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ON THE MEAN INACTIVITY TIME ORDERING WITH RELIABILITY APPLICATIONS

M. KAYID
Department of Mathematics
Faculty of Education (Suez)
Suez Canal University, Suez, Egypt

I. A. AHMAD
Department of Statistics and Actuarial Science
University of Central Florida
Orlando, FL 32816-2370
E-mail: iahmad@mail.ucf.edu

The purpose of this article is to study several preservation properties of stochastic comparisons based on the mean inactivity time order under the reliability operations of convolution and mixture. Characterizations and relationships with the other well-known orders are given. Some examples of interest in reliability theory are also presented. Finally, testing in the increasing mean inactivity time class is discussed.

1. INTRODUCTION

During the past several decades, various concepts of stochastic comparisons between random variables have been defined and studied in the literature, because they are useful in modeling for reliability and economics applications and as mathematical tools for proving important results in applied probability (see Shaked and Shanthikumar [20] for an exhaustive monograph on this topic).

For any life variable $X \geq 0$, the residual life variable $X_t = [X - t | X \geq t]$, where $t \in (0, l_X)$ and $l_X = \sup\{t : F_X(t) < 1\}$, is a nonnegative random variable representing

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the remaining life of X at age t. Hence, if $F(\cdot)$ is the distribution function of X and $\bar{F}(\cdot) \equiv 1 - F(\cdot)$ is its survival function, then the survival function of X_t is given by

$$\bar{F}_{X_t}(x) = \frac{\bar{F}(x+t)}{\bar{F}(t)}, \qquad x \geq 0, t \geq 0.$$

Given two random variables X and Y, X is said to be smaller than Y in the hazard rate order (denote by $X \leq_{\text{HR}} Y$) if

$$X_t \leq_{\text{st}} Y_t \quad \text{for all } t,$$

where the stochastic ordering ($\leq_{\text{st}}$) means that $\bar{F}_{X_t}(t) \leq \bar{F}_{Y_t}(t)$ for all t.

However, it is reasonable to presume that in many realistic situations, the random life variable is not necessarily related to the future but can also refer to the past. For instance, consider a system whose state is observed only at certain preassigned inspection times. If at time t the system is inspected for the first time and it is found to be "down," then the failure relies on the past (i.e., on which instant in $(0, t)$ it has failed). It thus seems natural to study a notion that is dual to the residual life, in the sense that it refers to past time and not to future (see Di Crescenzo and Longobardi [9]).

For any random variable X, let

$$X_{(t)} = [t - X | X < t], \qquad t \in \{x : F_X(x) < 0\},$$

denote a random variable whose distribution is the same as the conditional distribution of $t - X$ given that $X < t$. When the random variable X denotes the lifetime ($X \geq 0$, with probability 1) of a unit, $X_{(t)}$ is known as the inactivity time or reversed residual life (see, for instance, Chandra and Roy [8], Block, Savits, and Singh [7], Li and Lu [14], and Nanda, Singh, Misra, and Paul [17]).

Now we recall the definition of the mean inactivity time order ($\leq_{\text{MIT}}$), the increasing concave order ($\leq_{\text{ICV}}$), and the reversed hazard rate order ($\leq_{\text{RHR}}$).

Definition 1.1: *Let X and Y be two nonnegative random variables with absolutely continuous distribution functions F and G and densities f and g, respectively. X is said to be smaller than Y in the following:*

(i) the mean inactivity time order (denoted by $X \leq_{\text{MIT}} Y$) if

$$E[t - X | X < t] \geq E[t - Y | Y < t] \quad \textit{for all } t \in R^+$$

(ii) the increasing concave order (denoted by $X \leq_{\text{ICV}} Y$) if

$$\int_0^x F(u)\,du \geq \int_0^x G(u)\,du \quad \textit{for all } x \tag{1.1}$$

(iii) the reversed hazard rate order (denoted by $X \leq_{\text{RHR}} Y$) if

$$\frac{f(v)}{F(u)} \leq \frac{g(v)}{G(u)} \quad \textit{for all } u \leq v.$$

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For more details, one may refer to Shaked and Shanthikumar [20] and Muller and Stoyan [16] for the stochastic order ($\leq_{\text{ST}}$), hazard rate order ($\leq_{\text{HR}}$), reversed hazard rate order ($\leq_{\text{RHR}}$), increasing concave order ($\leq_{\text{ICV}}$) and Nanda et al. [17] for the mean inactivity time order ($\leq_{\text{MIT}}$) and other commonly used stochastic orders.

