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MONEY AND LIMITED ENFORCEMENT IN MULTILATERAL EXCHANGE

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We propose a model in which money performs an essential role in the process of exchange, despite the presence of a multilateral clearing house that collects resources from and distributes them to anonymous agents. Money improves the functioning of the clearing house, simultaneously keeping the incentives to contribute and guaranteeing the fine-tuning of allocations.

Keywords: Money, Essentiality, Multilateral Exchange

1. INTRODUCTION

It is well known, from the work of Kocherlakota (1998), that money can play a socially beneficial role in environments in which trade opportunities are restricted by two imperfections: (i) absence of commitment and (ii) anonymity. Environments in which trade happens in informationally isolated meetings among agents who do not have access to commitment devices are natural playgrounds for monetary theorists. This paper asks whether the separation of trade meetings is essential for money to play a socially beneficial role in the allocation of resources.

Specifically, we consider the model by Townsend (1980) with two types of agents and alternating endowments but we drop the *turnpike* assumption, allowing the agents to trade through a clearing house that first pools and then redistributes the endowments, ignoring the agents' identities. When collecting the resources, the clearing house cannot distinguish individual contributions to the pool, hence, individualized punishments cannot be administered, but can detect unexpected shortfalls from the aggregate contribution and apply collective punishments, shutting down trade altogether. We allow for the possibility that the clearing house may be unable to enforce unlimited collective punishments, assuming that the record of shortfalls is kept only for a limited amount of time, after which it is completely wiped out and trade may resume. When redistributing the resources

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back to the agents, the clearing house cannot tailor directly the individual amounts to the agents' characteristics, being unable to identify them. Hence, the agents' anonymity limits the workings of the clearing house both at the collection and at the redistribution stage. We consider two regimes for the clearing house that may operate without or with money. The cashless clearing house operates as a traditional Walrasian facility. When the clearing house uses money, contributions to the pool are rewarded with cash, withdrawals from the pool must be paid for in cash. Since the clearing house is centralized and irreplaceable, it has full control of the monetary aggregate.

We provide a complete characterization of the two regimes and compare their outcomes. The cash-based clearing house attains always at least the same set of allocations as the cashless one, and, in terms of ex ante welfare, the former does always at least as well as the latter. If past records are never wiped out, the two clearing houses achieve the same set of allocations; if the records are wiped out soon enough, the cash-based clearing house outperforms strictly the cashless one. Hence, monetary trade is never dominated and improves matters if the ability of the clearing house to enforce collective punishments is sufficiently limited. Money helps in two dimensions, inducing the incentive to contribute to the pool and allowing to fine-tune the allocations for different agents' types. The result that centralized trade works always at least as well and sometimes strictly better with money than without leads us to conclude that the separation of trade meetings is not essential to obtain a socially beneficial role for money.

Following the seminal paper by Kiyotaki and Wright (1989), trade meetings in monetary environments have been conceived mostly as bilateral. The *New Monetarist* framework developed by Lagos and Wright (2005),[1] featuring alternating Walrasian and bilateral markets, has stimulated researchers to ask whether centralized clearing is fully compatible with monetary trade. On the one hand, Aliprantis et al. (2007a, 2007b) and Araujo et al. (2012) have suggested that the presence of multilateral trading sessions may make money redundant. Araujo et al., in particular, have modeled centralized trade as a strategic market game with a finite number of agents. In such an environment, each agent's action affects prices, which can in turn be used to infer behavior and coordinate on individualized punishments. Our approach to centralized trade, instead, is non-strategic, in the spirit of the Walrasian tradition, featuring a clearing house that is unable to identify anonymous agents, but has unrestricted access to aggregate information, being centralized and irreplaceable. Thus, the agents can be disciplined only collectively for failing to contribute to the pool and the allocations cannot be tailored to their characteristics. Money helps the agents self-select into their preferred allocation while keeping the incentive to contribute to the pool. On the other hand, Hu et al. (2009) have argued that the efficient frontier can be reached in Lagos and Wright (2005) (even without policy intervention) when cooperative notions of trade are considered. We bypassed the hard question how to reconcile joint deviations with the agents' inability to commit, adopting the traditional notion of a centralized and irreplaceable clearing house. Other papers have considered similar environments in which, however, either money is inessential or the clearing house is totally ineffective. An instance of the former approach is the Bewley economy of Green and Zhou (2005), where agents are not anonymous and money is never essential. At the other end of the spectrum, Levine (1991) features a Bewley economy where even aggregate information is unavailable, precluding the working of a non-monetary system from the start. We stand on middle ground. In our paper, deviations can be detected, although they cannot be attributed to the individual deviator, and the clearing system is not altogether ineffective. The effectiveness of the punishments in our framework is graduated through the imperfect enforcement power of the clearing house due to the loss of records. The assumption that past records are wiped out after some time is reminiscent of the finite memory assumption adopted by Cole and Kocherlakota (2005) in a repeated game setting, and is in the spirit of Kocherlakota and Wallace (1998), where the records are updated with an exogenous time lag, and Cavalcanti and Wallace (1999), where they can be kept indefinitely but only for a subset of the population. Finally, there is an older literature, going back to Ostroy and Starr (1974, 1990), that has discussed how equilibrium allocations determined by the auctioneer in a Walrasian economy may be realized through a sequence of bilateral trades subject to limited commitment and informational constraints. We share the relevance of the inability to commit and the informational frictions as constraints for the execution of trades, but we integrate the auctioneer and the clearing house in a single multilateral facility that determines endogenously the equilibrium allocations. Other centralized models with cash-in-advance constraints, such as Magill and Quinzii (1996, Ch. 7), are not designed to address the question of the essentiality of money.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 characterizes the first best allocations. Section 4 analyzes the non-monetary system. Section 5 presents the monetary one. Section 6 compares the two and shows when money is essential. In Section 7, we also discuss a transfer scheme for the monetary regime that can Pareto improve upon the one presented in the main body of the paper. Section 8 concludes. The proofs are in the Supplementary Material.

