Proceedings of the Edinburgh Mathematical Society (2019) **62**, 609–623
doi:10.1017/S0013091518000603

THE SPECTRA OF ALGEBRAS OF GROUP-SYMMETRIC FUNCTIONS

DOMINGO GARCÍA[1], MANUEL MAESTRE[1] AND IGNACIO ZALDUENDO[2]

[1] *Departamento de Análisis Matemático, Universidad de Valencia, Valencia, Spain* (domingo.garcia@uv.es; manuel.maestre@uv.es)

[2] *Universidad Torcuato Di Tella. Av. Figueroa Alcorta 7350 (C1428BCW), Buenos Aires, Argentina* (izalduendo@utdt.edu)

(Received 7 November 2017; first published online 29 November 2018)

Abstract In the study of the spectra of algebras of holomorphic functions on a Banach space E, the bidual E'' has a central role, and the spectrum is often shown to be locally homeomorphic to E''. In this paper we consider the problem of spectra of subalgebras invariant under the action of a group (functions f such that $f \circ g = f$). It is natural to attempt a characterization in terms of the space of orbits $E''/_\sim$ obtained from E'' through the action of the group, so we pursue this approach here and introduce an analytic structure on the spectrum in some situations. In other situations we encounter some obstacles: in some cases, the lack of structure of $E''/_\sim$ itself; in others, problems of weak continuity and non-approximability of functions in the algebra. We also define a convolution operation related to the spectrum.

Keywords: symmetric polynomial; group of operators; holomorphic function

2010 *Mathematics subject classification:* Primary 46G20
Secondary 46E50

1. Introduction

In 1973, Nemirovskii and Semenov [**13**] initiated the study of functions on ℓ_p which are invariant under the permutation of variables, and the problem of their approximation by symmetric polynomials. Their results were generalized to the setting of rearrangement-invariant function spaces by González *et al.* [**12**]. Alencar *et al.* [**1**] studied the spectrum of the subalgebra of the ball algebra on ℓ_p of functions which are invariant under permutation of variables. Chernega *et al.* [**7**] studied permutation-invariant functions of bounded type, and particularly the spectrum of this algebra and convolution operations on the spectrum. Aron *et al.* [**6**] considered the general situation, where given a group G of linear automorphisms of a Banach space E and an algebra $\mathcal{H}$ of holomorphic functions on E, the algebra of group-symmetric functions

$$\mathcal{H}_G = \{f \in \mathcal{H} : f \circ g = f \text{ for all } g \in G\}$$

is studied. In several examples, symmetric functions were shown to arise as compositions of finite-variable functions with a small set of symmetric functions. Little was said regarding the relationship between the spectrum $\mathcal{M}$ of $\mathcal{H}$ and the spectrum $\mathcal{M}_G$ of $\mathcal{H}_G$, but it was proved that the restriction mapping $\rho : \mathcal{M} \longrightarrow \mathcal{M}_G$ is surjective.

Algebras of general holomorphic functions (which can be viewed as the case $G = \{I\}$) and their spectra, on the other hand, have been extensively studied, by Aron *et al.* [**5**] and by Aron *et al*, among others [**3**]. A central element in the theory is the Aron–Berner extension [**2**]: every holomorphic function f over an open subset of E can be extended to $\overline{f}$, holomorphic on an open subset of the bidual E'' of E, giving rise to evaluation characters outside the space E, and bringing into play the bidual and its properties. Thus E'' has an important role in all descriptions of $\mathcal{M}$.

As we shall see below, if $f \in \mathcal{H}_G$, $\overline{f} \circ g'' = \overline{f}$ for all $g \in G$, so $\overline{f}(z) = \overline{f}(w)$ whenever $w = g''(z)$. Thus we expect the role of E'' to be replaced by the orbit space $E''/_\sim$ obtained from E'' through the action of G. Our intention is to describe the spectrum $\mathcal{M}_G$ locally in terms of $E''/_\sim$. We explore this possibility and find some positive results as well as some obstructions.

The paper is organized as follows. In the second section, we describe some aspects of the orbit space; in particular, we identify areas of this space which are homeomorphic to open subsets of E''. The third section is the core of the paper. Here we introduce a mapping $\pi_G : \mathcal{M}_G \longrightarrow E''/_\sim$ and—in some situations—are able to introduce an analytic structure on $\mathcal{M}_G$. We find that lack of weak continuity and non-approximability of holomorphic functions, together with the absence of algebraic structure—and sometimes even of analytic structure—in $E''/_\sim$, are important obstacles. For some restricted areas of $\mathcal{M}_G$ we can dispense with approximability conditions. As in [**5**, **7**], we use a convolution operation on $\mathcal{M}_G$. We end this section with the study of another convolution operation involving affine operators over E''. In the last section, we mention a few examples and connections with prior work in [**4**].

2. The orbit space $E''/_\sim$

In what follows, E will be a Banach space, $U \subset E$ an open subset and $\mathcal{H}$ an algebra of holomorphic functions defined on U (i.e. $\mathcal{H}$ could be $\mathcal{H}_b(E)$, $\mathcal{H}(E)$, $A(B_E)$, $H^\infty(B_E)$, etc.). Also, G will denote a group of linear automorphisms of E which leave U fixed.

For each $g \in G$ we consider the bitranspose operator $g'' : E'' \longrightarrow E''$. G thus acts on E'' by

$$G \times E'' \longrightarrow E'' \quad \text{given by } (g, z) \mapsto g''(z) = z \circ g'.$$

We denote by $E''/_\sim$ the orbit space of this action, i.e. the quotient of E'' by the equivalence relation $z \sim w \Leftrightarrow w = g''(z)$ for some $g \in G$, and call q the quotient map $q : E'' \longrightarrow E''/_\sim$. We will also write $[z] = q(z)$. We consider in $E''/_\sim$ the quotient topology.

A few cautionary remarks are in order. The first is that $u''(z) = v''(z)$ does not imply that $u = v$. In fact, since all g are linear, we have $g''(0) = 0$ for any $g \in G$. We will therefore pay special attention to points $z \in E''$ for which $g''(z) = z$ implies $g = I$. The second remark is that $E''/_\sim$ has no algebraic structure, as the equivalence relation $\sim$ does not respect sums. An alternative to $E''/_\sim$ (with algebraic structure) would be the dual space $(E'_G)'$, where $E'_G = \{\gamma \in E' : \gamma \circ g = \gamma \text{ for all } g \in G\} = E' \cap \mathcal{H}_G$. This is a Banach space, but in many cases it is far too small for our purposes: E'_G does not separate points of $E''/_\sim$.

