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Transitivity of surface dynamics lifted to Abelian covers

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Abstract. A homeomorphism f of a manifold M is called H_1-transitive if there is a transitive lift of an iterate of f to the universal Abelian cover $\tilde{M}$. Roughly speaking, this means that f has orbits which repeatedly and densely explore all elements of $H_1(M)$. For a rel pseudo-Anosov map ϕ of a compact surface M we show that the following are equivalent: (a) ϕ is H_1-transitive, (b) the action of ϕ on $H_1(M)$ has spectral radius one and (c) the lifts of the invariant foliations of ϕ to $\tilde{M}$ have dense leaves. The proof relies on a characterization of transitivity for twisted $\mathbb{Z}^d$-extensions of a transitive subshift of finite type.

1. *Introduction*

There are many ways to characterize the complexity of a dynamical system on a manifold M. In this paper we focus on the characterization of H_1*-transitivity*. A homeomorphism f is called H_1-transitive when, roughly speaking, it has orbits which repeatedly and densely explore all of the elements of first homology. As is natural and commonly done, we formalize this notion by 'unwrapping' the manifold by passing to a covering space. For H_1-transitivity the appropriate lift is to the universal Abelian cover $\tilde{M}$ which is the covering space whose automorphism group is equal to $H_1(M; \mathbb{Z})$. Translations of orbits lifted to $\tilde{M}$ correspond to motion around homologically non-trivial loops in M. Thus, we adopt the following definition.

Definition 1.1. A homeomorphism $f : M \to M$ is called H_1-transitive if there is a lift $\tilde{g}$ of an iterate of f to the universal Abelian cover $\tilde{M}$ such that $\tilde{g}$ has an orbit which is dense in $\tilde{M}$.

Our main concern here is with a particular class of maps, rel pseudo-Anosov homeomorphisms of surfaces. These maps are an essential piece of Thurston's classification of isotopy classes of surface homeomorphisms and have many nice dynamical properties including a symbolic description by a transitive subshift of finite type. Rel pseudo-Anosov maps are characterized by the existence of a transverse pair of (mildly) singular, invariant foliations, each equipped with a transverse measure which expands or contracts under the map. Every non-trivial leaf of these foliations is dense in the surface M^2. For this class of maps we have the following equivalence between (a) the dynamical property of H_1-transitivity, (b) an algebraic condition on the spectral radius $\rho(\phi_*)$ of the induced action of the map ϕ on $H_1(M^2)$ and (c) a topological condition on the invariant foliations when lifted to the universal Abelian cover.

THEOREM 1.2. *Assume that $\phi : M^2 \to M^2$ is a rel pseudo-Anosov map. The following are equivalent:*
(a) *ϕ is H_1-transitive;*
(b) $\rho(\phi_*) = 1$*;*
(c) *there is a leaf of the lifted foliation $\tilde{\mathcal{F}}^u$ which is dense in the universal Abelian cover $\tilde{M}$.*

The proof of this theorem depends on Theorem 11.1 which gives a number of conditions which are equivalent to the total transitivity (all iterates are transitive) of a lifted rel pseudo-Anosov map $\tilde{\phi}$. These conditions include that $\tilde{\phi}$ is topologically mixing, that the periodic orbits of $\tilde{\phi}$ are dense in $\tilde{M}$ and that $\rho(\phi_*) = 1$ coupled with a condition on the rotation set of the Fried quotient of $(\tilde{\phi}, M^2)$. A version of Theorem 11.1 for the annulus was given in [**BGH93**] and for the torus in [**Par03**].

The proof of Theorem 11.1 in turn depends on Theorem 10.4 which characterizes total transitivity of a twisted skew product with group factor $\mathbb{Z}^d$ over a base subshift of finite type (Σ, σ). The twisted skew products considered here are maps $\tau : \Sigma \times \mathbb{Z}^d \to \Sigma \times \mathbb{Z}^d$ of the form

$$\tau(s, \mathbf{n}) = (\sigma(s), \Phi(\mathbf{n}) + h(s)),$$

where $\Phi : \mathbb{Z}^d \to \mathbb{Z}^d$ is the *twisting isomorphism* or just the *twisting*, and $h : \Sigma \to \mathbb{Z}^d$ is the *height function*. When the twisting is trivial ($\Phi = \mathrm{id}$) the map τ is called an *untwisted skew product* or just a *skew product*. The ergodic theory and topological dynamics of untwisted skew products with various bases and group components have been intensely studied for at least 50 years (see [**PP06**] for some history), and their use as symbolic models for dynamics lifted to covering spaces is well established. The transitive untwisted skew products over subshifts of finite type with group factor $\mathbb{Z}^d$ were characterized by Coudene [**Cou04**] and those with group factor $\mathbb{R}^d$ by Nitica [**Nit00**]. Twisted skew products are themselves a special case of the well-studied notion of a group extension (see, for example, [**Men05**]).

When a twisted skew product models the lift of a rel pseudo-Anosov map ϕ to the universal Abelian cover, the twisting isomorphism is the action of ϕ on first homology, or $\Phi = \phi_*$. Thus, to study maps which do not act trivially on homology we must consider the case of non-trivial twisting. The first observation in this study is that the coarse connection between the dynamics of a lift $\tilde{\phi}$ to $\tilde{M}$ and the action of ϕ_* on $H_1(M; \mathbb{R})$ implies that when $\rho(\phi_*) > 1$, there will be open sets in $\tilde{M}$ whose iterates under $\tilde{\phi}$ will go to infinity exponentially fast (see (10.5) below). Thus, $\rho(\phi_*) = 1$ is a necessary condition for transitivity (the case $\rho(\phi_*) < 1$ cannot occur because ϕ is a homeomorphism).

The next obvious necessary condition for the transitivity of $\tilde{\phi}$ can be informally expressed by 'orbits of $\tilde{\phi}$ must go to infinity in all directions'. Gottschalk and Hedlund were the first to notice that this condition can also be sufficient for transitivity [**GH55**].

