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SOME QUESTIONS CONCERNING HRUSHOVSKI'S AMALGAMATION CONSTRUCTIONS

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Abstract In order to construct a counterexample to Zilber's conjecture—that a strongly minimal set has a degenerate, affine or field-like geometry—Ehud Hrushovski invented an amalgamation technique which has yielded all the exotic geometries so far. We shall present a framework for this construction in the language of standard geometric stability and show how some of the recent constructions fit into this setting. We also ask some fundamental questions concerning this method.

Keywords: amalgamation construction; strongly minimal; envelopes; free amalgamation; collapse; standard system of geometries (codes)

AMS 2000 *Mathematics subject classification:* Primary 03C45

1. Introduction

In his book on stable groups Poizat [**27**] writes (with respect to our 1987 understanding of ω-stable fields):

> Nous n'avons pas fait de progrès décisif depuis le temps où nous chassions les aurochs avec une hache de pierre ; nous avons seulment acquis une meilleure comprehension de l'enjeu du problem.†

Sadly, there is no better way to describe the current situation with respect to the problem of classifying strongly minimal sets. Since the refutation of Zilber's Conjecture, Hrushovski's amalgamation construction of new strongly minimal sets, introduced in [**18**] and [**17**], stood in the way of any (naive) attempt of classification of their possible pregeometries. Indeed, our mastery of the techniques underlying these constructions is now better than it used to be, but our understanding of the fundamental questions they give rise to can hardly be said to have improved. To the best of my knowledge, there has not been any progress at all with respect to some questions that have already been asked in

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† We have not made any decisive progress since the time we were chasing aurochs with an ax of stone; we have only obtained a better understanding of the scope of the problem.

Hrushovski's original paper. Several survey papers deal with these amalgamation constructions (e.g. [**2**, **30**, **31**]). These papers are concerned mainly with the construction itself—developing axiomatic frameworks in which it can be carried out—and the known examples it gives rise to. The aim of the present paper is threefold, and different.

- Present a setting—given in the language of (standard) geometric stability theory—in which these construction can be understood.
- Show how new structures which have been recently constructed using these methods (most notably [**6**] and [**7**]) fit into this framework.
- Point out fundamental questions concerning these constructions, that to the best of my knowledge have not been addressed.

A central theme of this paper is that the major gap in our understanding of the scope of these constructions lies in the technically simpler part, usually known as the free construction, whereas the more involved part of the construction, known as the collapse, can be fairly well understood within a framework slightly generalizing the notion of smooth approximations.

The first part of this survey is an attempt to explain the latter statement. The starting point is Zilber's treatment of envelopes, introduced as part of his analysis of totally categorical theories [**33**]. These tools were subsequently generalized to the context of $\aleph_0$-categorical $\aleph_0$-stable structures in [**11**], and in a more developed form to the construction of smooth approximations of Lie coordinatizable structures in [**9**]. In the exposition we will cover the analogies of the collapse with the construction of [**9**] singling out the crucial differences, and how they can be dealt with, using the fusion over a vector space as a worked out example. This choice is not arbitrary. Of all known examples, this is the one (together with the closely related 'Red fields' of [**8**]) where all the model-theoretic phenomena discussed herein manifest in a non-trivial way.

The problem of classifying the combinatorial geometries obtained through these amalgamation constructions is the focus of the second part of the paper. The thesis I suggest to explore through a series of questions is that the right object to investigate towards an eventual such classification is the local geometry (or rather, the local geometries) of the regular types of infinite rank in the free construction. These questions seem to me to be crucial not only for the possibly too ambitious classification project, but mostly for our understanding of the scope of existing construction techniques and as a guide in our attempts to develop ones that will go beyond them, possibly reaching into genuinely unknown territory. Hopefully, the exposition of the first section will make at least the statement of the problems clearer.

A few words concerning what this survey will not cover are also in place. Although by no means intended for the experts only, some knowledge of the construction techniques this paper is concerned with will not hurt. A reasonably good understanding of any of the many papers on the subject, from Hrushovski's original papers, through the constructions of [**28**] or [**1**] and any of the above mentioned survey papers, should suffice to enable the reader to follow the present text. For the benefit of interested readers, who do not have such an acquaintance with these constructions, a quick survey is given in §§ 2.1–2.3. Due to the large number of ideas involved in the construction, it may seem that § 2 goes into the fine details of the collapse. However, this is not the case—many non-trivial technical points (mostly the more context dependent ones) are left out. Readers interested in the details are referred to [**13**]. This survey will also not deal with the construction of strictly stable structures in the style of [**20**]. Not because they are not interesting, but because they do not seem to fall into the model theoretic framework suggested herein, and I do not have anything new to offer on that matter. Finally, since the main concern of this paper is the construction of new structures of finite rank, the many subtle variants giving rise to, e.g. simple theories [**19**], will not be discussed. For these structures it is not even clear how to define what would a collapse (of finite rank) be, let alone produce one*.

Lastly, it should be said that no originality claims are made in this paper. Although it may not be obvious from the texts of [**18**] and [**17**], one need not be a detective to realize that their relation with the theory of envelopes as developed in [**9**] was clear to Hrushovski and possibly to others as well. Nevertheless, I hope that there is still room for a survey paper that points this out explicitly, and offers a separation of the ideas needed to overcome the technical difficulties in verifying the validity of the axiomatic framework—for individual instances of the general theory—from the ideas needed to produce a genuinely new construction.

1.1. Some technicalities

We spell out some permanent assumptions, bibliographical remarks and terminology choices.

Throughout this survey all theories making any appearance will be countable. It is often helpful, though by no means necessary, to assume the theories in question to be given in a relational language.

In what follows we will be working in models of a theory T_∞ which we will assume to be the result of a free amalgamation construction (this will be explained in more detail below). For the moment, it is enough to say that T_∞ is the theory of the (unique) countable universal model of some (countable) class $\mathcal{A}$ of finite models of a universal theory T in a language L, where universality is defined with respect to a distinguished class of L-embeddings (which will be called strong or self-sufficient embeddings). In many cases a natural expansion of T_∞ by definitions can give quantifier elimination (relative to T). Though in most places this will not matter, in the present survey T_∞ will be considered in L, which we will call its natural language.

As already mentioned, the fusion over a vector space will serve us as a guiding example throughout the second section. There are two texts dealing with this construction to which we will be referring. The exposition of [**15**] is closer in spirit to this survey, but does not contain the full result. The text of [**6**] does contain the full result but uses a totally different exposition, and in some places, the translation to the language used herein may be obscure. For that reason, and for the sake of readability, crucial lemmas

* It is not the simplicity of the structure that is the problem, but rather the fact that, in general, the one-dimensional sets in the resulting structures will not be weak linear geometries in the sense of [**9**].

from [**6**] will be given with a precise reference, together with a 'translation' in the text. Nonetheless, the reader may find a handy copy of [**6**] useful.

Unfortunately, there is no terminology which is generally accepted for dealing with the technicalities of Hrushovski's amalgamation constructions. When dealing with it here, I will adopt, for most purposes, the terminology of [**6**]. Partly since this will serve as the main example guiding us through the text, but mostly because I like it. As far as geometric stability theory terminology is concerned, less common definitions will be given in the text, but the reader will be assumed to be acquainted with such notions as orthogonality, almost orthogonality, regular types and their local geometries etc. A good source for readers wishing to refresh their memory with these definitions and their basic properties is [**25**].

2. From envelopes to pseudo-envelopes

One of the main obstacles for a model theorist trying to understand Hrushovski's amalgamation constructions is that the literature in the subject is given in specialized (and non-standard) terminology based on long technical definitions (codes, parasitic, primitive, pre-algebraic, strong and self-sufficient extensions and difference sequences to name just a few). The main aim of this section, is to show that the—usually technically more involved—stage of the construction known as 'the collapse' can be understood in standard geometric stability theoretic terms.

Though given in standard terms, the construction involves quite a few subtle techniques, the details of which non-expert readers may find tedious and not easy to follow. For that reason, the first three subsections are intended to give a self-contained general overview of the ideas appearing in the construction. These should hopefully give sufficient background for readers not interested in the fine details of the construction and wishing to skip directly to § 3, and a smooth introduction to the more technical parts for the others.

2.1. The free construction

In many texts concerning Hrushovski's amalgamation technique it is pointed out that the construction is carried out in two different steps: first a free amalgamation construction, which usually produces a structure of infinite rank, and then a collapse stage which produces one of finite rank. In practice, however, either the first stage is skipped altogether or the collapse looks more like a rehash (albeit with additional difficulties) of the free amalgamation stage, taking place in a more restrictive (universal) class. As such, the collapse may be seen more like an independent construction than a second stage necessarily following the free amalgamation. One of the aims of this section is to make clear the role of each of the two stages in the construction.

From the technical point of view, the free amalgamation construction (free construction, for short) is rather straightforward and seems to be quite well understood. What may be slightly less clear is that it provides the playground where all the rest of the work will be done. More importantly, if we view the construction of combinatorial geometries with prescribed properties as the ultimate goal*, this is the stage where this geometry is obtained (and later slightly modified to fit into a structure of finite rank). Zilber (e.g. in [**34**]) interprets this construction—or rather its complete axiomatization—as one characterized by

- a 'Generalized Schanuel' condition requiring that some predetermined integer valued pre-dimension function δ, defined on the set of finite models of a universal theory, $T^{\forall}$, be non-negative (informally, Zilber describes this condition as asserting that 'no over-determined system of equations has a solution');
- an existential-closedness condition (with respect to the universal theory determined by the Schanuel condition), which can be interpreted informally as 'every system of equations which is not over-determined has a solution'.

Reversing Zilber's point of view, the pre-dimension function, δ, can be thought of as suggesting, given a (finite) set of 'varieties' in n-space, what should be the transcendence degree of a generic point in their intersection—or, in other words, what is the dimension of their intersection.

For example, in the case of fusion of two strongly minimal theories T_1, T_2 (with quantifier elimination) over a totally categorical $T_0 = T_1 \cap T_2$, for a finite $A \models T_1^{\forall} \cup T_2^{\forall}$ the pre-dimension function is given by

$$\delta(A) = \mathrm{MR}_1(\mathrm{tp}(A)) + \mathrm{MR}_2(\mathrm{tp}(A)) - \mathrm{MR}_0(\mathrm{tp}(A)),$$

where MR_i denotes Morley rank in the sense of T_i (in other words, MR_i is the dimension of the T_i-locus of A, and $\delta(A)$ is the 'right' dimension of the intersection of these loci). So the Schanuel condition simply says that $\delta(A) \geqslant 0$ for all finite A, i.e. that there are no solutions to equations, which on dimension theoretical grounds should not have one.

The existential-closedness condition says that this is the only reason for a system of equations not to have a solution. To give a more accurate description of this axiom denote $L := L(T^{\forall})$ (for $T^{\forall}$ the universal theory appearing in the axiomatization). We define a class of distinguished '*strong*' (or *self-sufficient*) embeddings $f : A \hookrightarrow B$ for $A, B \models T^{\forall}$, by requiring that†

$$\delta(f(A)) = \min\{\delta(A') : f(A) \subseteq A' \subseteq B\}.$$

If $A \subseteq B$ we will write $A \leqslant B$ (and say that A is strong in B) if the identity is a strong embedding of A into B. In those terms the 'Schanuel condition' can be stated as $\emptyset \leqslant A$ for all A. We will denote $\mathcal{C}_0 := \{\emptyset \leqslant A \models T_\infty\}$. The existential-closedness axiom scheme can now be stated as the requirement that for every model M and $A \subseteq M$, if $A \leqslant B \in \mathcal{C}_0$ (finite) then there exists a strong embedding of B into M over A.

