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ARTICLE

# Hypergraph independence polynomials with a zero close to the origin

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**Abstract**

For each uniformity $k \geq 3$, we construct $k$ uniform linear hypergraphs $G$ with arbitrarily large maximum degree $\Delta$ whose independence polynomial $Z_G$ has a zero $\lambda$ with $|\lambda| = O\left(\frac{\log \Delta}{\Delta}\right)$. This disproves a recent conjecture of Galvin, McKinley, Perkins, Sarantis, and Tetali.

**Keywords:** Independence polynomial; hypergraph; zero-free region

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## 1. Introduction

A hypergraph $G = (V, E)$ is a set of vertices $V$ together with a set of edges $E \subset 2^V$. A hypergraph is $k$-uniform if every edge has size $k$. The degree of a vertex $v \in V$, denoted by $d(v)$, is the number of edges it appears in; in a hypergraph with maximum degree $\Delta$, each vertex appears in at most $\Delta$ edges.

An independent set in $G$ is a set of vertices $I \subset V$ such that $I$ contains no edge. Let $\mathcal{I}(G)$ denote the family of all independent sets in $G$. The independence polynomial of $G$ is defined by

$$Z_G(\lambda) = \sum_{I \in \mathcal{I}(G)} \lambda^{|I|}.$$

This polynomial plays an important role in mathematics, physics and computer science [1, 3, 4, 7–10]. A key property for understanding this polynomial is the largest radius of a disk-shaped zero-free region(ZFR), a region in $\mathbb{C}$ where $Z_G$ has no zero. We refer the reader to the introduction of [2] for a survey of how knowledge of the zeros of $Z_G$ can lead to interesting results about independent set.

When $G$ is a graph with maximum degree $\Delta$, the ZFRs for $Z_G$ are well-understood [6, 8]. Specifically, Shearer [8] showed that $Z_G$ has no zero inside the disk $|\lambda| < \frac{(\Delta-1)^{\Delta-1}}{\Delta^\Delta}$, and this bound is the best possible.

In a recent paper [2], Galvin, McKinley, Perkins, Sarantis and Tetali studied the zeros of $Z_G$ when $G$ is a general hypergraph with given maximum degree $\Delta$. They showed that $Z_G$ has no zero inside the disk $|\lambda| < \frac{\Delta^\Delta}{(\Delta+1)^{(\Delta+1)}}$. Furthermore, for each uniformity $k \geq 2$, they constructed a family of $k$-uniform hypergraphs with arbitrarily large maximum degree $\Delta$ such that $Z_G$ has a zero $\lambda$ with $|\lambda| < O_k\left(\frac{\log \Delta}{\Delta}\right)$, thereby showing that their bound is tight up to a logarithmic factor if we place no additional assumption on $G$.

A hypergraph is linear if each pair of edges intersect in at most one vertex. The aforementioned constructions in [2, Section 4] are far from linear, since they contain edges that intersect in $(k-1)$ vertices. In [2, Conjecture 3], Galvin, McKinley, Perkins, Sarantis and Tetali conjectured that their lower bound on the maximum radius of the zero-free disk for $Z_G$ can be improved under the additional assumption that $G$ is linear.

**Conjecture 1.1.** *For each $k \geq 2$, there exists a constant $C_k > 0$ such that the following is true. If $G$ is a $k$-uniform, linear hypergraph with maximum degree $\Delta$ and if*

$$|\lambda| \leq C_k \Delta^{-\frac{1}{k-1}}$$

*then $Z_G(\lambda) \neq 0$.*

This conjecture is motivated by results on asymptotic enumeration in [5]. Galvin, McKinley, Perkins, Sarantis and Tetali verified this conjecture when $G$ is a hypertree [2, Theorem 4],

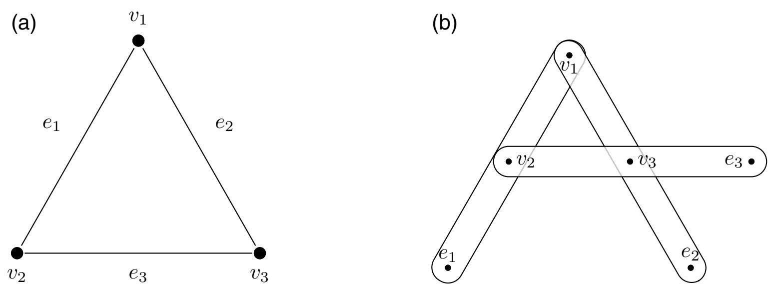
In this note, we disprove Conjecture 1.1 in any uniformity $k \geq 3$. Our counterexample shows that the radius of the disk-shaped ZFR $|\lambda| < \frac{\Delta^\Delta}{(\Delta+1)^{(\Delta+1)}}$ is tight up to a logarithmic factor even if we assume that $G$ is a $k$-uniform linear hypergraph.