Perhaps the most common way to formalize the 'all directions' condition for an untwisted skew product is to use the collection of displacements, $D(\tau)$, of periodic orbits of the base map (see §5.2). Here we primarily use the asymptotic average displacements as they are more tractable under the iterations and translations of maps required here. We maintain the usual terminology from lifted dynamics and call this average displacement the *rotation vector* of an orbit. The set of all rotation vectors is the *rotation set*, and we formulate the 'all directions' condition in Theorem 7.1 by requiring that zero be in the interior of the rotation set. This condition alone does not imply transitivity in the untwisted case, but requires the addition of a condition, the *finite lifting property*, which is the analog of the fact that the lift of a pseudo-Anosov map to any compact covering space is transitive.

The definition of the displacement set or rotation vector requires that each point in the base be assigned a well-defined displacement in the group factor or cover. This is only possible when the skew product is untwisted or $\phi_* = \mathrm{id}$ (see Remark 5.1). Thus, the weaker hypothesis of $\rho(\phi_*) = 1$ requires additional consideration. The first step is to note that a classic theorem of Kronecker implies that for some $N > 0$, $\mathrm{spec}(\phi_*^N) = \{1\}$, where spec indicates the spectrum (see the comment above Definition 10.3).

We call an isomorphism Ψ with $\mathrm{spec}(\Psi) = \{1\}$ a *generalized shear*, because over the reals one can find a basis in which Ψ is represented by its Jordan matrix of ones on the diagonal and perhaps also some ones on the super diagonal. However, in general one cannot conjugate to this Jordan form in $\mathrm{SL}(d, \mathbb{Z})$ and so we use instead the form given in Lemma 9.3. Using the basis of $\mathbb{Z}^d$ given by Lemma 9.3, we can treat a τ with generalized shear twisting as a sequence of untwisted skew products over countable state Markov shifts and so prove transitivity by induction. The $n = 0$ step of this argument requires the transitivity of the largest quotient on which τ is untwisted. The transitivity of this Fried quotient is obtained using Theorem 7.1 and the main induction step is handled by Lemma 10.1.

In the last section of the paper, after proving Theorem 1.2 we comment in §11.2 on some dynamical properties of the $\phi_* = \mathrm{id}$ case and in Proposition 11.6 on some topological properties of the lifted foliations.

2. *Topological transitivity and countable state Markov shifts*

In this paper M is always a compact, orientable surface perhaps with boundary. All homeomorphisms $h : M \to M$ are orientation preserving. If there is no coefficient ring given, homology is always with integer coefficients, and so $H_1(M) = H_1(M, \mathbb{Z}) \cong \mathbb{Z}^d$ for some $d \in \mathbb{N}$.

For a topological space X, the closure, interior and frontier are denoted by $\mathrm{Cl}(X)$, $\mathrm{Int}(X)$ and $\mathrm{Fr}(X)$, respectively. For any map f, its image is denoted $\mathrm{im}(f)$. For a homeomorphism h of X, the *forward orbit* of a point $x \in X$ is $o_+(x, h) := \{x, h(x), h^2(x), \ldots\}$, the *backward orbit* is $o_-(x, h) := \{\ldots, h^{-2}(x), h^{-1}(x), x\}$, and the *orbit* is $o(x, h) = o_+(x, h) \cup o_-(x, h)$.

2.1. *Transitivity and mixing.*

A homeomorphism h is called *topologically transitive* or just *transitive* if for every pair of open sets U_1 and U_2 there is an integer $n \in \mathbb{Z}$ with

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$h^n(U_1) \cap U_2 \neq \emptyset$, and h is called *topologically mixing* if for every pair of open sets U_1 and U_2 there is an integer $N \in \mathbb{Z}$ with $h^n(U_1) \cap U_2 \neq \emptyset$ for all $n \geq N$. If h^n is transitive for all $n > 0$, then h is called *totally transitive*.

It is standard that if h^n is transitive or topologically mixing for some $n > 0$, then h has the same property. Further, topologically mixing implies totally transitive and in many cases is equivalent to it. A standard result on transitivity is as follows.

LEMMA 2.1. *Assume that h is a homeomorphism of a complete, separable metric space X. The following are equivalent:*

(a) *h is transitive;*
(b) *there exists an orbit of h that is dense in X,* $\mathrm{Cl}(o(x, h)) = X$*;*
(c) *there is a dense, G_δ-subset $Y \subset X$ so that $y \in Y$ implies that* $\mathrm{Cl}(o_+(y, h)) = X$ *and* $\mathrm{Cl}(o_-(y, h)) = X$.

2.2. *Countable state Markov shifts.* We recall the basic definitions and properties of countable state Markov shifts. For more details see [**Kit98**]. A countable state Markov shift is built from a countable set of *states* $S = \{1, 2, 3, \ldots\}$. The *transition matrix* C is indexed by $S \times S$, and the entries of C are all zeros or ones and are denoted by $C_{i,j}$. An *allowable one-step transition* for C is a pair of states s_1 and s_2 so that $C_{s_1,s_2} = 1$. An *allowable n-step transition* or *allowable $n+1$-block* is a list of $n+1$-states $s_1 s_2 \ldots s_{n+1}$ with each pair $s_i s_{i+1}$ an allowable one-step transition. In general, if there is an allowable transition of any length between two states a and b, we say that there is an allowable transition between a and b, and this situation is denoted by $a \to b$.

The collection of bi-infinite sequences of states is $S^{\mathbb{Z}}$ and the *shift space*, Σ, defined by the matrix C is the subspace of sequences all of whose finite blocks represent transitions which are allowed by the matrix C or, equivalently,

$$\Sigma = \{s \in S^{\mathbb{Z}} \mid C_{s_i,s_{i+1}} = 1 \text{ for all } i \in \mathbb{Z}\}.$$

The *shift map* on Σ is the left shift $\sigma : \Sigma \to \Sigma$. The shift space and the shift together constitute the countable state Markov shift which is denoted by (Σ, σ). When S is a finite set, (Σ, σ) is called a *subshift of finite type* or a *topological Markov chain*.

A metric on S which gives it the discrete topology yields a product metric on $S^{\mathbb{Z}}$ and a subspace metric induced on Σ. Under the resulting topology Σ is a totally disconnected, separable, complete metric space and the shift map is a homeomorphism. If S is finite, Σ is compact, and otherwise it is not. All of the transition matrices here will have finite row and sum columns which implies that Σ is locally compact.