We will say that a point $z \in E''$ is *regular* if there exists a neighbourhood U of z such that $g''(U) \cap U = \emptyset$ for all $g \neq I$. Clearly, the set of regular points is open in E''. Also, if z is regular, its isotropy group $G_z = \{g \in G : g''(z) = z\}$ is trivial. We are interested in the set of regular points because on that set q is locally one-to-one. We have the following.

Remark 2.1. If $z \in E''$ is regular, q is a local homeomorphism at z.

Proof. The map $q : E'' \longrightarrow E''/_\sim$ is continuous by definition of the quotient topology. It is also open, as if $V \subset E''$ is open, $q(V)$ is open if and only if $q^{-1}(q(V))$ is open, but

$$\begin{aligned} q^{-1}(q(V)) &= \{g''(w) : g \in G \text{ and } w \in V\} \\ &= \bigcup_{g \in G} g''(V), \end{aligned}$$

and all g'' are surjective and thus open by the open mapping theorem. Now we need only see that q is one-to-one near z. Since z is regular, it has a neighbourhood U with $g''(U) \cap U = \emptyset$ for all $g \neq I$. Thus, if w_1 and w_2 are in U and $q(w_1) = q(w_2)$, $w_2 = g''(w_1)$ for some $g \in G$, and $w_2 \in g''(U) \cap U$. Thus, $g = I$ and $w_1 = w_2$. □

We note that in some cases (for example, if G is finite), z is regular whenever its isotropy group $G_z = \{g \in G : g(z) = z\}$ is trivial. This is not true in general, however, and may be false even for a compact group. For instance, take $G = S^1$, the unit sphere in $\mathbb{C}$, $E = \mathbb{C}$ and the action $(\lambda, z) \mapsto \lambda z$. Then the isotropy groups are trivial for all non-zero z; however, $E''/_\sim \simeq [0, \infty)$ is not locally homeomorphic to $\mathbb{C}$. This example also shows that on subsets not homeomorphic to open subsets of E'', the orbit space $E''/\sim$ may or may not have analytic structure. In fact, $E''/\sim$ may not be Hausdorff, as is shown by the following example.

Example 2.2. We take G (as in [6]) to be the group of operators on $E = c_0$ generated by $\gamma_m : c_0 \longrightarrow c_0$,

$$\gamma_m(e_j) = \begin{cases} \omega_m e_j & \text{if } j = m \\ e_j & \text{otherwise,} \end{cases}$$

where $\omega_m = \mathrm{e}^{2\pi \mathrm{i}/m}$. Now consider the following elements of $E'' = \ell^\infty$:

$$\begin{aligned} a &= (\omega_1^{-1}, \omega_2^{-1}, \omega_3^{-1}, \ldots, \omega_n^{-1}, \ldots) \\ I &= (1, 1, 1, \ldots, 1, \ldots). \end{aligned}$$

https://doi.org/10.1017/S0013091518000603 Published online by Cambridge University Press

Their orbits Ga and GI are disjoint, for $I \notin Ga$. However, $I \in \overline{Ga}$. Indeed, for any $\varepsilon > 0$, let m be large enough that $|1 - \omega_k^{-1}| < \varepsilon$ for all $k > m$. Then if we let

$$\begin{aligned} g_m(x) &= (\omega_1 x_1, \omega_2 x_2, \omega_3 x_3, \ldots, \omega_m x_m, x_{m+1}, \ldots), \\ \|I - g_m(a)\| &= \|(0, 0, 0, \ldots, 1 - \omega_{m+1}^{-1}, 1 - \omega_{m+2}^{-1}, \ldots)\| \\ &= \sup_{k>m} |1 - \omega_k^{-1}| < \varepsilon. \end{aligned}$$

Taking into account the proof of Remark 2.3, this shows that $E''/\sim$ is not Hausdorff.

In the following remark, we consider the strong operator topology (SOT) on G, i.e. $g_i \longrightarrow g$ if and only if $g_i(x) \longrightarrow g(x)$ for each $x \in E$.

Remark 2.3. If G is SOT-compact, $E''/_\sim$ is Hausdorff.

Proof. Let A_1 and A_2 be open subsets of $E''/_\sim$ and note that $A_1 \cap A_2 = \emptyset$ if and only if $q^{-1}(A_1) \cap q^{-1}(A_2) = \emptyset$. Thus, to separate two distinct $[z]$ and $[w]$ in $E''/_\sim$, we must separate the orbits of z and w in E''. Since G is endowed with the SOT topology, for every $x \in E''$, the mapping

$$G \longrightarrow E'' \quad \text{such that } g \mapsto g''(x)$$

is continuous. Thus, when G is compact, the orbits

$$Gz = \{g''(z) : g \in G\} \quad \text{and} \quad Gw = \{g''(w) : g \in G\}$$

are disjoint compact sets and may be separated by disjoint open subsets of E''. □

We define, as in [6], the action of G over $\mathcal{M}$

$$G \times \mathcal{M} \longrightarrow \mathcal{M} \quad (g, \varphi) \mapsto \varphi_g \quad \text{where } \varphi_g(h) = \varphi(h \circ g),$$

and let $V_g = \{\varphi_g : \varphi \in V\}$. In the next section, we will recall from [5] the topology defined on $\mathcal{M}$ and the (continuous) map $\pi : \mathcal{M} \longrightarrow E''$ such that $\pi(\varphi)(\gamma) = \varphi(\gamma)$ for all γ in E'. Define $\varphi \in \mathcal{M}$ to be *regular* if it has a neighbourhood V such that $V_g \cap V = \emptyset$ for all $g \neq I$.

Remark 2.4. If $\pi(\varphi)$ is regular, then φ is regular.