2. FUNDAMENTALS

Time, indexed by $t = 1, 2, \ldots$, is discrete and continues for ever. There is a single perishable good, x. The economy is populated by $2N$ agents, equally divided between two types, indexed by $i = 1, 2$. Agents of type 1 receive, as an endowment, one unit of the good at odd dates and zero at even dates, that is, $e_t^1 \in \{0, 1\}$, $e_t^1 = 0$ for $t = 2n$, $e_t^1 = 1$ for $t = 2n - 1$, with $n \in \mathbb{N}$, and agents of type 2 receive $e_t^2 = 1 - e_t^1$ for all $t \geq 1$. Agents' preferences over consumption of the good are represented by the following life-time utility:

$$\sum_{t=1}^{\infty} \beta^{t-1} u\left(x_t^i\right),$$

where $\beta \in (0, 1)$ represents the discount rate and $u\left(x_t^i\right)$ the period utility function defined over $x_t^i \in \mathbb{R}_+$, the units of the good consumed by an agent of type i at date t. The function $u : \mathbb{R}_+ \to \mathbb{R}$ is (at least twice) continuously differentiable, with $u'(x) > 0$, $u''(x) < 0$, for all $x \in \mathbb{R}_+$, and $u'(0) = +\infty$. We also require $u(0) > -\infty$.

Agents cannot commit themselves to future actions and are anonymous, that is, their identities are private information.

3. FIRST BEST ALLOCATIONS

We begin with a characterization of the first best allocations. Let $\mu_i \in [0, 1]$ be the weight given to the well-being of type i agents, with $\sum_{i=1,2} \mu_i = 1$. The first best allocations are those that maximize the weighted sum of lifetime utilities of the two types:

$$\underset{\left\{x_t^1\right\}_{t=1}^{\infty}, \left\{x_t^2\right\}_{t=1}^{\infty}}{Max} \quad \sum_{i=1,2} \mu_i \sum_{t=1}^{\infty} \beta^{t-1} u\left(x_t^i\right), \tag{1}$$

subject to feasibility,

$$\sum_{i=1,2} x_t^i \leq 1, \forall t \geq 1, \tag{2}$$

and non-negativity $x_t^i \geq 0$, $\forall t \geq 1, \forall i$. Let $\mu \equiv \mu_1$ and, hence, $1 - \mu = \mu_2$. The feasibility constraint, equation (2), can be taken at equality, since the objective function is strictly increasing in the choice variables for all $\mu \in [0, 1]$. Clearly, it is optimal to assign $x_t^1 = 0$ and $x_t^2 = 1$, $\forall t \geq 1$, when $\mu = 0$, and vice versa for $\mu = 1$. Consider now $\mu \in (0, 1)$. We can ignore the non-negativity constraints on consumption for any $\mu \in (0, 1)$, thanks to the Inada condition. Using equation (2) at equality into the objective function, equation (1), we can compute the necessary—and sufficient, given the strict concavity of the utility function—condition for an optimum as

$$\Omega\left(x_t^1, \mu\right) \equiv \mu u'\left(x_t^1\right) - (1 - \mu)\, u'\left(1 - x_t^1\right) = 0, \forall t \geq 1, \tag{3}$$

together with $x_t^2 = 1 - x_t^1, \forall t \geq 1$. Clearly from equation (3), the optimal x_t^1 (and therefore x_t^2) is constant over time, $x_t^1 = x^1, \forall t \geq 1$. Let $z : [0, 1] \to [0, 1]$ be the function $z(\mu)$ that identifies the first best allocation for type 1 for all values of $\mu \in [0, 1]$, and $1 - z(\mu)$ the first best allocation for type 2. The following lemma gives a complete characterization of $z(\mu)$. Notice that the symmetric allocation is on the Pareto frontier. This observation will turn out to be useful later on.

LEMMA 1. *a. For any μ, there exists a unique $z \in (0, 1)$ such that $\Omega(z, \mu) = 0$; b. The function $z(\mu)$ is (at least once) continuously differentiable in μ, with $\frac{\partial z(\mu)}{\partial \mu} > 0$, for any μ; c. $z(0) = 0$, $z(\frac{1}{2}) = \frac{1}{2}$, and $z(1) = 1$.*

Consider, for a moment, an economy in which agents could commit themselves to any future action. Given the utility function described in Section 2, the First and Second Fundamental Welfare Theorems would apply. As a consequence, all the first best allocations identified by the function $z(\mu)$ could be decentralized as a competitive equilibrium. Agents would deliver their negative excess demands and withdraw their positive excesses through a clearing house that would be able to execute trade without impediments. Given the inability of the agents to commit to future, the problem of allocating resources is non-trivial. In what follows, we will describe the workings of a multilateral clearing house, first without and, then, with the use of fiat money to facilitate the execution of trades.

4. NON-MONETARY REGIME

The non-monetary clearing house. We consider, first, a transaction technology that does not make use of cash. We will refer to it as the non-monetary, or cashless, clearing house. The non-monetary clearing house operates, at each date t, following a two-stage procedure. 1. The clearing house requires agents to deliver an amount $d \in [0, 1]$ of the good; 2. the clearing house allows any agent to withdraw an amount $w \in \{w_1, w_2\}$ of the good, whereby an agent of type i obtains w_i with probability $\psi \in [0, 1]$ and $w_{j \neq i}$ with the complementary probability.[2] The clearing house operates subject to feasibility, that is,

$$w_1 + w_2 \leq d. \tag{4}$$

If, at any point in time, any delivery at stage 1 is smaller than d, the clearing house stops without moving to stage 2. The clearing house can keep the records of any deviation only for a limited amount of time.[3] After T periods, with $T = 2n$, $n \in \mathbb{N} \cup \{0\}$, the records are completely wiped out.[4] Following such a loss of information, the clearing house can be re-started. Once re-started, the clearing house operates as before. Define the vector $\omega = (w_1, w_2)$. The clearing house, before the beginning of trade, at date 0, is programmed with the parameters (d, ω, ψ) to maximize ex ante welfare and ensure participation by the agents, taking into account that they are anonymous. Its workings are common knowledge at date 0. Notice that, given the symmetry and repetitiveness of the environment, where agents belonging to the same type are identical and receive identical endowments every other period, we restrict attention to facilities that treat agents identically by type and over time, that is, we look at allocations that are symmetric and stationary. This is consistent with our focus on allocations that maximize ex ante welfare.