* It may be interesting to know whether a bad group of rank 3 can be characterized in terms of the associated combinatorial geometry. But even if this is impossible, already the construction of a non-CM-trivial geometry, admitting a group configuration but not a field configuration will be a big step forward.

† The above definition is not quite accurate. In reality we should not allow all intermediate subsets A' to be considered. For the present discussion suffice it if we say that in the case of fusion over a vector space we restrict our attention only to sub-vector spaces.

We leave it as an exercise to the reader to verify that in the case of the fusion over a (totally categorical) vector space both axiom schemes are first-order axiomatizable. From now on we will denote the above axiom schemes by T_∞ and always assume that they are elementary. Naturally, the first step in the process is to show that T_∞ is consistent. This is done by proving that the class $(\mathcal{C}_0, \leqslant)$ has the amalgamation property and that '$\leqslant$' is transitive on $\mathcal{C}_0$* (so it admits a Fraïssé limit, which is easily verified to be a model of T_∞).

2.2. The geometric structure of T_∞

As mentioned in the previous subsection, T_∞ provides us with the setting in which the collapse will eventually take place. In this subsection we describe the features of this setting.

A key observation towards that end (see, for example, Lemma 6.2 of [**15**] for the fusion over a vector space) is that (if δ is chosen carefully enough) for any $A \subseteq B \models T^\forall$, if $\emptyset \leqslant B$, there exists a unique minimal $A \subseteq C \leqslant B$ which we will denote $\mathrm{cl}_B(A)$ (and call the self-sufficient closure of A in B—omitting B, if the context is clear). From now on we will assume that $\mathrm{cl}(A) \subseteq \mathrm{acl}_{T_\infty}(A)$ for all A. We will see below that in structures where this is not the case, it is not clear how to define the collapse, and since this is the main concern of this section, this assumption is natural. It may also be worth noting that if for every finite A there exists a finite B such that $A \subseteq B$ then the above assumption will always hold. For a discussion of a construction where this is not the case, see [**29**] (where in fact the self-sufficient closure is not well defined).

Under the assumption that cl is well-defined and algebraic it is not hard to check that $\mathrm{cl}(Ba) \cong \mathrm{cl}(Aa) \oplus_A \mathrm{cl}(BA)$ is a notion of independence (where $B \oplus_A C$ denotes the free amalgam of B with C over A). It follows that T_∞ is stable and this notion of independence must coincide with non-forking†. So we get, in fact, that T_∞ is superstable, and in order not to overload the exposition, we will assume (as is usually the case) that there are enough formulae in T_∞ to isolate each type in its U-rank, implying that T_∞ is ω-stable and U-rank is equal to Morley rank.

Consider a set $A = \mathrm{acl}_{T_\infty}(A) \subseteq M \models T_\infty$ and $\bar{b}$ such that:

- $\delta(\bar{b}/A) := \min\{\delta(A'\bar{b}) - \delta(A') : A' \subseteq A,\ \text{finite}\} = 0$;
- $\delta(\bar{b}'/A) > 0$ for all $\bar{b}' \subseteq \bar{b}$.

By our characterization of forking and the minimality of $\bar{b}$ we get that $\mathrm{tp}(\bar{b}/A)$ is minimal, and by our assumption it must be strongly minimal. Call such a type simply pre-algebraic. More generally, call $\bar{b}$ such that $\delta(\bar{b}/A) = 0$ pre-algebraic, and δ-minimal if $\delta(\bar{b}'/A) > \delta(\bar{b}/A)$ for all $\bar{b}' \subseteq \bar{b}$. Naturally, if $A \leqslant B$ and $B \setminus A$ is finite (more precisely, if there exists a finite set $\bar{b} \subseteq B$ such that $\mathrm{cl}(A\bar{b}) = B$) there exists a decomposition (essentially unique) of $\mathrm{tp}(B/A)$ into δ-minimal types. We conclude, that the δ-minimal types coordinatize T_∞.

* In most cases $A \leqslant B$ will be equivalent to $\delta(X \cap A) \leqslant \delta(X)$ for all $X \subseteq B$. In cases where this is not the case, the latter, stronger condition, may be the right definition of $A \leqslant B$.

† This will only be true under the additional assumption that if $A \leqslant B_1, B_2$ then the isomorphism type of the free amalgam is uniquely determined.

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That is, for every $A \leqslant B$ as above there are $\bar{b}_1, \ldots, \bar{b}_n \subseteq \bar{b}$ such that $\operatorname{acl}(A\bar{b}_1, \ldots, \bar{b}_n) = \operatorname{acl}(A\bar{b})$ and $\operatorname{tp}(\bar{b}_i / \operatorname{acl}(A\bar{b}_1, \ldots, \bar{b}_{i-1}))$ is δ-minimal for all $1 \leqslant i \leqslant n$. This fact is crucial in everything that follows, but in many cases it is so obvious that it is in fact not mentioned explicitly. In the fusion over a vector space this is essentially the content of Proposition 4.7 of [**15**].

To simplify the exposition further, we will assume the pre-algebraic types are dense in T_∞ and that up to non-orthogonality T_∞ has one* type p_ω of rank ω, which we will call the generic type of T_∞. Note that in bi-coloured fields, for example, there are other natural candidates for the title and the unique regular type of rank ω is usually called the 'coloured generic'. This suggests that a more appropriate name for p_ω might be 'the generic regular type of T_∞', but for the sake of clarity, we will use the shorter terminology. Collecting all of the above together, it is now easy to check that the simply pre-algebraic types together with p_ω form a complete set of representatives of the non-orthogonality classes of regular types in T_∞. We will see that the geometries of the regular types have a crucial role in our analysis. In particular, the geometry we set to construct is (essentially) the local geometry of p_ω, which is given by cl.

At this stage, we can already give a rough idea of the ideology behind the collapse: given $M \models T_\infty$ we would like to find a homogeneous, infinite $N \leqslant M$ such that $\dim_N(p) < \infty$ for every strongly minimal type in $S(M)$ based over N. The idea is that, if everything works out properly, N will inherit the coordinatization from M (by the homogeneity) and therefore will be ω-stable and unidimensional (because N is infinite it must have some non-algebraic type, and the requirement $\dim_N(p) < \infty$ should make all the strongly minimal type of T_∞ algebraic, so, by coordinatization, p_ω will be the only non-algebraic type). This would make $\operatorname{Th}(N)$ uncountably categorical, and $p_\omega(N)$ its unique strongly minimal set, as desired.

One way of avoiding trivial solutions to the above project can be to require that if $p \in S^{T_\infty}(N)$ is realized in N then $p \perp p_\omega$ (i.e. that p_ω is not realized in N; this will be slightly elaborated in the closing paragraphs of this subsection). This would imply that N is not a saturated model of its theory, but the homogeneity will be good enough a replacement.

It is now clear, that if we want any chance of having both any $M \models \operatorname{Th}(N)$ to inherit the coordinatization from T_∞ and $\operatorname{Th}(N)$ to be uncountably categorical, we must require that any strongly minimal $p \in S(N)$ which is realized in some $N \prec_{T_\infty} M$ be locally finite. Towards this end, we will have to prove something in the spirit of the following lemma.

Lemma 2.1. *Every strongly minimal set in T_∞ is locally finite. In particular, it is locally modular.*

The lemma is not true in all free constructions, but if some $p \in S(M)$ were not locally finite, in order for $N = \operatorname{acl}_{T_\infty}(N)$ to be unidimensional, we would have to require that $\dim_N(p) = 0$ (the alternative solution—requiring that p be the unique regular type of

* This will not, in general, be the case. However, as long as the strongly minimal types are dense in T_∞ the construction is practically unaltered.

Th(N)—is uninteresting). For such a requirement to be consistent delicate questions concerning definability of non-orthogonality to p will have to be addressed*.

It is worth noting, however, that even in the presence of non-locally modular strongly minimal types in T_∞ it may well be that a collapse of the locally modular types is still possible, resulting in potentially interesting new structures. A good test case where this approach seems possible to implement can be found in [**22**] where T_∞ is interpreted in a differentially closed field, with some of its strongly minimal sets non-orthogonal (in T_∞) to the field of constants. In this example, there is only one 'problematic' non-orthogonality class, so that the resulting collapsed structure will still be of rank ω, but will not be multidimensional. Similar ideas were already used in the constructions of [**21**].

The importance of the local modularity of the strongly minimal types of T_∞ will be discussed in more detail in the next subsection.

Despite being absolutely crucial for the construction, Lemma 2.1 is not stated explicitly in most texts dealing with the collapse. In the context of the fusion over a vector space the lemma is proved in Proposition 6.8 of [**15**]. In [**6**] local finiteness follows from Lemma 6.1 (which we slightly reformulate).

Lemma 2.2. *Let $M \models T_\infty$ and $N = \mathrm{acl}_{T_\infty}(N) \subseteq M$ be such that $\dim_N(p)$ is finite for every pre-algebraic type $p \in S(N)$, then any strongly minimal formula $\varphi \in L(N)$ has finitely many solutions in N.*

However, local modularity only acts behind the scenes of the text in [**6**]. In [**7**] it is even harder to track down these facts. The somewhat harder proof of local finiteness of strongly minimal sets is rather well hidden in the proof of Lemma 7.3, but is never stated explicitly (though it follows implicitly from the fact that the class K_μ has the amalgamation property).

2.3. An overview of the collapse

The ideology underlying the exposition is that once T_∞ has been constructed, given $M \models T_\infty$ the collapse can be described as obtaining (elementary) means of constructing subsets $N \leqslant M$, in which the dimensions of simply pre-algebraic types are well controlled (and in particular finite).

To better understand this, remark that our assumption that the pre-algebraic types are dense in T_∞ would, in most cases, imply that for any type $p \in S^{T_\infty}(A)$ and $a \models p|A$ if $d(a/A) > 0$ then MR(p) is infinite. Since, up to non-orthogonality, p_ω is the unique regular type of infinite rank, it follows that any forking extension of p_ω is domination equivalent to a pre-algebraic type. Since pre-algebraic types are totally categorical, any definable subset of M of finite rank has at most finitely many solutions in N (because the simply pre-algebraic ones are finite in N). In particular, any forking extension of p_ω will be algebraic in N, so p_ω, interpreted in N, will be a minimal type.

Such control over the behaviour of strongly minimal sets made its first appearance in the totally categorical case, where this process is precisely Zilber's construction of

* As long as p is locally modular, it is quite hopeful that such questions could be addressed—as was conjectured in [**29**]. But in the non-locally modular case the situation is much more delicate.

envelopes. Generalized later to the context of $\aleph_0$-categorical, $\aleph_0$-stable structures in [**11**] it reached its ultimate form in [**9**], as part of the characterization of smoothly approximable structures. In many ways, as will be made clear in this section, the collapse can be viewed as a generalization of these constructions. The geometrical ideas underlying the collapse appear already in these early works, and the main technical difficulty in their adaptation to the present context lies in its being out of the $\aleph_0$-categorical realm.