The purpose of this article is to study several preservation properties of stochastic comparisons based on the mean inactivity time order. Section 2 contains definitions, notations, and basic properties used throughout the article. Also in that section, we give some characterizations and relationships of the $\leq_{\text{MIT}}$ order and other well-known orders. In Section 3, we present some preservation results under the operations of convolution and mixture, as well as some examples of interest in reliability theory. Finally, in Section 4, we discuss hypothesis testing in the increasing mean inactivity time class (IMIT) defined as the class where $E[X_{(t)}]$ is increasing for all nonnegative t.

Throughout the article we will use the term "increasing" in place of "nondecreasing" and "decreasing" in place of "nonincreasing." $a/0$ is understood to be ∞ whenever $a > 0$. All integrals and expectations are implicitly assumed to exist whenever they are written.

2. DEFINITIONS, NOTATIONS, AND CHARACTERIZATIONS

Let X and Y have the distribution functions F and G, respectively; we denote the expected value of the random variables $X_{(t)}$ and $Y_{(t)}$ by $\alpha(t)$ and $\beta(t)$, respectively, where

$$\alpha(t) = \int_0^t \frac{F(u)\,du}{F(t)}, \qquad t > 0,$$

and

$$\beta(t) = \int_0^t \frac{G(u)\,du}{G(t)}, \qquad t > 0.$$

Observe that by the definition of $\leq_{\text{MIT}}$ order, $X \leq_{\text{MIT}} Y$ holds if and only if $\alpha(t) \geq \beta(t)$ for all $t \geq 0$. Actually, an equivalent condition for MIT order is given in Nanda et al. [17], and is the following.

Proposition 2.1: *Let X and Y be two continuous nonnegative random variables with absolutely continuous distribution functions F and G. Then, $X \leq_{\text{MIT}} Y$ if and only if*

$$\frac{\int_0^t F(u)\,du}{\int_0^t G(u)\,du} \text{ is decreasing in } t \in R^+. \tag{2.1}$$

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For any real number a, let a^- denote the negative part of a; that is, $a^- = a$ if $a \le 0$ and $a^- = 0$ if $a > 0$. Therefore, if $X \le_{\mathrm{MIT}} Y$, then

$$\frac{E[(X-t)^-]}{E[(Y-t)^-]} \text{ is decreasing in } t \text{ over } \{t : E[(Y-t)^-] > 0\}$$

or, equivalently, $X \le_{\mathrm{MIT}} Y$ if and only if

$$F(t)\int_0^t G(u)\,du \le G(t)\int_0^t F(u)\,du \quad \text{for all } t \in R^+.$$

Nanda et al. [17] proved that the reversed hazard rate order is stronger than the mean inactivity time order. In the following result, we prove that the mean inactivity time order is stronger than the increasing concave order.

THEOREM 2.1: *Let X and Y be two nonnegative random variables. If $X \le_{\mathrm{MIT}} Y$, then $X \le_{\mathrm{ICV}} Y$.*

PROOF: Let F and G be the distribution functions of X and Y, respectively. From (2.1), it follows that

$$\log \int_0^a F(u)\,du - \log \int_0^a G(u)\,du \quad \text{is decreasing in } a \in \{a : G(x) > 0\}.$$

Therefore,

$$\frac{F(a)}{\int_0^a F(u)\,du} \le \frac{G(a)}{\int_0^a G(u)\,du} \quad \text{for all } a \in \{a : G(x) > 0\}.$$

Now, the proof is similar to that of Theorem 3.A.13 in Shaked and Shanthikumar [20]. A straightforward computation gives

$$\begin{aligned} -\log\left\{\frac{\int_0^x F(u)\,du}{\int_0^\infty F(u)\,du}\right\} &= \int_x^\infty \left[\frac{F(a)}{\int_0^a F(u)\,du}\right] da \\ &\le \int_x^\infty \left[\frac{G(a)}{\int_0^a G(u)\,du}\right] da \\ &= -\log\left\{\frac{\int_0^x G(u)\,du}{\int_0^\infty G(u)\,du}\right\}. \end{aligned}$$

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Thus,

$$\frac{\int_0^x F(u)\,du}{\int_0^\infty F(u)\,du} \geq \frac{\int_0^x G(u)\,du}{\int_0^\infty G(u)\,du}.$$

