GÖDEL DIFFEOMORPHISMS

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Abstract. In 1932, von Neumann proposed classifying the statistical behavior of differentiable systems. Joint work of B. Weiss and the author proved that the classification problem is complete analytic. Based on techniques in that proof, one is able to show that the collection of recursive diffeomorphisms of the 2-torus that are isomorphic to their inverses is Π_1^0-hard via a computable 1-1 reduction. As a corollary there is a diffeomorphism that is isomorphic to its inverse if and only if the Riemann Hypothesis holds, a different one that is isomorphic to its inverse if and only if Goldbach's conjecture holds and so forth. Applying the reduction to the Π_1^0-sentence expressing "ZFC is consistent" gives a diffeomorphism T of the 2-torus such that the question of whether $T \cong T^{-1}$ is independent of ZFC.

§1. The problem. The isomorphism problem in ergodic theory was formulated by von Neumann in 1932 in his pioneering paper [17]. Motivated by the idea that physical systems are frequently modeled by differential equations and verified by empirical experiments, it proposes classifying the statistical behavior of measure preserving diffeomorphisms of smooth compact manifolds.

In modern language, a consequence of the Pointwise Ergodic Theorem is that measure isomorphism captures statistical behavior. Thus von Neumann's program is usually stated:

> *Classify the Lebesgue measure preserving diffeomorphisms of compact manifolds up to measure preserving isomorphism.*

There is an extensive literature about this problem (see, e.g., the textbooks [7, 8, 10, 13–16, 18, 19]). Much of the progress focused on invertible measure preserving transformations of a fixed standard measure space (i.e., $([0,1),\lambda)$). The Polish group of Measure Preserving Transformations, MPT, acts on itself by conjugacy and the associated equivalence relation is the same as measure isomorphism.

DEFINITION 1. A measure preserving system $(X,\mathcal{B},\mu,T)$ is a standard probability space with an invertible measure preserving transformation T. T is *ergodic* if and only all T-invariant measurable sets either have measure zero or one.

There is a natural association between measure preserving transformations of a standard measure space and invariant measures for homeomorphisms of compact separable metric spaces. Given such a homeomorphism, the set of invariant (probability) measures is compact convex set (in the weak topology) and hence spanned by its extreme points. The extreme points are exactly the *ergodic* measures. By Choquet's theorem every invariant measure is an integral of the ergodic measures. The upshot is that *the ergodic measures form the basic building blocks for invariant measures*. Halmos showed that the *ergodic* transformations are a dense $\mathcal{G}_\delta$ set in MPT.

For these reasons, work on the classification theorem has been focussed on the polish group action of MPT on the collection $\mathcal{E}$ of ergodic transformations.

§2. Some history. There is a huge literature on this showing how certain classes can be classified. For example, Ornstein showed that Kolmogorov's entropy is a complete invariant for Bernoulli shifts, and Halmos and von Neumann showed that two ergodic translations of compact groups are isomorphic if and only if the associated Koopman operators are unitarily equivalent.

However, the general problem proved intractable. Hjorth [12] showed that the equivalence relation of being conjugate in the group MPT is not Borel. Later Foreman et al. [3] extended this by showing that the equivalence relation of isomorphism of *ergodic* measure preserving transformations is complete analytic. Foreman and Weiss had earlier proved the action of MPT on $\mathcal{E}$ is *turbulent* (in the sense of Hjorth [11]) and Foreman showed that the problem of *Graph Isomorphism* can be reduced to the isomorphism problem of measure preserving transformations of $[0,1]$.

These results can be summarized by saying that the isomorphism problem for ergodic measure preserving transformations of $[0,1]$ is complete analytic and lies strictly above every S_∞-action in the ordering of Borel reducibility.

§3. The results being announced. The results above do not answer von Neumann's original question, since the transformations involved are abstract measure preserving transformations, not diffeomorphisms. Indeed in the same paper von Neumann asked the following question, called the *smooth realization problem*:

> *Is every measure preserving transformation of a standard measure space isomorphic to a measure preserving diffeomorphism of a compact manifold?*

This question is still open, the only known limitation being that a measure preserving diffeomorphism of a compact manifold has to have finite entropy, a result of Kushnirenko.

A final classical problem is to determine when "time running backwards" is the same as "time running forwards." When the acting group is $\mathbb{Z}$, this is the question of whether $T \cong T^{-1}$. Halmos and von Neumann conjectured

that this was always true and it was not until 1951 that Anzai gave an example where it failed. (See [1, 9].)

Let $\mathrm{Diff}^\infty(\mathbb{T}^2,\lambda)$ be the space of C^∞, invertible, measure-preserving diffeomorphisms of the 2-torus endowed with the C^∞-topology. Two diffeomorphisms S, T are *measure isomorphic* or *measure conjugate* if and only if there is an invertible measure preserving transformation ϕ mapping from the two torus to itself such that $\phi \circ T = S \circ \phi$ (a.e.). In this paper, we write this equivalence relation as $S \cong T$.