Roughly, the main steps in the collapse are

(1) prove that the free amalgam is coordinatized by (usually) a unique regular type of (in most cases) infinite rank and families of totally categorical strongly minimal sets;

(2) identify a good set of representatives of the non-orthogonality classes of strongly minimal sets and collect them in a 'standard system' $\mathcal{C}$ where each class is represented (essentially) once;

(3) show that for a function $\mu : \mathcal{C} \to \mathbb{N}$, μ-envelopes of $\emptyset$ (with respect to $\mathcal{C}$), i.e. maximal algebraically closed subsets $X \subseteq M$ satisfying $\dim_X J \leqslant \mu(J)$ for all $J \in \mathcal{C}$, are homogeneous (and in particular that, for some μ, they exist);

(4) find a good elementary analogue of envelopes for which the same construction works.

The coordinatization lemma which forms the first part of the process described above is usually straightforward and was dealt with in the previous subsections, as well as the somewhat trickier and crucial fact that all strongly minimal sets in T_∞ are totally categorical.

To obtain a consistent set of conditions of the form 'the dimension of J_a is at most $\mu(J_a)$' for a strongly minimal geometry J_a (definable in M), it is most convenient to make sure that such a condition is given only once for each non-orthogonality class of geometries. It is therefore important to be able to collect a (non-redundant) set of representatives for those classes. The technical term for such collections is systems of standard geometries. This is taken care of in step two in the process described above. The key to the construction of such systems of geometries is definability of orthogonality in T_∞. This requires a good understanding of forking and algebraic closure in the free amalgam, but since—as we have already seen—this geometry has an explicit description in the same terms used for the construction of the amalgam it should not, in general, be hard to obtain. Section 2.5 covers these questions in some detail.

The construction of envelopes can now follow, almost word by word, the one in § 3 of [**9**]. In this construction, as in the $\aleph_0$-categorical case, local modularity of the strongly minimal geometries in question plays an important role by supplying us with the 'uniqueness of parallel lines', assuring the homogeneity of envelopes. To better understand this, assume that $X = \mathrm{acl}_{T_\infty}(X) \subseteq M \models T_\infty$ is such that $\dim_X p = \mu(p)$ for some p, the generic type of a projective space (over a finite field) definable over X. If we want to have any chance of X being homogeneous, we must decide—given A, an affine space of

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p defined over X—whether A has a point in X or not. The 'uniqueness of parallel lines' assures, that if $X = \mathrm{acl}_{T_\infty}(X)$ is contained in a μ-envelope and does not contain a point of A then for any $a \in A$, Xa is contained in a μ-envelope too (so the answer is 'every affine space defined over X has a point'). In § 2.5 we will see that this is, essentially, all we have to require to assure the homogeneity of envelopes.

But, in general, envelopes will not be elementary. Given a family J of strongly minimal sets, the statement $\dim J_a = n$ will not be first order (the reason being that despite the total categoricity of J_a we cannot uniformly bound $|\mathrm{acl}_{T_\infty}(a) \cap J_a|$). This calls for a first-order approximation of $\dim J_a = n$, which in [**6**] was appropriately called pseudo-Morley sequence of length n. This leads naturally to the concept of pseudo-envelopes, which are introduced in § 2.6.

Since pseudo-Morley sequences approximate actual Morley sequences, the construction of pseudo-envelopes is not very far from the construction of envelopes, with one important exception. Strongly minimal sets in T_∞ are totally orthogonal, i.e. if $E_0 = \mathrm{acl}\, T_\infty(E_0)$ embeds into a μ-envelope, and a/E_0 is strongly minimal, then for any projective geometry I definable over $E_a := \mathrm{acl}_{T_\infty}(E_0 a)$ we have $\dim_{Ea} I > 0$ only if $I \not\perp E_0$. This is, of course, a very handy tool in the construction of envelopes. As in the smoothly approximable case, this property follows inductively from the coordinatization of T_∞, with the base case following from the fact that if $c = \mathrm{Cb}(\mathrm{tp}(a/E_0))$ then $\dim_{\mathrm{acl}(c)} \mathrm{tp}(a/E_0) = 0$. This last equality is not quite true of pseudo-Morley sequences, for which an appropriate correction must be proved. The strategy for overtaking this obstruction are discussed in § 2.7.

Choosing $\mu : \mathcal{J} \to \mathbb{N}$ for which this construction can be carried out, will assure that no strongly minimal set I definable over a μ-pseudo-envelope E has infinite dimension in E and by total categoricity (of strongly minimal sets in T_∞) in fact $|E \cap I| < \infty$. Homogeneity of pseudo-envelopes assures quantifier elimination (modulo T_∞) so E inherits the coordinatization from T_∞. This implies that E is unidimensional, and from the coordinatization, ω-stable. Hence E is uncountably categorical.

We now proceed to describe this construction in more detail. We will use the construction of the fusion over vector spaces as a main example, stressing the translation of the specialized terms of the construction into the standard geometric terminology.

2.4. Envelopes

We make a slight digression to discuss the notion of envelopes, as it was developed in the $\aleph_0$-categorical context.

A key step in Zilber's proof of the non-finite axiomatizability of totally categorical theories was noting the importance of envelopes. We remind readers of the following definition.

Definition 2.3. Let T be stable, $\mathcal{J} := (\varphi_1, \ldots, \varphi_k)$ a set of pairwise orthogonal strongly minimal sets definable over $\emptyset$ and $X \subseteq M \models T$ any set. A $\mathcal{J}$-envelope $E_{\mathcal{J}}(X)$ of X is a maximal set such that $E_{\mathcal{J}}(X) \mathop{\smile\!\!\!\!\!\!|}_X \mathcal{J}$.

In this terminology, Zilber proved the following theorem.

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Theorem 2.4. *Let φ be a strongly minimal set definable in a totally categorical theory T then*

(1) *$E_\varphi(X)$ is finite for all finite X;*

(2) *for any $m \in \mathbb{N}$ there exists a natural number s_m such that for all X, $E_\varphi(X)$ is m-elementary (i.e. for all $\bar{a} \in X$, $|\bar{a}| = m$, $E_\varphi(X) \models \exists \bar{x}\psi(\bar{x}, \bar{a})$ if and only if $\models \exists \bar{x}\psi(\bar{x}, \bar{a})$) provided that $|\mathrm{acl}(X) \cap \varphi| > s_m$;*

(3) *φ-envelopes are homogeneous.*

To obtain a possibly better understanding of the nature of envelopes, note that Zilber's theorem proves that if M is the unique countable model of a totally categorical structure, then it is smoothly approximable by envelopes. This means not only that $\mathrm{Th}(M)$ is the limit theory of its envelopes but also that $\mathrm{Aut}(M)$ is the limit (in the natural sense of pointwise convergence) of the groups of automorphisms of envelopes.

Zilber's use of envelopes was subsequently generalized to the context of $\aleph_0$-categorical $\aleph_0$-stable structures. In both cases the notion of coordinatization played a crucial role. Recall the following definition (say, in an $\aleph_0$-categorical $\aleph_0$-stable theory T).

Definition 2.5. If $p \in S(\emptyset)$ is non-algebraic and is $\emptyset$-definable, then p coordinatizes ψ if $\mathrm{acl}(x) \cap p \neq \emptyset$ for all $x \in \psi$.

The reader unfamiliar with this notion of coordinatization should have in mind the example of a definable set $\psi(x)$ of rank n and $\{\varphi(x, y) : y \in p\}$ an infinite definable normal family of rank $n - 1$ subsets of $\psi(x)$. Then by local modularity $\mathrm{MR}(p) = 1$ and the p-coordinate of $a \in \psi$ is $b \models p$ such that $a \in \varphi(x, b)$.

The main ingredients in Zilber's proof, appearing later also in [**11**] (in the absence of $\emptyset$-definable equivalence relations), were

- the theory is coordinatized by finitely many (one, in the totally categorical case) $\aleph_0$-categorical strongly minimal set(s);

- $\aleph_0$-categorical strongly minimal sets are locally modular;

- if ψ is a non-trivial, $\aleph_0$-categorical and non-modular strongly minimal set defined over $\emptyset$ there are an $\emptyset$-definable equivalence relation with finite classes F and a $\emptyset$-definable vector space V such that $(\psi/F, V)$ is an affine space;

- wrapping all of the above together, if $\mathcal{J}$ is a set of modular coordinatizing geometries, and T is primitive (i.e. does not have non-trivial $\emptyset$-definable equivalence relations), then E is a $\mathcal{J}$-envelope of $X = \mathrm{acl}(X)$ if and only if $E = \mathrm{acl}(E) \supseteq X$, every $\mathcal{J}$-affine space definable over E has a point in E and $\dim_E(J) = \dim_X(J)$ for all $J \in \mathcal{J}$.

Ideologically, it should be clear that the homogeneity of envelopes (which plays a crucial role in the proof) must come from their maximality. However, it should also be clear that maximality alone is not enough: envelopes have to be algebraically closed, but

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they also have to include realizations of types (non-orthogonal but) almost orthogonal to $\mathcal{J}$—as many as possible without increasing the $\mathcal{J}$-dimension. However, in general, given (a finite) $X = \mathrm{acl}(X)$ the order in which we add to X realizations of types $p \perp^a_X J$ (but non-orthogonal to J) is not immaterial. In the above setting, the additional ingredient, coming easily from local modularity, is the so-called 'uniqueness of parallel lines': for strongly minimal types $p_1 \not\perp p_2$ either $p_1 \perp^a p_2^{\otimes n}$ for all n or $p_2 \perp^a p_1^{\otimes n}$ for all n, provided that $p_1 \perp^a p_2$.

The construction of envelopes in the works of Zilber and Cherlin–Harrington–Lachlan was relatively easy (modulo the hard classification of locally finite pregeometries) due to the fact it used the simple definition of envelopes given above, where we only care about finitely many coordinatizing geometries (based on $\emptyset$). In [**9**] the notion of Lie coordinatizable structures was introduced, refining the definition of coordinatization and allowing definable (multidimensional) families of geometries, giving the structure a form of a tree (historically, in the stable case in which we are interested, a related definition of quasi-coordinatizable structures appears already in [**10**] with a close relative in [**23**], but the one in [**9**] is more accessible, and generalizes easily to the contexts we are interested in).

Definition 2.6. A stable structure M is coordinatized by Lie geometries if it carries a tree structure of finite height with a unique, $\emptyset$-definable root, such that for each $a \in M$ above the root either a is algebraic over its immediate predecessor in the tree ordering, or there exists $b < a$ and a b-definable projective geometry J_b fully embedded in M and either:

(1) $a \in J_b$; or

(2) there is $c \in M$ with $b < c < a$, and a c-definable affine geometry $(J_c; A_c)$ with vector part J_c, such that $a \in A_c$ and the projectivization of J_c is J_b.

We refer to § 2 of [**9**] for more details concerning the above definition (where it is given in the general, non-stable, context).