The cylinder sets form a countable base for the topology of Σ. A *central cylinder set of length two* is

$$[a, b]_0 := \{s \in \Sigma \mid s_0 = a \text{ and } s_1 = b\} \tag{2.1}$$

where $a \to b$ is an allowable one step transition for (Σ, σ).

The standard characterization of transitive subshifts of finite types also holds for countable state Markov shifts: the system (Σ, σ) is topologically transitive if and only if it is irreducible, i.e. for any pair of states a_1 and a_2 there is an allowable transition $a_1 \to a_2$. Also, if (Σ, σ) is transitive and has a fixed point, then it is topologically mixing (see [**Kit98**, Observation 7.2.2]). If (Σ, σ) is totally transitive, then as just noted, it certainly has a periodic point, say of period n, and so σ^n is topologically mixing, and so σ is too. Thus, for countable state Markov shifts, totally transitive is equivalent to topologically mixing.

3. *Abelian covering spaces and the rotation set*

Recall that regular, connected covering spaces of the surface M are in one-to-one correspondence with normal subgroups $G \lhd \pi_1(M)$. For the cover corresponding to G, the *deck group* (also called the group of cover automorphisms) is naturally identified with the quotient $\pi_1(M)/G$. We are exclusively concerned with *Abelian covers*. These are the covering spaces for which the deck group is Abelian. The largest such cover corresponds to $G = [\pi_1(M), \pi_1(M)]$, the commutator subgroup, in which case the deck group is $H_1(M)$. This covering space is called the *universal Abelian cover* and is denoted here by $\tilde{M}$. Any other Abelian cover can be obtained by moding out $\tilde{M}$ by the action of a subgroup $\Gamma \subset H_1(M)$. This quotient is denoted by $\tilde{M}_\Gamma := \tilde{M}/\Gamma$. Note that $\tilde{M}$ is a cover over $\tilde{M}_\Gamma$ with deck group Γ and $\tilde{M}_\Gamma$ is a cover over M with deck group $H_1(M)/\Gamma$.

Any homeomorphism $f : M \to M$ lifts to the universal Abelian cover. For smaller Abelian covers, f lifts to $\tilde{M}_\Gamma$ if and only if $f_*(\Gamma) = \Gamma$, where $f_* : H_1(M) \to H_1(M)$ is the induced action. For a cover $\tilde{M}_\Gamma$ to which f does lift, the induced action of f_* on the deck group $H_1(M)/\Gamma$ is denoted by f_Γ. If $\tilde{f}$ is a lift of f to $\tilde{M}_\Gamma$, a fundamental relation is

$$\tilde{f} \circ \delta_g = \delta_{f_\Gamma(g)} \circ \tilde{f}, \tag{3.1}$$

where δ_g is the deck transformation corresponding to $g \in H_1(M)/\Gamma$. Thus, $\tilde{f}$ commutes with all deck transformations precisely when f acts trivially on $H_1(M)/\Gamma$.

3.1. *Subgroups of finitely generated Abelian groups.*

If $\Gamma \subset \mathbb{Z}^d$ is a rank k subgroup, then there is a basis $\{u_1, \ldots, u_d\}$ of $\mathbb{Z}^d$ and positive integers $a_1, \ldots, a_k$, so that

$$\{a_1 u_1, \ldots, a_k u_k\} \tag{3.2}$$

is a basis for Γ. This is usually given as a simple consequence of the Smith normal form (for example, [**New72**]). This fundamental fact implies that Γ is *co-finite*, i.e. the quotient group $\mathbb{Z}^d/\Gamma$ is finite, if and only if Γ has rank d. In this case it follows that if K is the order of $\mathbb{Z}^d/\Gamma$, then $K\mathbb{Z}^d \subset \Gamma$ and further, if we form a matrix M using a basis for Γ as the columns, then the order of $\mathbb{Z}^d/\Gamma$ is $|\det(M)|$.

For a subset X of an Abelian group G, let $\langle X \rangle$ be the subgroup generated by X, and the positive semigroup generated by X is

$$\langle X \rangle_+ = \{nx \mid n \in \mathbb{N}, n > 0, \text{ and } x \in X\}.$$

It is easy to see that if G is a finite Abelian group and X is a subset, then $\langle X \rangle_+ = \langle X \rangle$.

The subgroup $\Gamma \subset \mathbb{Z}^d$ is called *pure* if whenever $g \in \Gamma$ is divisible in $\mathbb{Z}^d$, it is divisible in Γ, i.e. if $g \in \mathbb{Z}^d$ and $mg \in \Gamma$ for some $m \neq 0$, then $g \in \Gamma$. Thus Γ is pure if and only if the integers a_j in (3.2) are all equal to one, and Γ is pure if and only if the quotient $\mathbb{Z}^d/\Gamma$ is torsion-free or trivial. Also, the subgroup $\langle g \rangle$ generated by a single element g of $\mathbb{Z}^d$ is pure if and only if g is indivisible in $\mathbb{Z}^d$. In addition, if Γ is pure in $\mathbb{Z}^d$, then it is always a summand, i.e. there is a subgroup $H \subset \mathbb{Z}^d$ with $\mathbb{Z}^d = \Gamma \oplus H$ (this and all direct sums in this paper are internal). Equivalently, if Γ is pure, then any basis of Γ can be extended to a basis of $\mathbb{Z}^d$. Finally, since any subgroup H of $\mathbb{Z}^d$ is free and thus isomorphic to some $\mathbb{Z}^k$, if $\Gamma \subset H$ and Γ is pure in H ($g \in H$ with $mg \in \Gamma$ with $m \neq 0$ implies $g \in \Gamma$), then any basis of Γ can be extended to a basis of H. For a subset X of an Abelian group G, let $P(X)$ denote the smallest pure subgroup of G which contains X. The group $P(X)$ is commonly called the *purification* of X.

3.2. *The Fried cover.* Fried pointed out in [**Fri83, Fri86**] that for many dynamical applications it is best to work with covering spaces on which all lifts of f commute with all deck transformations. Such a cover is necessarily Abelian [**Fri83**, Lemma 1]. From (3.1) it follows that for such a cover f must act like the identity on the deck group and that the largest such cover corresponds to the subgroup $F' := \mathrm{im}(f_* - \mathrm{id}) \subset H_1(M)$.

