Combinatorics, Probability and Computing (2007) 16, 193–217.
doi:10.1017/S0963548306007942 First published online 20 September 2006 Printed in the United Kingdom

Multi-Path Matroids

JOSEPH E. BONIN[1] and OMER GIMÉNEZ[2†]

[1]Department of Mathematics, The George Washington University, Washington, DC 20052, USA
(e-mail: jbonin@gwu.edu)

[2]Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya,
Jordi Girona 1–3, 08034, Barcelona, Spain
(e-mail: omer.gimenez@upc.edu)

Received 19 October 2004; revised 6 December 2005

We introduce the minor-closed, dual-closed class of multi-path matroids. We give a polynomial-time algorithm for computing the Tutte polynomial of a multi-path matroid, we describe their basis activities, and we prove some basic structural properties. Key elements of this work are two complementary perspectives we develop for these matroids: on the one hand, multi-path matroids are transversal matroids that have special types of presentations; on the other hand, the bases of multi-path matroids can be viewed as sets of lattice paths in certain planar diagrams.

1. Introduction

In [2] it is shown how to construct, from a pair P,Q of lattice paths that go from $(0,0)$ to (m,r), a transversal matroid $M[P,Q]$ whose bases correspond to the paths from $(0,0)$ to (m,r) that remain in the region bounded by P and Q. The basic enumerative and structural properties of these matroids, which are called lattice path matroids, are developed in [2, 3]. This paper introduces multi-path matroids, a generalization of lattice path matroids that share many of their most important properties.

Section 2 starts by reviewing the definition of lattice path matroids as well as an alternative perspective on these matroids that uses collections of incomparable intervals in a linear order. This alternative perspective leads to the starting point for multi-path matroids: the linear order is replaced by a cyclic permutation. In addition to defining and providing examples of multi-path matroids, Section 2 also defines basic concepts that are used in the rest of the paper.

† Partially supported by Beca Fundació Crèdit Andorrà. Current address: Universitat Pompeu Fabra, Departament de Tecnologia, Passeig de Circunval.lació 8, 08003 Barcelona, Spain.

Section 3 shows that the dual and all minors of a multi-path matroid are multi-path matroids (lattice path matroids have the corresponding properties; transversal matroids do not). Proving these properties entails developing several alternative presentations for multi-path matroids. In particular, we show that the bases of a multi-path matroid can be viewed as certain sets of lattice paths in a diagram (such as that in Figure 4 on page 201) that has fixed global bounding paths and one or more pairs of starting and ending points.

The diagrams we develop in Section 3 are crucial tools in the next two sections. Section 4 shows that the Tutte polynomial of a multi-path matroid can be computed in polynomial time in the size of the ground set. This result stands in contrast to the following result of [5]: for any fixed algebraic numbers x and y with $(x-1)(y-1) \neq 1$, the problem of computing $t(M; x, y)$ for an arbitrary transversal matroid M is #P-complete. Our work on the Tutte polynomial is cast in the general framework of what we call computation graphs, which allow us to apply the idea of dynamic programming to this computation.

Section 5 shows that, as is true of lattice path matroids, internal and external activities of bases of multi-path matroids have relatively simple lattice-path interpretations. We also sketch a somewhat faster, although more complex, algorithm for computing the Tutte polynomial of a multi-path matroid via basis activities.

The final section addresses several structural properties of multi-path matroids. For instance, we show that multi-path matroids that are not lattice path matroids have spanning circuits and we make some comments about minimal presentations of multi-path matroids.

We close this introduction by recalling several key notions; see [9] for concepts of matroid theory not defined here. A *set system* is a multiset $\mathscr{A} = (A_j : j \in J)$ of subsets of a finite set S. A *transversal* of $\mathscr{A}$ is a set $\{x_j : j \in J\}$ of $|J|$ distinct elements such that x_j is in A_j for all j in J. A *partial transversal* of $\mathscr{A}$ is a transversal of a set system of the form $(A_k : k \in K)$ with K a subset of J. Edmonds and Fulkerson [6] showed that the partial transversals of a set system $\mathscr{A}$ are the independent sets of a matroid on S. This matroid $M[\mathscr{A}]$ is a *transversal matroid* and the set system $\mathscr{A}$ is a *presentation* of $M[\mathscr{A}]$. For a basis B of $M[\mathscr{A}]$, a *matching of B with $\mathscr{A}$* is a function $\phi : B \longrightarrow \mathscr{A}$ such that

(1) b is in $\phi(b)$ for each b in B and

(2) the number of elements of B that ϕ maps to any set X in $\mathscr{A}$ is at most the multiplicity of X in $\mathscr{A}$.

This terminology is suggested by the interpretation of set systems as bipartite graphs [9, Section 1.6]. In this paper, $\mathscr{A}$ will typically be an *antichain*, that is, no set in $\mathscr{A}$ contains another set in $\mathscr{A}$. Presentations of transversal matroids are generally not unique. A presentation $(A_1, A_2, \ldots, A_r)$ of the transversal matroid M *contains* the presentation $(A'_1, A'_2, \ldots, A'_r)$ of M if $A'_i \subseteq A_i$ for all i with $1 \leqslant i \leqslant r$. We let $[n]$ denote the set $\{1, 2, \ldots, n\}$.

2. Basic definitions

We start by reviewing lattice path matroids and an alternative perspective on these matroids. The majority of this section is devoted to defining multi-path matroids, providing illustrations, and defining notation and concepts that are used in the rest of the paper. Lattice path matroids were introduced in [2]; special classes of lattice path matroids had been studied earlier from other perspectives (see Section 4 of [3]).

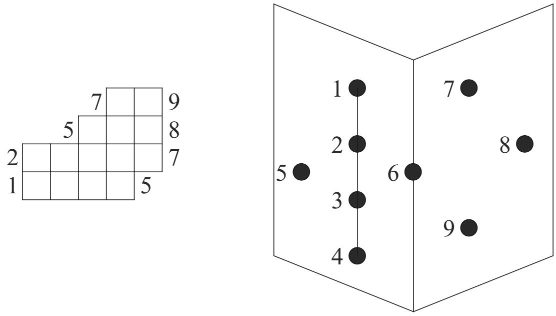

Figure 1. A lattice path presentation and geometric representation of a lattice path matroid.

A lattice path can be viewed geometrically as a path in the plane made up of unit steps East and North, or, more formally, as a word in the alphabet $\{E, N\}$, where E denotes the East step $(1, 0)$ and N denotes the North step $(0, 1)$. When viewed as a word in the alphabet $\{E, N\}$, a lattice path does not have fixed starting and ending points. Thus, one may identify different geometric lattice paths that arise from the same word; whether we identify such paths will depend on, and should be clear from, the context.

Fix lattice paths P and Q from $(0, 0)$ to (m, r) with P never going above Q. Let $\mathscr{P}$ be the set of all lattice paths from $(0, 0)$ to (m, r) that go neither above Q nor below P. For i with $1 \leqslant i \leqslant r$, let N_i be the set

$$N_i = \{j \ : \ \text{step } j \text{ is the } i\text{-th North step of some path in } \mathscr{P}\}.$$

The matroid $M[P, Q]$ is the transversal matroid on the ground set $[m + r]$ that has $(N_1, N_2, \ldots, N_r)$ as a presentation. Note that $M[P, Q]$ has rank r and nullity m. A *lattice path matroid* is any matroid isomorphic to such a matroid $M[P, Q]$.

Figure 1 shows a lattice path matroid of rank 4 and nullity 5. The sets N_1, N_2, N_3, and N_4 are $\{1, 2, 3, 4, 5\}$, $\{2, 3, 4, 5, 6, 7\}$, $\{5, 6, 7, 8\}$, and $\{7, 8, 9\}$. As this example suggests, the sets $N_1, N_2, \ldots, N_r$ are intervals in $[m + r]$, and both the left endpoints and the right endpoints form strictly increasing sequences. This motivates the following result from [3].

Theorem 2.1. *A matroid is a lattice path matroid if and only if it is transversal and some presentation is an antichain of intervals in a linear order on the ground set.*

The following result [2, Theorem 3.3] starts to suggest a deeper connection with lattice paths.

Theorem 2.2. *The map $R \mapsto \{i \ : \ \text{the } i\text{-th step of } R \text{ is North}\}$ is a bijection from $\mathscr{P}$ onto the set of bases of $M[P, Q]$.*

Multi-path matroids are the generalizations of lattice path matroids that result from using a cyclic permutation in place of the linear order in Theorem 2.1.

Fix a cyclic permutation σ of the set S. A σ*-interval* (or simply an *interval*) in S is a non-empty subset I of S of the form $\{f_I, \sigma(f_I), \sigma^2(f_I), \ldots, l_I\}$; this σ-interval is denoted $[f_I, l_I]$ and the elements f_I and l_I are called, respectively, the *first* and *last element of* I. Note

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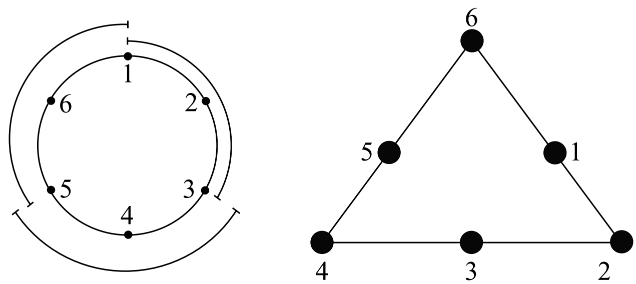

Figure 2. The 3-whirl $\mathcal{W}^3$ as a multi-path matroid.

that singleton subsets (which arise if f_I is l_I) as well as the entire set S (which arises if $\sigma(l_I)$ is f_I) are σ-intervals. If one views the elements of S placed around a circle in the order given by σ, then the σ-intervals are the sets of elements that can be covered by arcs of the circle; in the case of a σ-interval $[f_I, l_I]$ that is S, the arc has a gap between l_I and f_I.

We now define our main object of study.

Definition 1. A *multi-path matroid* is a transversal matroid that has a presentation by an antichain of σ-intervals in some cyclic permutation σ of the ground set.

The term 'multi-path' comes from the alternative perspective on these matroids given in Theorem 3.6. To distinguish the different types of presentations of interest in this paper, presentations of the type in Theorem 2.1 are *interval presentations*, while those of the type in Definition 1 are *σ-interval presentations*.

The first elements $f_{I_1}, f_{I_2}, \dots, f_{I_r}$ that arise from an antichain $\mathscr{I} = (I_1, I_2, \dots, I_r)$ of σ-intervals are distinct; thus, the rank of $M[\mathscr{I}]$ is r, the number of intervals. Also, for $\mathscr{I}$ to be an antichain of σ-intervals, the set S can be in $\mathscr{I}$ only if r is 1. However, Lemma 3.2 shows that the antichain condition in Definition 1 can be relaxed without changing the resulting class of matroids; in some cases this relaxation allows S to be in $\mathscr{I}$.