Proof. Suppose that $\pi(\varphi)$ is regular, and take U a neighbourhood as in the definition. Let $V = \pi^{-1}(U)$. Then V is a neighbourhood of φ. Note now that $\pi(V_g) = g''(\pi(V))$;

https://doi.org/10.1017/S0013091518000603 Published online by Cambridge University Press

indeed, for all $\gamma \in E'$

$$\begin{aligned}\pi(\psi_g)(\gamma) &= \psi(\gamma \circ g)\\ &= (\psi \circ g')(\gamma)\\ &= g''(\pi(\psi))(\gamma).\end{aligned}$$

Thus, if $V_g \cap V \neq \emptyset$,

$$\begin{aligned}\emptyset \neq \pi(V_g \cap V) \subset \pi(V_g) \cap \pi(V)\\ &= g''(\pi(V)) \cap \pi(V)\\ &= g''(U) \cap U\end{aligned}$$

and $g = I$. □

Note that where π is a local homeomorphism, for example, if E is symmetrically regular (see [**5**]), the converse also holds.

3. The structure of $\mathcal{M}_G$

In this section, we consider the problem of analytic structure on the spectrum of $\mathcal{H}_G$. We will compare the spectrum with the orbit space $E''/\sim$ via a mapping

$$\pi_G : \mathcal{M}_G \longrightarrow E''/\sim .$$

We will introduce a topology on $\mathcal{M}_G$ and under some conditions prove local injectivity of π_G. This will provide analytic structure to parts of $\mathcal{M}_G$ over areas of $E''/\sim$ which have such structure. As mentioned above, some subsets of $E''/\sim$ may lack analytic structure.

We begin by addressing the relationship between G-symmetry and the Aron–Berner extension [**2**]. Let $G'' = \{g'' : g \in G\}$ and denote the Aron–Berner extension of f by $\overline{f}$. Recall [**2**] that any function holomorphic on an open set $U \subset E$ may be extended to $\overline{f}$ holomorphic on an open subset of E'' containing U. This open subset in general depends on f. If $f \in H_b(E)$, $\overline{f}$ is defined on all of E'', producing an extension morphism $H_b(E) \longrightarrow H_b(E'')$, and an analogous situation occurs for $H^\infty(B)$. However, in general, there may not be an open subset of E'' to which all $f \in \mathcal{H}$ extend. Thus, the following proposition is to be understood for individual functions h.

Proposition 3.1. *For all $h \in \mathcal{H}$ and $g \in G$, the Aron–Berner extension of $h \circ g$ is*

$$\overline{h \circ g} = \overline{h} \circ g''.$$

Also, f is G-symmetric if and only if $\overline{f}$ is G''-symmetric.

Proof. We first see that $\overline{h \circ g} = \overline{h} \circ g''$: let $F = \overline{h} \circ g''$. In order to use the criteria in [**14**], we compare F with $\overline{h \circ g}$.

(i) $F = h \circ g$ over E: for any $x \in E$,

$$F(x) = \overline{h}(g''(x)) = \overline{h}(g(x)) = h(g(x)).$$

https://doi.org/10.1017/S0013091518000603 Published online by Cambridge University Press

(ii) For any $x \in E$, $DF(x) = D(\overline{h} \circ g'')(x) = D\overline{h}(g''(x)) \circ g'' = D\overline{h}(g(x)) \circ g''$, but since $g(x) \in E$, $D\overline{h}(g(x))$ is w^*-continuous, and g'' is $w^* - w^*$-continuous, so $DF(x)$ is w^*-continuous.

(iii) Let $z \in E''$ and suppose that x_α w^*-converges to z. Note that $g''(x_\alpha)$ w^*-converges to $g''(z)$. Now, since $\overline{h}$ is the Aron–Berner extension of h,

$$\begin{aligned} DF(z)(x_\alpha) = D\overline{h}(g''(z))(g''(x_\alpha)) \longrightarrow D\overline{h}(g''(z))(g''(z)) &= D(\overline{h} \circ g'')(z)(z) \\ &= DF(z)(z). \end{aligned}$$

Thus, by [14], $\overline{h \circ g} = \overline{h} \circ g''$.

Now, if f is G-symmetric, for all $g \in G$ we have

$$\overline{f} \circ g'' = \overline{f \circ g} = \overline{f},$$

and if $\overline{f}$ is G''-symmetric,

$$\overline{f \circ g} = \overline{f} \circ g'' = \overline{f},$$

and restriction to E gives $f \circ g = f$. □

This implies that for all $f \in \mathcal{H}_G$, one may define (at least on some open subset of $E/_\sim$),

$$\widetilde{f} : E''/_\sim \longrightarrow \mathbb{C} \quad \text{by } \widetilde{f}([z]) = \overline{f}(z),$$

where $\overline{f}$ is the Aron–Berner extension of f. This is well defined, as if $[w] = [z]$ and $w = g''(z)$, $\overline{f}(w) = \overline{f}(g''(z)) = (\overline{f} \circ g'')(z) = \overline{f \circ g}(z) = \overline{f}(z)$. For this reason, we will try to model $\mathcal{M}_G$ over $E''/_\sim$.

Consider the action of G on the spectrum of $\mathcal{H}$:

$$G \times \mathcal{M} \longrightarrow \mathcal{M} \quad (g, \varphi) \mapsto \varphi_g$$

where $\varphi_g(h) = \varphi(h \circ g)$. We denote by $\mathcal{O}_\varphi = \{\varphi_g : g \in G\}$ the orbit of φ and by ρ the restriction map $\rho : \mathcal{M} \longrightarrow \mathcal{M}_G$, which is known to be surjective [6].

In general, the orbit $\mathcal{O}_\varphi$ is contained in the fibre $\rho^{-1}(\rho(\varphi))$ and may be strictly smaller. We will suppose that the equality $\mathcal{O}_\varphi = \rho^{-1}(\rho(\varphi))$ holds; this is equivalent to asking that $\mathcal{H}_G$ 'separates orbits', i.e. if $\psi \notin \mathcal{O}_\varphi$, then $\psi(f) \neq \varphi(f)$ for some $f \in \mathcal{H}_G$.

Also, set (as in [5]) $\pi : \mathcal{M} \longrightarrow E''$, given by $\pi(\varphi)(\gamma) = \varphi(\gamma)$, and recall from the proof of Remark 2.4 that $g''(\pi(\varphi)) = \pi(\varphi_g)$.