The deck group of this cover, the quotient $H_1(M)/F'$, will frequently have torsion. For most of our applications we work only with the torsion-free part. Letting F be the purification of F', $F := P(F')$, we see that the largest cover with a free deck group on which all lifts commute with the deck has deck group $H_1(M)/F$. It is easy to check that $f_*(F) = F$ and so f always lifts to this cover. This leads to the following.

Definition 3.1. Given a compact surface M and homeomorphism $f : M \to M$, the Fried cover, $\tilde{M}_F$, of (f, M) is the covering space corresponding to the subgroup of $H_1(M)$ given by $F = P(\mathrm{im}(f_* - \mathrm{id}))$.

Remark 3.2. Note that $\mathbb{Z}^d / \mathrm{im}(f_* - \mathrm{id})$ is finite if and only if $\det(f_* - \mathrm{id}) \neq 0$ using the first paragraph of §3.1. This determinant is non-zero if and only if 1 is not an eigenvalue of f_*. Thus, the Fried cover deck group $\mathbb{Z}^d/P(\mathrm{im}(f_* - \mathrm{id}))$ is non-trivial if and only if $1 \in \mathrm{spec}(f_*)$. When $\mathbb{Z}^d/P(\mathrm{im}(f_* - \mathrm{id}))$ is trivial, by convention, the Fried cover is itself M.

3.3. *Rotation sets.* In this section we recall the generalized rotation vector of orbits of a homeomorphism f. This notion has its origins in Schwartzman's asymptotic cycles and is now a common tool (see, for example, [**Boy94**, §11] and the introduction of [**Jen01**]). The definition requires a covering space which has torsion-free deck group and on which all lifts of f commute with all deck transformations. Thus, it is natural to use the largest such cover, namely the Fried cover given in Definition 3.1. Assume that 1 is an eigenvalue of f_* and so the deck group, $H_1(M)/F$, of the Fried Cover $\tilde{M}$ is non-trivial, i.e. $H_1(M)/F = \mathbb{Z}^d$ for $d > 0$.

The definition of the rotation vector requires a means of measuring displacements in the Fried cover which is compatible with the deck action. If $H_1(M) = \mathbb{Z}^k$, a standard construction yields a continuous map on the universal Abelian cover $\tilde{\beta}' : \tilde{M} \to \mathbb{R}^k$ with $\tilde{\beta}' \circ \delta_{\mathbf{n}}(\tilde{x}) = \tilde{\beta}'(\tilde{x}) + \mathbf{n}$ for all $\mathbf{n} \in \mathbb{Z}^k$. We give one way to perform the construction. Pick a set of generators and a basis $\mathbf{n}_1, \ldots, \mathbf{n}_k$ for $H_1(M)$ and a corresponding co-basis $c_1, \ldots, c_k$ for $H^1(M; \mathbb{Z})$, i.e. $c_i(\mathbf{n}_j) = \delta_{ij}$ (Kronecker delta). Now treat c_i as an element

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of $H^1_{DR}(M, \mathbb{R})$ and assume that it is represented by the closed one-form ω_i. Lift ω_i to $\tilde{\omega}_i$ on $\tilde{M}$, and fix a basepoint $\tilde{z}_0 \in \tilde{M}$. The ith coordinate of $\tilde{\beta}'$ is $\tilde{\beta}'_i(\tilde{z}) = \int_{\tilde{\gamma}} \tilde{\omega}_i$, where $\tilde{\gamma}$ is any smooth arc in $\tilde{M}$ from $\tilde{z}_0$ to $\tilde{z}$. We then project $\tilde{\beta}'$ to obtain an equivariant map $\tilde{\beta} : \tilde{M}_F \to \mathbb{R}^d$.

Now for a given lift $\tilde{f}_F$ of f to the Fried cover $\tilde{M}_F$ and an $x \in M$, pick a lift $\tilde{x} \in \tilde{M}_F$ of x and $n \in \mathbb{Z}$ and let

$$B(x, n) = \tilde{\beta}(\tilde{f}_F^n(\tilde{x})) - \tilde{\beta}(\tilde{x}). \tag{3.3}$$

Since $\tilde{f}_F$ commutes with the deck group of $\tilde{M}_F$ and $\tilde{\beta}$ is equivariant, this definition is independent of the choice of $\tilde{x}$, but it does depend on the choice of $\tilde{f}$. It is immediate that B is an additive dynamic cocycle over (M, f). We let $\mathrm{rot}(x, \tilde{f}_F)$ be the element of $\mathbb{R}^d$,

$$\mathrm{rot}(x, \tilde{f}_F) := \lim_{n\to\infty} \frac{B(x, n)}{n}$$

when the limit exists, and define the rotation set in $\mathbb{R}^d$ as

$$\mathrm{rot}(\tilde{f}_F) = \{\mathrm{rot}(x, \tilde{f}_F) \mid x \in M\}.$$

It is immediate that

$$\mathrm{rot}(\delta_{\mathbf{n}} \circ \tilde{f}_F^q) = q\ \mathrm{rot}(\tilde{f}_F) + \mathbf{n}, \tag{3.4}$$

for all $q \in \mathbb{Z}$ and $\mathbf{n} \in \mathbb{Z}^d$. Since $B(x, 1)$ is continuous on M, it is bounded and thus is in L^1 of any f-invariant probability measure. This implies by the point-wise ergodic theorem that the rotation vector exists almost everywhere with respect to such measures.

Given a homeomorphism f of M and a lift $\tilde{f}$ to the universal Abelian cover $\tilde{M}$, there is a unique lift $\tilde{f}_F$ of f to the Fried cover which is the projection of $\tilde{f}$. Define

$$\mathrm{rot}_F(\tilde{f}) = \mathrm{rot}(\tilde{f}_F). \tag{3.5}$$

By convention if the Fried cover is trivial we let $\mathrm{rot}_F(\tilde{f})$ be the empty set.

4. *Twisted skew products*

Twisted skew products over subshifts of finite type provide a symbolic model for the lifts of rel pseudo-Anosov maps to Abelian covers. For the inductive arguments based on Lemma 10.1 we also need to consider a countable state Markov chain as the base shift.