Since $X \leq_{\text{MIT}} Y$ implies that

$$\frac{\int_0^x F(u)\,du}{\int_0^x G(u)\,du} \geq \lim_{x\to\infty} \frac{\int_0^x F(u)\,du}{\int_0^x G(u)\,du}$$

$$\geq \lim_{x\to\infty} \frac{F(x)}{G(x)} = 1,$$

it follows that (1.1) holds. ■

We have the following implications among some of the previous orders:

$$\begin{array}{ccccc} & & X \leq_{\text{MIT}} Y & & \\ & \nearrow & & \searrow & \\ X \leq_{\text{RHR}} Y & & & & X \leq_{\text{ICV}} Y. \\ & \searrow & & \nearrow & \\ & & X \leq_{\text{ST}} Y & & \end{array}$$

3. PRESERVATION RESULTS

Useful properties of the stochastic orders are their closure with respect to typical reliability operations like convolution or mixture (see Barlow and Proschan [5] and Shaked and Shanthikumar [20]). In this section, we present some preservation results for the *mean inactivity time order* ($\leq_{\text{MIT}}$). First, we recall the definition of some notions that will be used in the sequel.

DEFINITION 3.1: *Given two continuous random variables X and Y with densities f and g, respectively, X is said to be smaller than Y in the likelihood ratio order (denoted by $X \leq_{\text{lr}} Y$) if*

$$\frac{f(t)}{g(t)} \text{ decreases over the union of the supports of } X \text{ and } Y.$$

DEFINITION 3.2: *A probability vector $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i > 0$ for $i = 1, 2, \ldots, n$ is said to be smaller than the probability vector $\underline{\beta} = (\beta_1, \ldots, \beta_n)$ in the sense of the discrete likelihood ratio order, denoted by $\underline{\alpha} \leq_{\text{dlr}} \underline{\beta}$, if*

$$\frac{\beta_i}{\alpha_i} \le \frac{\beta_j}{\alpha_j} \quad \textit{for all } 1 \le i \le j \le n.$$

DEFINITION 3.3: *A function $g(x)$, $-\infty < x < \infty$, is said to be a Polya function of order 2 (PF_2) if the following hold:*

(a) $g(x) \ge 0$ *for* $-\infty < x < \infty$ *and*
(b)

$$\begin{vmatrix} g(x_1 - y_1) & g(x_1 - y_2) \\ g(x_2 - y_1) & g(x_2 - y_2) \end{vmatrix} \ge 0$$

for all $-\infty < x_1 < x_2 < \infty$ *and* $-\infty < y_1 < y_2 < \infty$ *or, equivalently,*

(b′) $\log[g(x)]$ *is concave on* $(-\infty,\infty)$.

The equivalence of (b) and (b′) is shown in Barlow and Proschan [5, Exercise 12, p. 79].

3.1. Convolution

As an important reliability operation, convolutions of a certain stochastic order are often paid much attention. The closure properties of $\le_{\rm ST}$, $\le_{\rm HR}$, $\le_{\rm RHR}$, and $\le_{\rm ICV}$ orders can be found in Shaked and Shanthikumar [20]. In Theorem 3.1, we establish the closure property of the $\le_{\rm MIT}$ order under the convolution operation. In general, if $X_1 \le_{\rm MIT} Y_1$ and $X_2 \le_{\rm MIT} Y_2$, where X_1 and X_2 are independent random variables and Y_1 and Y_2 are also independent random variables, then it is not necessarily true that $X_1 + X_2 \le_{\rm MIT} Y_1 + Y_2$. However, if these random variables have log-concave density, then it is true. This is shown in the following.

THEOREM 3.1: *Let X_1, X_2, and Y be three nonnegative random variables, where Y is independent of both X_1 and X_2, and let Y have density g. If $X_1 \le_{\rm MIT} X_2$ and g is log-concave, then $X_1 + Y \le_{\rm MIT} X_2 + Y$.*

PROOF: First, we note that for fixed $s \ge 0$ and $i = 1,2$,

$$\begin{aligned}
\Phi(i,t) &= \int_0^\infty F_{X_i+Y}(t-v)\,dv \\
&= \int_0^\infty \int_0^\infty F_i(t-v-u)\,dF_Y(u)\,dv \\
&= \int_0^\infty \int_{-\infty}^t F_i(z-v) f_Y(t-z)\,dz\,dv \\
&= \int_{-\infty}^t f_Y(t-z) \int_0^\infty F_i(z-v)\,dv\,dz \\
&= \int_{-\infty}^t f_Y(t-z)\psi(i,z)\,dz.
\end{aligned}$$