Deliveries and withdrawals. First, since the agents are anonymous, at the delivery stage, the clearing house cannot distinguish between an agent of type 1 or 2. We immediately have that the probabilities with which they can withdraw the respective amounts must be the same, $\psi = 1 - \psi$, that is,

$$\psi = \frac{1}{2}. \tag{5}$$

This is the crucial element that hinders the functioning of the clearing facility on the side of withdrawals. Another consequence of the anonymity of the agents has been already built into the way the clearing house works following a deviation at stage 1. The clearing house can detect whether somebody delivered less than d, but it cannot tell the identity of the deviator. Hence, it cannot punish directly a single individual. It can, however, punish all the agents simultaneously shutting down trade altogether.

The length of time, T, after which the records are wiped out and the clearing house opens up again, parameterizes the severity of the consequences of the lack of enforcement power by the clearing house. If $T = 0$, trade will resume so fast that deviators cannot be punished, and the consequences of the lack of enforcement are most severe. At the other extreme, for $T \to \infty$, the clearing house can never be re-opened, thus making the threat of punishment most effective.

Define $f_T(\beta) \equiv \sum_{j=1}^{\frac{T}{2}} \beta^{2j-1} = \beta \left(\frac{1-\beta^T}{1-\beta^2}\right)$, and $g_T(\beta) \equiv \sum_{j=0}^{\frac{T}{2}} \beta^{2j} = \frac{1-\beta^{T+2}}{1-\beta^2}$. For a general T, to ensure participation by agents who are in a position to deliver some amount of the good at any given date, the parameters (d, ω) will have to satisfy the following constraint:

$$\frac{g_T(\beta)}{2} \sum_{i=1,2} u(1 - d + w_i) + \frac{f_T(\beta)}{2} \sum_{i=1,2} u(w_i) \geq g_T(\beta)\, u(1) + f_T(\beta)\, u(0), \quad \textbf{(6)}$$

where the RHS is the payoff arising from the shutdown of trade of T periods following a deviation at some date, and the LHS is the expected utility, computed using equation (5), of abiding by the rules of trade for the same length of time.[5] The participation condition (6) is written from the point of view of an agent who is supposed to deliver some amount d at the current date, assuming that all other agents always participate in the trading arrangement. Should the agent expect all other agents not to participate at any point in time, his best response would be not to deliver anything. Hence, autarky is always an equilibrium of this trading arrangement. Notice that, due to the symmetry and stationarity of the environment, every agent, whenever in a position to decide whether to deliver some amount of the good, is faced with the same participation condition (6). This condition also guarantees that all agents favor the re-opening of the clearing house ex post.

Ex ante welfare, for $\mu \in [0, 1]$, which represents the weight of type 1 agents in the welfare function, is given by

$$W(d, \omega, \mu, \beta) = \frac{1}{2(1-\beta^2)} \left[\sum_{i=1,2} \mu_i \beta^{i-1} \sum_{i=1,2} u(1 - d + w_i) + \sum_{i=1,2} \mu_i \beta^{2-i} \sum_{i=1,2} u(w_i) \right]. \quad \textbf{(7)}$$

By strict concavity of the utility function, both equation (7) and the LHS of equation (6) are strictly higher with $w_1 = w_2 = w$, for any $d \in [0, 1]$ and $\mu \in [0, 1]$.

Since the utility function is strictly increasing, wasting resources can only decrease the agents' welfare, hence, from equation (4) taken at equality, we have that, in order to work in the best interest of the agents, while satisfying participation and feasibility, the clearing house should set

$$w_1 = w_2 = \frac{d}{2}. \tag{8}$$

This is the sense in which withdrawals are not tailored to the individuals' types. With equation (8), the participation constraint (6) becomes

$$g_T(\beta)\, u\left(1-\frac{d}{2}\right) + f_T(\beta)\, u\left(\frac{d}{2}\right) \geq g_T(\beta)\, u(1) + f_T(\beta)\, u(0), \tag{9}$$

and equation (7) reduces to

$$\mu\left[u\left(1-\frac{d}{2}\right)+\beta u\left(\frac{d}{2}\right)\right] + (1-\mu)\left[u\left(\frac{d}{2}\right)+\beta u\left(1-\frac{d}{2}\right)\right]. \tag{10}$$

Hence, we have reduced the problem of the clearing house to the choice of $d \in [0, 1]$, so as to maximize (10) and satisfy (9).

We begin with a characterization of the allocations that satisfy the participation constraint (9), which we will call sustainable. Then, we will choose, among the sustainable allocations, those that maximize the agents' ex ante welfare (10).

Sustainable allocations. We call sustainable the allocations satisfying the participation constraint.