We now define

$$\pi_G : \mathcal{M}_G \longrightarrow E''/\sim \quad \text{by } \pi_G(\alpha) = [\pi(\varphi)], \text{ if } \alpha = \rho(\varphi).$$

This is well defined, as if $\rho(\psi) = \rho(\varphi)$, since $\mathcal{H}_G$ separates orbits, $\psi = \varphi_g$ for some $g \in G$, and thus $\pi(\psi) = \pi(\varphi_g) = g''(\pi(\varphi))$, so $[\pi(\psi)] = [\pi(\varphi)]$.

In [5] the convolution operation between $\varphi \in \mathcal{M}$ and $z \in E''$ was defined by

$$(\varphi * z)(h) = \varphi(\overline{h} \circ t_z), \quad \text{where } t_z(x) = x + z,$$

and it was proved that the sets $V_{\varphi,U}$ given by

$$V_{\varphi,U} = \{\varphi * z : z \in U\},$$

with $\varphi \in \mathcal{M}$ and U neighbourhoods of zero in E'', form a basis for a topology in $\mathcal{M}$ whenever the space E is symmetrically regular, i.e. when every continuous symmetric operator from E to E' is weakly compact. All reflexive Banach spaces are symmetrically regular, as well as c_0. The space ℓ_1 is not.

For the same purpose, we will consider the subsets of $\mathcal{M}_G$: $\rho(V_{\varphi,U})$. We note that if $\rho(\psi) = \rho(\varphi)$, then $\psi = \varphi_g$ for some $g \in G$, and

$$(\psi * z)(f) = (\varphi_g * z)(f) = \varphi_g(\overline{f} \circ t_z) = \varphi(\overline{f} \circ t_z \circ g);$$

however, for $f \in \mathcal{H}_G$,

$$(\overline{f} \circ t_z \circ g)(x) = \overline{f}(g(x) + z) = (\overline{f} \circ g'')(x + g''^{-1}(z)) = (\overline{f} \circ t_{g''^{-1}(z)})(x),$$

thus $(\psi * z)(f) = \varphi(\overline{f} \circ t_{g''^{-1}(z)}) = (\varphi * g''^{-1}(z))(f)$. Therefore, $\rho(\psi * z) = \rho(\varphi * g''^{-1}(z))$. Thus, if $\rho(\psi) = \rho(\varphi)$,

$$\rho(\{\varphi * z : z \in U\}) = \rho(\{\psi * z : z \in g''(U)\}).$$

We can now prove the following.

Theorem 3.2. *If E is symmetrically regular, the family*

$$\rho(V_{\varphi,U})$$

with $\varphi \in \mathcal{M}$ and U neighbourhoods of zero in E'' forms a basis for a topology on $\mathcal{M}_G$.

Proof. The sets $\rho(V_{\varphi,U})$ cover $\mathcal{M}_G$ by the surjectivity of ρ. We must verify that if $\beta \in \rho(V_{\varphi,U}) \cap \rho(V_{\psi,V})$ then, for some $\psi_0 \in \mathcal{M}$ and a neighbourhood W of zero in E'', we have

$$\beta \in \rho(V_{\psi_0,W}) \subset \rho(V_{\varphi,U}) \cap \rho(V_{\psi,V}).$$

Since $\beta \in \rho(V_{\varphi,U}) \cap \rho(V_{\psi,V})$, there are $\varphi_0 \in V_{\varphi,U}$ and $\psi_0 \in V_{\psi,V}$ such that $\rho(\psi_0) = \beta = \rho(\varphi_0)$. Since ψ_0 and φ_0 are in the same orbit, $\psi_0 = {\varphi_0}_g$ for some $g \in G$. Now consider

$$[V_{\varphi,U}]_g = \{(\varphi * z)_g : z \in U\}.$$

We note that for any $h \in \mathcal{H}$, $(\varphi * z)_g(h) = (\varphi * z)(h \circ g) = \varphi(\overline{h \circ g} \circ t_z)$, but

$$(\overline{h \circ g} \circ t_z)(x) = (\overline{h} \circ g'' \circ t_z)(x) = \overline{h}(g''(x + z)) = \overline{h}(g(x) + g''(z)) = (\overline{h} \circ t_{g''(z)} \circ g)(x);$$

thus, $(\varphi * z)_g(h) = \varphi(\overline{h} \circ t_{g''(z)} \circ g) = \varphi_g(\overline{h} \circ t_{g''(z)}) = (\varphi_g * g''(z))(h)$. Therefore,

$$[V_{\varphi,U}]_g = \{(\varphi * z)_g : z \in U\} = \{\varphi_g * g''(z) : z \in U\} = \{\varphi_g * w : w \in g''(U)\} = V_{\varphi_g, g''(U)}.$$

So $\psi_0 = \varphi_{0_g} \in V_{\varphi_g, g''(U)} \cap V_{\psi,V}$. Since E is symmetrically regular, we have, by [5], that there is a neighbourhood W of zero in E'' with

$$V_{\psi_0,W} \subset V_{\varphi_g,g''(U)} \cap V_{\psi,V}.$$

Then

$$\rho(V_{\psi_0,W}) \subset \rho(V_{\varphi_g,g''(U)} \cap V_{\psi,V}) \subset \rho(V_{\varphi_g,g''(U)}) \cap \rho(V_{\psi,V}).$$

But note that $\rho(V_{\varphi_g,g''(U)}) = \rho([V_{\varphi,U}]_g) = \rho(V_{\varphi,U})$, so

$$\beta = \rho(\psi_0) \in \rho(V_{\psi_0,W}) \subset \rho(V_{\varphi,U}) \cap \rho(V_{\psi,V}). \qquad \square$$

Theorem 3.3. *If E is symmetrically regular and $\mathcal{H}_G$ is contained in the closed subalgebra of $\mathcal{H}$ generated by E', then π_G is one-to-one on the sets $\rho(V_{\varphi,U})$.*

Proof. If $\pi_G(\rho(\varphi * z)) = \pi_G(\rho(\varphi * w))$, then $[\pi(\varphi * z)] = [\pi(\varphi * w)]$; thus, there is a $g \in G$ for which

$$\pi(\varphi * w) = g''(\pi(\varphi * z)) = \pi((\varphi * z)_g).$$

Thus, for any $\gamma \in E'$,

$$(\varphi * w)(\gamma) = (\varphi * z)_g(\gamma) = (\varphi * z)(\gamma \circ g).$$

If f is in the closed subalgebra generated by E', $(\varphi * w)(f) = (\varphi * z)(f \circ g)$. If, further, $f \in \mathcal{H}_G$,

$$(\varphi * w)(f) = (\varphi * z)(f \circ g) = (\varphi * z)(f).$$

Thus, $\rho(\varphi * w) = \rho(\varphi * z)$, and π_G is one-to-one. $\square$

Our result is of course related to the weak continuity of holomorphic functions, so we have an analogous result in those terms for evaluation maps.