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As shown in Proposition 2.1, the assertion follows if we prove that $\Phi(i,t)$ is TP_2 in (i,t) (Joag-Dev, Kochar, and Proschan [12]). By the assumption $X_1 \le_{\text{MIT}} Y_1$, we can say that $\psi(i,z)$ is TP_2 in (i,z). Moreover, since Y has log-concave density, $f_Y(t-z)$ is TP_2 in (t,z). Therefore, by the basic composition formula (Karlin [13]), it follows that $\Phi(i,t)$ is TP_2 in (i,t). This completes the proof. ■

COROLLARY 3.1: *If $X_1 \le_{\text{MIT}} Y_1$ and $X_2 \le_{\text{MIT}} Y_2$, where X_1 is independent of X_2 and Y_1 is independent of Y_2, then the following statements hold:*

(i) If X_1 and Y_2 have log-concave densities, then $X_1 + X_2 \le_{\text{MIT}} Y_1 + Y_2$.
(ii) If X_2 and Y_1 have log-concave densities, then $X_1 + X_2 \le_{\text{MIT}} Y_1 + Y_2$.

PROOF: The following chain of inequalities, which establish (i), follows from Theorem 3.1:

$$X_1 + X_2 \le_{\text{MIT}} X_1 + Y_2 \le_{\text{MIT}} Y_1 + Y_2.$$

The proof of (ii) is similar. ■

THEOREM 3.2: *If $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ are sequences of independent random variables with $X_i \le_{\text{MIT}} Y_i$ and X_i and Y_i have log-concave densities for all i, then*

$$\sum_{i=1}^{n} X_i \le_{\text{MIT}} \sum_{i=1}^{n} Y_i \qquad (n = 1,2,\ldots).$$

PROOF: We will prove the theorem by induction. Clearly, the result is true for $n = 1$. Assume that the result is true for $p = n - 1$; that is,

$$\sum_{i=1}^{n-1} X_i \le_{\text{MIT}} \sum_{i=1}^{n-1} Y_i. \tag{3.1}$$

Note that each of the two sides of (3.1) has a log-concave density (see, e.g., Karlin [13, p. 128]). Appealing to Corollary 3.1, the result follows. ■

3.2. Mixture

Let now $X(\theta)$ be a random variable having distribution function F_θ and let Θ_i be a random variable having distribution G_i, for $i = 1,2$, and support R^+. The following is a closure of MIT order under mixture.

THEOREM 3.3: *Let $\{X(\theta),\ \theta \in R^+\}$ be a family of random variables independent of Θ_1 and Θ_2. If $\Theta_1 \le_{\text{lr}} \Theta_2$ and if $X(\theta_1) \le_{\text{MIT}} X(\theta_2)$ whenever $\theta_1 \le \theta_2$, then $X(\Theta_1) \le_{\text{MIT}} X(\Theta_2)$.*

PROOF: Let F_i be the distribution function of $X(\Theta_i)$, with $i = 1,2$. We known that

$$F_i(x) = \int_0^\infty F_\theta(x)\, dG_i(\theta).$$

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Because of Proposition 2.1, we should prove that $\Phi(i,t) = \int_0^\infty F_i(t-x)\,dx$ is TP_2 in (i,t). However, actually

$$\begin{aligned}\Phi(i,t) &= \int_0^\infty F_i(t-x)\,dx \\ &= \int_0^\infty \int_0^\infty F_\theta(t-x)\,dG_i(\theta)\,dx \\ &= \int_0^\infty g_i(\theta) \int_0^\infty F_\theta(t-x)\,dx\,d\theta \\ &= \int_0^\infty g_i(\theta)\,\psi(\theta,t)\,d\theta.\end{aligned}$$

By the assumption $X(\theta_1) \le_{\mathrm{MIT}} X(\theta_2)$ whenever $\theta_1 \le \theta_2$, we have that $\psi(\theta,t)$ is TP_2 in (θ,t), and from the assumption $\Theta_1 \le_{\mathrm{lr}} \Theta_2$, it follows that $g_i(\theta)$ is TP_2 in (i,θ). Thus, again, the assertion follows from the basic composition formula. ■

Suppose that X_i, $i = 1,\ldots,n$, is a collection of independent random variables. Suppose that F_i is the distribution function of X_i. Let $\underline{\alpha} = (\alpha_1,\ldots,\alpha_n)$ and $\underline{\beta} = (\beta_1,\ldots,\beta_n)$ be two probability vectors. Let X and Y be two random variables having the respective distribution functions F and G defined by

$$F(x) = \sum_{i=1}^n \alpha_i F_i(x) \quad \text{and} \quad G(x) = \sum_{i=1}^n \beta_i F_i(x). \tag{3.2}$$