DEFINITION 1. *An allocation that satisfies the participation constraint (9) is sustainable without cash.*

For the purpose of comparing the current system with the monetary one, it is convenient to make a change of variable, defining $y \equiv 1 - \frac{d}{2} \in \left[\frac{1}{2}, 1\right]$. Let $\Gamma_T(\beta) : (0, 1) \rightrightarrows \left[\frac{1}{2}, 1\right]$ be the correspondence that satisfies equation (9) for any given T, identifying sustainable allocations as β varies over its domain of definition. Thus, an allocation y is sustainable without cash if and only if $y \in \Gamma_T(\beta)$. Next, we obtain the relevant properties of $\Gamma_T(\beta)$. First, no-trade allocations are obviously always sustainable. Second, clearly, when the clearing house can be restarted immediately, that is, $T = 0$, the only allocation that can be sustained is the autarkic one. This occurs because the punishment for failing to deliver the good is ineffective. Lemmas A2 and A3 in the Supplementary Material identify all the sustainable allocations for any $T > 0$. Here, we give an informal description. When $T = \infty$, we have the harshest possible punishment, and all the feasible allocations can be sustained when agents are sufficiently patient, while only a strict subset of the feasible allocations can be sustained when agents are impatient enough. Indeed, if the agents are more patient, the threat of a future punishment is harsher, thus extending the set of sustainable allocations. When T is finite, there are two possibilities. If the records are wiped out sufficiently infrequently—that is, $T > \overline{T} \equiv \left\lfloor \frac{2[u(1)-u(\frac{1}{2})]}{2u(\frac{1}{2})-u(1)-u(0)} \right\rfloor$—[6] and, thus, the threat of future punishment is

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severe enough, the situation is fairly similar to the previous case, with an infinite T. If, on the other hand, the records are wiped out sufficiently frequently—that is, $T \leq \overline{T}$—so that the threat of the punishment, in turn, is not too severe, the set of allocations that can be sustained is constrained, but it always includes some non-autarkic allocations. The following lemma contains the complete characterization of the sustainable allocations for any given T.

LEMMA 2. $\Gamma_T(\beta)$ *is non-empty, compact, convex-valued and continuous in* β *for any* T.

It is interesting to notice, as an aside, that the set of sustainable allocations becomes larger for larger values of T.[7] This is intuitive, since the punishment associated with a deviation becomes more severe when trade is shut down for a longer period.

Finally, notice that, by concentrating on sustainable allocations, we are not leaving out any relevant allocation. Indeed, the allocations we have identified—including the autarkic ones—constitute all the (stationary) equilibria of the non-monetary regime, since the trading system induces participation by the agents and the clearing of transactions.

Welfare. After having characterized all the sustainable allocations, we turn to the choice of the allocations that maximize ex ante welfare, equation (10), among the ones that can be sustained. The allocation $y \in \Gamma_T(\beta)$ is chosen to maximize

$$W(y, \mu, \beta) = \frac{1}{1-\beta^2}\{\mu[u(y) + \beta u(1-y)] + (1-\mu)[u(1-y) + \beta u(y)]\}. \quad \textbf{(11)}$$

Let $y_T^*(\mu, \beta) \equiv \arg\max\{(11) \mid y \in \Gamma_T(\beta)\}$, $W_T^*(\mu, \beta) \equiv \max\{(11) \mid y \in \Gamma_T(\beta)\}$. Since equation (11) is continuous and Lemma 2 applies, the Theorem of the Maximum implies that $y_T^*(\mu, \beta)$ and $W_T^*(\mu, \beta)$ are continuous functions of μ and β, for any T.

5. MONETARY REGIME

The monetary clearing house. We now consider a trading system that uses cash. The monetary clearing house is subject to the same informational constraints as the non-monetary one and works as follows. At date 0, an amount M_0 of divisible fiat money is available equally to agents of type 2. At each date t, the clearing house works in three stages: (1) agents can deliver one unit of the good in return for $\frac{1}{v_t}$ units of money and may receive a lump-sum transfer of money $\eta_t \in \mathbb{R}_+$ conditional on whether there was a delivery; (2) the clearing house collects lump-sum transfers of money, $\tau_t \in \mathbb{R}$, equally from (to) all agents and, in the case of negative transfers, destroys the corresponding amount; and (3) agents can obtain v_t units of the good for every unit of money inserted in the clearing house.[8] In the case of negative transfers, if some agent does not deliver the required amount, the clearing house stops operating. The clearing house can resume its functions following a wipeout of the records after T periods, with $T = 2n$, $n \in \mathbb{N} \cup \{0\}$,

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exactly as before. Once re-started, the clearing house distributes a new currency and operates as before, while the old currency is no longer accepted by the clearing house.[9] This helps discipline the agents, depriving a deviator of the benefit of having extra currency once the clearing house opens up again. The assumption plays an important role in what follows and, we believe, goes to the heart of a system that works with a clearing house which is centralized and irreplaceable.[10] This is the crucial difference with the related approaches to multilateral trade in monetary settings mentioned in the introduction such as Araujo et al. (2012) where the market price plays the role of a signal allowing the agents to infer whether a deviation has occurred. In our model, the presence of a centralized clearing house allows not only to use observations on aggregate variables to make inferences on individual deviations, but also to make bolder threats of collective punishment, including the replacement of the trading instrument if needed, since all traded objects go through the clearing house in any case.

The lump-sum transfers, η_t and τ_t, are expressed in real terms, that is, in consumption units, and are chosen to maximize ex ante welfare. The workings of the clearing house, including v_t, η_t, and τ_t at all t, are common knowledge at date 0. For the moment, we deal only with the case that does not discriminate between the types, that is, in which $\eta_t = 0$ for all t. Later, in Section 7, we will analyze the discriminatory case in which $\eta_t > 0$ for an agent who delivers something at time t. Such a scheme works despite the presence of anonymity, since the transfers η_t are obtained by an agent only if he makes a delivery in the current period, and only agents with an endowment in the current period can deliver anything to the clearing house. This particular discriminatory scheme, which—whenever feasible—improves upon the non-discriminatory one, exploits the extreme nature of the endowment process and may not work in more general environments, where, however, other discriminatory schemes may be devised that are still sensitive to the amount delivered to the clearing house, in the spirit of the schemes considered by Andolfatto (2010) and Wallace (2014).