Proposition 3.4. *If all $f \in \mathcal{H}_G$ are weakly continuous and $\varphi = e_x$ is the evaluation at x, then $\pi_G : \rho(V_{\varphi,U}) \longrightarrow E''/\sim$ is one-to-one.*

Proof. By weak continuity of $f \in \mathcal{H}_G$, given ε and a, there are $\gamma_1, \dots, \gamma_k \in E'$ such that

$$|\gamma_i(b) - \gamma_i(a)| < 1 \text{ for } i = 1, \dots, k \text{ implies } |\overline{f}(b) - \overline{f}(a)| < \varepsilon.$$

If, as above, we have $(\varphi * w)(\gamma) = (\varphi * z)_g(\gamma)$ for all $\gamma \in E'$, and if $\varphi = e_x$,

$$\overline{\gamma}(x + w) = \overline{\gamma}(g''(x + z)).$$

Thus, for any γ, $|\overline{\gamma}(x + w) - \overline{\gamma}(g''(x + z))| < 1$, and for any $\varepsilon > 0$

$$|\overline{f}(x + w) - \overline{f}(g''(x + z))| < \varepsilon.$$

Hence, $\overline{f}(x + w) = \overline{f}(g''(x + z)) = \overline{f \circ g}(x + z) = \overline{f}(x + z)$, so $\rho(\varphi * w) = \rho(\varphi * z)$. $\square$

https://doi.org/10.1017/S0013091518000603 Published online by Cambridge University Press

The following example shows the non-injectivity of π_G in the absence of weak continuity and lack of density of the algebra generated by linear forms.

Example 3.5. We take $E = \ell_2$, $\mathcal{H} = H_b(\ell_2)$, and consider the group G generated by $g(x) = ix$. G has four elements: $x \mapsto ix$, $x \mapsto -x$, $x \mapsto -ix$ and $x \mapsto x$.

Consider the 2-homogeneous polynomial $P : \ell_2 \longrightarrow \mathbb{C}$

$$P(x) = \sum_{n=1}^{\infty} x_n^2.$$

Note that P^2 is G-symmetric, because

$$(P^2 \circ g)(x) = \left[\sum_{n=1}^{\infty} (ix_n)^2\right]^2 = [-P(x)]^2 = P^2(x).$$

By [**10**], there is a character $\varphi \in \mathcal{M}$ which is zero on all odd-degree homogeneous polynomials, and $\varphi(P) = 1$. Now, let U be a neighbourhood of zero, $z \in U$, with $P(z) \neq 0$ and $w = g(z) = iz$. We have $(\varphi * w) = (\varphi * z)_g$ over E', i.e. $\pi_G(\rho(\varphi * w)) = \pi_G(\rho(\varphi * z))$, because since $(\gamma \circ t_w)(x) = \gamma(x + w) = \gamma(x) + \gamma(w)$,

$$\begin{aligned}(\varphi * w)(\gamma) &= \varphi(\gamma + \gamma(w)) = \varphi(\gamma) + \gamma(w) = \gamma(w) = i\gamma(z)\\ (\varphi * z)_g(\gamma) &= (\varphi * z)(\gamma \circ g) = \varphi(\gamma \circ g) + (\gamma \circ g)(z) = \gamma(g(z)) = i\gamma(z).\end{aligned}$$

However, $\rho(\varphi * w) \neq \rho(\varphi * z)$ as, when applied to $P^2 \in \mathcal{H}_G$, $(\varphi * w)(P^2) = \varphi(P^2 \circ t_w)$, but

$$\begin{aligned}(P^2 \circ t_w)(x) = P^2(x + w) &= \left[\sum_{n=1}^{\infty} (x + w)_n^2\right]^2\\ &= \left[\sum_{n=1}^{\infty} x_n^2 + 2\sum_{n=1}^{\infty} x_n w_n + \sum_{n=1}^{\infty} w_n^2\right]^2,\end{aligned}$$

so $(\varphi * w)(P^2) = [\varphi(P) + 0 + P(w)]^2 = (1 + P(w))^2 = (1 - P(z))^2$. Analogously, though, $(\varphi * z)(P^2) = [\varphi(P) + 0 + P(z)]^2 = (1 + P(z))^2$, which is different, having chosen $P(z) \neq 0$.

Over certain parts of the spectrum, however, π_G may be one-to-one without imposing conditions regarding approximability or weak continuity. To see this, we will use the following *barycentre operator*

$$b : E \longrightarrow E, \quad \text{given by} \quad b(x) = \int_G g(x)\,\mathrm{d}\mu(g),$$

where μ is the Haar measure if G is compact in the SOT topology (if G is an ascending union of compact subgroups, b can also be defined [**6**]). $b(x)$ is the barycentre of the orbit of x under the group action. Note that owing to the invariance of the Haar measure, one has $g \circ b = b = b \circ g$ for any $g \in G$, and thus also $b \circ b = b$. The barycentre operator is

https://doi.org/10.1017/S0013091518000603 Published online by Cambridge University Press

continuous and linear. We use $b'' : E'' \longrightarrow E''$ to denote its bitranspose, but will reserve the notation b' for the composition operator

$$b' : \mathcal{H} \longrightarrow \mathcal{H} \quad \text{given by} \quad b'(h) = h \circ b.$$

We note that b' is multiplicative, and its image is contained in $\mathcal{H}_G$. However its image is not $\mathcal{H}_G$, but a smaller subalgebra. We have the following.