In the following examples, S is the set $[n]$ and σ is the cycle $(1, 2, \dots, n)$. Since a linear order can be 'wrapped around' to obtain a cycle, every lattice path matroid is a multi-path matroid. The converse is not true, as the first example shows.

Example 1. The 3-whirl $\mathcal{W}^3$ is an excluded-minor for the class of lattice path matroids [3]; however, Figure 2 shows that the 3-whirl is a multi-path matroid. The three intervals are $I_1 = \{1, 2, 3\}$, $I_2 = \{3, 4, 5\}$, and $I_3 = \{5, 6, 1\}$. A similar construction shows that all whirls are multi-path matroids.

Once the linear order on the ground set is fixed, there is only one interval presentation of a lattice path matroid $M[P, Q]$ since P and Q correspond to, respectively, the greatest and least bases in lexicographic order. (See also [3, Theorem 5.6].) In contrast, even lattice path matroids can have multiple σ-interval presentations, as the next example shows.

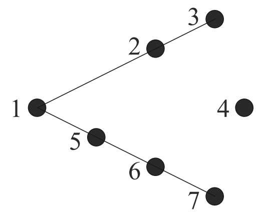

Figure 3. A multi-path matroid that has multiple minimal presentations.

Example 2. All uniform matroids are lattice path matroids. The following set systems are different σ-interval presentations of the uniform matroid $U_{3,6}$ of rank 3 on the set [6]:

$$(\{1,2,3,4\},\{2,3,4,5\},\{3,4,5,6\}),$$
$$(\{1,2,3,4,5\},\{2,3,4,5,6\},\{3,4,5,6,1\}).$$

A presentation $\mathscr{A}$ of a transversal matroid M is *minimal* if no other presentation of M is contained in $\mathscr{A}$. Interval presentations of lattice path matroids are minimal [3, Theorem 6.1]. We next show that multi-path matroids that are not lattice path matroids can have multiple minimal presentations that are σ-interval presentations.

Example 3. The following set systems are σ-interval presentations of the matroid shown in Figure 3 and both are minimal:

$$(\{5,6,7,1,2\},\{2,3,4\},\{4,5,6,7\}),$$
$$(\{6,7,1,2,3\},\{2,3,4\},\{4,5,6,7\}).$$

The rest of this section contains observations about multi-path matroids as well as definitions that are used in later sections.

Many constructions, such as minors, involve subsets of the ground set; in such settings, we use the following definition. The cyclic permutation σ on S induces a cyclic permutation σ_X on each subset X of S defined as follows: for x in X, the image $\sigma_X(x)$ is the first element in the list $\sigma(x),\sigma^2(x),\sigma^3(x),\ldots$ that is in X. Thus, σ_X is formed from σ by skipping over the elements that are not in X.

There is an induced cycle on the σ-intervals in an antichain $\mathscr{I}=(I_1,I_2,\ldots,I_r)$ of σ-intervals. Indeed, the last elements $l_{I_1},l_{I_2},\ldots,l_{I_r}$ are distinct since $\mathscr{I}$ is an antichain, so a cyclic permutation Σ on $\mathscr{I}$ is given by $\Sigma(I_j)=I_k$ if $\sigma_X(l_{I_j})=l_{I_k}$ where X is $\{l_{I_1},l_{I_2},\ldots,l_{I_r}\}$. Likewise the cycle σ_Y on $Y=\{f_{I_1},f_{I_2},\ldots,f_{I_r}\}$ induces a cyclic permutation Σ' on $\mathscr{I}$. The assumption that $\mathscr{I}$ is an antichain gives the equality $\Sigma=\Sigma'$. We use Σ to denote the cyclic permutation of $\mathscr{I}$ induced in this manner from σ. For instance, in Example 1, Σ is (I_1,I_2,I_3).

Fix an element x in a σ-interval I. It will be useful to consider the two parts in which $I-x$ naturally comes. The *first part of* $I-x$ is the empty set if x is f_I, or the σ-interval $[f_I,\sigma^{-1}(x)]$ if x is not f_I. Similarly, the *last part of* $I-x$ is the empty set if x is l_I, or the σ-interval $[\sigma(x),l_I]$ if x is not l_I. From the set $I-x$ alone, references to the first and last parts could be ambiguous (for instance, if x is f_I or l_I), but x will be clear from the context, so no confusion should result.

Note that an element x in S is a loop of $M[\mathscr{I}]$ if and only if x is in no interval in $\mathscr{I}$. Thus, if x is a loop, then the intervals in $\mathscr{I}$ are intervals in the linear order $x < \sigma(x) < \sigma^2(x) < \cdots < \sigma^{-1}(x)$, so $M[\mathscr{I}]$ is a lattice path matroid. Likewise, note that if some first element f_I is not in $\Sigma^{-1}(I)$, then $M[\mathscr{I}]$ is a lattice path matroid.

3. Minors, duals, and the lattice path interpretation

This section shows that the class of multi-path matroids is closed under minors and duals. (Analogous properties hold for lattice path matroids but not for arbitrary transversal matroids.) We also develop several alternative descriptions of multi-path matroids, some of which involve lattice paths and so account for the name. The lattice path interpretations as well as closure under contractions enter into the proof of closure under duality.

We start with a simple lemma that applies to all transversal matroids.

Lemma 3.1. *Assume X and Y are in a set system $\mathscr{A}$ with $X \subseteq Y$. Let z be in X and let $\mathscr{A}'$ be obtained from $\mathscr{A}$ by replacing one or more occurrences of Y by $Y - z$. Then $M[\mathscr{A}] = M[\mathscr{A}']$.*

Proof. Note that it suffices to prove the result in the case that one occurrence of Y is replaced by $Y - z$, and for this it suffices to show that for any basis B of $M[\mathscr{A}]$ and matching $\phi : B \longrightarrow \mathscr{A}$, we can find a matching $\phi' : B \longrightarrow \mathscr{A}'$. Clearly there is such a matching ϕ' if z is not in B, or if z is in B but $\phi(z)$ is not Y. Thus assume that z is in B and $\phi(z)$ is Y. If X is not in the image of ϕ, then the map ϕ' that agrees with ϕ except that $\phi'(z)$ is X is the required matching. Now assume $\phi(x)$ is X for some x in B. Since x is in X and therefore in Y, the following map ϕ' is the required matching:

$$\phi'(w) = \begin{cases} X, & \text{if } w = z, \\ Y - z, & \text{if } w = x, \\ \phi(w), & \text{otherwise.} \end{cases}$$ □

It is well known and easy to see that if $\mathscr{A} = (A_1, A_2, \ldots, A_r)$ is a presentation of a transversal matroid M on S, then any single-element deletion $M\backslash x$ is transversal and $\mathscr{A}' = (A_1 - x, A_2 - x, \ldots, A_r - x)$ is a presentation of $M\backslash x$. Since deleting $\emptyset$ from any set system in which it appears does not change the associated transversal matroid, we may assume that $\emptyset$ is not in $\mathscr{A}'$. Note that if $\mathscr{A}$ is an antichain of σ-intervals, then the sets in $\mathscr{A}'$ are σ_{S-x}-intervals, but there may be containments among these sets. This issue is addressed through the next lemma, which gives a relaxation of the antichain criterion in Definition 1.

Lemma 3.2. *Assume the transversal matroid M has a presentation by a multiset $\mathscr{A}$ of σ-intervals that satisfies the following condition:*

(C) *if $I \subseteq J$ for $I, J \in \mathscr{A}$, then either f_J or l_J is in I.*

Then M is a multi-path matroid and $\mathscr{A}$ contains a σ-interval presentation of M.

Proof. If $\mathscr{A}$ is an antichain, there is nothing to prove, so assume I and J are in $\mathscr{A}$ and $I \subseteq J$. By condition (C) and symmetry, we may assume f_J is in I. By replacing J if

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needed, we may assume no σ-interval in $\mathscr{A}$ whose first element is f_J properly contains J. Let $\mathscr{A}'$ be the set system obtained from $\mathscr{A}$ by replacing J by the σ-interval $J - f_J$, or eliminating J if $J - f_J$ is empty. Lemma 3.1 gives the equality $M[\mathscr{A}] = M[\mathscr{A}']$; we will show that $\mathscr{A}'$ satisfies condition (C). The presentation of M by σ-intervals that results from applying this modification as many times as possible must be an antichain, which proves the lemma.

To show that $\mathscr{A}'$ satisfies condition (C), first note that the only pairs of intervals that potentially could contradict condition (C) must include $J - f_J$. Let K be another interval in $\mathscr{A}'$. If the containment $K \subseteq J - f_J$ holds, then K is a subset of J but does not contain f_J; it follows from condition (C) applied to J and K in $\mathscr{A}$ that l_J (which is also l_{J-f_J}) must be in K, as needed. Now assume the containment $J - f_J \subseteq K$ holds. If l_K is in $J - f_J$, there is nothing to show, so assume this is not the case. Since J is the largest set in $\mathscr{A}$ that has f_J as its first element, f_K is not f_J. If $\sigma^{-1}(f_J)$ were in K, then J and K would contradict condition (C) for $\mathscr{A}$. Thus, the first element of K must be $\sigma(f_J)$, so f_K is in $J - f_J$, as needed. □

It is easy to check that if $(I_1, I_2, \ldots, I_r)$ is an antichain of σ-intervals, then the set system of σ_{S-x}-intervals $(I_1 - x, I_2 - x, \ldots, I_r - x)$ satisfies condition (C) of Lemma 3.2. This observation along with the remarks before that lemma prove the following theorem.

Theorem 3.3. *The class of multi-path matroids is closed under deletion.*

To show that the class of multi-path matroids is closed under contractions, we give presentations of single-element contractions (Lemma 3.4) that we then show satisfy condition (C) of Lemma 3.2.

Lemma 3.4. *Let the antichain $\mathscr{I}$ of σ-intervals be a presentation of M, and let Σ be the cycle $(I_1, \ldots, I_t, I_{t+1}, \ldots, I_r)$ where $I_1, \ldots, I_t$ are the σ-intervals that contain a given element x. A presentation of the contraction M/x is given by:*

(a) *$\mathscr{I}$, for $t = 0$,*

(b) *$(I_2, I_3, \ldots, I_r)$, for $t = 1$,*

(c) *$\mathscr{I}' := \big((I_1 \cup I_2) - x, (I_2 \cup I_3) - x, \ldots, (I_{t-1} \cup I_t) - x, I_{t+1}, \ldots, I_r\big)$, for $t > 1$.*

Proof. Part (a) holds since x is a loop of M if t is 0. If t is positive, then x is not a loop, so the bases of M/x are the subsets B of $S - x$ such that $B \cup x$ is a basis of M. Part (b) follows since matchings $\phi : B \cup x \longrightarrow \mathscr{I}$ map x to I_1. For part (c), we need to show that for subsets B of $S - x$, there is a matching $\phi : B \cup x \longrightarrow \mathscr{I}$ if and only if there is a matching $\phi' : B \longrightarrow \mathscr{I}'$.