The following result gives conditions under which X and Y are comparable with respect to the MIT order. One can refer to Ahmed [3] and Ahmed and Kayid [4] for a similar preservation property of the mean residual life order ($\le_{\mathrm{MRL}}$) and the Laplace transform of residual life order ($\le_{\mathrm{Lt-rl}}$), respectively. Definition, properties, and applications of $\le_{\mathrm{MRL}}$ order and $\le_{\mathrm{Lt-rl}}$ order can be found, for instance, in Shaked and Shanthikumar [20], Belzunce, Ortega, and Ruiz [6], and Gao, Belzunce, Hu, and Pellerey [10].

THEOREM 3.4: *Let $X_1,\ldots,X_n$ be a collection of independent random variables with corresponding distribution functions $F_1,\ldots,F_n$, such that $X_1 \le_{\mathrm{MIT}} X_2 \le_{\mathrm{MIT}} \ldots \le_{\mathrm{MIT}} X_n$ and let $\underline{\alpha} = (\alpha_1,\ldots,\alpha_n)$ and $\underline{\beta} = (\beta_1,\ldots,\beta_n)$ such that $\underline{\alpha} \le_{\mathrm{dlr}} \underline{\beta}$. Let X and Y have distribution functions F and G defined in Eqs. (3.2). Then, $X \le_{\mathrm{MIT}} Y$.*

PROOF: Again, because of Proposition 2.1, we need to establish that

$$\frac{\int_0^\infty \sum_{i=1}^n \beta_i F_i(x-u)\,du}{\int_0^\infty \sum_{i=1}^n \alpha_i F_i(x-u)\,du} \le \frac{\int_0^\infty \sum_{i=1}^n \beta_i F_i(y-v)\,dv}{\int_0^\infty \sum_{i=1}^n \alpha_i F_i(y-v)\,dy} \quad \text{for all } 0 < y < x. \tag{3.3}$$

Multiplying by the denominators and canceling out equal terms, it can be shown that inequality (3.3) is equivalent to

$$\sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{j=1}^{n}\beta_i\alpha_j\int_0^\infty F_i(x-u)\,du\int_0^\infty F_j(y-v)\,dv \le \sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{j=1}^{n}\beta_i\alpha_j\int_0^\infty F_i(y-u)\,du\int_0^\infty F_j(x-v)\,dv$$

or, equivalently

$$\sum_{\substack{i=1\\ i<j}}^{n}\sum_{j=1}^{n}\left[\beta_i\alpha_j\int_0^\infty F_i(x-u)\,du\int_0^\infty F_j(y-v)\,dv + \beta_j\alpha_i\int_0^\infty F_j(x-u)\,du\int_0^\infty F_i(y-v)\,dv\right]$$
$$\le \sum_{\substack{i=1\\ i<j}}^{n}\sum_{j=1}^{n}\left[\beta_i\alpha_j\int_0^\infty F_i(y-v)\,dv\int_0^\infty F_j(x-u)\,du + \beta_j\alpha_i\int_0^\infty F_j(x-v)\,dv\int_0^\infty F_i(y-u)\,du\right].$$

Now, for each fixed pair (i,j) with $i<j$, we have

$$\left[\beta_i\alpha_j\int_0^\infty F_i(y-v)\,dv\int_0^\infty F_j(x-u)\,du + \beta_j\alpha_i\int_0^\infty F_j(y-v)\,dv\int_0^\infty F_i(y-u)\,du\right]$$
$$-\left[\beta_i\alpha_j\int_0^\infty F_i(x-u)\,du\int_0^\infty F_j(y-v)\,dv + \beta_j\alpha_i\int_0^\infty F_j(x-u)\,du\int_0^\infty F_i(y-v)\,dv\right]$$
$$=(\beta_i\alpha_j-\beta_j\alpha_i)\left[\int_0^\infty F_i(y-v)\,dy\int_0^\infty F_j(x-u)\,dx - \int_0^\infty F_i(x-u)\,dx\int_0^\infty F_j(y-v)\,dy\right],$$

which is nonnegative because both terms are nonpositive by assumption. This completes the proof. ∎

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3.3. Applications

To demonstrate the usefulness of the above results in recognizing MIT-ordered random variables, we consider the following examples.