Proposition 3.6. *If $\alpha \in \mathcal{M}_G$ is equal to $\alpha \circ b'$, then π_G is one-to-one on a neighbourhood of α.*

Proof. Suppose $\alpha = \rho(\varphi)$. Then

$$\alpha = \alpha \circ b' = \rho(\varphi) \circ b' = \rho(\varphi \circ b')$$

so, for a neighbourhood U of zero in E'', $\rho(V_{\varphi\circ b',U})$ is a neighbourhood of α in $\mathcal{M}_G$. The map π_G is one-to-one on this neighbourhood: suppose $\pi_G(\rho((\varphi \circ b') * w)) = \pi_G(\rho((\varphi \circ b') * z))$, i.e. $[\pi((\varphi \circ b') * w)] = [\pi((\varphi \circ b') * z)]$. There is then an element $g \in G$ such that

$$(\varphi \circ b') * w = g''((\varphi \circ b') * z)) \quad \text{over } E'. \tag{1}$$

Now, for all $\gamma \in E'$,

$$\begin{aligned} [(\varphi \circ b') * w](\gamma) &= (\varphi \circ b')(\overline{\gamma} \circ t_w) = \varphi(\overline{\gamma} \circ t_w \circ b) \text{ and} \\ g''[(\varphi \circ b') * z](\gamma) &= (\varphi \circ b')(\overline{\gamma \circ g} \circ t_z) = \varphi(\overline{\gamma} \circ g' \circ t_z \circ b), \end{aligned}$$

but for all $x \in E$,

$$\begin{aligned} (\overline{\gamma} \circ t_w \circ b)(x) &= \overline{\gamma}(b(x) + w) = \gamma(b(x)) + w(\gamma) \text{ and} \\ (\overline{\gamma} \circ g' \circ t_z \circ b)(x) &= \overline{\gamma}(g''(b(x) + z)) = \overline{\gamma}(g(b(x)) + g''(z)) \\ &= \gamma(b(x)) + g''(z)(\gamma). \end{aligned}$$

Applying φ,

$$\begin{aligned} [(\varphi \circ b') * w](\gamma) &= \varphi(\gamma \circ b) + w(\gamma) \text{ and} \\ g''[(\varphi \circ b') * z](\gamma) &= \varphi(\gamma \circ b) + g''(z)(\gamma). \end{aligned}$$

Thus, (1) implies that $w(\gamma) = g''(z)(\gamma)$ for all $\gamma \in E'$, so $w = g''(z)$. Now, if $f \in \mathcal{H}_G$,

$$\begin{aligned} [(\varphi \circ b') * w](f) &= [(\varphi \circ b') * g''(z)](f) \\ &= (\varphi \circ b')(\overline{f} \circ t_{g''(z)}) \\ &= \varphi(\overline{f} \circ t_{g''(z)} \circ b), \end{aligned}$$

https://doi.org/10.1017/S0013091518000603 Published online by Cambridge University Press

but for any $x \in E$,

$$\begin{aligned}(\overline{f} \circ t_{g''(z)} \circ b)(x) &= \overline{f}(b(x) + g''(z)) \\ &= \overline{f}(g(b(x)) + g''(z)) \\ &= (\overline{f} \circ g'')(b(x) + z) \\ &= \overline{f \circ g}(b(x) + z) \\ &= \overline{f}(b(x) + z) \\ &= (\overline{f} \circ t_z \circ b)(x),\end{aligned}$$

so $[(\varphi \circ b') * w](f) = \varphi(\overline{f} \circ t_z \circ b) = (\varphi \circ b')(\overline{f} \circ t_z) = [(\varphi \circ b') * z](f)$. Thus, $\rho((\varphi \circ b') * w) = \rho((\varphi \circ b') * z)$, and π_G is one-to-one. □

We note that $\varphi \circ b'$ is not a regular point of $\mathcal{M}$, as $g''(\pi(\varphi \circ b')) = \pi(\varphi \circ b')$ for any $g \in G$.

The convolution $(\varphi, (b', z)) \mapsto (\varphi \circ b') * z$ suggests a wider family of convolution products related to affine operators on E''—such as $t_z \circ b$—which we now explore.

Definition 3.7. We say an affine operator $t : E'' \longrightarrow E''$ is *G-related* if $tG \subset Gt$, i.e. for every $u \in G$ there is a $v \in G$ such that $t \circ u'' = v'' \circ t$.

Examples of such operators include:

(i) all $g'' : E'' \longrightarrow E''$, with $g \in G$;

(ii) b'';

(iii) for any $z \in E''$, $t^z : E'' \longrightarrow E''$ given by $t^z(w) = b''(w) + z$.

We denote the set of all G-related operators by $N(G)$ and note that $N(G)$ is a semigroup containing G. Also, any $t \in N(G)$ produces a well-defined

$$\widetilde{t} : E''/\sim \longrightarrow E''/\sim$$

by setting $\widetilde{t}([z]) = [t(z)]$. The equivalence relation $s \sim t \Leftrightarrow s = g'' \circ t$ for some $g \in G$ is a semigroup congruence; thus, $N(G)/G$ has semigroup structure.

Proposition 3.8.

(i) *If $\varphi \in \mathcal{M}$ and $t \in N(G)$,*

$$(\varphi \star t)(h) = \varphi(\overline{h} \circ t)$$

defines a character in $\mathcal{M}$, and $\varphi \star I = \varphi$.

(ii) *If $f \in \mathcal{H}_G$, then $\overline{f} \circ t \in \mathcal{H}_G$; $\overline{f} \circ t$ depends only on the class $\widehat{t} \in N(G)/G$ of t. Thus, if $\alpha \in \mathcal{M}_G$ and $\widehat{t} \in N(G)/G$,*

$$(\alpha \star \widehat{t})(f) = \alpha(\overline{f} \circ t)$$

defines a character in $\mathcal{M}_G$.

