between A_0 and B_0, when $p = p_0$. For the circuit beyond A_7B_7 evidently has infinite impedance and does not affect C_3: again, C_2 is short-circuited by L_3 in series with C_3 and evidently the remaining circuit has infinite impedance. Now let us insert the alternator at A_7 (and remember that $A_{11}B_{11}$ are permanently joined). Inspection shows that Y is infinite and that X is zero. But we saw that if Y is infinite then q_{r-1} must be zero if Z_0 is to be infinite. Now consider the circuit from A_0B_0 to A_7B_7 and suppose the remainder removed. Then C_2 is short-circuited by L_3, C_3 and so the impedance between A_0 and B_0 is infinite: this means (since $r = 7$) that q_6 is zero and so the necessary condition is satisfied. We may note that the system up to A_7B_7 is a rejector circuit from one end and an acceptor circuit from the other. If the alternator is inserted at A_5, inspection will show that $Y = 0$ and that $X = 1/(p_0C)$: but if $Y = 0$ the condition that Z_0 should be infinite is that $q_{r-2} = 0$. Consequently (since now $r = 5$) q_3 should be zero, and inspection shows that this is so.

(6) *Analogy with torsional oscillations.*

It has been stated previously that the equations of motion for a system of fly-wheels fixed to an elastic shaft are the same as the equations for the currents in the network of Fig. 2. If the comparison is made, it will be found that the twist in any portion of the shaft corresponds to the current in the corresponding condenser and that the displacement of any fly-wheel corresponds to the current in the corresponding inductance. The five-stage filter illustrated in Fig. 6 is analogous to six equal fly-wheels fixed at even distances along a light uniform shaft which is mounted freely in bearings. This is illustrated in Fig. 7.

When considering Fig. 6 we saw that when $p = p_0$, the current in C_3 was zero and likewise the currents in L_2 and L_5, and that the current in L_3 equals the current in L_4 in both magnitude and phase. Now the current in L_6 lags a quarter period behind the P.D. between A_{11} and B_{11}, and hence behind the P.D. between A_9 and B_9: but the current through C_4 leads this P.D. by a quarter period and so the currents in L_4 and L_6 are in antiphase with one another. Similarly the currents in L_1 and L_3 are in antiphase.

Now let us translate these results to the system of Fig. 7 which we shall suppose to have been set in motion by applying an alternating couple, of frequency $p_0 = 1\Big/\sqrt{\frac{I_1}{\lambda_1}}$ for a short time only to the wheel of moment of inertia I_1, fixed to a shaft of strength λ_1. If the couple is then removed, each wheel of this frictionless

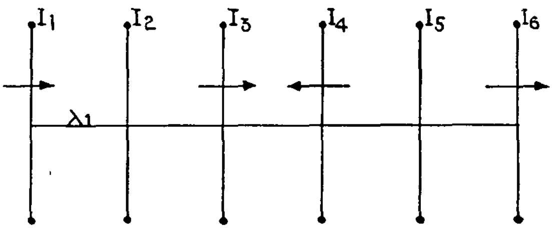

Fig. 7.

system will continue to oscillate at the frequency p_0. The system is exactly analogous to Fig. 6 and we can find the mode of vibration from the information we now have about the currents in Fig. 6.

Thus, since the current in C_3 is zero the angle of twist of the middle section of shaft must also be zero and therefore I_3 and I_4 must oscillate always as one body. Again the current in L_2 and L_5 is zero and therefore I_2 and I_5 should be at rest. Further the currents in L_1 and L_6 were in phase with one another and in antiphase to the current in L_3 and L_4, hence I_1 and I_6 should oscillate in phase with one another but in antiphase with I_3 and I_4. Arrowheads are marked in Fig. 7 to suggest this condition of motion. Evidently this motion is possible, for it gives zero angular momentum to the system and could have been initiated, for example, by twisting I_1 and I_6 through a clockwise angle θ and I_3 with I_4 through an anticlockwise angle θ and then releasing the system. We should then have the familiar two node vibration with nodes at I_2 and I_5.

Now consider the case when $p^2 = 2p_0^2$ in Fig. 6. The current through C_5 is twice as large as that through L_6 and consequently the currents through L_5 and L_6 must be in antiphase with one another. The current in C_4 is zero so that the current in L_4 and L_5 is the same in magnitude and phase and equal to half the current through C_3. Proceeding to the beginning of the circuit we find the currents are arranged symmetrically about the middle member. Now translating this to Fig. 7 we find I_2 and I_3 oscillating as one body, likewise I_4 and I_5. The displacement of I_2, I_3 and I_6 are always identical and in antiphase with I_1, I_4 and I_5. Again the motion is a possible one for a free system as the total angular momentum is zero: it is a three node vibration with nodes at the middle of the first, third and fifth section of the shaft.

Lastly consider the case when $p^2 = 3p_0^2$ in Fig. 6. The current in L_5 is twice the current in L_6 and is in antiphase with it whilst the current in C_3 is zero. If we proceed in a similar manner to consider all the members of the network, we find the currents distributed symmetrically on either side of the middle condenser. Translating this to Fig. 7, we find I_3 and I_4 moving as one body and with amplitude θ and that I_1 and I_6 move in phase with them also with amplitude θ. But I_2 and I_5 move with amplitude 2θ, and in antiphase with the other fly-wheels. This is again a possible condition for a free motion and gives the four node vibration with nodes one-third the way along the first and fifth section of the shaft (each distance measured from the terminal mass) and again at two-thirds along the second and fourth section (distances measured from the second and from the fifth mass). The single node and the five node vibrations are followed less readily because of the non-integral values of p^2.