Definition 4.1. A twisted skew product is constructed from:

(a) a countable state Markov shift (Σ, σ) called the *base shift*;

(b) a finitely generated Abelian group G called the *group component*;

(c) a function $h : \Sigma \to G$ called the *height function* which is required to be constant on central cylinder sets of length two, and so $h(s) = h(s_0, s_1)$;

(d) and an isomorphism $\Psi : G \to G$ called the *twisting automorphism*.

The twisted skew product built from these ingredients is the map $\tau : \Sigma \times G \to \Sigma \times G$ given by

$$\tau(s, g) = (\sigma(s), \Psi(g) + h(s)). \tag{4.1}$$

In the special case when the twisting isomorphism $\Psi = \mathrm{id}$, the map τ is called a skew product or, for emphasis, an untwisted skew product. The group component most commonly considered here is $G = \mathbb{Z}^d$ and in this case the twisting automorphism Ψ is given by a twisting matrix $A \in \mathrm{SL}(d, \mathbb{Z})$.

Remark 4.2. Condition (c) in the definition of a twisted skew product was adopted so that the skew could be easily identified with a countable Markov shift as in §4.2. In the somewhat more general case when h depends on length $(\ell + 1)$-blocks, one can pass to the ℓ-block presentation of (Σ, σ) (see [**Kit98**, p. 27]), and the corresponding height function will depend on length-two central blocks. Since G is a discrete group, when h is continuous on a subshift of finite type, there will always be an ℓ with h constant on central length ℓ cylinder sets.

Skew products as just defined are also called G-extensions of the countable state Markov shift (Σ, σ). The height function h is also commonly called the *cocycle* since in the untwisted case it generates an additive cocycle over (Σ, σ) as in §5.2 below. We sometimes call a skew product a twisted or untwisted extension of the base shift as is appropriate.

The iterate of a twisted skew product is itself a twisted skew product. Specifically, if τ is as in (4.1), then

$$\tau^k(s, g) = (\sigma^k(s), \Psi^k(g) + \hat{h}^{(k)}(s)), \tag{4.2}$$

where

$$\hat{h}^{(k)}(s) = \Psi^{k-1}(h(s)) + \Psi^{k-2}(h(\sigma(s))) + \cdots + \Psi(h(\sigma^{k-2}(s))) + h(\sigma^{k-1}(s)).$$

In particular, the twisting automorphism of τ^k is Ψ^k.

Note that G acts by addition on the second component of $\Sigma \times G$. For a $g \in G$, we write this action of g as

$$T_g(s, g') = (s, g + g'). \tag{4.3}$$

A simple calculation shows that

$$\tau \circ T_g = T_{\Psi(g)} \circ \tau \tag{4.4}$$

which is the obvious analog of (3.1). Thus, τ commutes with the action of G if and only if $\Psi = \text{id}$, i.e. when τ is an untwisted skew product.

4.1. *Quotients, the finite lifting property and the Fried quotient.* Many of the standard constructions for covering spaces with a deck group G such as quotients and subcovers have analogs for twisted skew products with group component G. Given a twisted skew product τ as in (4.1), if $\Gamma \subset G$ is a subgroup with $\Psi(\Gamma) = \Gamma$, then τ descends to a twisted skew product $\tau_\Gamma : \Sigma \times (G/\Gamma) \to \Sigma \times (G/\Gamma)$, defined by

$$\tau_\Gamma(s, g + \Gamma) = (\sigma(s), \Psi(g) + h(s) + \Gamma). \tag{4.5}$$

Further, the projection $\pi : G \to G/\Gamma$ induces a semiconjugacy, id $\times\pi$, from τ to τ_Γ. Thus, in particular, if τ is transitive, so is any quotient τ_Γ.

More generally, if $\Gamma_1 \subset \Gamma_2 \subset G$ are two Ψ-invariant subgroups, then it is easy to check that

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma_2/\Gamma_1 & \longrightarrow & G/\Gamma_1 & \longrightarrow & G/\Gamma_2 & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \Gamma_2/\Gamma_1 & \longrightarrow & G/\Gamma_1 & \longrightarrow & G/\Gamma_2 & \longrightarrow & 0
\end{array}$$

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commutes, where the vertical maps are the natural maps induced by Ψ. Thus, the induced skew product on $\Sigma \times (G/\Gamma_1)$ is semiconjugate to that on $\Sigma \times (G/\Gamma_2)$ and, furthermore, the Noether isomorphism $(G/\Gamma_1)/(\Gamma_2/\Gamma_1) \cong G/\Gamma_2$ induces a conjugacy of the twisted skew products on $\Sigma \times (G/\Gamma_1)/(\Gamma_2/\Gamma_1)$ and $\Sigma \times (G/\Gamma_2)$.

Rel pseudo-Anosov homeomorphisms have a very useful property which was pointed out and used by Fried [**Fri82a**]. These maps are transitive and their lift to any covering space with finite deck group is also transitive. We now define the analogous property for a twisted skew product. We restrict ourselves now to the case where the group component of the skew product is a free Abelian group $\mathbb{Z}^d$.

Definition 4.3. The skew product $\tau : \Sigma \times \mathbb{Z}^d \to \Sigma \times \mathbb{Z}^d$ with twisting automorphism, Ψ, is said to have the *finite transitivity property* (ftp) if for every Ψ-invariant, co-finite subgroup $\Gamma \subset \mathbb{Z}^d$ the quotient map $\tau_\Gamma : \Sigma \times (\mathbb{Z}^d/\Gamma) \to \Sigma \times (\mathbb{Z}^d/\Gamma)$ is transitive.

Note that if τ has the ftp then so does any quotient. This follows from the conjugacy induced by the Noether isomorphism given above. Note also that the base shift is itself a quotient of τ and so when τ has the ftp, its base shift is always transitive.

Using (4.4) and (3.1), untwisted skew products correspond to covering spaces on which all lifts commute with the deck group. As with covering spaces, it is often useful to pass to a quotient on which the projection is untwisted. The construction given here is the exact analog of that for covering spaces given in §3.2. Specifically, if τ is a twisted skew product with group component $\mathbb{Z}^d$ and twisting isomorphism Ψ, the largest untwisted quotient corresponds to the Ψ-invariant subgroup $F' = \mathrm{im}(\Psi - \mathrm{id})$, and the largest untwisted quotient with torsion-free group component corresponds to the purification $F = P(F')$.