Assume first that $\phi : B \cup x \longrightarrow \mathscr{I}$ is a matching. Let $\phi(x)$ be I_h. For $1 \leqslant i \leqslant r$ with $i \neq h$, let b_i be the element $\phi^{-1}(I_i)$. The necessary matching ϕ' is given by

$$\phi'(b_i) = \begin{cases} (I_i \cup I_{i+1}) - x, & \text{if } 1 \leqslant i < h, \\ (I_{i-1} \cup I_i) - x, & \text{if } h < i \leqslant t, \\ I_i, & \text{if } t < i \leqslant r. \end{cases}$$

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Now assume $\phi' : B \longrightarrow \mathscr{I}'$ is a matching. For $1 \leqslant i < t$, let b_i be the element of B that is matched to $(I_i \cup I_{i+1}) - x$. Since x is in $I_1, I_2, \ldots, I_t$, to complete the proof it suffices to construct an injection $\psi : \{b_1, b_2, \ldots, b_{t-1}\} \longrightarrow \{I_1, I_2, \ldots, I_t\}$ with each b_i in $\psi(b_i)$. Toward this end, classify $b_1, b_2, \ldots, b_{t-1}$ as follows: b_i is a *leader* if it is in the first part of $I_i - x$, otherwise b_i is a *trailer*. Note that if b_i is a leader, then b_i is in the first part of $I_j - x$ for every j with $1 \leqslant j \leqslant i$. Similarly, if b_i is a trailer, then b_i is in the last part of $I_j - x$ for every j with $i+1 \leqslant j \leqslant t$. Define ψ as follows: scan $b_1, b_2, \ldots, b_{t-1}$ in this order and for each leader b_i, let $\psi(b_i)$ be the first set among $I_1, I_2, \ldots, I_t$ that is not already in the image of ψ; then scan $b_{t-1}, b_{t-2}, \ldots, b_1$ in this order and for each trailer b_i, let $\psi(b_i)$ be the last set among $I_1, I_2, \ldots, I_t$ not already in the image of ψ. Clearly ψ is injective and b_i is in $\psi(b_i)$ for all i. □

With this lemma, we can now complete our work on contractions.

Theorem 3.5. *The class of multi-path matroids is closed under contraction.*

Proof. We use the notation of Lemma 3.4. It suffices to show that M/x is a multi-path matroid. This follows easily from parts (a) and (b) of Lemma 3.4 if t is at most 1, so assume t exceeds 1. To show that M/x is a multi-path matroid, it suffices to show that $\mathscr{I}'$ satisfies condition (C) of Lemma 3.2. To consider the sets in $\mathscr{I}'$ as σ_{S-x}-intervals, we need only specify the endpoints of any set that is $S - x$. If the σ-interval I_h is $S - x$, where $t < h \leqslant r$, we take I_h to be the σ_{S-x}-interval $[\sigma(x), \sigma^{-1}(x)]$. If $(I_i \cup I_{i+1}) - x$ is $S - x$, we take this to be the σ_{S-x}-interval $[f_{I_i}, \sigma^{-1}(f_{I_i})]$. Note that there are only three possible containments among the sets in $\mathscr{I}'$:

(i) $(I_i \cup I_{i+1}) - x \subseteq (I_j \cup I_{j+1}) - x$ with $1 \leqslant i, j < t$,
(ii) $(I_i \cup I_{i+1}) - x \subseteq I_h$ with $1 \leqslant i < t$ and $t < h \leqslant r$, and
(iii) $I_h \subseteq (I_i \cup I_{i+1}) - x$ with $1 \leqslant i < t$ and $t < h \leqslant r$.

In case (i), note that if f_{I_j} is not in $(I_i \cup I_{i+1}) - x$, then $j < i$. It follows that $l_{I_{j+1}}$ is in the σ-interval $[\sigma(x), \sigma^{-1}(l_{I_{i+1}})]$, so $l_{I_{i+1}}$ is not in $(I_j \cup I_{j+1}) - x$. This contradicts the assumed containment, so f_{I_j} is in $(I_i \cup I_{i+1}) - x$ and condition (C) holds in case (i). Note that the containment in case (ii) holds only if I_h is $[\sigma(x), \sigma^{-1}(x)]$, so condition (C) clearly holds in this case also. Lastly, consider the containment in case (iii). Since x is not in I_h, if f_{I_i} were not in I_h, then I_h would be either contained in or disjoint from $[\sigma(f_{I_i}), \sigma^{-1}(x)]$, so either $I_h \subseteq I_i$ or $I_h \subseteq I_{i+1}$ would hold. That both conclusions are contrary to $\mathscr{I}$ being an antichain shows that f_{I_i} is in I_h, so condition (C) of Lemma 3.2 holds. Thus, M/x is a multi-path matroid. □

We now give an alternative perspective on multi-path matroids that accounts for the name, extends the path interpretation of lattice path matroids, and plays a pivotal role in much of the rest of this paper. Figure 4 illustrates these ideas with the 3-whirl (Example 1 in Section 2). Assume $M[\mathscr{I}]$ has rank r and nullity m. Fix an element x of $M[\mathscr{I}]$. (In Figure 4, x is 1.) Let the cyclic permutation Σ of $\mathscr{I}$ be $(I_1, I_2, \ldots, I_{k-1}, I_k, \ldots, I_r)$ where the intervals I_j with $x \in I_j$ and $x \neq f_{I_j}$ are $I_1, I_2, \ldots, I_{k-1}$. Note that the linear order $I_1, I_2, \ldots, I_r$, which plays an important role below, has been specified unless k is 1 or $k-1$ is r. For $k = 1$, let I_1 be the interval I in $\mathscr{I}$ that minimizes the size of the interval $[x, f_I]$. For $k - 1 = r$, let I_1 be the interval I in $\mathscr{I}$ that minimizes the size of the interval $[x, l_I]$.

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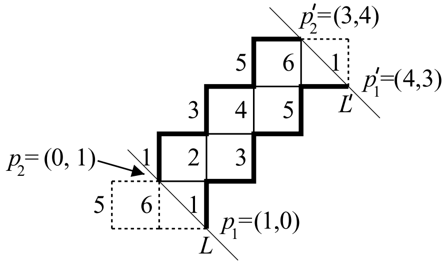

Figure 4. The diagram $D(\mathscr{W}^3, 1)$ with the labels on the North steps.

Consider the subsets $\{p_1, p_2, \ldots, p_k\}$ and $\{p_1', p_2', \ldots, p_k'\}$ of $\mathbb{Z}^2$ where $p_i = (k-i, i-1)$ and $p_i' = p_i + (m, r)$. Let L and L' be the lines of slope -1 that contain these sets. Let P be the lattice path from p_1 to p_1' formed from the sequence $x, \sigma(x), \sigma^2(x), \ldots, \sigma^{-1}(x)$ by replacing each element l_{I_j}, for $I_j \in \mathscr{I}$, by a North step and replacing the other elements by East steps. Let Q be the lattice path from p_k to p_k' formed from $x, \sigma(x), \sigma^2(x), \ldots, \sigma^{-1}(x)$ by replacing each element f_{I_j}, for $I_j \in \mathscr{I}$, by a North step and replacing the other elements by East steps. Note that P never goes above Q. The lines L and L' and the paths P and Q bound the region of interest. Label the North and East steps in this region as follows: steps that are adjacent to the points $p_1, p_2, \ldots, p_k$ are labelled x, those one step away from $p_1, p_2, \ldots, p_k$ are labelled $\sigma(x)$, and so on. The resulting diagram, which we denote by $D(\mathscr{I}, x)$, depends on both $\mathscr{I}$ and x. (To simplify the example, the diagram shown in Figure 4 omits the labels on the East steps.) The diagram $D(\mathscr{I}, x)$ captures the set system $\mathscr{I}$: each interval among $I_k, I_{k+1}, \ldots, I_r$ is the set of labels on the North steps in one row; each interval I_i among $I_1, I_2, \ldots, I_{k-1}$ also appears in this way, but split into two parts, with x and the elements in the last part of $I_i - x$ appearing among the lowest $k-1$ rows and with the elements in the first part of $I_i - x$ appearing among the highest $k-1$ rows. Theorem 3.6, which is a counterpart of Theorem 2.2, shows the significance of $D(\mathscr{I}, x)$.

Theorem 3.6. *Fix an element x in a multi-path matroid $M[\mathscr{I}]$. A set B is a basis of $M[\mathscr{I}]$ if and only if there is a lattice path R such that:*

(i) *R goes from a point p_i to the corresponding point p_i',*
(ii) *R uses East and North steps of the diagram $D(\mathscr{I}, x)$, and*
(iii) *the labels on the North steps of R are the elements of B.*

Proof. Let $b_1, b_2, \ldots, b_r$, in this order, be the labels on the North steps of a path R that satisfies conditions (i) and (ii). Thus, $b_1, b_2, \ldots, b_r$ are contained, respectively, in r consecutive intervals in the cycle $(I_1, I_2, \ldots, I_r)$; also, $b_1, b_2, \ldots, b_r$ are distinct since the North and East steps of R are labelled, in order, $x, \sigma(x), \sigma^2(x), \ldots, \sigma^{-1}(x)$. It follows that $\{b_1, b_2, \ldots, b_r\}$ is a basis of $M[\mathscr{I}]$.

For the converse, we use the notation established when defining the diagram $D(\mathscr{I}, x)$. All references to an order on the ground set S are to the linear order $x < \sigma(x) < \sigma^2(x) < \cdots < \sigma^{-1}(x)$. Assume B is a basis of $M[\mathscr{I}]$ and let $\phi : B \longrightarrow \mathscr{I}$ be a matching. To complete the proof, it suffices to prove the following claim.

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The elements of B, listed in order as $b_1, b_2, \ldots, b_r$, are in the sets

$$I_{k-t}, I_{k-t+1}, \ldots, I_{k-1}, I_k, \ldots, I_r, I_1, I_2, \ldots, I_{k-t-1},$$

respectively, for some t with $0 \leqslant t \leqslant k-1$.

Indeed, the required path R takes East steps from p_{k-t} until a North step labelled b_1 is reached; after taking that North step, East steps are taken until a North step labelled b_2 is reached, and so on.