Remark 2.7.

(1) It may well be that in the stable context the term 'coordinatization by linear geometries' is more appropriate than the one given.

(2) For totally categorical theories the two definitions of coordinatization coincide.

(3) For an $\aleph_0$-categorical, $\aleph_0$-stable theory T the new definition is more subtle. For simplicity, assume that $M \models T$ is coordinatizable (in the sense of Definition 2.5) by one strongly minimal type. The coordinatization induces an equivalence relation '$\sim$' on M. As every $\sim$-equivalence class is itself $\aleph_0$-categorical, $\aleph_0$-stable, and of lower rank than $\mathrm{MR}(M)$, it is now easy to see how to obtain a tree of coordinatizing geometries on M. Indeed, if M is primitive, i.e. admits no non-trivial $\emptyset$;-definable equivalence relations then the definitions coincide.

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Gathering the results form § 2.2, the coordinatization of T_∞ by δ-minimal types and the modularity of the strongly minimal ones among them (Lemma 2.1) we start getting the impression that the structures we are interested in can be viewed as a fairly natural, somewhat overgrown, version of (stable) Lie coordinatizable structures.

There are, however, two important differences.

(1) The depth of the graph* of coordinatizing geometries is not, as a rule, finite.

(2) The generic type p_ω is not, in general, projective.

In the collapse of T_∞ the presence of the non-projective generic type can, for all practical purposes, be ignored. Neglecting p_ω causes no harm because, in some sense, it is a grown up type that can take care of itself: p_ω is approximated by types of finite rank (and usually even by strongly minimal types); if we handle the strongly minimal types well enough, compactness will take care of p_ω. In fact, if instead of working in a saturated model of T_∞ we work in a large existentially closed $\bar{M} \models T_\infty$, the generic type will not appear in the coordinatizing tree of $\bar{M}$†.

As for the fact that the tree of (projective) coordinatizing geometries is infinite—in itself it does not cause a serious problem. It is, however, a witness to the fact that T_∞ is not $\aleph_0$-categorical, which is the main additional difficulty with respect to Lie coordinatizable structures. Instead of working with complete types, we will have to content ourselves with approximating formulae which will require harder work in order to obtain uniformity results needed to make our strategy work.

2.5. Systems of standard geometries

We will now proceed slightly deeper into the description of the strategy of the collapse. We will be working in an ω-stable theory T_∞ coordinatized by locally finite strongly minimal types and a unique generic type p_ω of rank ω. Our goal is to construct a homogeneous infinite $N = \mathrm{acl}_{T_\infty}(N) \subseteq M$ in which every strongly minimal type $p \in S(N)$ has at most finitely many realizations. As we have already seen, in this setting it will be enough to control the behaviour of simply pre-algebraic types.

In order to obtain such control we will adapt a tool form [**9**]. The idea of [**9**] (appearing already in earlier works of Lachlan) is to (definably) collect representatives of non-orthogonality classes of rank 1 types in systems of standard geometries. A standard geometry is, basically, a definable family whose domain is a complete type and whose range is a set of pairwise orthogonal rank 1 projective geometries. A system of standard geometries is a collection of standard geometries where each non-orthogonality class is uniquely represented. In the present context, since T_∞ is not $\aleph_0$-categorical, things are more complicated, and we will have to extract (and preserve) explicitly much of the information that comes for free in standard systems of geometries. The resulting objects are known, maybe not very informatively, as codes (2-codes in [**17**]).

* Indeed, in general this graph will not be a tree, but it is nonetheless easily stratified, which is all we will need.

† Recall that T_∞ is given in its natural language. In particular, if $\bar{a} \leqslant M$, $0 < \delta(\bar{a})$ there exists $\bar{a}' \models \mathrm{qftp}(\bar{a})$ with $\bar{a}' \not\leqslant M$. Hence M is existentially closed, if and only if $\delta(\bar{a}) = 0$ for all $\bar{a} \leqslant M$

In this subsection we will discuss the guidelines in the construction of codes, avoiding the unpleasant technical details and definitions. The first thing to observe is that if $p_1, p_2 \in S(B)$ are strongly minimal and $p_1 \not\perp p_2$ then there are two options.

- p_1, p_2 are both modular so $p_1 \not\perp^a_B p_2$ with a B-definable finite-to-finite correspondence, f between them. If p_1, p_2 are strictly minimal, then f is bijective.
- p_1 or p_2 are affine, i.e. locally modular non-modular, in which case their projectivization is based on B, reducing the problem to the case where p_1 is affine and p_2 projective. In that case p_1 does not have a solution in B and by the uniqueness of parallel lines, $\dim_{\mathrm{acl}(Ba)} p_2 = \dim_{\mathrm{acl}(B)} p_2$ for any $a \models p_1|_B$.

This leads to the following observations.

(1) If we want to control the dimension of $p \in S(N)$ in N it suffices to control the dimension in N of any representative of the non-orthogonality class of p.

(2) The representatives of the non-orthogonality classes of strongly minimal types in T_∞ which are easier to handle are modular (preferably even strictly minimal).

(3) Since, ultimately, we want to be able to axiomatize $\mathrm{Th}(N)$ the choice of representatives should be done in a uniform way.

Which raises, naturally, the question of definability of orthogonality in T_∞. Recall the following definition.

Definition 2.8. Let $p \in S(B)$ be a stationary type. We say that orthogonality to $p := p_a$ (for some finite a such that $\mathrm{Cb}(p) \subseteq \mathrm{dcl}(a)$) is definable if for every definable family $R(\bar{x}, \bar{y})$ there exists a formula $O_a(\bar{y})$ such that

$$O_a(\bar{y}) = \{b : R(\bar{x}, \bar{b}) \perp p\}.$$

The definition of orthogonality is uniform if there exists $\theta \in \mathrm{tp}(a)$ such that the same is true of all $a' \models \theta$.

To the best of my knowledge the question of definability of orthogonality in the context of amalgamation constructions has been addressed explicitly only in [**21**] (in a different setting) where it is the main technical tool required for the construction, and in § 7 of [**15**]. In the original works of Hrushovski's non-orthogonality of simple pre-algebraic n-types amounted to the action of $\mathrm{Sym}(n)$ on the variables of p, as can be inferred, from the proof of the Algebraic Amalgamation Lemma of [**18**]. Again, slightly changed in formulation, this lemma is stated as follows.

Lemma 2.9. *Let $A \leqslant B_1, B_2 \leqslant M \models T_\infty$, B_1/A simply pre-algebraic. Let $E = B_1 \oplus_A B_2$ be the free amalgam and assume that $p \in S(E)$ is pre-algebraic with $\mu > \delta(A)$ pairwise disjoint realization in E then either $p = \mathrm{tp}(B_1/A)$ for some enumeration of $B_1 \setminus A$ or $\dim_{B_2} p = \dim_E p$.*

In this and in similar cases definability of orthogonality trivializes. In the case of the fusion over a vector space V_F, in some sense, V-linear dependence replaces some of the roles of equality in fusions over $\{=\}$. It is therefore natural that in this setting a similar claim is true, but with the action of $\mathrm{AGL}_n(F)$ (the group of invertible affine transformations of V_F^n) on p. This is proved, essentially, in Lemma 6.4 of [**6**]. The uniform definability of orthogonality follows easily. Reformulating (and slightly weakening), it says the following.

Lemma 2.10. *Let A, B_1, B_2, E be as above. Let $p \in S(B_2)$ be a simply pre-algebraic modular type. There exists a natural number $\lambda(p)$ such that if there is a $\lambda(p)$-Morley-sequence in p contained in E then either $\dim_{B_2} p = \dim_E p$ or there is $H \in \mathrm{GL}_n(F)$ and $a \in B_2$ such that $\mathrm{tp}((H(B_1) + a)/A) = p$ (for some enumeration of $B_1 \setminus A$).*

Note that in both cases the above lemmas give us more information than the definability of orthogonality. They give us the number $\lambda(p)$ which is a bound on the length of Morley sequences in E of any strongly minimal $p \in S(E)$ which is not based on B_2 (or B_1). These bounds will be most significant in the next subsection.

Having obtained the definability of orthogonality, we now have a better chance of getting definable control over non-orthogonality classes. Following (2) of the guidelines described in § 2.3 it is usually useful to definably separate affine types from modular types. In most worked out examples (including, maybe somewhat surprisingly, [**7**]) all strongly minimal types are trivial, whence modular (in this latter example, this is a key feature in making the collapse possible). In the case of the fusion over a vector space (as in the case of the Red fields of [**8**]) this is not the case, and some work is needed. This appears as Lemma 7.8 of [**15**]. In [**6**] this corresponds to the distinction made between the so-called coset types and other types, which is taken care of in C(v) of the definition of codes [**6**, § 2].

For technical reasons it is more convenient to be able to definably collect simply pre-algebraic types. This means that for every simply pre-algebraic type $p \in S(B)$ we want a formula $\varphi(x, b) \in p$ and $\psi \in \mathrm{tp}(b/\emptyset)$ such that for every $b' \models \psi$ the formula $\varphi(x, b')$ is stationary with a simply pre-algebraic generic type. This is easily seen to be equivalent to definability of strong minimality in T_∞. In some cases, this may not be needed at all (see [**16**]) and in general it will suffice to have a rank preserving expansion of T_∞ with this property. It is not known if in the absence of such an expansion the construction cannot be carried out. It is also worth mentioning that for fusion constructions this technical point is the only place the definable multiplicity property (DMP) of the fused theories is needed—it gives definability of strong minimality in T_∞. It should not be hard to check that the converse is also true. In the fusion over a vector space, definability of strong minimality in T_∞ is taken care of by C(i)–C(iii) of the definition of codes (and the subsequent proof that codes exist). The proof of this part in the definition of codes amounts to not much more than definability of Morley rank and degree in each of the fused theories and is very close in spirit to the first-order axiomatization of T_∞ as discussed in § 2.1.

Finally, we have to decide how to choose representatives of the non-orthogonality classes (of strongly minimal types). If we enumerate all $\varphi(x, y)$ such that whenever $\varphi(x, b)$

is non-empty it isolates a simply pre-algebraic type, then using induction and uniform definability of orthogonality, it is not hard to make sure that if $\varphi_i(x,b) \perp \varphi_j(x,d)$ for some b and all $j < i, d$ we can find a restriction $\tilde{\varphi}_i(x,y)$ of $\varphi_i(x,y)$ such that $\tilde{\varphi}_i(x,b) \sim \varphi_i(x,b)$ and for all b' for which $\tilde{\varphi}_i(x,b')$ is non-empty the same is true (i.e. $\tilde{\varphi}_i(x,b') \perp \varphi_j(x,d)$). It is harder to control non-orthogonality within different instances of $\varphi(x,y)$ itself. As a rule, assuring that exactly one representative is chosen for each non-orthogonality class is non-trivial and has to be done in T^{eq}, as this is done in § 2.5 of [**9**] (and see also Lemma A.2 of [**15**] for the case of the fusion over sub-languages). A different approach, which was taken in [**6**] (following the simpler treatment in [**17**]), is to make sure that given a strongly minimal type $p := p_a$, a formula $\varphi(x,a) \in p$ is chosen in such a way that

(1) for all 'unavoidable' $p' \not\perp p$ there is some a' such that $\varphi(x,a') \in p'$;

(2) for all other $p' \not\perp p$ there is no such a'.