In the monetary regime, the agents are allowed to optimize against their budget constraints while deciding whether to participate in the transfer scheme. We will characterize, first, the allocations that satisfy the optimality and market clearing conditions, provisionally ignoring the participation decision. Then, we will consider the allocations that satisfy the participation conditions. We will call the allocations that satisfy optimality and participation, sustainable. Finally, we will choose the sustainable allocations that maximize ex ante welfare.

Optimality and market clearing. The maximization problem of agent i is

$$\underset{\{x_t^i\}_{t=1}^{\infty},\{m_t^i\}_{t=1}^{\infty}}{\text{Max}} \sum_{t=1}^{\infty} \beta^{t-1} u\left(x_t^i\right), \tag{12}$$

$$\text{s.t.} \quad x_t^i + v_t m_t^i = e_t^i + v_t m_{t-1}^i + \tau_t, \ \forall t \geq 1, \tag{13}$$

$$m_t^i \geq 0, \forall t \geq 1, \tag{14}$$

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where m_t^i is the amount of money owned by agent i in period t. Notice that lump-sum transfers are not indexed by the agent's type. This is coherent with the assumption that the agents are anonymous and indistinguishable from the point of view of the clearing house that permits trade. Given the Inada condition on the utility function we do not need to worry about the non-negativity constraint on consumption. Market clearing for the good requires

$$\sum_{i=1,2} x_t^i = 1, \forall t \geq 1, \tag{15}$$

while market clearing for money is implied by Walras Law. We consider, first, the allocations that solve the maximization problem above and satisfy equation (15), ignoring for the moment the participation constraint.

The first-order conditions for an optimum are

$$\beta^{t-1} u'\left(x_t^i\right) = \lambda_t^i, \forall t \geq 1, \tag{16}$$

for the consumption choice, where $\lambda_t^i > 0, \forall t \geq 1$ is the multiplier of the constraint in equation (13), and

$$-v_t \lambda_t^i + v_{t+1} \lambda_{t+1}^i + \theta_t^i = 0, \forall t \geq 1, \tag{17}$$

for the choice of money holdings, where $\theta_t^i \geq 0, \forall t \geq 1$ is the multiplier of the non-negativity constraint on money holdings, equation (14). There is also the complementary slackness condition for the non-negativity constraint on money holdings, equation (14),

$$\theta_t^i m_t^i = 0, \forall t \geq 1.$$

We consider the following candidate for a solution of the maximization problem: $m_t^1 = 0$ when $t = 2n, n \in \mathbb{N}$, $m_t^1 > 0$ when $t = 2n - 1, n \in \mathbb{N}$, and vice versa for agents of type 2. In other words, the candidate solution requires the agents to demand a positive amount of money when they receive their endowment of the good and spend entirely their money holdings before receiving any new endowment. In this situation, equations (16) and (17) give

$$u'\left(x_t^i\right) = \beta \frac{v_{t+1}}{v_t} u'\left(x_{t+1}^i\right),$$

for $i = 1, t = 2n - 1, n \in \mathbb{N}$, and $i = 2, t = 2n, n \in \mathbb{N}$. In the other cases $u'\left(x_t^i\right) \geq \beta \frac{v_{t+1}}{v_t} u'\left(x_{t+1}^i\right)$.

The stock of money in any period t is given by $M_t = M_{t-1} + \frac{2N\tau_t}{v_t}$. Define $\pi_t \equiv \frac{2N\tau_t}{v_t M_{t-1}}$, for all $t \geq 1$, thus, $M_t = (1 + \pi_t) M_{t-1}$, for all $t \geq 1$. As in the case of the non-monetary clearing house we look at symmetric and stationary allocations. Hence, we look at situations in which $\pi_t = \pi \in [\beta - 1, \infty)$, for all $t \geq 1$.[11] Stationarity implies also that $v_t (1 + \pi) = v_{t-1}$ for all $t \geq 1$. There are always circumstances in which cash is not valued, that is, $v_t = 0$ for all $t \geq 1$, and, therefore, agents do not trade. Henceforth, we concentrate on the case in which cash has value at all times and some trade can occur.

The stationary allocation of consumption that satisfies market clearing, equation (15), is cyclical of order two and entails for an agent of type 1 x units of consumption in odd periods and $1-x$ in even periods and vice versa for type 2 agents. The allocation must satisfy

$$\Phi(x, \pi, \beta) \equiv u'(x) - \frac{\beta}{1+\pi} u'(1-x) = 0. \tag{18}$$

Lemma A5 in the Supplementary Material establishes that, for any admissible inflation or deflation rate and discount rate, a unique non-autarkic allocation exists that satisfies equation (18). Define $\widetilde{x}(\pi, \beta) : [\beta - 1, \infty) \times (0, 1) \to \left[\frac{1}{2}, 1\right]$ as the (at least once) continuously differentiable function, $\widetilde{x} = \widetilde{x}(\pi, \beta)$, such that the values $(\widetilde{x}, \pi, \beta)$ satisfy equation (18). Lemma A6 in the Supplementary Material characterizes the behavior of the solutions as the monetary policy parameter varies over its feasible range. In particular, it establishes that $\widetilde{x}(\pi, \beta)$ is monotonically increasing in π. Define the correspondence $\widetilde{\Gamma}(\pi, \beta) : [\beta - 1, \infty) \times (0, 1) \rightrightarrows \left[\frac{1}{2}, 1\right]$ as $\widetilde{\Gamma}(\pi, \beta) \equiv [\widetilde{x}(\pi, \beta), 1]$, which contains all the relevant allocations that satisfy optimality and market clearing.