(iii) *If $\alpha = \rho(\varphi)$, then $\alpha \star \widehat{t} = \rho(\varphi \star t)$.*

https://doi.org/10.1017/S0013091518000603 Published online by Cambridge University Press

Proof. All are easy. We check only (ii): for any $u \in G$, since $t \in N(G)$, there is a $v \in G$ with $t \circ u'' = v'' \circ t$; thus

$$(\overline{f} \circ t) \circ u'' = \overline{f} \circ v'' \circ t = \overline{f} \circ t.$$

Also, if $\widehat{s} = \widehat{t}$ and $s = g'' \circ t$,

$$\overline{f} \circ s = \overline{f} \circ g'' \circ t = \overline{f \circ g} \circ t = \overline{f} \circ t. \qquad \square$$

Proposition 3.9. *If $\mathcal{H}_G$ is contained in the closed subalgebra of $\mathcal{H}$ generated by E', then π_G is one-to-one over the set*

$$W_\alpha = \{\alpha \star \widehat{t} : \widehat{t} \in N(G)/G\}.$$

Proof. If $\pi_G(\alpha \star \widehat{s}) = \pi_G(\alpha \star \widehat{t})$, then $\pi_G(\rho(\varphi \star s)) = \pi_G(\rho(\varphi \star t))$ and $[\pi(\varphi \star s)] = [\pi(\varphi \star t)]$, where $\alpha = \rho(\varphi)$. Thus, for some $g \in G$, $\pi(\varphi \star s) = g''(\pi(\varphi \star t)) = \pi(\varphi \star t) \circ g'$. Therefore, for all $\gamma \in E'$ we have $(\varphi \star s)(\gamma) = (\varphi \star t)(\gamma \circ g)$,

$$\text{which, by definition of } \star \text{ is } \varphi(\overline{\gamma} \circ s) = \varphi(\overline{\gamma \circ g} \circ t) = \varphi(\overline{\gamma} \circ g'' \circ t),$$
$$\text{which may be written: } (\varphi \circ s')(\gamma) = (\varphi \circ t' \circ g')(\gamma).$$

Thus $\varphi \circ s'$ and $\varphi \circ t' \circ g'$ are continuous algebra morphisms which coincide on E'. Since the closed algebra generated by linear forms contains $\mathcal{H}_G$, for all $f \in \mathcal{H}_G$ we have

$$(\varphi \circ s')(f) = (\varphi \circ t' \circ g')(f)$$
$$\text{which is } \varphi(\overline{f} \circ s) = \varphi(\overline{f} \circ g'' \circ t) = \varphi(\overline{f} \circ t)$$
$$\text{and, by definition of } \star, \ (\varphi \star s)(f) = (\varphi \star t)(f).$$

Then $\rho(\varphi \star s) = \rho(\varphi \star t)$ and $\alpha \star \widehat{s} = \alpha \star \widehat{t}$. Thus, π_G is one-to-one. $\square$

4. Examples

In the finite-dimensional setting, $E = \mathbb{C}^n$ is symmetrically regular and any space of holomorphic functions is the closed algebra spanned by the coordinate functionals. Thus, our results in Theorems 3.2 and 3.3 apply. In [4], the algebra $\mathcal{H}_G(U)$ and its spectrum are studied, particularly for finite unitary reflection groups and certain subgroups of permutations. The approach is different, however, and makes use of *proper* holomorphic functions $g : \Omega \subset \mathbb{C}^n \longrightarrow \Omega' \subset \mathbb{C}^k$ in the definition of $\mathcal{H}_g(U)$:

$$\mathcal{H}_g(U) = \{f \in \mathcal{H}(U) : \text{ if } z, w \in U \text{ with } g(z) = g(w) \text{ then } f(z) = f(w)\}.$$

It is proved in [4] that if $G \subset GL(n, \mathbb{C})$ is a finite unitary reflection subgroup, then $\mathcal{H}_G(U) = \mathcal{H}_g(U)$ for some proper map g.

This point of view allows the description of $\mathcal{H}_G(U)$ as $\mathcal{H}(\widetilde{U})$ for another open subset $\widetilde{U}$ of $\mathbb{C}^n$. For example, in the case of the symmetric group S_2 of order 2 consisting of all permutations of the set $\{1,2\}$, $\widetilde{U}$ is the symmetrized bidisc

$$\mathbb{G}_2 = \{(s,p) \in \mathbb{C}^2 : |s - s\overline{p}| + |p|^2 < 1\}.$$

It is natural to ask whether given a proper holomorphic mapping g, the algebra $\mathcal{H}_g(U)$ coincides with $\mathcal{H}_G(U)$ for some group G. This question was answered in the negative in [**4**, Example 5.4] taking the proper holomorphic mapping $g(z) = ((z_1+z_2)^2, (z_1 z_2)^2)$.

Let us study now an example with $E = c_0$, the Banach space of null sequences.

Example 4.1. We take G (as in Example 2.2) to be the group of operators on $E = c_0$ generated by $\gamma_m : c_0 \longrightarrow c_0$,

$$\gamma_m(e_j) = \begin{cases} \omega_m e_j & \text{if } j = m \\ e_j & \text{otherwise,} \end{cases}$$

where $\omega_m = \mathrm{e}^{2\pi \mathrm{i}/m}$. If G_n is the group generated by $\{\gamma_k : k \leq n\}$ then

$$G = \bigcup_{n \in \mathbb{N}} G_n$$

is the union of an ascending chain of compact groups. Also, each γ_k preserves the norm, so $G(B)$ is bounded for every bounded subset B of c_0. Thus, by [**6**, Theorem 2.5], there exists a continuous projection operator $\sigma : \mathcal{H}_b(c_0) \longrightarrow \mathcal{H}_{b_G}(c_0)$.

Also, note that $\mathcal{H}_{b_G}(c_0)$ 'separates orbits'. Indeed, if $\psi \notin \mathcal{O}_\varphi$, then by [**6**, Lemma 4.1] there is an $f \in \mathcal{H}_{b_G}(c_0)$ such that $\psi(f) \neq \varphi(f)$.

Since approximable polynomials are dense in $\mathcal{H}_b(c_0)$ (see [**11**, Propositions 1.59 and 2.8]) and c_0 is symmetrically regular, by Theorem 3.3 we have that $\pi_G : \mathcal{M}_G \longrightarrow \ell_\infty / \sim$ is locally one-to-one.

Finally, in this case it is clear that for fixed $n \in \mathbb{N}$, every point $z = (z_1, \ldots, z_n, 0, \ldots)$ is regular since there exists a neighbourhood U of z such that $g''(U) \cap U = \emptyset$ for all $g \neq I$. On the other hand, the point

$$a = (\omega_1^{-1}, \omega_2^{-1}, \omega_3^{-1}, \ldots, \omega_n^{-1}, \ldots)$$

in Example 2.2 is not regular: $\|a - \gamma_m(a)\| = |1 - \omega_m|$ can be as small as required, for large m.

A natural question is whether every closed subalgebra $\mathcal{A}$ of an algebra of group-symmetric functions $\mathcal{H}_G$ is also an algebra of group-symmetric functions. The answer is in general negative, as we show in the next example.