Definition 4.4. If τ is a twisted skew product with group component $\mathbb{Z}^d$ and twisting isomorphism Ψ, the *Fried quotient* of τ is the quotient τ_F for $F = P(\mathrm{im}(\Psi - \mathrm{id}))$.

4.2. *Twisted skew products, countable state Markov shifts and lifted transitions.* A twisted skew product can be identified with, or more precisely is conjugate to, a countable state Markov shift in a natural way. Given τ as in (4.1), assume that the states of the base shift Σ are $S = \{1, 2, 3, \ldots\}$. Define the states of a new countable Markov shift as $\hat{S} = S \times \mathbb{Z}^d$, and the allowable one-step transitions for the new shift $\hat{S}$ are $(a, \mathbf{n}) \to (b, \mathbf{m})$, where $a \to b$ is allowable for Σ and $\mathbf{m} = \Psi(\mathbf{n}) + h(a, b)$. Now let $\hat{\Sigma} \subset \hat{S}^{\mathbb{Z}}$ be the collection of sequences from $\hat{S}^{\mathbb{Z}}$ all of whose transitions are allowable and $\hat{\sigma}$ be the left shift on $\hat{\Sigma}$. The conjugacy between $(\hat{\Sigma}, \hat{\sigma})$ and $(\Sigma \times \mathbb{Z}^d, \tau)$ is given by $(\ldots, (s_{-1}, \mathbf{n}_{-1}), (s_0, \mathbf{n}_0), (s_1, \mathbf{n}_1), \ldots) \mapsto (s, \mathbf{n}_0)$. We often identify a skew product and its corresponding countable state Markov shift with little further mention.

Given an allowable one-step transition $a \to b$ for (Σ, σ), the *lifted* or *induced* transition for τ starting at $g \in G$ is

$$(a, g) \to (b, \Psi(g) + h(a, b)). \tag{4.6}$$

More generally, if $a \to c$ is an allowable n-step transition, its lifted transition for τ is constructed by concatenating the lifts of each one-step transition.

After identifying a skew product with a countable state Markov shift the criterion for transitivity given in §2.2 can be used. Thus, τ is transitive if and only if for any states $(a, \mathbf{n})$

and $(b, \mathbf{m})$, there is an allowable transition $(a, \mathbf{n}) \to (b, \mathbf{m})$. Equivalently, τ is transitive if and only if for any $(a, \mathbf{n})$ and $(b, \mathbf{m})$, there is $s \in \Sigma$ and an $n > 0$ so that $s_0 = a$ and $s_n = b$ and $\tau^n(s, \mathbf{n}) = (\sigma^n(s), \mathbf{m})$.

5. *Untwisted skew products*

In this section we review some of the standard constructions associated with untwisted skew products.

5.1. *Lifting untwisted transitions and transitivity.*

For untwisted skew products lifted transitions transform nicely under the action of G from (4.3). In this case,

$$(a_1, g_1) \to (a_2, g_2) \quad \text{implies } (a_1, g_1 + g) \to (a_2, g_2 + g), \tag{5.1}$$

for all $g \in G$. This implies, in particular, that if $(s, 0)$ is a periodic point for τ, then for all $g \in G$, (s, g) is too. In addition,

$$(a_1, g_1) \to (a_2, g_2) \text{ and } (a_2, g_3) \to (a_3, g_4) \quad \text{imply } (a_1, g_1) \to (a_3, g_4 + g_2 - g_3). \tag{5.2}$$

If τ is untwisted, we thus have that τ is transitive if and only if for any $j, k \in S$ and $\mathbf{m} \in \mathbb{Z}^d$, there is an allowable transition $(j, 0) \to (k, \mathbf{m})$.

5.2. *The height cocycle and the displacement set.*

A main tool in the study of untwisted skew products is the cocycle giving the total height or displacement of an orbit in the group component. This object has many names and notation, including the Fröbenious element and the total displacement. In the covering space context it gives a coordinate for the Abelian Nielsen class of a periodic point (see [**Boy94**]) or the twisted Lefschetz coefficient (see [**Fri83**]).

Given an untwisted skew product τ with height function h, let

$$h(s, n) = h(s) + h(\sigma(s)) + \cdots + h(\sigma^{n-1}(s)),$$

and so for any $g \in G$,

$$\tau^n(s, g) = (\sigma^n(s), g + h(s, n)).$$

Note that we are 'overloading' the symbol h. Other common notation for $h(s, n)$ includes $h^n(s)$ and $h^{(n)}(s)$. The notation we choose emphasizes the valuable and frequently used fact that for an untwisted skew product the height function h induces an additive cocycle over the base shift (Σ, σ):

$$h(s, n + m) = h(s, n) + h(\sigma^n(s), m).$$

Remark 5.1. It is worth noting here an important difference between twisted and untwisted skew products. If η is a twisted skew product, then one can define an additive G-valued cocycle for η itself by $E((s, g), n) = \pi_2(\tau^n(s, g)) - g$ where π_2 is projection onto the group factor. However, this will only descend to a cocycle on the base shift Σ when the value of E is independent of the element $g \in G$ and by virtue of (4.4), this only happens when $\Psi = \text{id}$, i.e. when η is untwisted. This basic fact is the reason that twisted skew products present additional difficulties over untwisted skew products.

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For untwisted skew products a special role is played by the value of the cocycle on periodic points of the base shift. Given an untwisted skew product τ, define the *displacement set* $D(\tau)$ as

$$D(\tau) = \{h(p, n) \mid p \in \mathrm{Fix}(\sigma^n),\ n > 0\}. \tag{5.3}$$

For future reference we note that if $p \in \mathrm{Fix}(\sigma^n)$, then $h(\sigma^k(p), n) = h(p, n)$ for all k.