Example 3.1: Let $X_{\underline{\lambda}}$ denote the convolution of n exponential distributions with parameters $\lambda_1, \ldots, \lambda_n$, respectively. Assume without loss of generality that $\lambda_1 \le \cdots \le \lambda_n$. Since exponential densities are log-concave, Theorem 3.4 implies that $X_{\underline{\lambda}} \le_{\text{MIT}} Y_{\underline{\mu}}$ whenever $\lambda_i \ge \mu_i$ for $i = 1, \ldots, n$.

Example 3.2: Let $X_i \sim \exp(\lambda_i)$, $i = 1, \ldots, n$, be independent random variables. Let X and Y be $\underline{\alpha}$ and $\underline{\beta}$ mixtures of X_i's. An application of Theorem 3.4 is immediately $X \le_{\text{MIT}} Y$ for every two probability vectors $\underline{\alpha}$ and $\underline{\beta}$ such that $\underline{\alpha} \le_{\text{dlr}} \underline{\beta}$.

Another application of Theorem 3.4 is contained in following example.

Example 3.3: Let $X_{\underline{\lambda}}$ and $X_{\underline{\mu}}$ be as given in Example 3.1. For $0 \le q \le p \le 1$ and $p + q = 1$, we have

$$pX_{\underline{\lambda}} + qX_{\underline{\mu}} \le_{\text{MIT}} qX_{\underline{\lambda}} + pX_{\underline{\mu}}.$$

4. TESTING IN THE IMIT CLASS

In the literature, many nonparametric classes of distributions have been defined (cf. Barlow and Proschan [5] and Ross [18]). In particular, the decreasing reversed hazard rate (DRHR) class of distributions has been studied by many researchers in the recent past (cf. Shaked and Shanthikumar [20], Block et al. [7], and Sengupta and Nanda [19]). Recently, Chandra and Roy [2] introduced a new nonparametric class called the increasing mean inactivity time class of life distributions (abbreviated as the IMIT class).

Recall that a random variable X having distribution function $F(\cdot)$, density $f(\cdot)$, and reversed hazard rate function $\tilde{r}(t) = f(x)/F(x)$ is said to have the following:

1. Increasing mean inactivity time (IMIT) if $E[X_{(t)}]$ is increasing in $t > 0$
2. Decreasing reversed hazard rate (DRHR) if $\tilde{r}(t)$ is decreasing in $t > 0$

Block et al. [7] have shown that there exists no nondegenerate life distribution that has increasing reversed hazard rate (IRHR) over the domain $[0, \infty)$. Also, Nanda et al. [17] proved that there exists no nonnegative random variable for which $E[X_{(t)}]$ decreases over the domain $[0, \infty)$ and the DRHR property is stronger than the IMIT property; that is,

$$\text{DRHR} \Rightarrow \text{IMIT}.$$

On the other hand, in the context of reliability, "ageless notion" is equivalent to the phenomenon that age has no effect on the residual survival function of a unit. Ageless has thus been equivalently described as constant failure rate, constant mean residual life, and exponential survival distribution. Note that the exponential distribution defines this notion; hence, testing any aging notion is done via testing exponentiality versus the class at hand. This applies to many classes, such as increasing hazard rate (IHR), increasing hazard rate average (IHRA), new better than used (NBU), new better than used in expectation (NBUE), harmonic new better than used in expectation (HNBUE), and decreasing mean residual lifetime (DMRL); see Ahmad [1], Ahmad and Mugdadi [2], and Mugdadi and Ahmad [15] for recent developments and references.

In order to do testing for the IMIT class (or the DRHR for that matter), one observes that there is no boundary distribution at all (i.e., there is no distribution where MIT (or RHR) is constant). The exponential distribution is easily seen to have IMIT and DRHR. Hence, to do testing for IMIT (or DRHR), we test $H_0 : F = F_0$ against $H_1 : F$ is IMIT (or DRHR) and not F_0, where F_0 is known (up to a set of parameters). One obvious choice of F_0, of course, is the exponential. Thus, we address $H_0 : F$ is exponential (μ) against $H_1 : F$ is IMIT and not exponential.

Note that $F \in$ IMIT if and only if

$$F^2(t) \geq f(t) \int_0^t F(u)\, du \quad \text{for all } t \geq 0;$$

we thus take as a measure of departure from H_0,

$$\delta^{(1)} = \frac{1}{3} - \int_0^\infty f^2(t) \left(\int_0^t F(u)\, du \right) dt. \tag{4.1}$$

Note that if F is $\exp(\mu)$, then $\delta^{(1)} = \frac{1}{12}$.