We prove the claim by first constructing a matching for a different set system. For i with $1 \leqslant i \leqslant k-1$, let X_i be the first part of I_i with respect to x and let Y_i be $I_i - X_i$. Note that the set $\phi^{-1}(\{I_1, I_2, \ldots, I_{k-1}\})$ is the disjoint union of two subsets whose elements are, in order, say, $b'_1, b'_2, \ldots, b'_t$ and $b''_1, b''_2, \ldots, b''_{k-1-t}$, where each b'_i is in the subset Y_j of the set $I_j = \phi(b'_i)$ while each b''_i is in the subset X_j of the set $I_j = \phi(b''_i)$. Thus, $0 \leqslant t \leqslant k-1$. Let $\mathscr{I}'$ be the set system that consists of the intervals

$$Y_{k-t}, Y_{k-t+1}, \ldots, Y_{k-1}, I_k, \ldots, I_r, X_1, X_2, \ldots, X_{k-t-1}.$$

We also let $Z_1, Z_2, \ldots, Z_r$, respectively, denote these intervals. Let $\Phi : B \longrightarrow \mathscr{I}'$ be given by

$$\Phi(b) = \begin{cases} Y_{k-1-t+i}, & \text{if } b = b'_i \text{ with } 1 \leqslant i \leqslant t, \\ X_i, & \text{if } b = b''_i \text{ with } 1 \leqslant i \leqslant k-1-t, \\ \phi(b), & \text{if } b \in \phi^{-1}(\{I_k, I_{k+1}, \ldots, I_r\}). \end{cases}$$

The inclusions $Y_1 \subset Y_2 \subset \cdots \subset Y_{k-1}$ and $X_{k-1} \subset X_{k-2} \subset \cdots \subset X_1$ imply that Φ is a matching.

Finally, to prove the claim it suffices to show that the i-th element b_i of B is in Z_i. If this statement were false, then either $b_i < f_{Z_i}$ or $b_i > l_{Z_i}$. The first option would imply that the i elements $b_1, b_2, \ldots, b_i$ can be in only $i-1$ sets, namely $Z_1, Z_2, \ldots, Z_{i-1}$; the second option would imply that the $r-i+1$ elements $b_i, b_{i+1}, \ldots, b_r$ can be in only $r-i$ sets, namely $Z_{i+1}, Z_{i+2}, \ldots, Z_r$. Both conclusions are contradicted by the matching Φ, so the claim and the theorem follow. □

Unlike the map given in Theorem 2.2, the correspondence between paths and bases in Theorem 3.6 is not bijective. For example, the two paths in Figure 4 indicated by thick lines correspond to the basis $\{1, 3, 5\}$. While some bases correspond to a single path, in general each basis corresponds to a family of paths that arise from a single word in the alphabet $\{E, N\}$ but starting at different points among $p_1, p_2, \ldots, p_k$.

Note that rotating the diagram $D(\mathscr{I}, x)$ by 180° about the point $(\frac{m+k-1}{2}, \frac{r+k-1}{2})$ gives the diagram $D(\mathscr{I}, \sigma^{-1}(x))$, using the cycle σ^{-1} in place of σ.

Reflecting the diagram $D(\mathscr{I}, x)$ in the line $y = x$ interchanges the East and North steps. Let $D^*(\mathscr{I}, x)$ denote this reflected diagram. A set X is the set of labels on the North steps of a path in $D^*(\mathscr{I}, x)$ that satisfies conditions (i) and (ii) of Theorem 3.6 if and only if X is the complement of a basis of $M[\mathscr{I}]$. Thus, as illustrated in Figure 5, $D^*(\mathscr{I}, x)$ is a lattice path representation of the dual matroid $M^*[\mathscr{I}]$. Some argument is required, however, to show that $M^*[\mathscr{I}]$ is a multi-path matroid since the set of σ-intervals one obtains from $D^*(\mathscr{I}, x)$ need not be an antichain; in particular, the ground set S may be among these σ-intervals. For instance, the East steps of a column of $D(\mathscr{I}, x)$ (for example,

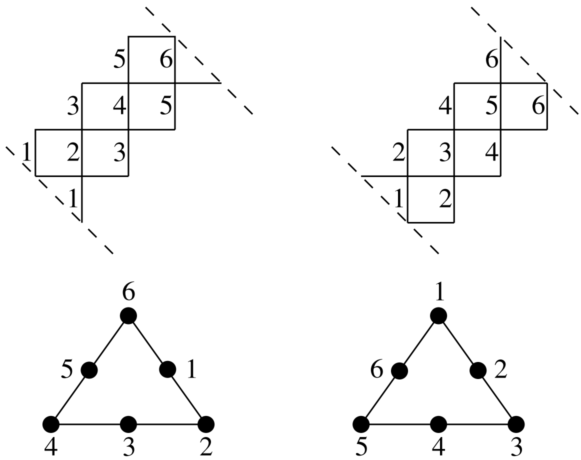

Figure 5. The dual of the 3-whirl $\mathcal{W}^3$ via flipping diagrams about the line $y = x$.

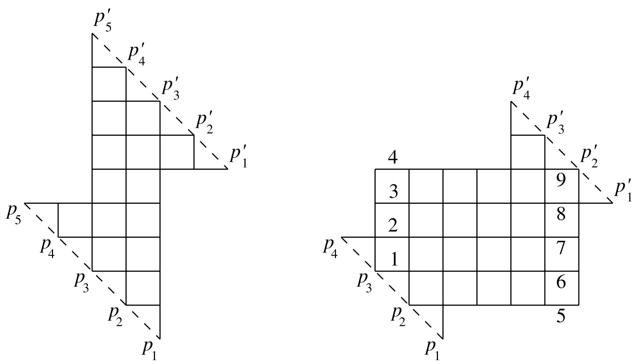

Figure 6. Two diagrams $D(\mathcal{I}, x)$ that, after reflection in the line $y = x$, give set systems that are not antichains.

the column between p_1 and p_2, or that between p_2 and p_3, in the first diagram in Figure 6) may be labelled with all elements of S. Also, the first part of an interval that includes x, say between p_i and p_{i+1}, must be joined with the corresponding last part between p'_i and p'_{i+1}, and this union may be S; the second diagram in Figure 6 illustrates this point with the column between p_2 and p_3 and that between p'_2 and p'_3 (the East steps in these columns are labelled). One way to address this problem, in the spirit of the proofs of Theorems 3.3 and 3.5, would be to show how to modify the set system that corresponds to $D^*(\mathcal{I}, x)$ to obtain a presentation of $M^*[\mathcal{I}]$ by an antichain of σ-intervals. Instead, we introduce a more general type of diagram (which plays a key role in Section 4) and show that for such a diagram D, the sets of labels of the North steps of the paths of D that satisfy conditions (i) and (ii) of Theorem 3.6 are the bases of a multi-path matroid. To avoid excess terminology, we also call these more general objects, which we define below, diagrams; this should create no confusion.

A *diagram* D is a 5-tuple (k, m, r, P, Q), where k is a positive integer, m and r are non-negative integers, P is a lattice path from $(k-1, 0)$ to $(k-1+m, r)$, and Q is a lattice

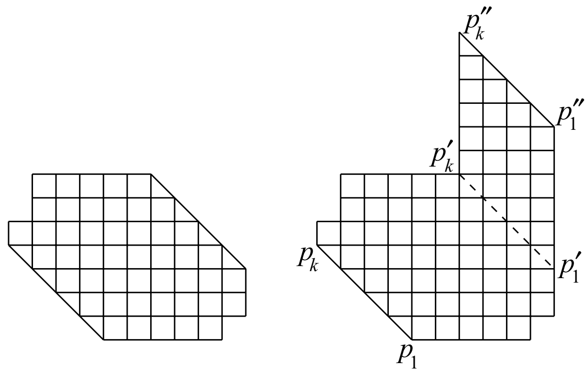

Figure 7. The diagram $D = (5,6,3,E^5NEN^2,NEN^2E^5)$ and its extension $D' = (5,6,9,E^5NEN^8,NEN^2E^5N^6)$.

path from $(0,k-1)$ to $(m,k-1+r)$ that never goes below P. For i with $1 \leqslant i \leqslant k$, let p_i be $(k-i,i-1)$ and let p'_i be $p_i + (m,r)$. Let L and L' be the lines of slope -1 that contain the points p_i and p'_i, respectively. The *region* $R(D)$ of a diagram D is the set of points in $\mathbb{R}^2$, including the boundary, enclosed by the paths P and Q and the lines L and L'. The *edges* of D are the segments between lattice points in D that are distance 1 apart. Assign label i to an edge in D if it is the i-th step in some lattice path that starts at a point on L; thus, edges are labelled with the elements of $[m+r]$. A *b-path* is a lattice path contained in the region $R(D)$ that starts at a point p_i and ends in the corresponding point p'_i. Thus, any b-path contains r North steps and m East steps, and the edges are labelled, in order, $1,2,\ldots,m+r$. The *label-set* of a b-path T is the set of labels on the North steps of T. Let $\mathscr{B}(D)$ be the set of all label-sets of b-paths in D. We now show that $\mathscr{B}(D)$ is the set of bases of a multi-path matroid, which we denote by $M[D]$. (To recover multi-path matroids in complete generality, replace the labels $1,2,\ldots,m+r$ with the elements $x,\sigma(x),\ldots,\sigma^{-1}(x)$, respectively. In much of the rest of the paper, we favor the notational simplicity gained by letting $[m+r]$ be the ground set of $M[D]$.)

Theorem 3.7. *For any diagram $D = (k,m,r,P,Q)$, the collection $\mathscr{B}(D)$ of subsets of $[m+r]$ is the set of bases of a multi-path matroid.*

Proof. By Theorem 3.5, it is sufficient to prove that $\mathscr{B}(D)$ is the set of bases of a contraction of some multi-path matroid $M[\mathscr{I}]$. Toward this end, let D' be the diagram $(k,m,r+k+1,PN^{k+1},QN^{k+1})$. (See Figure 7.) For i with $1 \leqslant i \leqslant k$, let p_i and p'_i be as above and let p''_i be $p'_i + (0,k+1)$. Denote the rows of D', from the bottom up, by $R_1,R_2,\ldots,R_{r+2k}$. Let σ be the cycle $(1,2,\ldots,m+r+k+1)$. Let $\mathscr{I}$ consist of the following sets: I_j, for j with $1 \leqslant j < k$, is the union of the set of labels on the North steps of row R_j and that of row $R_{r+k+j+1}$; the set I_j, for j with $k \leqslant j \leqslant k+r+1$, consists of the labels on the North steps in row R_j. Each set I_j is a σ-interval and D' is the diagram $D(\mathscr{I},1)$ for $\mathscr{I}$. We claim that $\mathscr{I}$ is an antichain. Note that each set I_j has at most $m+k$ elements and so is a proper subset of $[m+r+k+1]$. The sets $I_k,I_{k+1},\ldots,I_{k+r+1}$ form an antichain since we have $f_{I_k} < f_{I_{k+1}} < \cdots < f_{I_{k+r+1}}$ and $l_{I_k} < l_{I_{k+1}} < \cdots < l_{I_{k+r+1}}$ for these intervals in

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the usual linear order on $[m+r+k+1]$. A similar argument shows that $I_1, I_2, \ldots, I_{k-1}$ form an antichain. Now consider I_h and I_j with $1 \leqslant h < k \leqslant j \leqslant r+k+1$. At least one of 1 and $m+r+k+1$ is not in I_j, so $I_h \nsubseteq I_j$. The containment $I_j \subset I_h$ would imply that either $I_j \subseteq [f_{I_h}, m+r+k+1]$ or $I_j \subseteq [1, l_{I_h}]$ holds. The first inclusion contradicts the inequality $1 \leqslant f_{I_j} < f_{I_h} \leqslant m+r+k+1$ that is evident from the diagram D'; the second containment contradicts the inequality $1 \leqslant l_{I_h} < l_{I_j} \leqslant m+r+k+1$ that is also evident from D'. Thus, $\mathscr{I}$ is an antichain of σ-intervals.