Example 4.2. Consider in the Fréchet algebra $\mathcal{H}(\mathbb{C}^2)$ the subalgebra $\mathcal{A}$ that is the closure in $\mathcal{H}(\mathbb{C}^2)$ of the algebra generated by $\{z_1 + z_2, z_1^2, z_2^2\}$. It is very intuitive that neither z_1 nor z_2 belong to $\mathcal{A}$, but let us present a proof of this fact. Consider P a polynomial generated by $\{z_1 + z_2, z_1^2, z_2^2\}$. Clearly, it is a linear combination of polynomials of the form $Q(z_1, z_2) = (z_1 + z_2)^m z_1^n z_2^p$ with $m, n, p \in \mathbb{N} \cup \{0\}$. Thus,

$Q(z_1, z_2) = c_{(1,0)}(Q)z_1 + c_{(0,1)}(Q)z_2 + \sum_{(\alpha,\beta)\in\Delta,\alpha+\beta\geq 2} c_{(\alpha,\beta)}(Q)z_1^\alpha z_2^\beta$ with Δ a suitable finite set and

$$c_{(1,0)}(Q) = \frac{1}{2\pi i}\int_{|\lambda|=1} \frac{Q(\lambda, 0)}{\lambda^2}\,\mathrm{d}\lambda = \begin{cases} 1 & \text{if } m = 1,\ n = 0,\ p = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Analogously,

$$c_{(0,1)}(Q) = \frac{1}{2\pi i}\int_{|\mu|=1} \frac{Q(0, \mu)}{\mu^2}\,\mathrm{d}\mu = \begin{cases} 1 & \text{if } m = 1,\ n = 0,\ p = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Hence $P(z_1, z_2) = d(P)(z_1 + z_2) + \sum_{(\alpha,\beta)\in\Gamma,\alpha+\beta\geq 2\Gamma} c_{(\alpha,\beta)}(P)z_1^\alpha z_2^\beta$ with Γ another suitable finite set. Now, if $f \in \mathcal{A}$, there exists a sequence (P_k) uniformly convergent to f on compact subsets of $\mathbb{C}^2$ and we have that $c_{(1,0)}(f) = \lim_{k\to\infty} c_{(1,0)}(P_k) = \lim_{k\to\infty} c_{(0,1)}(P_k) = c_{(0,1)}(f)$. As a consequence, $z_j \notin \mathcal{A}$ for $j = 1, 2$. Let us assume now that there exists a group G of automorphisms of $\mathbb{C}^2$ such that $\mathcal{A} = \mathcal{H}_G(\mathbb{C}^2)$. If $g \in G$ then

$$g(z_1, z_2) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \left(\alpha z_1 + \beta z_2,\ \gamma z_1 + \delta z_2\right).$$

Applying this to the function $f(z_1, z_2) = z_1 + z_2$ we will get that $\alpha + \gamma = 1$ and $\beta + \delta = 1$. On the other hand, applying it to the function $f(z_1, z_2) = z_1^2$ we get $\alpha^2 = 1$ and $\beta = 0$. Finally, for the function $f(z_1, z_2) = z_2^2$ we get $\delta^2 = 1$ and $\gamma = 0$. All of these combined imply that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In consequence, $G = \{I\}$; however, $\mathcal{A} \subsetneq \mathcal{H}(\mathbb{C}^2)$, a contradiction.

Acknowledgements. The first and second authors were supported by MINECO and FEDER projects MTM2014-57838-C2-2-P and MTM2017-83262-C2-1-P, and the project Prometeo/2017/102 of the Generalitat Valenciana. The third author was supported by PIP 112201301 00422 (CONICET).

References

1. R. Alencar, R. Aron, P. Galindo and A. Zagorodnyuk, Algebras of symmetric holomorphic functions on ℓ_p, *Bull. Lond. Math. Soc.* **35** (2003), 55–64.
2. R. Aron and P. Berner, A Hahn-Banach extension theorem for analytic mappings, *Bull. Soc. Math. France* **106**(1) (1978), 3–24.
3. R. Aron, B. Cole and T. Gamelin, Spectra of algebras of analytic functions on a Banach space, *J. Reine Angew. Math.* **415** (1991), 51–93.
4. R. Aron, J. Falcó, D. García and M. Maestre, Algebras of symmetric holomorphic functions of several complex variables, *Rev. Mat. Complut.* **31** (2018), 651–672.
5. R. Aron, P. Galindo, D. García and M. Maestre, Regularity and algebras of analytic functions in infinite dimensions, *Trans. Amer. Math. Soc.* **348**(2) (1996), 543–559.
6. R. Aron, P. Galindo, D. Pinasco and I. Zalduendo, Group-symmetric holomorphic functions on a Banach space, *Bull. Lond. Math. Soc.* **48**(5) (2016), 779–796.

7. I. Chernega, P. Galindo and A. Zagorodnyuk, Some algebras of symmetric analytic functions and their spectra, *Proc. Edinb. Math. Soc.* **55** (2012), 125–142.
8. I. Chernega, P. Galindo and A. Zagorodnyuk, The convolution operation on the spectra of algebras of symmetric analytic functions, *J. Math. Anal. Appl.* **395** (2012), 569–577.
9. I. Chernega, P. Galindo and A. Zagorodnyuk, The multiplicative convolution operation on the spectra of algebras of symmetric analytic functions, *Rev. Mat. Complut.* **27** (2014), 575–585.
10. D. Deghoul, Construction de caractères exceptionnels sur une algèbre de Fréchet, *C. R. Acad. Sci. Paris* **312**, Série I (1991), 579–580.
11. S. Dineen, *Complex analysis on infinite dimensional spaces*, Springer Monographs in Mathematics (Springer-Verlag, London, 1999).
12. M. González, R. Gonzalo and J. Jaramillo, Symmetric polynomials on rearrangement invariant function spaces, *J. Lond. Math. Soc.* **59**(2) (1999), 681–697.
13. A. S. Nemirovskii and S. M. Semenov, On polynomial approximation of functions on Hilbert space, *Mat. USSR Sb.* **21** (1973), 255–277.
14. I. Zalduendo, A canonical extension for analytic functions on Banach spaces, *Trans. Amer. Math. Soc.* **320**(2) (1990), 747–763.