In a series of papers [4–6] Foreman and Weiss were able to prove:

THEOREM 2. *The collection*:

$$\{T : T \cong T^{-1} \text{ and } T \text{ is ergodic}\} \subseteq Diff^\infty(\mathbb{T}^2,\lambda)$$

is a complete analytic set.

COROLLARY 3. *The measure isomorphism relation on ergodic members of* $Diff^\infty(\mathbb{T}^2,\lambda)$ *is a complete analytic equivalence relation*.

The previous result that Graph Isomorphism is reducible to the isomorphism relation on abstract measure preserving transformations carries over to diffeomorphisms, nearly verbatim. However it is an open problem at this time whether the equivalence relation "isomorphism for diffeomorphisms" is *strictly* above every S_∞-action.

Proving Theorem 2 required building a reduction of ill-founded trees to the smooth transformations isomorphic to their inverse. The techniques are general enough to settle other open problems dating to the 1960's. In a forthcoming paper Foreman and Weiss prove:

- There are measure-distal diffeomorphisms of the 2-torus of height 3. In fact there are measures-distal homeomorphisms of arbitrary countable ordinal height.
- For all (compact) Choquet simplices K there is a Lebesgue measure preserving diffeomorphism of the 2-torus having K as its simplex of invariant measures.

Conversations with M. Magidor, J. Steel, T. Carlson, H. Towsner, J. Avigad, T. Slaman, and a helpful question from S. Friedman led to the following result [2].

THEOREM 4. *There is a computable function*

$$F : \{Codes \text{ } for \text{ } \Pi^0_1 - sentences\}$$
$$\rightarrow \{Codes \text{ } for \text{ } computable \text{ } diffeomorphisms \text{ } of \text{ } \mathbb{T}^2\}$$

such that:

1. *For a code m for a Π^0_1 sentence, m is the code for a true sentence if and only if $F(m)$ is the code for a computable T, where T is measure theoretically conjugate to T^{-1}; and*
2. *For $m \neq n$, $F(m)$ is not conjugate to $F(n)$.*

The diffeomorphisms in the range of F are ergodic.[1]

Thus there is a *systematic* way of associating to each problem P that can be expressed as a Π_1^0-statement an ergodic computable diffeomorphism T_P of the 2-torus so that:

1. P is true if and only if $T_P \cong T_P^{-1}$

   and

2. for $P \neq P'$, T_P is not conjugate to $T_{P'}$.

It is known that many classical problems, such as the Riemann Hypothesis and Goldbach's Conjecture as well as "Con (T)" for recursively axiomatizable T be stated in a Π_1^0-way. Hence by varying the Π_1^0 statements we get corresponding corollaries (among others):

COROLLARY 5. *There are nonconjugate computable diffeomorphisms of the 2-torus* $T_{RH}, T_{GC}, T_{ZFC}, T_{sc\ cardinal}$ *such that*:

> RIEMANN HYPOTHESIS: $T_{RH} \cong T_{RH}^{-1}$ *if and only if the Riemann Hypothesis is true.*
> GOLDBACH'S CONJECTURE: $T_{GC} \cong T_{GC}^{-1}$ *if and only if Goldbach's Conjecture is true.*
> ZFC: *The statement "$T_{ZFC} \cong T_{ZFC}^{-1}$" is independent of ZFC (assuming ZFC is consistent).*
> ZFC + *there is a supercompact cardinal*: *The statement*
>
> $$T_{sc\ cardinal} \cong T_{sc\ cardinal}^{-1}$$
>
> *is independent of "ZFC + there is a supercompact cardinal" (assuming that theory is consistent).*

§4. Isn't this a triviality? A skeptic might object: "Suppose T_0 and T_1 are measure preserving diffeomorphisms and $T_0 \cong T_0^{-1}$ but $T_1 \not\cong T_1^{-1}$. Why not simply define

$$T_{RH}(x) = \begin{cases} T_0(x) & \text{if Riemann Hypothesis is true,} \\ T_1(x) & \text{if Riemann Hypothesis is false.} \end{cases}$$

Then T_{RH} is equal to either T_0 or T_1 depending on the truth of the Riemann Hypothesis. Hence it satisfies Corollary 5."

Theorem 4 precludes this "cheating" for two reasons. One is that the association of a computable diffeomorphism T to a code for a Π_1^0-statement ϕ is itself *computable*. Thus there is a computer program that takes the code for ϕ and produces an algorithm that computes T_ϕ. The theorem provides a deterministic computer program that computes T_{RH}; it is not diffeomorphism chosen arbitrarily according to an unknown truth value.

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[1] By *code* for a Π_1^0-sentence we mean its Goedel number. A *code* for a computable diffeomorphism is the c.e. code for the algorithm computing it.

The second reason is item 2: we are assigning nonisomorphic diffeomorphisms to different Π_1^0-statements. So $T_{RH} \not\cong T_{GC}$.

§5. How hard is the proof? The reductions built in the proof of Theorem 2 are extremely concrete. Indeed in [4] it was shown that in the context of Theorem 4 the reduction F is primitive recursive. Gaebler showed that the definition and proof that F is indeed a reduction can be done in ACA_0.

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