In § 2 of [**6**] this is achieved in C(vi) and C(vii) of the definition of codes. Luckily, the 'unavoidable' types can only appear in finitely many guises, allowing for a definable control over them. Specifically, if $\varphi_c(x,y)$ is a code then we have the following.

C(vi) For all b and m the set defined by $\varphi_c(x+m,b)$ is encoded by φ_c.

C(vii) There is a subgroup G_c of $\mathrm{GL}_n(F)$ such that

- for all $H \in G_c$ and all non-empty $\varphi_c(x,b)$ there is a unique b^H such that $\varphi_c(Hx,b) \equiv \varphi(x,b^H)$;
- if $H \in \mathrm{GL}_n(F) \setminus G_c$ then no non-empty $\varphi_c(Hx,b)$ is encoded by ϕ_c.

And the fact that this can be achieved is not a difficult consequence of compactness.

Since we are following [**6**] as a main example, in the present discussion we will adopt the approach which does not lead us into the imaginary world.

Gathering all this together we obtain a collection $\mathcal{C}$ of families of strongly minimal sets $\{\varphi_i(x,y)\}_{i<\omega}$ such that for every simply pre-algebraic type p there exists a unique $i < \omega$ and some b such that $\varphi_i(x,b) \not\perp p$ and the set $\{b' : \varphi_i(x,b') \not\perp p\}$ is uniformly definable. This gives us an analogue of a representative set of all standard systems of geometries [**9**, Definition 2.5.6], which play a central role in the construction of envelopes in Lie coordinatizable structures. The significant difference (which causes many technical problems) from the $\aleph_0$-categorical case is that the domain of our standard systems of geometries (in this context usually better known as codes) is not a complete type.

We note that the complexity of the construction of a standard system of geometries (codes) depends mostly on the possible complexity of non-orthogonality and not (directly) on the geometry of the strongly minimal sets. Thus the codes appearing in [**17**], e.g. are relatively simple, whereas those appearing in [**7**] are as complicated as those appearing in fusion over a vector space, despite of the fact than in both cases all strongly minimal types are trivial.

2.6. Pseudo-envelopes

From now on we fix a set $\mathcal{C}$ of representatives of all standard system of geometries satisfying all the properties discussed in the previous subsection. We remind the reader of the following definition from [**9**].

Definition 2.11.

(1) An approximation to a geometry of a given type is a finite or countable dimensional geometry of the same type.

(2) A dimension function is a function defined on $\mathcal{C}$, with values isomorphism types of approximations to canonical projective geometries of the given type. (This is actually determined by a dimension, and the type.)

(3) If μ is a dimension function, then a μ-envelope is a subset E satisfying the following three conditions:

  (a) E is algebraically closed in M (not M^{eq});

  (b) for $c \in M \setminus E$, there is a standard system of geometries J with domain A and an element $b \in A \cap E$ for which $\mathrm{acl}(Ec) \cap J_b$ properly contains $\mathrm{acl}(E) \cap J_b$;

  (c) for J a standard system of geometries defined in A and $b \in A \cap E$, $J_b \cap E$ has the isomorphism type given by $\mu(J)$.

The two main properties of envelopes which are of interest to us are their existence and homogeneity for every dimension function μ. Our aim is to suggest a definition of envelopes suitable for the present context, such that for some finite μ (i.e. such that $\mu(J)$ is finite for all $J \in \mathcal{C}$) the same will be true.

A major concern in the present context is that we want envelopes to be uncountably categorical (in the original setting the aim was to show envelopes were finite, provided μ was) so we have to take care that the good properties of our envelopes will be preserved under taking elementary extensions and substructures. We will tackle this last problem by assuring that our envelopes have a reasonable first-order axiomatization.

As we have already hinted above, a major problem in carrying out the plan of constructing envelopes is that the domain of our system of standard geometries are not complete types. This is a serious problem for the following reason. If $J \in \mathcal{C}$ and $a \in \mathrm{dom}(J)$, then J_a is a totally categorical geometry. Hence, there is a finite set of formulae $\varphi_1(x, a_1), \dots, \varphi_{k_a}(x, a_k)$ generating the structure on J_a. As long as the φ_i can be chosen uniformly in a, there will be no problem specifying in a first-order way that for all $a' \in \mathrm{dom}(J) \cap A$ we have $\dim_{J_{a'}} N = \mu(J)$ (once we specify $\mathrm{dom}(J)$ we would only have to say that there is a Morley sequence of length $\dim \mu(J)$ in $J_a(N)$ whose algebraic closure is the whole of $J_a(N)$). But if the structure of J_a is not uniform in a, this will not be a first-order statement*. For future reference it will be useful to introduce the following definition.

* This is equivalent to the non-definability of strict minimality in T_∞^{eq}.

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Definition 2.12. A system of standard geometries $\mathcal{C}$ admits a uniform structure, if for any $J \in \mathcal{C}$ there are $\varphi_1(x,y),\dots,\varphi_{n_J}(x,y)$ such that for all $a \in \mathrm{dom}(J)$ the structure of J_a is generated by $\varphi_1(x,a),\dots,\varphi_{n_J}(x,a)$.

Our first goal is therefore to define pseudo-μ-envelopes in T_∞ where the type of the geometry approximating J_a will not be specified, but only a good enough definable approximation thereof. Fortunately, [**9**] provides a characterization of envelopes which is easier to handle in our context.

Fact 2.13. *Let M be a Lie coordinatizable structure. If $E = \mathrm{acl}(E) \subseteq M$ is maximal such that for every $J \in \mathcal{C}$ and $a \in \mathrm{dom}(J)$ the geometry $J_a(E)$ embeds into $\mu(J)$ then E is a μ-envelope.*

It is now natural to make the following definition.

Definition 2.14. Let $\mu : \mathcal{C} \to \mathbb{N} \cup \{\infty\}$. An algebraically closed structure E is a pseudo-μ-envelope if it is maximal such that for every $J \in \mathcal{C}$ and every $a \in \mathrm{dom}(J) \cap E$ there is no pseudo-$\mu(J)$-Morley sequence of length $\mu(J)$ in $J_a(E)$.

It remains, of course, to define pseudo-n-Morley sequences, which is a rather tedious and technical job. We will not give the whole set of requirements (which may vary from structure to structure) but review only the most important ones.

Fix a stationary type p and denote $p^{\otimes n}$ the type of an independent set of n realizations of p. A pseudo-n-Morley sequence in p is a formula $\psi_n \in p^{\otimes n}$ such that for $(a_1,\dots,a_n) \models \psi_n$:

(1) $\mathrm{Cb}(p) \subseteq \mathrm{dcl}(a_{i_1},\dots,a_{i_k})$ for some k depending only on p (but not n) and for all $1 \leqslant i_1 < \dots < i_k \leqslant n$. In particular, n has to be large enough;

(2) $(a_{\sigma(1)},\dots,a_{\sigma(n)}) \models \psi_n$ for any $\sigma \in \mathrm{Sym}(n)$;

(3) if $b \models p|_{(a_1,\dots,a_n)}$ then $(a_1,\dots,a_{i-1},b,a_{i+1},\dots,a_n) \models \psi_n$ for all $1 \leqslant i \leqslant n$; also $(a_1,\dots,a_n,b) \models \psi_{n+1}$;

(4) if $n-1$ is large enough then $(a_1,\dots,a_{n-1}) \models \psi_{n-1}$;

(5) there are no obvious dependencies in $a_1,\dots,a_n$ over $\mathrm{Cb}(p)$ (e.g. they are pairwise disjoint and in the case of the fusion over a vector spaces they are linearly independent with respect to that vector space);

(6) in most cases it will be more convenient to choose ψ_n quantifier free in the natural language of T_∞.

In [**18**] pseudo-Morley sequences (in simply pre-algebraic types) are just (long enough) sequences of pairwise disjoint realizations of the type. In [**17**] the definition is slightly more complicated and is practically equivalent to (1)–(5), which are given as part of the definition of 2-codes. In [**6**] so-called difference sequences, as defined in § 3 are pseudo-Morley sequences (if we restrict our attention, as we may, to modular types) with quite a few additional properties which are irrelevant at the moment, and will be referred to below. Since the definition of difference sequences involves many technical details, we will not give it here.

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2.7. Overtaking obstructions

Recall that we fixed a system $\mathcal{C}$ of standard geometries. By definition, for every function μ (and any definition of pseudo-μ-Morley sequence) pseudo-μ-envelopes (with respect to $\mathcal{C}$) exist. Our problem is to show that they are homogeneous (and elementary).

We will do it by giving an explicit axiomatization of the theory of pseudo-μ-envelopes (for appropriately chosen μ). The first approximation to an axiomatization of pseudo-μ-envelopes would be:

(1) $T_\infty^\forall$;

(2) for every model E, every $J \in \mathcal{C}$ and $a \in \mathrm{dom}(J) \cap E$ there is no pseudo-$(\mu(J)+1)$-Morley sequence in $J_a(E)$;

(3) every model is (self-sufficient and) algebraically closed in the sense of T_∞;

(4) for every model E, every $J \in \mathcal{C}$ and every $a \in \mathrm{dom}(J) \cap E$ there exists a pseudo-$\mu(J)$-Morley sequence in $J_a(E)$;

(5) for every model E, every affine geometry definable over E has a point in E.

Remark 2.15. In general, it is more natural to define the envelope of a set X, in which case something has to be said concerning the realizations of p_ω in the envelope of X. In the present context, the most natural (conforming also with the standard definition) would be to require that $E(X) \mathop{\smile\!\!\!\!\!\!\,|}_X a$ for all $a \models p_\omega|_X$. As we have already said before—since p_ω can take care of itself—we will ignore this part in our treatment of envelopes.

Like the situation in [**9**], these properties (striking out 'pseudo' whenever it appears) characterize μ-envelopes, in the present context as well. However, as stated, they will not, in general, be consistent. To better understand the problem consider E_0 satisfying (1)–(3). Let $J \in \mathcal{C}$ and $a \in \mathrm{dom}(J) \cap E_0$ be such that the longest pseudo-Morley sequence in $J_a(E_0)$ has length $\nu(J_a) < \mu(J)$. Let $b \models J_a|_{E_0}$ and $E := \mathrm{acl}(E_0 b)$. Does E satisfy (3)? By the construction of our system of standard geometries, for all $J' \in C$ and all $a' \in \mathrm{dom}(J) \cap E_0$ we have

$$\dim_E J'_a \leqslant \dim_E J'_a + 1 \tag{$*$}$$

and strict inequality holds unless $J = J'$ and $J_a \not\perp J_{a'}$. Since $E_0 = \mathrm{acl}_{T_\infty}(E_0)$ it follows from the above that $J'_a(E_0) = J'_{a_0}(E_0)$ for all other J' and $a' \in \mathrm{dom}(J) \cap E_0$ implying that (3) holds of $J'_{a'}$. In fact, since our standard geometries are all modular, if $J = J'$ and $J_a \not\perp J_{a'}$ we must have equality in $(*)$. It is usually not hard to check in that case—with a reasonable definition of pseudo-Morley sequences—that for every such a' we have in E sequences of length $\nu(J_a) + 1 \leqslant \mu(J)$. However, it is not at all clear (and in fact, not in general true, see the example of § 4 of [**3**]) that the same holds of any $a \in \mathrm{dom}(J) \cap E$ (but not necessarily $a \in E_0$).