Let Z consist of the last $k+1$ elements of $[m+r+k+1]$. We now show that $\mathscr{B}(D)$ is the set of bases of the contraction of the multi-path matroid $M[\mathscr{I}]$ by Z. Since Z is independent in $M[\mathscr{I}]$, the bases of $M[\mathscr{I}]/Z$ are the subsets B of $[m+r]$ for which $B \cup Z$ is a basis of $M[\mathscr{I}]$. Note that the last $k+1$ steps in any lattice path whose label set is $B \cup Z$ are North steps that go from a point p_i' to the corresponding point p_i''. Thus, $B \cup Z$ is a basis of $M[\mathscr{I}]$ if and only if $B \cup Z$ is the label set of a path in D' that goes from some point p_i to the corresponding point p_i'' through the point p_i'. It follows that B is a basis of $M[\mathscr{I}]/Z$ if and only if B is the label set of a b-path in D, that is, if and only if B is in $\mathscr{B}(D)$, as claimed. □

That arbitrary diagrams define multi-path matroids allows us to give another perspective on certain minors. (Since the proof of Theorem 3.7 uses Theorem 3.5, this does not replace our earlier work.) Let M be the multi-path matroid on $[m+r]$ that is represented by a diagram D. Let X and Y be disjoint subsets of $[m+r]$ where Y is independent, X is coindependent (*i.e.*, the complement of a spanning set), and $X \cup Y$ consists of the last k elements of $[m+r]$. From the formulation of minors in terms of bases, it follows that the bases of the minor $M\backslash X/Y$ correspond to the paths in D whose last k steps are determined: these steps are East or North according to whether their labels are in X or Y, respectively. The initial segments of paths in D whose last k steps are as specified make up a smaller diagram D' that represents the minor $M\backslash X/Y$. This observation, which is behind the proof of Theorem 3.7, plays an important role in the next section.

The next theorem summarizes the results in this section. The assertion about duality follows from Theorem 3.7 and the remarks before that theorem.

Theorem 3.8. *The class of multi-path matroids is dual-closed, minor-closed, and properly contains the class of lattice path matroids.*

4. Tutte polynomial

The Tutte polynomial has received considerable attention, in part due to its many striking properties (*e.g.*, it is the universal deletion–contraction invariant) and its many important evaluations (*e.g.*, the chromatic and flow polynomials of a graph, the weight enumerator of a linear code, and the Jones polynomial of an alternating knot). (See [4, 12].) In this section, we show that the Tutte polynomial of a multi-path matroid can be computed in polynomial time. This result stands in contrast to the hardness results known for computing the Tutte polynomial of an arbitrary member of many classes of matroids [5, 7, 8, 10, 11]. We cast our work on the Tutte polynomial in a broader framework; we introduce what we call computation graphs, which allow us to apply dynamic programming.

The Tutte polynomial $t(M; x, y)$ of a matroid M on the ground set S can be defined in a variety of ways, perhaps the most basic of which is the following:

$$t(M; x, y) = \sum_{A \subseteq S} (x-1)^{r(S)-r(A)} (y-1)^{|A|-r(A)}. \tag{4.1}$$

The following recurrence relation is more suited to our work. The Tutte polynomial $t(M; x, y)$ is 1 if M is the empty matroid; otherwise, for any element e of M,

$$t(M; x, y) = \begin{cases} x\, t(M/e; x, y) & \text{if } e \text{ is an isthmus,} \\ y\, t(M\backslash e; x, y) & \text{if } e \text{ is a loop,} \\ t(M/e; x, y) + t(M\backslash e; x, y) & \text{otherwise.} \end{cases} \tag{4.2}$$

As stated, both of these formulations require roughly $2^{|S|}$ computations. We take advantage of the fact that, for a multi-path matroid, the recurrence relation can be applied in a manner that involves minors that are easily recognized to be equal; more precisely, the number of different minors that need to be considered turns out to be polynomial in $|S|$, and this allows us to organize the computation in a way that runs in polynomial time. Before turning to multi-path matroids, we establish a general framework for computations of this type.

Let M be a matroid on the set $[n]$. To use the recurrence relation (4.2), it suffices to consider what we will call the *initial minors* of M, that is, the matroids formed by deleting or contracting, in turn, $n, n-1, \ldots, h+2, h+1$, where at the stage at which an element is deleted, it is not an isthmus, and at the stage at which an element is contracted, it is not a loop. The ground set of an initial minor is an initial segment $[h]$ of $[n]$. Note that if $M\backslash X/Y$ is an initial minor, then Y is independent and X is coindependent.

We define a *computation graph* G for the matroid M to be an edge-labelled directed graph with label set $\{c, d\}$ that satisfies the following conditions.

(1) Each vertex u represents an initial minor M_u of M. Every initial minor of M is represented by at least one vertex.

(2) Let u be a vertex and let h be the greatest element of the initial minor M_u. If there is a d-edge from u to v, then $M_u\backslash h = M_v$. If there is a c-edge from u to w, then $M_u/h = M_w$. In addition,

   (a) if h is an isthmus of M_u, then u is the tail of exactly one c-edge and no d-edge,
   (b) if h is a loop of M_u, then u is the tail of exactly one d-edge and no c-edge,
   (c) otherwise u is the tail of exactly one c-edge and one d-edge.

(3) There are two distinguished vertices v_M and $v_\emptyset$; these are the unique vertices that represent the matroid M and the empty matroid, respectively.

By point (1), to construct a computation graph G for a matroid M by using some representation (*e.g.*, a multi-path diagram), apart from the trivial cases in point (3) we are not required to determine whether different representations give the same minor. Note that the restrictions imposed on the edges imply that u is at distance h from $v_\emptyset$ if and only if M_u has h elements; let V_h be the set of such vertices u. Then $\{V_0, V_1, \ldots, V_n\}$ is a partition of the vertices of G and any edge that has its tail in V_h has its head in V_{h-1}.

Recurrence relation (4.2) allows us to compute $t(M; x, y)$ from the computation graph G. There is a trade-off between several factors that enter into the computation graph: having fewer vertices allows us to compute the Tutte polynomial more quickly, but getting fewer vertices requires recognizing that many initial minors (perhaps with different representations) are equal. A typical application of these ideas would yield a computation graph with polynomially many vertices without determining all instances of equal initial minors. The following lemma helps quantify these observations.

Lemma 4.1. *For a matroid M of positive rank r and positive nullity m, the Tutte polynomial $t(M; x, y)$ can be computed from a computation graph G on v vertices in $\mathcal{O}(vrm)$ operations.*

Proof. Partition the vertices of G into blocks $V_0, \ldots, V_{m+r}$, as described before; since G has no oriented cycles this can be done with $\mathcal{O}(v)$ operations. Assign to every vertex u the Tutte polynomial $t(M_u; x, y)$ in the following manner. First assign 1 to the unique vertex $v_\emptyset$ in V_0, then compute the Tutte polynomials for all vertices in V_1, then those for all vertices of V_2, and so on. To compute the Tutte polynomial $t(M_u; x, y)$ for u in V_h, apply recurrence relation (4.2): by condition (2) in the definition of a computation graph, the edges for which u is the tail indicate which of the three cases of the recurrence to use, and the Tutte polynomials of $M_u \backslash h$ and M_u / h have already been computed because they correspond to vertices of V_{h-1}. Thus, for every vertex u we just need to add two polynomials or multiply a polynomial by x or y, and this can be done in $\mathcal{O}(rm)$ operations since $t(M; x, y)$ has at most $rm + r + m$ coefficients. Hence we can compute the Tutte polynomial of every initial minor of M, including M itself, in $\mathcal{O}(v + vrm)$, that is, $\mathcal{O}(vrm)$, operations. □

We now focus on the multi-path matroid $M[\mathscr{I}]$, or M, on $[m+r]$, where σ is the cycle $(1, 2, \ldots, m+r)$. Let D be the diagram $D(\mathscr{I}, 1)$. We first study the initial minors $M \backslash X / Y$ that arise in constructing a computation graph for M, and to do so we work with the diagrams introduced in Section 3. In particular, we show how to obtain a diagram D' for any initial minor $M \backslash X / Y$. The resulting diagrams need not arise from presentations by antichains of σ-intervals.

It follows from the basis formulation of deletion and contraction that a subset B of $[m+r] - (X \cup Y)$ is a basis of $M \backslash X / Y$ if and only if $B \cup Y$ is a basis of M (recall that X and Y are, respectively, coindependent and independent). These bases, by Theorem 3.6, correspond to b-paths where the last $q = |X \cup Y|$ steps are determined: steps corresponding to elements of Y are North and steps corresponding to elements of X are East. Let a and b be the smallest and largest integers i such that there is a path from p_i to p'_i in D with the last q steps as specified by X and Y. (See Figure 8.) For $a \leqslant i \leqslant b$, let p''_i be the point $p'_i - (|X|, |Y|)$; thus any path from p_i to p'_i whose last q steps are as specified by X and Y goes through the point p''_i. Let P' be the lattice path in D from p_a to p''_a that no path in D from p_a to p''_a goes below; similarly, let Q' be the lattice path in D from p_b to p''_b that no path in D from p_b to p''_b goes above. Let D' be the diagram that has $p_a, \ldots, p_b$ as starting points, $p''_a, \ldots, p''_b$ as ending points, and P' and Q' as the bottom and top border. Thus, if D is (k, m, r, P, Q), then D' is $(b - a + 1, m - |X|, r - |Y|, P', Q')$.