Participation. Consider now the decision of an agent whether to pay the taxes or not. This decision is relevant only if $\tau < 0$, that is, $\pi \in [\beta - 1, 0)$. If an agent decides to stick to the taxation regime operated by the monetary clearing house, will get the implied allocation $\widetilde{x} = \widetilde{x}(\pi, \beta)$, while if it decides not to pay the taxes the clearing house will stop operating for T periods, shutting down trade for all the agents. Consider an agent who is receiving an endowment in the current period. If he decides not to pay taxes in the current period, he anticipates that the clearing house will stop for some time and will eventually resume its activity but with a new currency. Hence, it is pointless, for an agent who has decided not to pay taxes, to deliver anything to the clearing house. Therefore, for such an agent the taxation τ and the implied deflation rate π will have to satisfy the following participation constraint:

$$g_T(\beta) u(\widetilde{x}) + f_T(\beta) u(1-\widetilde{x}) \geq g_T(\beta) u(1) + f_T(\beta) u(0), \tag{19}$$

where $f_T(\beta)$ and $g_T(\beta)$ have been defined before. Consider an agent who does not receive an endowment in the current period. If he decides not to pay taxes, the consequence for him will be the impossibility to consume in the current period, followed by autarky for T periods. Hence, for this type of agent, the participation constraint is given by

$$g_T(\beta) u(1-\widetilde{x}) + f_T(\beta) u(\widetilde{x}) \geq f_T(\beta) u(1) + g_T(\beta) u(0). \tag{20}$$

Rearranging equations (19) and (20), and noticing that $g_T(\beta) = \frac{1-\beta^{T+2}}{1-\beta^2} > \beta\frac{1-\beta^T}{1-\beta^2} = f_T(\beta)$ for any $\beta \in (0, 1)$ and $T \geq 0$, it is easy to see that whenever an allocation satisfies equation (19), it also satisfies equation (20). Therefore, to ensure agents' participation, we can ignore the latter constraint and work only with the former. The constraint (19) places a lower bound on the deflation rates that can be achieved in a monetary economy and is the same as the participation condition (9) of the non-monetary economy. In the monetary economy, however, it applies only in the case of taxes, that is, for a deflation. Since the constraint is the same as before, we keep using $\Gamma_T(\beta)$ to denote the allocations that satisfy the participation constraint.

Sustainable allocations. We call sustainable the allocations that satisfy optimality and the participation constraint.

DEFINITION 2. *An allocation that satisfies equation (18) for any* $\pi \in [\beta - 1, \infty)$ *and* $\beta \in (0, 1)$*, and equation (19) when* $\pi \in [\beta - 1, 0)$*, is sustainable with cash.*

Define the correspondence $\Gamma_T^M(\beta) : (0,1) \rightrightarrows \left[\frac{1}{2}, 1\right]$ as

$$\Gamma_T^M(\beta) \equiv \left(\left(\widetilde{\Gamma}(\beta - 1, \beta) \setminus \widetilde{\Gamma}(0, \beta)\right) \cap \Gamma_T(\beta)\right) \cup \widetilde{\Gamma}(0, \beta),$$

mapping values of β into allocations that are sustainable with cash. In other words, an allocation x is sustainable with cash if and only if $x \in \Gamma_T^M(\beta)$. The next lemma provides a complete characterization of allocations that are sustainable with cash.

LEMMA 3. $\Gamma_T^M(\beta)$ *is non-empty, compact, convex-valued and continuous in* β *for any* T.

A monetary equilibrium is an allocation, a price, and a taxation scheme at all times, such that the allocation satisfies problem (12) for every agent given prices and taxation, markets clear at all times, taxation and the stock of money are compatible, and agents are willing to participate in the taxation scheme. Notice that the allocations $x \in \Gamma_T^M(\beta)$ constitute all the (stationary) monetary equilibrium allocations of our economy. To see that this is so, just observe that the bounded sequence of consumption $(x, 1 - x)$ and money holdings $(m, 0)$ repeating itself identically every other period for agents of type 1 and vice versa for agents of type 2, satisfies market clearing, the necessary condition for an optimum and the transversality condition, hence, it constitutes an unconstrained monetary equilibrium for every π. The values of $\pi \in [\beta - 1, 0)$ which would imply non-participation in the lump-sum taxation scheme are excluded by the imposition of equation (19).

Welfare. We move to the maximization of the ex ante welfare. We know that monetary equilibrium allocations and inflation rates are related by a function $\widetilde{x}(\pi, \beta)$ which is strictly increasing. Although it would be natural, economically, to think of π as the variable chosen to maximize ex ante welfare, it is equivalent and more convenient for the purpose of the comparison with the non-monetary trading system, to let the allocation be the choice variable. Since $\widetilde{x}(\pi, \beta)$ is invertible, one can always derive the implied inflation or deflation rate. Therefore, $x \in \Gamma_T^M(\beta)$ is chosen to maximize

$$W^{M}(x,\mu,\beta)=\frac{1}{1-\beta^{2}}\{\mu[u(x)+\beta u(1-x)]+(1-\mu)[u(1-x)+\beta u(x)]\}. \quad \textbf{(21)}$$

Let $x_T^{M*}(\mu,\beta)\equiv \arg\max\{(21)\,|\,x\in\Gamma_T^M(\beta)\}$, $W_T^{M*}(\mu,\beta)\equiv\max\{(21)\,|\,x\in\Gamma_T^M(\beta)\}$. Since equation (21) is continuous and Lemma 3 applies, the Theorem of the Maximum implies that $x_T^{M*}(\mu,\beta)$ and $W_T^{M*}(\mu,\beta)$ are continuous functions of μ and β, for any T.

6. COMPARISON OF THE REGIMES

Set inclusion. We begin the comparison of the non-monetary and monetary regimes with the observation that the set of equilibrium allocations obtained under the monetary regime cannot be smaller than the one obtained under the non-monetary regime.[12]

PROPOSITION 1. $\Gamma_T(\beta)\subseteq\Gamma_T^M(\beta)$, *for any* β *and* T.