**Theorem 1.2.** *For each uniformity $k \geq 3$ and $\Delta > 100k^2$, there exists a $k$-uniform linear hypergraph $G$ with maximum degree $\Delta$, such that $Z_G$ has a negative real zero $\lambda$ with $\lambda \in [-\frac{6k\log\Delta}{\Delta}, 0]$.*

## 2. The counterexample

We begin by describing a general construction.

**Definition 2.1.** *Let $G = (V, E)$ be a hypergraph. We define $S_G$ as the hypergraph whose vertex set is $V \sqcup E$, and whose edge set is $\{e \cup \{e\} : e \in E\}$.*




A graph $G$. The 3-uniform hypergraph $S_G$.

We begin by showing a basic property of $S_G$.

**Proposition 2.2.** *If $G = (V, E)$ is a $(k-1)$ uniform linear hypergraph with maximum degree $\Delta$, then $S_G$ is a $k$ uniform linear hypergraph with maximum degree $\Delta$.*

**Proof.** Each edge in $S_G$ has the form $e \cup \{e\}$ for some $e \in E$, and $|e \cup \{e\}| = |e| + 1 = k$. So $S_G$ is $k$ uniform.

For any pair of distinct edges $e_1 \cup \{e_1\}$ and $e_2 \cup \{e_2\}$ in $S_G$, we have

$$|(e_1 \cup \{e_1\}) \cap (e_2 \cup \{e_2\})| = |e_1 \cap e_2| \leq 1.$$

Thus $S_G$ is linear.

Finally, each vertex in $V$ has the same degree in $G$ and $S_G$, while each element in $E$ has degree 1 in $S_G$. Therefore, the maximum degree of $S_G$ is the same as the maximum degree of $G$. $\square$

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In the next two lemmas, we give an explicit formula for the independence polynomial $Z_{S_G}$ of $S_G$ and prove that it has a zero close to the origin whenever $G$ satisfies a mild expansion property.

**Lemma 2.3.** *Let $G=(V,E)$ be a hypergraph. For each set of vertices $S\subset V$, let $E(S)$ denote the edges of $G$ with at least one vertex in $S$, and write $e(S)=|E(S)|$. Then we have*

$$Z_{S_G}(\lambda)=\sum_{S\subset V}\lambda^{|V|-|S|}(1+\lambda)^{e(S)}.$$

**Proof.** Classifying the independence sets of $S_G$ based on their intersections with $V\subset V(S_G)$, we have

$$Z_{S_G}(\lambda)=\sum_{S\subset V}\ \sum_{I\in\mathcal{I}(S_G)\,:\,I\cap V=V\setminus S}\lambda^{|I|}.$$

A set of vertices $I\subset V\sqcup E$ with $I\cap V=V\setminus S$ is independent in $S_G$ if and only if $J:=I\cap E$ is contained in $E(S)$. So we have

$$\sum_{I\in\mathcal{I}(S_G)\,:\,I\cap V=V\setminus S}\lambda^{|I|}=\lambda^{|V\setminus S|}\sum_{J\subset E(S)}\lambda^{|J|}=\lambda^{|V\setminus S|}(1+\lambda)^{e(S)}$$

and the lemma follows. □

**Lemma 2.4.** *Let $G=(V,E)$ be a hypergraph with $n\geq 3$ vertices. Assume that $n$ is odd. Furthermore, assume that for some $\alpha\in[3\log n,n]$, we have $e(S)\geq\alpha\,|S|$ for any $S\subset V$. Then $Z_{S_G}$ has a negative real zero in the interval*

$$\left[-\frac{3\log n}{\alpha},0\right].$$

**Proof.** Set $\lambda_0=-\frac{3\log n}{\alpha}$. As $Z_{S_G}(0)=1$, it suffices to show that $Z_{S_G}(\lambda_0)<0$.

By Lemma 2.3, we have the identity

$$Z_{S_G}(\lambda_0)=\sum_{S\subset V}\lambda_0^{|V|-|S|}(1+\lambda_0)^{e(S)}.$$

We isolate the term with $S=\emptyset$ and obtain

$$Z_{S_G}(\lambda_0)=\lambda_0^{|V|}\left(1+\sum_{S\subset V,S\neq\emptyset}\lambda_0^{-|S|}(1+\lambda_0)^{e(S)}\right)$$

Since $0\leq 1+\lambda_0\leq e^{\lambda_0}$, for each $S\subset V$ we can estimate

$$\left|\lambda_0^{-|S|}(1+\lambda_0)^{e(S)}\right|\leq|\lambda_0|^{-|S|}\,e^{\lambda_0 e(S)}\leq\alpha^{|S|}e^{-3\log n\cdot|S|}.$$

Thus we have

$$\left|\sum_{S\subset V,S\neq\emptyset}\lambda_0^{-|S|}(1+\lambda_0)^{e(S)}\right|\leq\sum_{k=1}^{n}\binom{n}{k}\alpha^k e^{-3\log n\cdot k}\leq\sum_{k=1}^{n}\frac{1}{k!}\left(\frac{\alpha}{n^2}\right)^k.$$

where the second inequality uses the estimate $\binom{n}{k}\leq\frac{n^k}{k!}$.