In the situation described in §4.1 where $\Gamma_1 \subset \Gamma_2$ are subgroups of $\mathbb{Z}^d$, then the natural map $\pi : \mathbb{Z}^d / \Gamma_1 \to \mathbb{Z}^d / \Gamma_2$ yields a simple relationship between the displacement sets of τ_{Γ_1} and τ_{Γ_2}. Specifically, if the height functions of τ_{Γ_1} and τ_{Γ_2} are h_1 and h_2, respectively, then for a periodic point $p \in \Sigma$, we have $h_2(p, n) = \pi(h_1(p, n))$. It then follows that

$$\begin{gathered}\langle D(\tau_{\Gamma_2})\rangle_+ = \pi(\langle D(\tau_{\Gamma_1})\rangle_+),\\ \langle D(\tau_{\Gamma_2})\rangle = \pi(\langle D(\tau_{\Gamma_1})\rangle) = \langle D(\tau_{\Gamma_1})\rangle / (\langle D(\tau_{\Gamma_1})\rangle \cap \Gamma_2).\end{gathered} \tag{5.4}$$

5.3. *The rotation set.* Given an additive cocycle it is natural to compute the asymptotic average values. In the case of the displacement cocycle this average is the analog of the rotation vector of a lifted homeomorphism and so for the sake of uniform terminology we use that name here.

We continue to restrict to the case of an untwisted skew product τ and in addition we require that the group factor be torsion-free, $\mathbb{Z}^d$. If the height cocycle for τ is $h(s, n)$, for $s \in \Sigma$ define its *rotation vector* as the vector in $\mathbb{R}^d$ given by

$$\mathrm{rot}(s) = \lim_{n\to\infty} \frac{h(s, n)}{n}, \tag{5.5}$$

if the limit exists. For any invariant probability measure μ on (Σ, σ) if the height function is integrable with respect to μ ($h \in L^1(\mu)$), then by the point-wise ergodic theorem the limit in (5.5) exists almost everywhere with respect to μ. The collection of all rotation vectors for τ is called the *rotation set* and is denoted by

$$\mathrm{rot}(\tau) = \{\mathrm{rot}(s) \mid s \in \Sigma\}.$$

Note that for a $p \in \mathrm{Fix}(\sigma^k)$, $\mathrm{rot}(p) = h(p, k)/k$, and for all $q > 0$ and $\mathbf{p} \in \mathbb{Z}^d$,

$$\mathrm{rot}(T_{\mathbf{p}} \circ \tau^q) = q\ \mathrm{rot}(\tau) + \mathbf{p}, \tag{5.6}$$

where $T_{\mathbf{p}}$ is the action of $\mathbf{p}$ given in (4.3).

As noted in Remark 5.1, the height function does not descend to a cocycle on the base shift when a skew product in non-trivially twisted. Thus, in the twisted case there is not generally a usable notion of rotation vector. However, valuable information on the twisted skew product can be obtained using the rotation set of the largest torsion-free quotient on which τ descends to an untwisted skew product. In analogy with Definition 3.5 we have the following.

Definition 5.2. If τ is a twisted skew product, let $\mathrm{rot}_F(\tau) = \mathrm{rot}(\tau_F)$, where τ_F is the Fried quotient of τ defined in §4.4. When $\mathbb{Z}^d/F$ is the trivial group we adopt the convention that $\mathrm{rot}_F(\tau) = \emptyset$.

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6. *Symbolic models for lifted pseudo-Anosov maps*

6.1. *Pseudo-Anosov maps.* We briefly review a few properties of rel pseudo-Anosov maps of relevance here. For more information see [**FLP91, CB88**]. A *pseudo-Anosov* homeomorphism ϕ of a surface M is characterized by the existence of a pair $(\mathcal{F}^u, \mu^u)$, $(\mathcal{F}^s, \mu^s)$ of transverse, ϕ-invariant measured foliations, one expanding and the other contracting. A homeomorphism ϕ of a compact surface M is called *pseudo-Anosov relative to the finite set A* if it has all of the usual properties of pseudo-Anosov maps but, in addition, its invariant foliations have one-prong singularities on the set A. For brevity of terminology, if ϕ is pseudo-Anosov relative to some finite (or empty) set, it is called *rel pseudo-Anosov*.

A rel pseudo-Anosov map always has a Markov partition containing a finite number of rectangles $\{R_1, \ldots, R_k\}$. By subdividing the partition if necessary we may assume that for each i, j, the intersection $R_i \cap \phi(R_j)$ has at most one component. The transition matrix C is a $k \times k$ matrix defined by $C_{ij} = 1$ if $R_i \cap \phi(R_j) \neq \emptyset$ and $C_{ij} = 0$ otherwise. Let Σ denote the subshift of finite type constructed from the matrix C; this subshift is always transitive (irreducible) and topologically mixing. There is a semiconjugacy α from (Σ, σ) to (M, ϕ) which is bounded to one, is bijective on dense, G_δ sets and the image of a cylinder set in Σ is a topological disk in M, and thus the pseudo-Anosov map is also transitive and topologically mixing.

6.2. *Twisted skew products corresponding to a lifted pseudo-Anosov.* We now describe the construction of a twisted skew product which is a symbolic model for the lift of the pseudo-Anosov map ϕ to an Abelian cover. The process is quite standard, but we need some details of the construction below. In addition, the case when ϕ is not isotopic to the identity does not seem to have been described previously in the literature.

Assume now that ϕ is a rel pseudo-Anosov map of the compact surface M with universal Abelian cover $\tilde{M}$ with Markov partition, transition matrix C, subshift of finite type (Σ, σ), and semiconjugacy α as in §6.1 A skew product corresponding to the lift $\tilde{\phi}$ will have base shift (Σ, σ), group factor $H_1(M) = \mathbb{Z}^d$, and twisting automorphism ϕ_*. The height function h measures how much a lifted rectangle moves in the cover and is defined as follows.

Fix a fundamental domain $\tilde{M}_0'$ for $\tilde{M}$ and one lift $\tilde{R}_j$ of each rectangle R_j such that $\tilde{R}_j \cap \tilde{M}_0 \neq \emptyset$ for all j and $\cup \tilde{R}_j$ is a connected set. Let $\tilde{M}_0 = \cup \tilde{R}_j$. Note that $\tilde{M}_0$ is also a fundamental domain for $\tilde{M}$. Now since C is a (0, 1) matrix, if $a \to b$ is an allowable one-step transition for Σ, we have

$$\tilde{\phi}(\tilde{R}_a) \cap \delta_{\mathbf{n}}(\tilde{R}_b) \neq \emptyset \tag{6.1}$$

for exactly one $\mathbf{n} \in \mathbb{Z}^d$. Define $h : \Sigma \to \mathbb{Z}^d$ as constant on a length two cylinder set $[a, b]_0$ by $h(s) = \mathbf{n}$ for all $s \in [a, b]_0$, where $\mathbf{n} \in \mathbb{Z}^d$ is the unique deck element for which (6.1) holds.