Since (1)–(3) are obviously not strong enough (as they say nothing on the maximality of pseudo-envelopes) and (1)–(4) are inconsistent we have to correct our axioms into something of the following flavour.

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(4′) For every model E, every $J \in \mathcal{C}$, $a \in \operatorname{dom}(J) \cap E$ and $k < \mu(j)$, if $b_1, \ldots, b_k$ is a pseudo-k-Morley sequence in $J_a(E)$ either

(a) there exists $b_{k+1} \in J_a(E)$ such that $(b_1, \ldots, b_k, b_{k+1}) \models \psi^J_{k+1}$; or

(b) for every generic $b \in J_a$ there exists $J' \in \mathcal{C}$ and $a' \in \operatorname{dom}(J')$ such that in $\operatorname{acl}_{T_\infty}(Eb)$ there exists a pseudo-$(\mu(J')+1)$-Morley sequence in $J'_{a'}$.

Our problem now is to show that (4′)(b) can be stated in a first-order way. Towards this end we will introduce the only new definition of this text (see [**15**, Appendix A] for more details).

Definition 2.16.

(1) Let $a \in \operatorname{dom}(J)$ for some $J \in \mathcal{C}$. Say that J_a admits an obstruction if there exists some $B = \operatorname{acl}(B)$ such that for every generic $b \in J_a$ there exists some $J' \in \mathcal{C}$ and $d \in (\operatorname{dom}(J') \cap \operatorname{acl}(Bb)) \setminus (\operatorname{dom}(J') \cap B)$ such that ($J'_d \perp J$ and) $\dim_{\operatorname{acl}(Bb)} J'_d > 0$. Call $\dim_{\operatorname{acl}(Ba)} J'_d$ the size of the obstruction.

(2) J_a admits bounded obstruction if there is a bound on the size of the obstructions it admits.

(3) J admits bounded obstructions if J_a does for all $a \in \operatorname{dom}(J)$ and the bound is uniform (in a).

It follows almost immediately from the definition of Lie coordinatizable structures, that any family of standard geometries does not admit any obstructions at all (i.e. all obstructions are uniformly bounded by 0). The same proof works in the present context. As explained in § 2.5, the problem with translating the proof to the case of pseudo-envelopes is that if $c \in \operatorname{dom}(J)$, it may well be that $\operatorname{acl}(c)$ contains fairly long (unboundedly as c varies over $\operatorname{dom}(J)$) pseudo-Morley sequences (so it will not be appropriate to call these, 'pseudo-obstructions', since they actually cause much more trouble than obstructions—but we will do it nonetheless!).

In [**13**] this problem is bypassed, by assuming the existence of a system of standard geometries admitting uniform structure. As we have already said, this assumption is not valid in T_∞ for most amalgamation constructions. The solution is to prove enough uniformity of structure in order to uniformly bound the size of pseudo-obstructions. In practice, what we will prove is the existence of a function $c : \mathcal{C} \to \mathbb{N}$ and for every $J \in \mathcal{C}$ a natural number $B(J)$ such that for every $J' \in \mathcal{C}$ the size of a possible pseudo-obstruction to J witnessed by an element from J' is bounded by $B(J) + c(J')$. Before we proceed to explain how this can be achieved, we show why this would be enough.

Choose any enumeration of $\mathcal{C}$. Let $E_0 \models$ (1)–(3), $b \in \operatorname{dom}(J^i)$ and $a \in J^i_b$ generic over E_0. By our assumption the size of a J_i obstruction witnessed by some J' is bounded by $B(J^i) + c(J')$. Choose any function $\mu^* : \mathcal{C} \to \mathbb{N}$ which is finite to one, and set $\mu(j) = \mu^*(j) + c(J^j)$. So there are only finitely many $j \in \mathbb{N}$ such that

$$B(J^i) + c(J^j) \geqslant \mu(j) = \mu^*(j) + c(J^j).$$

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So for such a choice of μ, for a given $J \in \mathcal{C}$ there are only finitely many $J' \in \mathcal{C}$ which may witness the existence of a pseudo-obstruction, and axiom scheme (4′)(b) readily becomes elementary.

So we now turn back to the proof of the above form of uniform boundedness of pseudo-obstructions. The idea is to show that if $E = \mathrm{acl}(E)$, $J \in \mathcal{C}$, $a \in \mathrm{dom}(J) \cap E$ and $b \in J_a \setminus E$ then there exists a function $\Delta : \mathcal{C} \to \mathbb{N}$ such that if $b_1, \dots, b_{\Delta(J')}$ is a pseudo-Morley sequence in $J'_c \cap \mathrm{acl}(Ea)$ for some $J' \in \mathcal{C}$ and $c \in \mathrm{dom}(J')$ then either $J'_c \not\perp E$ or $J'_c \not\perp \mathrm{acl}(ab)$. This is enough, since if $J'_c \not\perp E$ then $c \in E$ (because $E = \mathrm{acl}(E)$) and therefore it must be that $J' = J$ and $c = b$, so this is not an obstruction. If, on the other hand, $J'_c \not\perp \mathrm{acl}(ab)$, we may assume that $c \notin E$ so there is a uniform bound $c(J')$ on the length of a pseudo-Morley sequence in J' containing only elements which meet $\mathrm{acl}(b)$ (otherwise this will contradict either the assumption that $c \notin \mathrm{acl}(b) \subseteq E$ or the fact that $\mathrm{acl}(b) \leqslant M$). So all but $c(J')$ of the elements in the sequence witnessing the pseudo-obstruction must lie in $\mathrm{acl}(ab) \setminus \mathrm{acl}(b)$, and it remains to check that the number of such elements is bounded by a function of $|a|$ (which depends, among others, on the definition of pseudo-Morley sequences). In the case of fusion over a vector space this bound is simply $|a|$.

So finally, we are reduced to showing that indeed (with the same notation) $J'_c \not\perp E$ or $J'_c \not\perp \mathrm{acl}(ab)$. The key observation is that the geometries (but not the pre-geometries!) of strongly minimal types in T_∞ are determined by the quantifier free formulae (in the natural language of T_∞), and that the quantifier free structure of standard geometries is uniform*.

To state this more precisely, it will be useful to introduce for a stationary type p the notation $\mathrm{qf}\, p^{\otimes n} := \{\varphi \in p^{\otimes n} : \varphi \text{ quantifier free}\}$. With this notation, Lemma 5.4 in [**6**], translated into a more standard model theoretic language, says the following.

Lemma 2.17. *Let $M \leqslant N$ and $J \in \mathcal{C}$. There exists a function $\lambda(n, J)$ such that for $b \in \mathrm{dom}(J)$ and distinct $a_1, \dots, a_{\lambda(n,J)} \in J_b(N)$ either $J_b \not\perp M$ or there are $1 \leqslant i_1 < \cdots < i_n \leqslant \lambda(n, J)$ such that $(a_{i_1}, \dots, a_{i_n}) \models \mathrm{qf}\, J_b^{\otimes n}|_M$.*

This is a simple corollary of the total categoricity of strongly minimal sets in T_∞. The same lemma appears, and has an as important role in the construction of the bad field of [**7**]. In the latter case the proof is somewhat harder, mostly due to the fact that local finiteness of the strongly minimal sets in T_∞ is far from obvious.

The desired conclusion now follows rather easily, but since it uses quite heavily the specific construction, we will avoid the details. In [**6**] this appears as Lemma 7.3 and in [**7**] it is Lemma 9.1, with a similar proof.

It should be clear by now that we can use the same results also to obtain the analogous correction of (5).

(5′) Every affine space (A, V) definable over E has a point in E, unless for every generic $b \in A$ one of possibly finitely many (definable families) of obstructions occur.

* This statement is not quite precise: see, for example, [**16**]. Moreover, it depends heavily on $\mathcal{L}(T^\forall)$, and if T_∞ happens to have quantifier elimination in that language, will generally not be true. Suffice it if we say that in all known examples there exists $\mathcal{L}' \subseteq \mathcal{L}(T_\infty)$ such that the geometry of all strongly minimal sets is determined by their $\mathcal{L}'$ structure and in which we have uniformity of structure.

So in order to check that this axiom is consistent with (1)–(4′) we simply have to make sure that if no obstruction occur and (A, V) does not have a point in E then $\mathrm{acl}(Eb)$ still satisfies (1)–(4′) for any choice of $b \in A$ generic over E. This is an almost immediate corollary of the uniqueness of parallel lines (see § 3 of [**9**] for the details).

2.8. Concluding remarks

We have shown in the last subsection that for μ growing fast enough, pseudo-μ-envelopes are first-order axiomatizable. Denoting the resulting theory T_μ it is now a straightforward exercise to ascertain that models of T_μ are homogeneous. The proof proceeds by back and forth. The point is that if E is a pseudo-envelope, $A_i = \mathrm{acl}_{T_\infty}(A_i)$ (for $i = 1, 2$) are small and $f : A_1 \to A_2$ is a partial isomorphism, then for every $a \in E$ there exists $\bar{a} \subseteq E$ such that $a \in \bar{a}$ and $\bar{a}/A_1$ is prealgebraic. The analysis of the previous subsection shows that the only way there could not be a solution of $f(\mathrm{tp}(\bar{a}/A1))$ in E is if there is an obstruction witnessed already in $\mathrm{acl}_{T_\infty}(f(\mathrm{Cb}(\bar{a}/A_1) \cup \{\bar{a}\}))$ (where $f(a)$ is a any generic realization of $f(\mathrm{tp}(\bar{a}/A_1)))$, which cannot be, as this would imply an obstruction already existed in A_1.

Therefore, T_μ eliminates quantifiers (relative to T_∞) whence it inherits the coordinatization of T_∞ and consequently has a unique non-algebraic regular type (all the regular types in T_∞ orthogonal to p_ω have dimension bounded by μ in all models—hence they are algebraic). With the coordinatization T_μ inherits also the ω-stability of T_∞, so pseudo-envelopes are uncountably categorical, and T_μ is the collapse of T_∞.

The most natural question at this stage is of course to understand the geometry of the strongly minimal set (corresponding to the unique non-algebraic type) obtained in this way. To answer this question, it will be more convenient to work in a saturated $E \models T_\mu$, such that $E \leqslant M \models T_\infty$ (recall that for the construction of T_μ we worked in an existentially closed model of T_∞, so the resulting envelope could not be saturated). Choosing a transcendence base $I \leqslant E$, we get that E is the pseudo-μ-envelope of I. It is now not hard to check, that any finitary property of the geometry of p_ω will be preserved in the geometry of T_μ, provided that μ is large enough. More precisely, consider any set $a_1, \dots, a_k \in I$ then

$$\mathrm{cl}_{p_\omega}(a_1, \dots, a_k) \cap E = \mathrm{acl}_{T_\mu}(a_1, \dots, a_k) \setminus \mathrm{acl}_{T_\mu}(\emptyset).$$

This readily implies, that at least for generic μ (in any reasonable definition of the term) the geometrical properties of p_ω are reflected in the unique generic type of T_μ. For example, it is clear that if p_ω is n-ample for some n (see [**26**] for the definition) then so is T_μ. Note, however, that in general (see, for example, [**18**, § 5.2]) μ may be recoverable from the geometry of T_μ, so that the geometry of p_ω will not be isomorphic to that of T_μ.