Let $X_1, \dots, X_n$ denote a random sample from F. A nonparametric estimate of $\delta^{(1)}$ is

$$\hat{\delta}^{(1)} = \frac{1}{3} - \frac{1}{n(n-)(n-2)a} \sum \sum_{i \neq j \neq l} \sum k\left(\frac{X_i - X_j}{a} \right)(X_i - X_j) I(X_i > X_l), \tag{4.2}$$

where $k(\cdot)$ is a known probability density function which is bounded and symmetric with mean 0 and finite variance σ_k^2 and $a = a_n$ are positive constant such that $a \to 0$, $an \to \infty$ as $n \to \infty$.

The following theorem gives the large sample behavior of $\hat{\delta}^{(1)}$ both under H_0 and in general.

Theorem 4.1: *If $na^4 \to 0$ as $n \to \infty$, if f has a bounded second derivative, and if $V(\psi_n(X_1)) < \infty$, where $\psi_n(X_1)$ is given in Eq. (4.10), then $\sqrt{n}(\hat{\delta}^{(1)} - \delta^{(1)})$ is asymptotically normal with mean 0 and variance $\lim_{n\to\infty} V(\psi_n(X_1))$. Under H_0, the variance is $\frac{71}{2160}$.*

Proof: First, it is easy to see that with $g_n(x) = E(1/a)k((x - X)/a)$,

$$\begin{aligned} E\hat{\delta}^{(1)} &= \frac{1}{3} - \frac{1}{a} Ek\left(\frac{X_1 - X_2}{a} \right)(X_2 - X_3) I(X_1 > X_3) \\ &= \frac{1}{3} - \int g_n(x) \left\{ xF(x) - \int_0^x uf(u)\, du \right\} f(x)\, dx. \end{aligned} \tag{4.3}$$

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We can also write

$$\hat{\delta}^{(1)} = \frac{1}{3} - \int f^2(x) \left\{ xF(x) - \int_0^x uf(u)\,du \right\} f(x)\,dx. \quad \textbf{(4.4)}$$

Thus,

$$|E\hat{\delta}^{(1)} - \delta^{(1)}| \leq \int |g_n(x) - f(x)|\, f(x) \left| xF(x) - \int_0^x uf(u)\,du \right| dx.$$

However,

$$|g_n(x) - f(x)| \leq \frac{a^2}{2} |f''(x)| \int u^2 k(u)\,du = 0(a^2). \quad \textbf{(4.5)}$$

Hence,

$$\sqrt{n}|E\hat{\delta}^{(1)} - \delta^{(1)}| = 0(a^2\sqrt{n}) = o(1) \quad \text{by assumption.}$$

Next, let us look at $\sqrt{n}(\hat{\delta}^{(1)} - E\hat{\delta}^{(1)})$, writing Θ_n for $E\hat{\delta}^{(1)}$ and

$$\phi_n(X_1, X_2, X_3) = a^{-1}k\left(\frac{X_1 - X_2}{a}\right)(X_1 - X_3)I(X_1 > X_3). \quad \textbf{(4.6)}$$

Then, using a standard decomposition, we have

$$\sqrt{n}(\hat{\delta}^{(1)} - E\hat{\delta}^{(1)}) = \sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}\psi_n(X_i) + \frac{1}{n(n-1)(n-2)}\sum\sum_{i\neq j\neq l}\sum \xi_n(X_i, X_j, X_l)\right], \quad \textbf{(4.7)}$$

where

$$\begin{aligned}\psi_n(X_i) = {} & E[\phi_n(X_1, X_2, X_3)|X_1] \\ & + E[\phi_n(X_2, X_1, X_3)|X_1] \\ & + E[\phi_n(X_3, X_2, X_1)|X_1] - 3\theta_n\end{aligned}$$

and

$$\xi_n(X_1, X_2, X_3) = \phi_n(X_1, X_2, X_3) - \psi_n(X_1) - 2\theta_n.$$

Now, by the Layaponoulf central limit theorem, the first term of Eq. (4.7) is asymptotically normal if

$$L_n = \frac{E(\psi_n(X_1))^3}{\sqrt{n}[V(\psi_n(X_1))]^{3/2}} \to 0 \quad \text{as } n \to \infty.$$

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However, using the fact that $g_n(x) = f(x) + (a^2/2)f''(x)\sigma_k^2$, we easily see that for large n,

$$E[\phi_n(X_1, X_2, X_3)|X_1] = E[\phi_n(X_2, X_1, X_3)|X_1]$$
$$= X_1 f(X_1)F(X_1) - f(X_1)\int_0^{X_1} xf(x)\,dx + O_P(a^2), \quad \textbf{(4.8)}$$

and

$$E[\phi_n(X_3, X_2, X_1)|X_1] = \int_{X_1}^{\infty} xf^2(x)\,dx - X_1\int_{X_1}^{\infty} f^2(x)\,dx + O_P(a^2). \quad \textbf{(4.9)}$$