Although both systems have to ensure that agents have the incentive to deliver some resources, they do so in quite different ways. The cashless system works thanks to the threat of collective punishment, while the cash-based system only needs such a threat to induce agents to pay taxes. The participation constraint is identical in the two regimes, but in the monetary system it applies to a more limited set of circumstances. Thus, the cash-based system can always sustain at least the same allocations as the cashless system. We will see below that there are robust cases in which it can sustain strictly more. Before doing that, we turn to the welfare comparison of the two regimes.

Welfare comparison. Since the set of allocations that constitute a monetary equilibrium is never smaller than the set of allocations that constitute a non-monetary equilibrium and the welfare function is the same in the two cases, clearly, the maximized welfare in the monetary equilibrium cannot be smaller than in the non-monetary equilibrium, $W_T^{M*}(\mu,\beta)\geq W_T^{*}(\mu,\beta)$ for any μ, β, and T. Interestingly, in our framework there is a one-to-one correspondence between set inclusion and welfare dominance. Thus, the question of the essentiality of money can be stated indifferently in terms of welfare or allocations.

PROPOSITION 2. *For any* μ, $W_T^{*}(\mu,\beta)<W_T^{M*}(\mu,\beta)$, *iff* $\Gamma_T(\beta)\subset\Gamma_T^M(\beta)$ *for some* β *and* T.

The reader may now wonder whether the strict inclusion holds generally. The answer is negative. Indeed, one can find examples of economies in which it is never true for any β. Below, we provide first a necessary, then a sufficient condition for the strict inclusion to hold.

Strict inclusion. The next proposition shows that $T<\infty$ is necessary to have the strict inclusion.

PROPOSITION 3. $\Gamma_{\infty}(\beta) = \Gamma_{\infty}^{M}(\beta)$ *for any* β.

Since there is a one-to-one correspondence between allocations and welfare, it follows that also ex ante welfare is the same, $W_{\infty}^{*}(\mu, \beta) = W_{\infty}^{M*}(\mu, \beta)$, for any μ and β. Hence, for money to be strictly essential, it necessarily has to be that the record of past deviations is wiped out in finite time and, thus, the ability of the clearing house to credibly enforce the threat of punishing shortfalls in the contributions to the pool is less than perfect.[13] This is only a necessary condition, though, and the reader may still wonder whether the strict inclusion ever really happens. The next proposition provides a sufficient condition for the existence of an interval of values of β such that the inclusion is, indeed, strict, in economies with a finite T.

PROPOSITION 4. *If* $T < \overline{T}$,[14] *there exists an interval* $B_T \subseteq (0, 1)$ *with non-empty interior, such that* $\Gamma_T(\beta) \subset \Gamma_T^{M}(\beta)$ *if* $\beta \in B_T$.

Hence, if the records of past deviations are wiped out sufficiently frequently, and, thus, the enforcement power of the clearing house is sufficiently limited, the money-based trading regime achieves a strictly larger set of allocations. The strict inclusion occurs when the allocation that constitutes a monetary equilibrium without taxation cannot be reached under the cashless trading system because the punishment is not sufficiently effective and, thus, the set of allocations that can be sustained without cash is very limited, while the cash-based economy is not subject to a participation constraint in the absence of taxation. The possibility of re-starting the clearing house in finite time following a complete wipe-out of the records constitutes precisely a limit to the effectiveness of punishment. Since there is a one-to-one correspondence between allocations and welfare, we immediately have that, if $T < \overline{T}$, for $\beta \in B_T$, $W_T^{*}(\mu, \beta) < W_T^{M*}(\mu, \beta)$, for any μ. This completes our argument. The monetary trading system, even without discriminatory taxation, allows to achieve always at least the same allocations and the same welfare as the non-monetary one, and in some cases it allows to achieve a strictly larger set of allocations which are also strictly better from the point of view of the agents' ex ante welfare. The monetary system is more flexible than the non-monetary one, since only the allocations corresponding to deflationary price sequences, obtained through taxation, need to be induced via the threat of collective punishment, while all other allocations can be selected optimally by the agents themselves. When enforcement is sufficiently imperfect, such higher flexibility emerges fully and determines the strict superiority of the cash-based system.

Notice that, instead of imposing an upper bound on T, the strict inclusion result could be obtained imposing a restriction on the concavity of the utility function, making consumption smoothing less important and, thus, the threat of future punishment less effective. An example of a class of utility functions that would work is $u(x) = \frac{x^{1-\alpha}}{1-\alpha}$, with $\alpha \in \left(0, \frac{\ln(T+2) - \ln(T+1)}{\ln 2}\right)$, for given T. For instance, with $\alpha = 0.1$, the upper bound would be $\overline{T} = 12$, and the allocation $\widetilde{x} = (1 + \beta^{10})^{-1}$ could be achieved by a monetary but not cashless clearing house for $T < 12$ and any $\beta \in (0, 1)$.

Notice, also, that the model can easily be extended to capture some form of absence of double coincidence of wants, as follows. Suppose the economy has $L \geq 3$ goods and L types of agents, indexed by $l = 1, \ldots, L$, with each type l interested in consuming good $l + 1$ and having an endowment of one unit of good l (modulo L). Suppose also there are $2N$ agents per type and N of them receive the endowment in odd periods, while the others in even periods. The rest remains the same. All the results derived in Sections 4, 5, and 6 apply to such a model literally, provided we limit attention to symmetric situations in which the different types are treated equally.