So the role of the free construction is to provide us with the theory T_∞ where the collapse is taking place and the local geometry (of its unique regular type of infinite rank) is the one ultimately determining the geometry of the collapse. It is a thorough (though more often than not implicit) analysis of the geometric properties of T_∞ which tells us how to perform the collapse, which can be viewed as a smooth approximation of T_∞.

It is perhaps not just a coincidence that both the free construction and the collapse are (or at least can be) obtained by means of amalgamation. But this similarity is misleading: the two constructions are genuinely different. I hope that this section of the paper has made it clear that there is a good model theoretic theory underlying the collapse. I hope that the next section will help me convince that the true challenge, the *terra incognita*, lies in the realm of free constructions.

3. Some old open questions

This section is dedicated to the presentation of some, in my view, important open questions related to Hrushovski's construction. Some of the problems may be quite hard, but others seem accessible, possibly even easy. Throughout, all theories will be in countable languages.

3.1. Geometry

The questions in this subsection are motivated mainly by the following problem.

Problem 3.1. *Is there a classification of strongly minimal geometries?*

But also by the challenge of constructing a bad group of finite Morley rank. Already the first step in Problem 3.1 seems rather hard.

Problem 3.2. *Formulate a conjecture on the classification of strongly minimal geometries.*

A reasonable approach to the problem would be a classification of strongly minimal sets according to the complexity of their geometries. A possible measure of complexity can be found in Pillay's hierarchy of n-ample geometries proposed in [**26**]. Pillay's conjectural hierarchy has two major drawbacks.

- It does not offer any gauge of complexity for structures interpreting infinite fields (as such structures are n-ample for all n).
- All known strongly minimal structures either interpret an infinite field or fall into one of the three first classes in the infinite hierarchy.

A partial remedy to the first of these problems could be to classify the strongly minimal geometries in terms of (countable) collections of 'basic' types of geometries (say the n-ample non-$(n+1)$-ample geometries together with the geometries of algebraically closed fields). But this does not settle the question of the sort of classification such collections of geometries may give us.

Recovering the geometry of a strongly minimal set from a set of reducts of lower complexity may be an over ambitious goal, as the following example shows. Consider T, a strongly minimal fusion (over $\{=\}$, say) of two algebraically closed fields, T_1, T_2. The natural choice of a set of geometries associated with T is given by the geometries of T_1, T_2. However, recovering the geometry of T from the data might call for additional information concerning the μ-function used in the collapse (see, for example, [**18**, § 5]).

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In the above example the more natural geometry to associate with $T_\mu = T_1 \oplus_\mu T_2$ is the local geometry of p_ω in the corresponding free fusion $T_\infty := T_1 \oplus_\infty T_2$. In addition, it is easy to verify that if $T_\mu = T_1 \oplus_{\mu_1} T_2 \oplus_{\mu_2} T_3$ the local geometry of $T_\infty := T_1 \oplus_\infty T_2 \oplus_\infty T_3$ is fully determined by T_1, T_2, T_3 (and in particular does not depend on the order in which the theories are fused). So we make the following definition.

Definition 3.3. Let $\{T_i\}_{i \in I}$ be a countable collection of strongly minimal sets in countable (disjoint) languages, $\{\mathcal{G}_i\}_{i \in I}$ their associated geometries. The free amalgam $\mathcal{G} := \bigoplus_{i \in I} \mathcal{G}_i$ is the local geometry of $T_\infty := \bigoplus_{i \in I} T_i$.

A combinatorial geometry $\mathcal{G}$ is a *free amalgam*, if it is obtained as the free amalgam of n-ample (possibly unbounded n) strongly minimal geometries and the geometries of algebraically closed fields.

To clarify the above definition we need a couple of observations.

Remark 3.4.

(1) Recall that if T_1, T_2 are strongly minimal geometries then $T_2 \oplus_\infty T_2$ always exists, regardless of DMP, which is only needed for the collapse.

(2) The pre-dimension function for the free fusion of countably many theories at one go is given by $\delta(A) := |A| + \sum_{i \in I}(\mathrm{MR}_i(A) - |A|)$. By requiring that $\delta(A) \geqslant 0$ (which is an elementary property) we assure that the support of this last sum is finite.

(3) The standard proof of ω-stability of the resulting theory $\bigoplus_{i \in I} T_i$ goes through unaltered.

Thus, we have a canonical way of associating with a given collection of strongly minimal geometries (possibly with repetitions*) a combinatorial geometry (which need not, in general, be that of a strongly minimal set). It may therefore be reasonable to try and classify the geometries of strongly minimal sets using those of free amalgams of strongly minimal geometries.

This raises naturally, the question already hinted to in the end of the previous section, concerning the relation between the local geometry of T_∞ and the geometry of T_μ. As already mentioned, the geometry of T_μ can be viewed as obtained from the one in T_∞ by reducing to an appropriate subset of realizations of p_ω. Although this is not quite a satisfactory characterization, the following can be asked.

Problem 3.5. *Is every strongly minimal geometry an infinite homogeneous subset (with the induced closure operator) of a free amalgam?*

It may well be that the definition of freely amalgamated geometries will have to be modified to encompass even the known constructions. The obvious test cases are the following.

* What is the geometry of the fusion of an algebraically closed field with itself?

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Problem 3.6.

(1) *Let $T := T_1 \oplus_{T_0}^{\infty} T_2$ be the (free) fusion of the strongly minimal theories T_1, T_2 over their common totally categorical vector space T_0. If T_1, T_2 satisfy the conclusion of the previous problem, does T?*[*]

(2) *Let T be a theory of finite Morley rank with DMP. Let $D(T)$ be a strongly minimal theory interpreting it, as obtained in [**14**]. If every strongly minimal set definable in T satisfies the conclusion of the previous problem, does $D(T)$?*

(3) *As a concrete example of (2) above, let T be the theory consisting of E, an equivalence relation with infinitely many classes, each of which supporting (uniformly) a stricture of an algebraically closed field. Let X be the imaginary sort corresponding to E and T' the theory T fused with a function $F : E \to X$ which is a bijection on the E-classes. So T' is a strongly minimal theory. Does its geometry come from an amalgamated one?*

(4) *What about the geometries associated with the (almost strongly minimal) coloured fields?*

A positive answer (to my mind, not very likely) to the following question would suggest a more natural formulation of Problem 3.5.

Problem 3.7. *Let $T_\mu := T_1 \oplus_\mu T_2$. Is the geometry of T_μ locally isomorphic to that of T_∞ (i.e. are the two geometries isomorphic after localizing at a set S with $\dim(S) = \operatorname{codim}(S) = \aleph_0$).*

More generally, we can ask the following question.

Problem 3.8. *Is there a canonical way of associating with any strongly minimal geometry an amalgamated geometry in such a way that the geometry associated to 'basic' geometries is basic?*

Problem 3.7 generalizes a question form [**18**]. In § 5 of the paper continuum many non-isomorphic strongly minimal flat geometries are constructed, and the question arises as to whether these geometries are non-locally isomorphic. It is also asked whether a continuum of non-locally isomorphic flat strongly minimal geometries exists. To the best of my knowledge these questions are still open.

A related problem, generalizing Problem 3.7, which may be of interest (although it does not seem to have immediate implications on the classification problem) is the following.

Problem 3.9. *Let $\{G_i\}_{i \in I}$ and $\{F_j\}_{j \in J}$ be countable collections of strongly minimal geometries, $\mathcal{G}$, $\mathcal{F}$ their respective free amalgams. Suppose there is no bijection $f : I \to J$ such that $\mathcal{G}_i \cong \mathcal{F}_{f(i)}$ for all $i \in I$. Can it be that $\mathcal{G} \cong F$. Can they be locally isomorphic if the basic geometries are not?*

[*] Recall that under the same assumptions, if in addition T_1, T_2 are one based, then $T_1 \oplus_{T_0}^{\mu} T_2$ is 1-based [**15**] whereas $T_1 \oplus_\mu T_2$ (the fusion over $\{=\}$) is not.

Note that in the above problem $\mathcal{F}$, $\mathcal{G}$ need not be amalgamated geometries, as it is not assumed that $\{\mathcal{G}_i\}_{i\in I}$ and $\{F_j\}_{j\in J}$ are 'basic' geometries.

Returning to the question of the number of non-locally isomorphic strongly minimal geometries, I do not know of any published proof of the existence of continuum many such. We give here a short proof, based on an idea of Hrushovski's. Recall, from the introduction of [**17**]:

> The geometry [of the fusion] can be viewed as 'relatively flat' over the geometries of the strongly minimal sets, however. In particular, it can be shown that if G is a connected group definable in the strongly minimal amalgam of D_1, D_2 then there exist connected groups G_i definable over D_i and definable surjective group homomorphism $f : G \to G_1 \times G_2$ with finite (central) kernel. It would be good to have the sharper result with the arrow reversed $f : G_1 \times G_2 \twoheadrightarrow G$.

Already from the easier of the above assertions it would follow that if F_1, F_2 are strongly minimal fields of distinct positive characteristics p_1, p_2 (respectively) then any strongly minimal group of characteristic p_i interpretable in their fusion $F_1 \oplus_\mu F_2$ is a finite cover of $(F_i, +)$. For every prime p denote $K_p := \mathrm{acl}(\mathbb{F}_p)$. By induction it would follow that for any (possibly infinite) set S of primes the free fusion $K_S := \bigoplus_{p\in S} K_p$ interprets strongly minimal groups of characteristic p precisely for the primes $p \in S$. By Theorem C of [**12**], this gives a continuum of non-locally isomorphic strongly minimal geometries, as S varies over subsets of the primes. So we prove the following claim.

Claim 3.10. *If G is a connected group definable in the strongly minimal fusion $T : D_1 \oplus_\mu D_2$ then there exist connected groups G_i definable over D_i and definable surjective group homomorphism $f : G \to G_1 \times G_2$ with finite kernel.*

A sketch of the proof. The proof is not hard, so we give only a sketch. The idea is to pull back the group configuration of G to group configurations in D_1 and D_2. So let $Q := \{a, b, c, x, y, z\} \subseteq G$ be a set of G-generics forming a group configuration (e.g. $y = a^{-1}x$, $c = ab$ and $z = b^{-1}a^{-1}x$). For simplicity we may assume that $Q \leqslant M \models D_1 \oplus_\mu D_2 =: T$. By assumption, $y \in \mathrm{acl}_T(ax) \setminus (\mathrm{acl}_T(x) \cup \mathrm{acl}_T(a))$, whence $d(y/ax) = 0$ and if $\hat{y}_i$ are L_i-transcendence bases for $y/(a, x)$, we get $|\hat{y}_1| + |\hat{y}_2| = n$ (for $n = \mathrm{length}(y)$, otherwise we would get a contradiction to the assumption that $a, x \in G$ are independent generics).