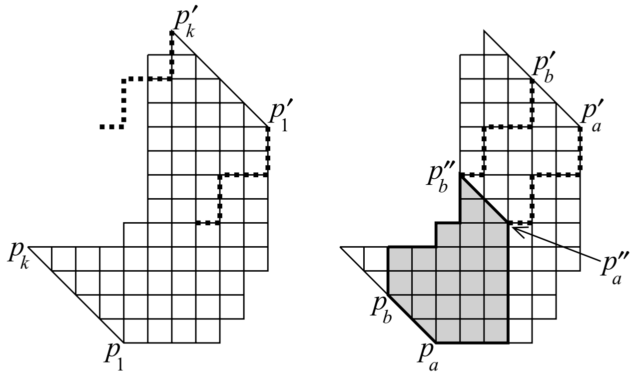

Figure 8. The shaded region in the second diagram represents the initial minor $M/\{15,14,11,10\}\backslash\{13,12,9\}$.

Lemma 4.2. *Let $D=(k,m,r,P,Q)$ be the diagram $D(\mathscr{I},1)$ of a multi-path matroid M on $[n]$.*

(1) *We can construct from D a diagram D' corresponding to an initial minor $M\backslash X/Y$ in $\mathcal{O}(n)$ operations.*

(2) *We can construct from D at most $(n+1)\big(min(r,m)+1\big)(k^2+k)/2$ different diagrams D' corresponding to initial minors of M. In particular, M has at most this many initial minors.*

Proof. The description above for constructing D' from D has two parts: find a and b, and then find P' and Q'. We sketch how to do these two steps. Since X and Y are coindependent and independent, a and b exist; find them by comparing the last $|X\cup Y|$ steps of P and Q with the steps specified by X and Y. (See Figure 8. The dotted paths are those specified by X and Y.) Specifically, let $\mathrm{N}(W,i)$ be the number of North steps among the last i steps of a path W and let $P_{X,Y}$ be the path specified by X and Y; then a is

$$\max\{\mathrm{N}(P_{X,Y},i)-\mathrm{N}(P,i)\,:\,0\leqslant i\leqslant|X\cup Y|\}+1.$$

A similar formula gives b, so a and b can be computed in $\mathcal{O}(n)$ operations. Construct P' (respectively, Q') by going from p_a to p''_a (respectively, from p_b to p''_b), taking East (respectively, North) steps whenever possible. This also takes $\mathcal{O}(n)$ operations.

Assertion (2) follows by noting that a diagram is completely determined by (i) the size of $X\cup Y$, (ii) the size of either X or Y, (iii) the points p''_a and p''_b, and that these two points are determined by the two numbers $a\leqslant b$ between 1 and k. □

We now show how to compute the Tutte polynomial of a multi-path matroid M in polynomial time from its diagram $D=D(\mathscr{I},1)$. We start by constructing a computation graph for M whose vertices correspond to the diagrams of initial minors of M that are obtained from D as described before Lemma 4.2. Start with a graph that consists of just one vertex v_M that corresponds to the diagram D and iterate the following process.

- Choose a vertex v other than $v_\emptyset$ with outdegree 0. Let M', on $[h]$, and D' be the corresponding initial minor and diagram.

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- Compute the diagrams D_d and D_c corresponding to $M'\backslash h$, if h is not an isthmus, and M'/h, if h is not a loop, as described before Lemma 4.2.
- Find a vertex v_d (respectively, v_c) in the computation graph that corresponds to D_d (respectively, D_c); if there is no such vertex, add a new vertex to the computation graph. Add a d-edge (respectively, a c-edge) from v to v_d (respectively, v_c).

Stop when the only vertex of outdegree 0 is $v_\emptyset$, which corresponds to the empty matroid. The resulting graph G is clearly a computation graph for M. The same initial minor M' can be represented more than once in G since different diagrams can represent it, but each diagram D' appears just once and all diagrams have been derived from D. By part (2) of Lemma 4.2, the number v of vertices of G is $\mathcal{O}(n \min(r, m) k^2)$. By Lemma 4.1, we can compute $t(M; x, y)$ from G in $\mathcal{O}(rm\mathrm{v})$ operations. So now we need only show that this construction of the computation graph can be done in polynomial time.

We show that we can construct G in $\mathcal{O}(\mathrm{v}\, n \log \mathrm{v})$ operations. Consider the operations required for each iteration of the algorithm (each expansion of a vertex v of outdegree 0). First we compute D_d and D_c in $\mathcal{O}(n)$ operations and then we check whether they are already in the graph. Comparing two diagrams (*i.e.*, 5-tuples) requires $\mathcal{O}(n)$ operations; by using a suitable ordering of the vertices, a binary search using $\mathcal{O}(\log \mathrm{v})$ comparisons suffices to determine whether a given diagram is already in the graph. Thus we need $\mathcal{O}(n \log \mathrm{v})$ operations for any of the v iterations, so G can be constructed in $\mathcal{O}(\mathrm{v}\, n \log \mathrm{v})$ operations.

Hence the number of operations needed to construct this computation graph and obtain the Tutte polynomial from it is $\mathcal{O}\big(\mathrm{v}(rm + n \log \mathrm{v})\big)$. Three remarks allow us to simplify the expression for the number of operations required; first, $r + m$ is n; second, since the diagram D represents an antichain of σ-intervals, the inequality $k \leqslant n$ follows; lastly, $\log \mathrm{v}$ is $\mathcal{O}(\log n)$ because v depends polynomially on n. Thus, the work in this section gives the following theorem.

Theorem 4.3. *We can compute the Tutte polynomial of a multi-path matroid on n elements in $\mathcal{O}(n^6)$ operations.*

5. Basis activities

Another formulation of the Tutte polynomial is given by basis activities, which are also of independent interest. In this section, we describe the internal and external activities of bases of multi-path matroids in terms of lattice paths in diagrams and we sketch an alternative approach to computing the Tutte polynomial of a multi-path matroid through basis activities.

The Tutte polynomial of M can be written as

$$t(M; x, y) = \sum_{B \in \mathscr{B}(M)} x^{i(B)} y^{e(B)}, \tag{5.1}$$

where $\mathscr{B}(M)$ is the collection of bases of M and the exponents $i(B)$ and $e(B)$ are defined as follows. Fix a linear order $<$ on the ground set S of M and let B be a basis of M. An element u in $S - B$ is *externally active with respect to* B if there is no element v in B with $v < u$ for which $(B - v) \cup u$ is a basis. An element u in B is *internally active with respect to* B if there is no element v in $S - B$ with $v < u$ for which $(B - u) \cup v$ is a basis.

The *internal activity* $i(B)$ of a basis B is the number of elements that are internally active with respect to B. The external activity of B, denoted $e(B)$, is defined similarly. Note that $i(B)$ and $e(B)$ depend not only on B but also on the order. Equation (5.1) says that the coefficient of $x^i y^e$ in $t(M;x,y)$ is the number of bases of M with internal activity i and external activity e. In particular, the number of such bases is independent of the order.

We will use the following lemma, which is well known and easy to prove.

Lemma 5.1. *Fix a linear order on the ground set S of a matroid M and its dual M^*. An element u is internally active with respect to the basis B of M if and only if u is externally active with respect to the basis $S-B$ of M^*.*

Throughout this section we use the notation and terminology we establish in the next several paragraphs. We assume that the ground set of the multi-path matroid $M[\mathscr{I}]$ is $[m+r]$ and that σ is the cycle $(1,2,\ldots,m+r)$. We study the internal and external activities of the bases of $M[\mathscr{I}]$ relative to the linear order $1<2<\cdots<m+r$. Let D be the diagram $D(\mathscr{I},1)=(k,m,r,P,Q)$; recall that P and Q are, respectively, the bottom and top border of the diagram.

For any subset X of $[m+r]$ the *representation* $\Pi(X,p)$ *of X starting at the lattice point p* is the path of $m+r$ steps that starts at p whose u-th step is N if u is in X, and E otherwise. We say that a path is *valid* if it is entirely contained in the diagram D. Thus, Theorem 3.6 states that the bases of $M[\mathscr{I}]$ are the sets B such that, for some p_i, the path $\Pi(B,p_i)$ is valid and ends at the corresponding point p'_i. Note that if $\Pi(B,p_i)$ and $\Pi(B,p_j)$ are both valid paths for $i<j$, then all paths $\Pi(B,p_t)$ with $i<t<j$ are also valid.

For v,u in $[m+r]$ with $v\leqslant u+1$, we use $[v,u]\Pi(X,p_i)$ to denote the path that starts at the beginning of the v-th step of $\Pi(X,p_i)$ and follows this path until the end of the u-th step. Similarly, $(v,u]\Pi(X,p_i)$, $[v,u)\Pi(X,p_i)$, and $(v,u)\Pi(X,p_i)$ are defined in the obvious way; for instance $(v,u)\Pi(X,p_i)$ is $[v+1,u-1]\Pi(X,p_i)$. In particular, $(v,u)\Pi(X,p_i)$ is defined when $v\leqslant u-1$, and $(u-1,u)\Pi(X,p_i)$ consists of the single point that is common to steps $u-1$ and u of $\Pi(X,p_i)$.

The following lemma gives the conditions under which an element u of a basis B can be replaced by an element v to yield a basis B'.

Lemma 5.2. *Let B be a basis of a multi-path matroid $M[\mathscr{I}]$ with $u\in B$, $v\notin B$. Let $\Pi(B,p_i)$ be a valid path. Let B' be $(B-u)\cup v$.*

(1) *If $v<u$, then B' is a basis if and only if either*

  (a) *the path $(v,u)\Pi(B,p_i)$ does not touch the top border Q, or*

  (b) *neither $[1,v)\Pi(B,p_i)$ nor $(u,m+r]\Pi(B,p_i)$ touches the bottom border P.*

(2) *If $u<v$, then B' is a basis if and only if either*

  (a) *the path $(u,v)\Pi(B,p_i)$ does not touch P, or*

  (b) *neither $[1,u)\Pi(B,p_i)$ nor $(v,m+r]\Pi(B,p_i)$ touches Q.*

Proof. By duality it suffices to prove the first claim. By Theorem 3.6, B' is a basis if and only if B' has a valid representation. Compare the paths $\Pi(B',p_i)$ and $\Pi(B',p_{i-1})$ with

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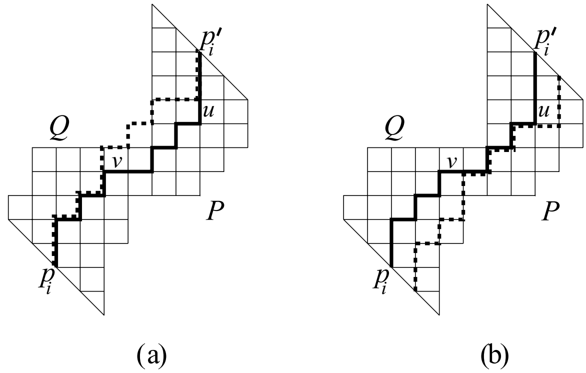

Figure 9. Paths $\Pi(B', p_i)$ (dotted line) and $\Pi(B, p_i)$ in part (a), and $\Pi(B', p_{i-1})$ (dotted line) and $\Pi(B, p_i)$ in part (b).