Hence, for n large enough,

$$\psi_n(X_1) = \eta(X_1) + O_P(a^2), \quad \textbf{(4.10)}$$

where

$$\eta(X_1) = 2X_1 f(X_1)F(X_1) - 2f(X_1)\int_0^{X_1} xf(x)\,dx + \int_{X_1}^{\infty} xf^2(x)\,dx - X_1\int_{X_1}^{\infty} f^2(x)\,dx.$$

Hence, $V(\psi_\alpha(X_1)) = V(n(X_1)) + 0(a^4)$ and $E|\psi(X_1)|^3 \le C_3$. Therefore, $L_n \to 0$, provided that $na^4 \to 0$.

Finally,

$$E\left\{\frac{\sqrt{n}}{n(n-1)(n-2)}\sum\sum_{i\neq j\neq l}\sum \xi_n(X_i, X_j, X_l)\right\}^2$$
$$= (n-1)^{-1}E\xi_n^2(X_1, X_2, X_3)$$
$$= o\left(\frac{1}{na}\right) = o(1). \quad \textbf{(4.11)}$$

The null variance is obtained by substituting the exponential into $\eta(X_1)$. The result now follows. ■

To conduct the test, calculate $\sqrt{n}\Big/\sqrt{\frac{71}{2160}}(\hat{\delta}^{(1)} \quad -\frac{1}{12})$ and reject H_0 if this is much larger than Z_α. Of course, we must choose k and a to carry out the test. The choice of k is not crucial and the standard normal will do fine. The choice of a is crucial and there are different ways to do that (cf. Wand and Jones [21]). The easiest and highly practical rule is the normal scale rule (Wand and Jones [21, p. 60]) with $a = cn^{-1/\alpha}$, with α an integer greater than 1 and c is the sample standard deviation.

To assess the goodness of this test, one can evaluate its limiting Pitman efficacy for an alternative that is IMIT but not exponential and compare it to similar values of other tests for this problem. Because there are no other tests known for this problem, this comparison is left to future work on the topic. The asymptotic Pitman efficacy of a test based on a measure of departure from H_1 equal to $\delta^{(1)}$ is given by

$$\mathrm{APE}(\delta_\theta^{(1)}) = \frac{1}{\sqrt{\frac{71}{2160}}} \left| \frac{d}{d\theta}\, \delta_\theta^{(1)} \right|_{\theta \to \theta_0}. \quad \textbf{(4.12)}$$

Two of the distributions that are IMIT but are not exponential are as follows:

1. The linear failure rate:

$$\bar{F}_\theta(t) = e^{-t-(\theta/2)t^2}, \qquad t \ge 0, \theta \ge 0$$

2. The Makeham:

$$\bar{F}_\theta(t) = e^{-t-\theta(e^{-t}+t-1)}, \qquad t \ge 0, \theta \ge 0.$$

Note that the exponential is attained at $\theta = 0$ in both cases. The APE of the above test is

$$\frac{1}{\sqrt{\frac{71}{2160}}} \left\{ 2\int_0^\infty f_{\theta_0}(t) f'_{\theta_0}(t) \left(\int_0^t F_{\theta_0}(u)\, du \right) dt + \int_0^\infty f^2_{\theta_0}(t) \left(\int_0^t F'_{\theta_0}(u)\, du \right) dt \right\},$$

where

$$F'_{\theta_0}(u) = \frac{d}{d\theta} F_{\theta_0}(u)|_{\theta \to \theta_0} \quad \text{and} \quad f'_{\theta_0}(t) = \frac{d}{d\theta} f_\theta(t)|_{\theta \to \theta_0}.$$

Direct calculation gives a value of APE equal to 0.732 and 0.244 for the above two alternatives, respectively. These efficacy values are to be compared to those of any other procedure that might be developed for this problem. These values, however, are very close to the values obtained in standard life testing problems such as testing for decreasing mean residual lifetime (DMRL), where the Hollander and Proschan [11] test has values 0.866 and 0.242, respectively. Note, however, that these two problems are not compatible.

The interested reader might want to choose other distributions as null distributions and develop the null value of $\delta^{(1)}$ and perform the test after the null variance is calculated.

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