We can also find $\hat{a}_i \subseteq a$ such that $\mathrm{MR}_i(\hat{a}_i) = \mathrm{MR}_i(a/x, y) = \mathrm{MR}_i(y/a, x)$. Similarly, we can find $\hat{x}_i \subseteq x$ with the same property. Using the fact that $\mathrm{tp}_T(x/y) = \mathrm{tp}_T(z/y)$ we find $\sigma \in \mathrm{Aut}(M/y)$ (some saturated $M \models T$) such that $\sigma(z) = x$, so we get that $\sigma(b)$ is interalgebraic with a over x, y. Pulling back $\hat{a}_i$ and $\hat{x}_i$ we can find $\hat{b}_i$ and $\hat{z}_i$ such that the triple (b, z, y) has the same properties with respect to $(\hat{b}_i, \hat{z}_i, \hat{y}_i)$ as (a, x, y) had with respect to $(\hat{a}_i, \hat{x}_i, \hat{y}_i)$. By similar arguments we can find a corresponding $\hat{c}_i \subseteq c$ with the same properties (with respect to (a, b, c) and (c, z, x)).

Thus, setting $\hat{Q}_i := \{\hat{a}_i, \hat{b}_i, \hat{c}_i, \hat{x}_i, \hat{y}_i, \hat{z}_i\}$ we get that either $Q \in \mathrm{acl}_i(\hat{Q}_i)$, or $Q/\hat{Q}_i$ is a group configuration in L_i. Because $|\hat{y}_1| + |\hat{y}_2| = n$ whereas $\mathrm{MR}_1(y) + \mathrm{MR}_2(y) > n$ it cannot be that both $Q \in \mathrm{acl}_1(\hat{Q}_1)$ and $Q \in \mathrm{acl}_2(\hat{Q}_2)$, whence at least one actually gives a group configuration. Let G_1, G_2 be the corresponding groups (if $Q \in \operatorname{acl}_i(\hat{Q}_i)$ we set $G_i = \{e\}$, the trivial group).

The claim now follows easily (using the fact that $\hat{Q}_i$ is a set of D_i-independent generic elements). □

Remark 3.11. By similar arguments it can be shown that the only strongly minimal fields interpretable in $T = D_1 \oplus_\infty D_2$ are the ones already interpretable in either D_1 or D_2 (use the field configuration and the fact that we know it corresponds to the action of a two-dimensional group on a one-dimensional homogeneous space).

Obtaining the sharper characterization of groups definable in $D_1 \oplus_\mu D_2$ as suggested in [**17**] would, of course, call for a closer analysis of the situation.

It may be worth pointing out, though, that a similar argument would show that if M is a structure of finite Morley rank (with DMP) and D a strongly minimal set interpreting M, as obtained by the construction of [**14**], then any group interpretable in D is a finite cover of one already definable in M. It seems that the same goes for Ziegler's fusion of structures of finite Morley rank [**32**], but here the situation is somewhat more delicate. The problem is, that in order to carry the argument through, some care should be taken in handling imaginaries.

Moreover, if $T_1 := D_1 \oplus D_2$ is the free fusion of D_1, D_2 (i.e. $\mu = \infty$) the same should be true. Indeed, in that case, if G is a (connected) group definable in T and Q is a self-sufficient quadrangle as above, then $a \subseteq \operatorname{acl}_T(xy) \iff a \subseteq \langle xy \rangle$ where $\langle\cdot\rangle$ is the transitive closure of closing under acl_1 and acl_2. Hence this is a special case of the one treated in the proof of Claim 3.10. Interestingly, enough, though, despite of the fact that the claim remains true for the (strongly minimal) fusion over a vector space, it will no longer be true of the free fusion over a vector space. Using [**15**] it should be a fairly easy exercise to classify the groups of finite rank in this last case.

Finally, note that if p_ω is the (regular) generic type of T_∞, any group configuration in $p_\omega^{\otimes n}$ would produce a group configuration in T_μ for large enough μ, and therefore must originate in group configurations in D_1, D_2, as in the previous claim.

All this (together with the fact that connected CM-trivial groups of finite Morley rank are nilpotent [**24**]) support, the general feeling expressed in the introduction to [**5**]:

> Hrushovski's amalgamation techniques, as they stand at present, produce theories which are CM-trivial or at least CM-trivial over the data (the latter notion has not been made precise yet). So [**24**] suggests that these methods will not produce new simple groups of finite Morley rank.

Today this statement is as well accepted as it was when written in 1998, but, unfortunately, also as ambiguous as it was then. The main problem with making it more precise is that technically it is probably wrong. Let M be a countable (saturated) structure of finite Morley rank. Assume without loss that $T := \operatorname{Th}(M)$ eliminates quantifiers and is given in a relational language. Let $\mathcal{C}$ be the class of all finite models of $T^\forall$. For $A \in \mathcal{C}$ define $\delta(A) = \operatorname{MR}(\operatorname{tp}(f(A)))$ for some elementary embedding $f : A \to M$. By quantifier elimination $\delta(A)$ is well defined. Note also that quantifier elimination assures that if

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$A \subseteq B \in \mathcal{C}$ then $\delta(A) \leqslant \delta(B)$ so $A \subseteq B \iff A$ is self-sufficient in B. Thus $(\mathcal{C}, \subseteq)$ has the amalgamation property, and its Fraïssé limit is M.

Therefore, if we translate 'Hrushovski's amalgamation techniques' into 'Amalgamation constructions in a universal (or inductive) class with pre-dimension', we get that any structure of finite Morley rank can be reconstructed in such a way. Possibly more precise (and certainly more cumbersome) would be the following.

Definition 3.12. A stable pre-dimensioned class is a universal theory T in a language L and a function $\delta : \mathcal{C} \to \mathbb{N}$, where $\mathcal{C}$ is the set of all finitely generated models of T, satisfying

(1) δ is L-isomorphism invariant;

(2) $\delta(\emptyset) = 0$;

(3) δ is sub-modular;

(4) setting, for $A \subseteq B \in \mathcal{C}$, the relation $A \leqslant B \iff \delta(A) \leqslant \delta(A')$ for all $A \subseteq A' \subseteq B$, makes $(C, \leqslant)$ into an amalgamation class;

(5) if $A \leqslant B_1$, $A \leqslant B_2$, $B_i \in \mathcal{C}$ then there is at most one structure $D \in \mathcal{C}$ (up to isomorphism) which is an $\leqslant$-amalgam for B_1 with B_2 over A, satisfying $\delta(D/B_2) = \delta(B_1)$—moreover, if there is no $A \subsetneq B_1' \subseteq B_1$ such that $B_1' \hookrightarrow B_2$ then such D (the free amalgam of B_1 with B_2 over A) does exist;

(6) the Fraïssé limit of $(\mathcal{C}, \leqslant)$ is a saturated model of its own theory.

By Corollary 3.4.5 of [**31**] stable pre-dimensioned classes produce ω-stable theories (if $|\mathcal{C}| = \aleph_0$). In those terms, stable pre-dimensioned classes produce structures which are flat (in the sense of Hrushovski's, as quoted above) over the data, in the sense that any group configuration in the Fraïssé limit of $\mathcal{C}$ arises from one in $\mathcal{C}$. More precisely, if $\mathcal{U}$ is the Fraïssé limit of $\mathcal{C}$ and $Q \subseteq \mathcal{U}$ is an algebraic quadrangle, there is $C \in \mathcal{C}$, $R \leqslant C$ and an embedding $f : C \hookrightarrow \mathcal{U}$ satisfying $f(R) = Q$.

More generally, if we define, in the natural way, for a stable pre-dimensioned class $\mathcal{C}$, replacing the notion of acl with that of cl-closure, we can readily define δ-quadrangles in $\mathcal{C}$, or whether $\mathcal{C}$ is n-ample, non-$(n+1)$-ample (what about the interpretability of an infinite field), properties which will all be carried out to the Fraïssé limit of $\mathcal{C}$.

The considerable amount of work invested over the years around Hrushovski's amalgamation constructions improved our understanding of their mechanism and our mastery of their manipulation; it changed significantly our view on the diversity of possible phenomena in strongly minimal theories, but it has not taught us how stretch these constructions further. Of all the many variants of these constructions (ultimately producing structures of finite rank) that I know, only the one of [**4**] uses a pre-dimension function that is not an immediate derivations of the ones introduced in Hrushovski's original works. So to my taste, the true challenge in this field of constructions, is to find new stable pre-dimensioned classes. Ones which are n-ample non-$(n+1)$-ample for large (i.e. greater than 1!) n, ones in which algebraic quadrangles do not arise from known ranked groups,

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ones with new pre-dimension functions. And what about the collapse? I believe it will take care of itself.

3.2. The collapse

The problems in this subsection are of a less general nature than the ones in the previous subsection, but since they are more closely related to the discussion of the previous section, it may be appropriate to present them here.

To the best of my knowledge, Cherlin's question (quoted in [**17**], but not appearing in any of Cherlin's own papers) concerning the existence of a maximal minimal theories has not been completely settled in the strongly minimal case. Of course, the work of [**17**] gives a negative answer for strongly minimal theories with DMP, or even to ones with a strongly minimal expansion with DMP. It is therefore quite natural to ask the following question.

Problem 3.13. *Does every strongly minimal set have a strongly minimal expansion with DMP?*

It seems plausible that if we restrict ourselves to rank preserving expansions the answer to the question is negative (an adaptation of the construction of [**16**] to the example of § 3 in [**14**] could do the trick) but in its full generality the question seems to me completely open.

Another, possibly of more interest, approach to Cherlin's question could be as follows.

Problem 3.14. *Let T_1, T_2 be strongly minimal theories (with quantifier elimination) in (countable) disjoint languages. Is there a strongly minimal completion of $T_1^{\forall} \cup T_2^{\forall}$.*

Of course, a positive answer to Problem 3.13 will also answer positively this last question, but it may not be the easiest way of answering it. As already mentioned earlier, there is no problem construction an ω-stable completion $T_\infty \supseteq T_1^{\forall} \cup T_2^{\forall}$ of rank ω, and the natural question is whether this theory can be collapsed. As pointed out in § 2.3, in that generality T_∞ need not have definability of strong minimality, though it will have a weaker version thereof, namely for a strongly minimal $\varphi(x, b)$ there is $n \in \mathbb{N}$ and $\theta \in \text{tp}(b)$ such that $\text{MR}(\varphi(x, b)) = 1$ and its multiplicity at most n, for all $b' \models \theta$. This in itself suffices for the construction of envelopes (since T_∞ will still not have obstructions), but when trying to construct pseudo-envelopes things become more delicate. Apparently, a closer look at pseudo-obstructions and the ways they can arise will be needed.

In all probability if the fusion of strongly minimal sets can be carried out without DMP, the same techniques could be adapted to complete the characterization, started in [**14**], of structures of finite Morley rank interpretable in strongly minimal structures, and in particular answer the following fairly natural question.

Problem 3.15. *Is every uncountably categorical theory interpretable in a strongly minimal one? What if we require the interpretation to be rank preserving?*

In fact, I do not know the answer to the above question even for almost strongly minimal theories.

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