$\Pi(B, p_i)$. (See Figure 9.) Since $\Pi(B', p_i)$ is above $\Pi(B, p_i)$, only Q may prevent $\Pi(B', p_i)$ from being valid, and in that case no $\Pi(B', p_j)$ with $j \geqslant i$ would be valid; similarly, only P may prevent $\Pi(B', p_{i-1})$ from being valid, and in that case no $\Pi(B', p_j)$ with $j \leqslant i-1$ would be valid. Thus B' is a basis if and only if either $\Pi(B', p_i)$ or $\Pi(B', p_{i-1})$ is valid. These two conditions are equivalent to conditions (1(a)) and (1(b)), as Figure 9 illustrates. □

With the help of this basis exchange lemma we now characterize the internally and externally active elements.

Theorem 5.3. *Let B be a basis of a multi-path matroid $M[\mathscr{I}]$ and let $\Pi(B, p_i)$ be a valid path.*

(I) *An element u in B is internally active if and only if either*

(a) $[u] \subseteq B$, *or*

(b) *the u-th step of $\Pi(B, p_i)$ lies in the top border Q and $(u, m+r]\Pi(B, p_i)$ touches the bottom border P.*

(II) *An element u not in B is externally active if and only if either*

(a) $[u] \cap B = \emptyset$, *or*

(b) *the u-th step of $\Pi(B, p_i)$ lies in P and $(u, m+r]\Pi(B, p_i)$ touches Q.*

Proof. By duality, we only need to prove part (I).

Note that u is internally active if $[u] \subseteq B$. Thus, let V be $[u] - B$ and assume that V is not empty. Sufficiency follows because if u satisfies condition (I(b)), then it satisfies neither conditions (1(a)) nor (1(b)) of Lemma 5.2 for any v in V.

To prove the converse assume that u is internally active. Let v be $\max(V)$. Since u is internally active, $(B - u) \cup v$ is not a basis, so by condition (1(a)) of Lemma 5.2 the path $(v, u)\Pi(B, p_i)$ touches Q. This path has only North steps, by the choice of v, so its ending point has to touch Q. This proves that the u-th step of $\Pi(B, p_i)$ lies in Q, so the first part of condition (I(b)) holds.

For the second part, let v be $\min(V)$. By condition (1(b)) of Lemma 5.2, since $(B - u) \cup v$ is not a basis, at least one of the paths $[1, v)\Pi(B, p_i)$ and $(u, m + r]\Pi(B, p_i)$ touches P. We show that if the first path touches P, then so does the second, hence either way $(u, m + r]\Pi(B, p_i)$ touches P, which proves the second part of condition (I(b)). Indeed, the minimality of v implies that $[1, v)\Pi(B, p_i)$ has only North steps. So if $[1, v)\Pi(B, p_i)$ touches P, then it has to touch it from the beginning, that is, p_i has to be p_1. Hence $(u, m + r]\Pi(B, p_i)$ touches P at its ending point p'_1. □

Theorem 5.3 shows that we can find the internal and external activities of a basis by looking at just one of its representations in the diagram D. This reduces the problem of counting the number of bases with given internal and external activities to the problem of counting the number of lattice paths of a certain kind in D. In the remainder of this section we sketch a polynomial-time algorithm that computes this number of bases. Note that by equation (5.1) this yields a different approach to computing the Tutte polynomial of a multi-path matroid; this approach is slightly quicker than that in the previous section, but it requires keeping track of more details.

The algorithm uses the characterization of activities in Theorem 5.3. Of the conditions in that result, conditions (b) are somewhat more difficult to deal with; we introduce the notion of pseudo-activities to count the steps that are active by conditions (b). Let R be a valid path in the diagram D that ends in one of the points $p'_1, \ldots, p'_k$. Let s be one of its steps and let R_s be the path that starts at the end of step s and follows R until its end. We say that s is *pseudo-internally active* in R if it is a North step that lies in the top border Q and the path R_s touches the bottom border P. Similarly we say that s is *pseudo-externally active* in R if it is an East step that lies in P and R_s touches Q. Note that, unlike activities, pseudo-activities are not defined for bases, but for paths that end at one of the points $p'_1, \ldots, p'_k$ (*e.g.*, the final segments of paths that correspond to bases).

Let p be a lattice point of the diagram D and let p'_i be one of the ending points $p'_1, \ldots, p'_k$. Let a and b be integers with $0 \leqslant a \leqslant r$ and $0 \leqslant b \leqslant m$. Let τ_P and τ_Q be variables that can take on the values *true* and *false*. We define $\Gamma(p, p'_i, a, b, \tau_P, \tau_Q)$ to be the number of valid lattice paths starting at p and ending at p'_i (consisting of one point if $p = p'_i$), with a pseudo-internally active steps and b pseudo-externally active steps, and touching P if and only if τ_P is *true*, and touching Q if and only if τ_Q is *true*. In particular, if either τ_P or τ_Q is false and either a or b is greater than zero, then $\Gamma(p, p'_i, a, b, \tau_P, \tau_Q) = 0$. The function Γ satisfies an easily verified, multi-part recurrence relation of which we mention just two parts. Let γ be $\Gamma(p, p'_i, a, b, \tau_P, \tau_Q)$ and let p_E and p_N be, respectively, $p + (1, 0)$ and $p + (0, 1)$. If p is in neither P nor Q, then

$$\gamma = \Gamma(p_N, p'_i, a, b, \tau_P, \tau_Q) + \Gamma(p_E, p'_i, a, b, \tau_P, \tau_Q).$$

If p and p_N are in Q, if p is not in P, and if τ_Q is *true*, then

$$\gamma = \Gamma(p_N, p'_i, \bar{a}, b, \tau_P, \tau_Q) + \Gamma(p_E, p'_i, a, b, \tau_P, \mathit{true}) + \Gamma(p_E, p'_i, a, b, \tau_P, \mathit{false})$$

where $\bar{a}$ is $a - 1$ if τ_P is *true* and a if τ_P is *false*, and $\Gamma(p_N, p'_i, \bar{a}, b, \tau_P, \tau_Q)$ is taken to be 0 if $\bar{a} < 0$ (note, for instance, that this term is also 0 if $\bar{a} > 0$ and τ_P is *false*). In this way we get a recurrence relation that can be expressed in six parts; in each part, γ is a sum of at most three evaluations of Γ, each involving one of the points that p leads to, namely, p_N or p_E.

With this multi-part recurrence relation we can compute all values of Γ using a dynamic programming algorithm not unlike that in the proof of Lemma 4.1. Fix an ending point p'_i. Consider a point p that is t steps away from p'_i and assume we know all values of Γ at 6-tuples involving points that are fewer than t steps away from p'_i. In particular, we know all values of Γ at 6-tuples involving p_E and p_N, so with the recurrence relations we can compute any particular value of Γ involving p in constant time. This shows that if we compute the values of Γ for t from 1 to $r+m$ in this order, then we obtain all values of Γ in $\mathcal{O}(N)$ operations, where N is the number of 6-tuples in the domain of Γ. In such 6-tuples $(p, p'_i, a, b, \tau_P, \tau_Q)$, the point p is one of the $mr + k(m+r+1)$ or fewer lattice points in the diagram D, the point p'_i is one of k points, a can range from 0 to r, and b can range from 0 to m, so $N \leqslant 4\big(mr + k(m+r+1)\big)k(r+1)(m+1)$. Thus, N is $\mathcal{O}(n^5)$ where n is $m+r$ (as noted above, $k \leqslant n$ since D represents an antichain of σ-intervals).

We show finally that we can compute the number of bases of internal activity i and external activity e from Γ. This yields a two-step algorithm for computing the Tutte polynomial of a multi-path matroid: first compute all values of Γ, and then obtain the coefficient of each term $x^i y^e$ in the Tutte polynomial. This algorithm requires $\mathcal{O}(n^5)$ operations and so is somewhat faster than that in Section 4.

Lemma 5.4. *The number of bases of $M[\mathcal{I}]$ with internal activity i and external activity e can be found in time $O(k(i+e))$ knowing the values of Γ.*

Proof. We give an algorithm that counts the bases with internal activity i and external activity e that contain the element 1. Note that the remaining bases are the complements of the bases of the dual with internal activity e and external activity i that contain the element 1, so we can compute their number with the same algorithm. Indeed, this requires minimal additional computation since the values of Γ_D for the diagram D give those of Γ_{D^*}, where D^* is the diagram obtained by reflecting D in the line $y = x$; specifically, $\Gamma_{D^*}(p, p'_i, a, b, \tau, \tau') = \Gamma_D(\overline{p}, \overline{p}'_i, b, a, \tau', \tau)$, where $\overline{p}$ and $\overline{p}'_i$ are the images of p and p'_i under the reflection.

Note that any basis has a unique valid representation that touches the top border Q, so this gives a one-to-one correspondence between bases and certain paths.

For $t > 0$ and j in $[k]$, we define $\beta(j,t)$ to be the number of bases B such that

(1) the internal activity is i and the external activity is e,
(2) $[t] \subseteq B$ and $t+1 \notin B$, and
(3) the path $\Pi(B, p_j)$ is the unique valid representation that touches Q.

Let R be the path N^tE that starts at p_j and let s be the last step of R. By conditions (2) and (3), R coincides with $[1, t+1]\Pi(B, p_j)$, where B is any of the bases that $\beta(j,t)$ is counting; clearly if R is not valid, then $\beta(j,t)$ is 0. The first t North steps of R are internally active elements in B, but the step s may or may not be externally active. If s does not lie in the bottom border P, then by Theorem 5.3 it is not externally active, so

$$\beta(j,t) = \sum_{(\tau_P,\tau_Q)\in T_P\times T_Q} \Gamma(p, p'_j, i-t, e, \tau_P, \tau_Q),$$

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where p is the point $p_j + (1, t)$ where R ends, T_P is $\{true, false\}$ and T_Q is $\{true\}$ if R does not touch Q, and $\{true, false\}$ if R touches it. The two possibilities for T_Q arise from the requirement in condition (3) that the paths touch Q, so if R does not touch it, then the remaining part of the path has to. Notice that many terms in this sum may be 0; for instance, if either $i - t$ or e is greater than 0, then every evaluation of Γ where τ_P or τ_Q is *false* is 0.

Now assume that s lies in P. (This can happen only if j is 1.) Note that s is externally active if and only if the remaining part of the path touches Q. Therefore

$$\beta(j, t) = \sum_{(\tau_P, \tau_Q) \in T_P \times T_Q} \Gamma(p, p'_j, i - t, e - \delta(\tau_Q), \tau_P, \tau_Q),$$

where p is the point $p_j + (1, t)$, the set T_Q is $\{true, false\}$ or $\{true\}$ according to whether R does or does not touch Q, the set T_P is $\{true\}$, and $\delta(\tau_Q)$ is 1 if τ_Q is *true*, and 0 otherwise. Note that since s lies in P, the paths we are counting start on P, so Γ can be nonzero only if τ_P is *true*.