7. DISCRIMINATORY TRANSFERS

Consider the case in which a positive transfer is given to an agent who delivers something, while nothing is given to agents who do not deliver anything. The term η_t now appears on the RHS of the budget constraint (13):

$$x_t^i + v_t m_t^i = e_t^i + v_t m_{t-1}^i + \eta_t + \tau_t, \ \forall t \geq 1.$$

The rest remains the same. Specifically, we consider the case in which $\eta_t = -\tau_t$ when $e_t^i > 0$, $\eta_t = 0$ when $e_t^i = 0$, and $\tau_t < 0$. Through this scheme, the monetary clearing house effectively sterilizes the taxation for one of the two types of agents every period, thus, implementing an asymmetric (or discriminatory) transfer scheme across types of agents. The scheme works despite the presence of anonymity, since the transfers η_t are conditional on a delivery being made in the current period, and only agents with an endowment in the current period can deliver anything to the clearing house. This would limit the applicability of such a scheme in case of a more general endowment process, but other types of discriminatory transfers could still be devised to tailor the transfer, at least in part, to the deliveries.[15] An argument mirroring exactly the proof of Proposition 2 in Townsend (1980) establishes that, fixing τ_t appropriately, any allocation $z(\mu)$ on the first best frontier—identified in Lemma 1—can be achieved, provided the agents are exogenously assumed to abide by the transfer scheme. However, since in our economy, agents cannot commit to stick to the proposed transfer scheme, we need to check which first best allocations satisfy the participation constraints. An agent who is in a position to make a delivery in the current period might decide not to pay taxes, in which case, given the way the clearing house works, it is a dominant strategy not to deliver anything. This gives rise to the following participation constraint:

$$h_T(\beta)\, u(z) \geq g_T(\beta)\, u(1) + f_T(\beta)\, u(0), \tag{22}$$

where $h_T(\beta) \equiv \sum_{j=0}^{T} \beta^j$ and $f_T(\beta)$, $g_T(\beta)$ are defined as before. On the other hand, an agent who does not receive an endowment in the current period will participate in the taxation scheme if

$$h_T(\beta)\, u(1-z) \geq f_T(\beta)\, u(1) + g_T(\beta)\, u(0). \tag{23}$$

As noted before, very little can be sustained when the threat of the punishment is most ineffective, that is, when $T = 0$. In such a case, the only allocation that is able to ensure participation by both types of agents is the one that gives always all the good to type 1, that is, $z = 1$. Consider the case $T > 0$. Lemma A8 in the Supplementary Material establishes that the set of allocations that satisfies equations (22) and (23) simultaneously is non-empty. Hence, using discriminatory transfers one can Pareto improve upon the monetary equilibrium considered in the text. This strengthens our argument.

8. CONCLUSION

We have proposed a model of multilateral trade in which a clearing house cannot identify agents, who are anonymous, and may have limited enforcement power. Money improves the functioning of the clearing house, simultaneously keeping the incentives to contribute and guaranteeing the fine-tuning of allocations to the agents desiderata. Since multilateral trade works always at least as well and sometimes better with money than without, we conclude that the separation of trade meetings, unlike informational and enforcement limitations, is not necessary to obtain a socially beneficial role of money. Notice, finally, that the argument of this paper can be extended to an economy with several consumption goods. Imagine a static economy—similar to the one described in Ostroy and Starr (1974)—with several agents and several goods where the clearing house is still subject to anonymity. The clearing house organizes trade in two rounds: first, it collects goods; then, it delivers them. It has the power to shut down the system entirely between the two stages to induce people to supply the goods in the first place. As long as there is some overlap in the consumption needs of the agents, the clearing system will in general be unable to implement the first best allocation, being subject, at the delivery stage, to a set of incentive constraints arising from the agents' anonymity. As before, the clearing house would have trouble distinguishing agents and, hence, providing them with the appropriate deliveries. Money would help implement a better allocation, constituting a message system between agent and clearing house, that allows to relax the incentive constraints.

SUPPLEMENTARY MATERIAL

To view supplementary material for this article, please visit http://dx.doi.org/10.1017/S1365100518000822.

NOTES

1. Recent papers in this literature include Awaya and Fukai (2017), Nosal et al. (2011), Li and Rocheteau (2011) and Sun (2011).
2. All deliveries are collected before withdrawals to avoid temporary rationing.

3. In our Walrasian setting, the wipe-out of the records is interpreted as a technical feature of the clearing house. In a repeated game setting, it would correspond to a restriction on the type of strategies the agents can use.

4. The assumption that the records are wiped out after an even number of periods seems appropriate in this environment where agents go through period-two cycles. It leads to a tighter participation constraint than the alternative possibility with an odd number of periods. The participation constraint for an odd number of periods coincides always with the one with $T = \infty$.

5. After the facility re-starts the payoff reverts to the same value, hence, it drops out of the inequality.

6. For any $w \in \mathbb{R}_+$, $\lfloor w \rfloor \in \mathbb{N}$ denotes the largest natural number not greater than w.

7. This is formally proved in the Supplementary Material, in Lemma A4.

8. As before, to avoid the possibility of any temporary mismatch of deliveries and withdrawals, the former precede the latter.

9. In our framework, money, which is never replaced in equilibrium, is durable, unlike, for instance, in the cash-in-advance two-period model of Magill and Quinzii (1996, Ch. 7), where a centralized Walrasian facility forces agents to convert their endowments in cash at the beginning of every period and replaces the old with new currency at the end of every period.

10. A clearing house that does not replace money after a deviation would face a harder task than the one considered here, but would not necessarily be unable to guarantee the essentiality of money. Since, for the purpose of showing that money can be essential with centralized clearing, it is enough to identify one feasible monetary clearing house that does the job, we limit attention to the one in the text.

11. We exclude $\pi < \beta - 1$, since it would be at odds with the existence of a monetary equilibrium.

12. Henceforth, we restrict attention to the comparison of the equilibria with trade. There are always no-trade equilibria which we ignore.

13. With T odd, $\Gamma_T(\beta) = \Gamma_T^M(\beta)$ for any β and T.

14. $\overline{T} \equiv \left\lfloor \frac{2\left[u(1)-u\left(\frac{1}{2}\right)\right]}{2u\left(\frac{1}{2}\right)-u(1)-u(0)} \right\rfloor \in \mathbb{N}$, as in Section 5.

15. These transfers are reminiscent of the taxation scheme considered by Andolfatto (2010) and Wallace (2014).

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