We assume that $\alpha\leq n$, so $\left(\frac{\alpha}{n^2}\right)^k<\frac{1}{n}$ for each $k\geq 1$. This leads to

$$\left|\sum_{S\subset V,S\neq\emptyset}\lambda_0^{-|S|}(1+\lambda_0)^{e(S)}\right|\leq\frac{1}{n}\sum_{k=1}^{n}\frac{1}{k!}<\frac{e}{n}<1.$$

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Thus we conclude that

$$1+\sum_{S\subset V, S\neq\emptyset}\lambda_0^{-|S|}(1+\lambda_0)^{e(S)}>0$$

so $Z_{S_G}(\lambda_0)<0$, as desired. □

Our main theorem is an easy corollary of this result. Indeed, for any $(k-1)$ uniform hypergraph $G$ we can show that $e(S)\geq\frac{\delta(G)|S|}{k-1}$ for any $S\subset V$. So when $\delta(G)\geq\Delta(G)-1$, one can take $\alpha=\frac{\Delta(G)-1}{k-1}$ in Lemma 2.4. We give an explicit construction of such a hypergraph.

**Lemma 2.5.** *For any uniformity $k\geq 2$ and $\Delta\geq k$, there exists a $k$ uniform, $\Delta$ regular linear hypergraph $H_{k,\Delta}$ on at most $2k\Delta$ vertices.*

**Proof.** We take a prime $p$ in $[\Delta, 2\Delta]$, which exists by Chebyshev's theorem. Let $H_{k,\Delta}$ be the hypergraph on the vertex set $V=[k]\times\mathbb{Z}_p$ with an edge $\{(i, a+id): i\in[k]\}$ for each pair $(a,d)\in\mathbb{Z}_p\times[\Delta]$. The number of vertices in $H_{k,\Delta}$ is $|V|=kp\leq 2\Delta k$.

Any vertex $(i,x)\in V$ is contained precisely in the edges corresponding to $(a,d)\in\mathbb{Z}_p\times[\Delta]$ with $a\equiv x-id \pmod p$. As there is exactly one $a\in\mathbb{Z}_p$ corresponding to each $d\in[\Delta]$, $H_{k,\Delta}$ is $\Delta$ regular.

The size of the intersection between two distinct edges corresponding to $(a,d)$ and $(a',d')$ is the number of solutions $i\in[k]$ to the linear congruence equation $a+id\equiv a'+id' \pmod p$. As $k\leq\Delta\leq p$, there is at most one solution $i\in[k]$ to this linear congruence equation, so every pair of hyperedges in $H_{k,\Delta}$ intersect in at most one vertex. Therefore, $H_{k,\Delta}$ is a linear hypergraph. □

**Proof of Theorem 1.2.** Let $H=H_{(k-1),\Delta}$ be the $(k-1)$ uniform linear hypergraph constructed in Lemma 2.5. If $H$ has an even number of vertices, we remove an arbitrary vertex $v$ of $H$ together with any edge containing the vertex. Thus, we obtain a $(k-1)$ uniform linear hypergraph $H$ with an odd number of vertices, maximum degree $\Delta$, and minimum degree at least $(\Delta-1)$. By Proposition 2.2, $G=S_H$ is a $k$ uniform linear hypergraph with maximum degree $\Delta$.

Let $n\leq 2k\Delta$ be the number of vertices in $H$. For any vertex subset $S$ of $H$, the $(k-1)$ uniformity of $H$ implies that the number of edges in $G$ with at least one vertex in $S$ is lower bounded by

$$e(S)\geq\frac{1}{k-1}\sum_{v\in S}d(v)\geq\frac{\Delta-1}{k-1}|S|\,.$$

By our assumptions $\Delta>100k^2$ and $n\leq 2k\Delta$, we can check that

$$\frac{\Delta-1}{k-1}>\frac{\Delta}{k}>10\sqrt{\Delta}>10\log\Delta>3\log(2k\Delta)\geq 3\log(n).$$

Furthermore, we have $\frac{\Delta-1}{k-1}\leq\Delta-1<n$. So we can apply Lemma 2.4 with $\alpha=\frac{\Delta-1}{k-1}$. We conclude that $Z_G$ has a negative real zero $\lambda$ with

$$-\lambda\leq\frac{3\log n}{(\Delta-1)/(k-1)}\leq\frac{3k\log(2\Delta k)}{\Delta}\leq\frac{6k\log\Delta}{\Delta}$$

where the last two inequalities follow from $\Delta>100k^2$. So $G$ satisfies the requirements of Theorem 1.2. □

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