To obtain the number of bases with internal activity i and external activity e containing 1 we add up all the values of β. Since the number t is bounded by i, we can do the computations in $O(ki)$ operations. The same algorithm applied to the dual matroid needs $O(ke)$ operations, hence we can compute the total number of bases of $M[\mathscr{I}]$ of internal activity i and external activity e from Γ in $O(k(i + e))$ operations. □

6. Further structural properties

This final section treats a variety of properties of multi-path matroids and their presentations.

Every connected lattice path matroid with at least two elements has a spanning circuit [3, Theorem 3.3]. The analogous property holds for multi-path matroids, as we now show. By the result just cited, it suffices to focus on multi-path matroids that are not lattice path matroids.

Theorem 6.1. *A multi-path matroid $M[\mathscr{I}]$ that is not a lattice path matroid has a spanning circuit. Furthermore, every element is in some spanning circuit.*

Proof. Since multi-path matroids of rank less than 2 are lattice path matroids, we are assuming that the rank is at least 2. The set $F = \{f_I \, : \, I \in \mathscr{I}\}$ of first elements is a proper subset of the ground set S. By the comments at the end of Section 2, the first element f_I of any interval I in $\mathscr{I}$ is in both I and $\Sigma^{-1}(I)$; also, $M[\mathscr{I}]$ has no loops. From these observations, it is immediate to check that $F \cup x$, for any x in $S - F$, is a spanning circuit of $M[\mathscr{I}]$. □

Corollary 6.2 follows from Theorem 6.1 since multi-path matroids that are not lattice path matroids have no loops, and loopless matroids with spanning circuits are connected.

Corollary 6.2. *Every multi-path matroid that is not a lattice path matroid is connected.*

From Corollary 6.2, or directly from Theorem 6.1, it follows that, in contrast to the class of lattice path matroids, the class of multi-path matroids is not closed under direct

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sums. For example, recall that the 3-whirl $\mathscr{W}^3$ is a multi-path matroid but not a lattice path matroid. Therefore the direct sum $\mathscr{W}^3 \oplus \mathscr{W}^3$ is neither a lattice path matroid (since $\mathscr{W}^3$ is a restriction) nor a multi-path matroid. Of course, one could consider the class of matroids whose connected components are multi-path matroids; such matroids can be realized with a simple variation on Definition 1, having the intervals being intervals in the cycles in the cycle decomposition of an arbitrary permutation of the ground set.

The next theorem gives some indication of how close multi-path matroids are to lattice path matroids.

Theorem 6.3. *The restriction of a multi-path matroid to a proper flat is a lattice path matroid.*

Proof. Let M be the multi-path matroid $M[\mathscr{I}]$. The class of lattice path matroids is closed under direct sums [2, Theorem 3.6], so it suffices to prove the assertion for proper flats F for which $M|F$ is connected. The assertion is easily seen to hold for flats of rank 2 or less. Let F be a proper flat of rank 3 or more for which $M|F$ is connected. By Theorem 6.1 and the corresponding result for lattice path matroids [3, Theorem 3.3], the restriction $M|F$ has a spanning circuit C. It follows from Hall's matching theorem that a circuit C' of a transversal matroid has non-empty intersection with exactly $|C'| - 1$ of the sets in any presentation; therefore the inequality $|C| - 1 = r(F) < r(M)$ implies that C is disjoint from at least one interval I in $\mathscr{I}$. Thus, F is disjoint from I, so F is a flat of the deletion $M\backslash I$. Observe that $M\backslash I$ is a lattice path matroid: by Lemma 3.1, the presentation $(J\backslash I\ :\ J \in \mathscr{I}, J \neq I)$ of $M\backslash I$ by intervals in $\sigma(l_I), \sigma^2(l_I), \ldots, \sigma^{-1}(f_I)$ contains a presentation of $M\backslash I$ by an antichain of intervals. Since $M\backslash I$ is a lattice path matroid, so is $M|F$. □

Theorem 6.3 allows one to carry over certain results about lattice path matroids to multi-path matroids. For instance, the description of the circuits of lattice path matroids [3, Theorem 3.9] applies to the non-spanning circuits of multi-path matroids. We mention several other results that are counterparts of results for lattice path matroids and that may prove useful for the further study of multi-path matroids. Let $M[\mathscr{I}]$ be a multi-path matroid of rank r on the set S.

(1) Let $I_{i_1}, I_{i_2}, \ldots, I_{i_h}$ be the intervals in $\mathscr{I}$ that have non-empty intersection with a fixed connected flat F of $M[\mathscr{I}]$ of rank greater than 1. Then $\{I_{i_1}, I_{i_2}, \ldots, I_{i_h}\}$ is a Σ-interval in $\mathscr{I}$ and h is $r(F)$.

(2) Statement (1) implies that there are at most r connected flats of a fixed rank greater than 1 in $M[\mathscr{I}]$. Whirls show that this bound cannot be improved.

(3) Statement (1) also implies that any flat of $M[\mathscr{I}]$ is covered by at most two connected flats.

(4) The elements in any connected flat of $M[\mathscr{I}]$ form a σ-interval in S.

(5) If X_1, X_2, and X_3 are connected flats of $M[\mathscr{I}]$, and if no two sets among X_1, X_2, X_3 are disjoint, then either one of X_1, X_2, X_3 is contained in the union of the other two, or $X_1 \cup X_2 \cup X_3$ is S.

(Compare statements (1) and (4) with [3, Theorem 3.11]; compare statements (2) and (3) with [3, Corollary 3.12].)

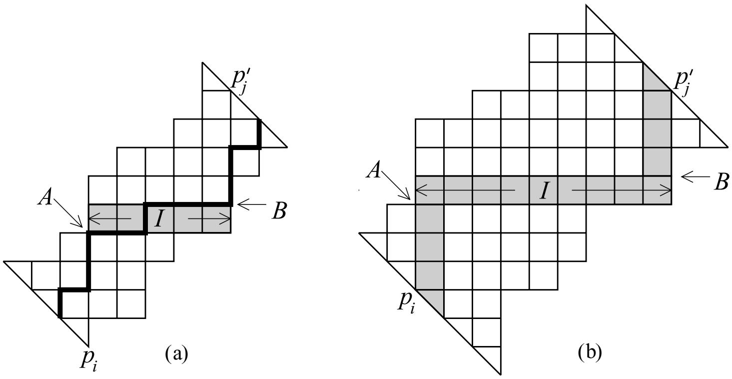

Figure 10. The cases (a) $i \leqslant j$ and (b) $i > j$ in the proof of Theorem 6.4.

Our final topic is minimal presentations of multi-path matroids. Example 3 in Section 2 gives distinct σ-interval presentations of a multi-path matroid that are also minimal presentations. The next theorem shows that any minimal σ-interval presentation is also a minimal presentation. Note that the converse is not true: for example, the presentation $(\{1,4\},\{2,4\},\{3,4\})$ of $U_{3,4}$ is minimal but these sets are not σ-intervals for any cycle σ on $[4]$.

Theorem 6.4. *The sets in a minimal σ-interval presentation of a multi-path matroid are cocircuits of the matroid. Any minimal σ-interval presentation of a multi-path matroid is a minimal presentation.*

Proof. Assume that the multi-path matroid M has rank r and that $\mathscr{I}$ is a minimal σ-interval presentation of M. Each set in a presentation of a transversal matroid is the complement of a flat of the matroid. Since cocircuits are the least non-empty complements of flats, a presentation by cocircuits is necessarily minimal, so the second assertion of the theorem follows from the first. Let I be in $\mathscr{I}$. Since the complement of I is a flat, the first assertion follows if we show that this complement contains $r-1$ independent elements. In terms of lattice paths, we need to show that there is a lattice path in some diagram $D(\mathscr{I},x)$ that connects a pair of corresponding points p_h and p'_h and has only one North step that is labelled by an element of I. This statement is trivial if r is 1, so assume r exceeds 1.

Since $\mathscr{I}$ is an antichain and r exceeds 1, some element, say x, of M is not in I. Let A and B be, respectively, the lower left and upper right points in the row of $D(\mathscr{I},x)$ that represents I. (See Figure 10.) Let i be the least positive integer for which there is a path in $D(\mathscr{I},x)$ from p_i to A. Note that there is a path in $D(\mathscr{I},x)$ from p_h to A if and only if $h \geqslant i$. Similarly, let j be the greatest integer for which there is a path from B to p'_j. Thus, there is a path in $D(\mathscr{I},x)$ from B to p'_h if and only if $h \leqslant j$. It follows that if $i \leqslant j$, then there is a path in $D(\mathscr{I},x)$ that connects any pair of corresponding points p_h and p'_h with $i \leqslant h \leqslant j$ and that has only one North step labelled by an element of I, as desired. We complete the proof by showing that the alternative, the inequality $i > j$, contradicts the assumption that $\mathscr{I}$ is a minimal σ-interval presentation. The inequality $i > j$ forces i to be greater than 1. If there were a path

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in $D(\mathscr{I},x)$ of the form N^aEQ from p_i to A, then the path $N^{a+1}Q$ from p_{i-1} to A would also be in $D(\mathscr{I},x)$, contrary to the choice of i, so there is only one path from p_i to A and this path consists of all North steps. Similarly, $j<k$ and the unique path from B to p'_j consists of all North steps. From these conclusions, it follows that for any path in $D(\mathscr{I},x)$, say from p_h to p'_h, that uses the North step labelled f_I in the row corresponding to I, or any North step immediately above this one, we have $h \geqslant i$ and the same sequence of steps, but instead going from p_{h-1} to p'_{h-1}, remains in $D(\mathscr{I},x)$. Thus, by deleting f_I from I, deleting $\sigma(f_I)$ from $\Sigma(I)$ if $f_{\Sigma(I)}=\sigma(f_I)$, deleting $\sigma^2(f_I)$ from $\Sigma^2(I)$ if $f_{\Sigma^2(I)}=\sigma^2(f_I)$, *etc.*, we obtain a smaller σ-interval presentation of M, that, as desired, contradicts the assumed minimality of $\mathscr{I}$. □

Let M be a matroid of rank r and nullity m. Since any hyperplane contains at least $r-1$ of the $r+m$ elements of M, any cocircuit has at most $m+1$ elements. From this observation, the following corollary of Theorem 6.4 is evident.

Corollary 6.5. *The sets in any minimal σ-interval presentation of a multi-path matroid of nullity m have at most $m+1$ elements.*

Acknowledgements

The authors thank Anna de Mier and Marc Noy for useful discussions about the material in this paper and the exposition. Thanks also go to the referee, who read the manuscript with great care and offered helpful comments and insights, including the connection between Γ_D and Γ_{D^*} that is mentioned in the proof of Lemma 5.4.

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