Using (3.1), changing the chosen fundamental domain from $\tilde{M}_0$ to some $\delta_{\mathbf{m}}(\tilde{M}_0)$ will change the height function h by a constant to $h' = (\phi_* - I)(m) + h$. In addition, a given rel pseudo-Anosov map is modeled by many (closely related) subshifts of finite type. Thus, there are actually many skew products which correspond to $\tilde{\phi}$.

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6.3. *The semiconjugacy.* We now construct a semiconjugacy $\tilde{\alpha}$ from $(\Sigma \times \mathbb{Z}^d, \tau)$ onto $(\tilde{M}, \tilde{\phi})$. In making the construction it is initially easier to work with the countable state Markov shift described in §4.2 which is naturally conjugate to τ. Recall that this shift $(\hat{\Sigma}, \hat{\sigma})$ has states $\hat{S} = \{1, 2, \ldots, k\} \times \mathbb{Z}^d$ and so sequences $\hat{s} \in \hat{\Sigma}$ are given as $\hat{s} = \ldots, (s_{-1}, \mathbf{n}_{-1}), (s_0, \mathbf{n}_0), (s_1, \mathbf{n}_1), \ldots$. Define $\tilde{\alpha} : \hat{\Sigma} \to \tilde{M}$ by

$$\tilde{\alpha}(\hat{s}) = \bigcap_{j \in \mathbb{Z}} \phi^{-j}(\delta_{\mathbf{n}_j}(R_{s_j})). \tag{6.2}$$

The basic properties of the Markov partition for ϕ coupled with (3.1) ensure that the intersection in (6.2) consists of exactly one point, $\tilde{\alpha}$ is continuous, onto, bounded to one and $\tilde{\alpha} \circ \hat{\sigma} = \tilde{\phi} \circ \tilde{\alpha}$. Further, $\tilde{\alpha}$ is bijective on dense, G_δ sets and the image of a cylinder set in $\hat{\Sigma}$ is a topological disk in $\tilde{M}$. Since the cylinder sets form a basis for the topology of $\tilde{M}$, $\tilde{\alpha}$ is transitive or topologically mixing if and only if $\tilde{\phi}$ has these properties.

As in §4.2 we can identify the Markov model for τ with τ and so we can also consider the semiconjugacy from $(\Sigma \times \mathbb{Z}^d, \tau)$ to $(\tilde{M}, \tilde{\phi})$. We also denote this semiconjugacy as $\tilde{\alpha}$. With this identification the construction of $\tilde{\alpha}$ gives that for each $\mathbf{n} \in \mathbb{Z}^d$,

$$\tilde{\alpha}(\Sigma \times \{\mathbf{n}\}) = \delta_{\mathbf{n}}(\tilde{M}_0). \tag{6.3}$$

6.4. *Dynamical correspondence of skew product and lifted pseudo-Anosov.* Using §§6.3 and 3.3 we now have maps

$$\Sigma \times \mathbb{Z}^d \xrightarrow{\tilde{\alpha}} \tilde{M} \xrightarrow{\tilde{\beta}} \mathbb{R}^d. \tag{6.4}$$

An understanding of the metric properties of these maps is necessary to compare the dynamics of τ and $\tilde{\phi}$ on these unbounded spaces. Recall that a map between pseudo-metric spaces $q : (X, d) \to (X', d')$ is called a *quasi-isometry* if there are numbers $a > 1$ and $b > 0$ with

$$\frac{1}{a} d(x_1, x_2) - b \le d'(q(x_1), q(x_2)) \le a d(x_1, x_2) + b$$

for all $x_1, x_2 \in X$. A quasi-isometry yields a correspondence of the metric structure on 'large scales'.

Define a pseudo-metric d_1 on $\Sigma \times \mathbb{Z}^d$ using the norm on the group component, $d_1(t, t') = \|\pi_2(t) - \pi_2(t')\|$. For a topological metric d on the surface M, let $\tilde{d}$ be its lift to the universal Abelian cover. As a consequence of (6.3), $\tilde{\alpha} : (\Sigma \times \mathbb{Z}^d, d_1) \to (\tilde{M}, \tilde{d})$ is a quasi-isometry. In addition, the map $\tilde{\beta}' : \tilde{M} \to \mathbb{R}^d$ from §3.3 is also a quasi-isometry from $(\tilde{M}, \tilde{d})$ to $\mathbb{R}^d$ with the standard metric. Thus, all of the natural equivariant methods of measuring large-scale displacements on $\Sigma \times \mathbb{Z}^d$ and $\tilde{M}$ are comparable.

We also need to compare the corresponding linear structures on the group factor of $\Sigma \times \mathbb{Z}^d$ and on the vector space $\mathbb{R}^d$. Specifically, the construction of $\tilde{\alpha}$ and $\tilde{\beta}$ yields for all $(s, \mathbf{n}) \in \Sigma \times \mathbb{Z}^d$,

$$|\tilde{\beta} \circ \tilde{\alpha}(s, \mathbf{n}) - \mathbf{n}| \le \sqrt{d} \tag{6.5}$$

(the $\sqrt{d}$ is the diameter of the unit cube in $\mathbb{R}^d$).

Passing to Fried quotients, if $H_1(M)/F = \mathbb{Z}^k$, then the maps in (6.4) descend to

$$\Sigma \times \mathbb{Z}^k \xrightarrow{\tilde{\alpha}_F} \tilde{M}_F \xrightarrow{\tilde{\beta}_F} \mathbb{R}^k,$$

with $\tilde{\alpha}_F$ a semiconjugacy from τ_F to $\tilde{\phi}_F$. Both $\tilde{\alpha}_F$ and $\tilde{\beta}_F$ are quasi-isometries of the projected pseudo-metrics and the analog of (6.5) for $\tilde{\beta}_F \circ \tilde{\alpha}_F$ holds as well.

The next proposition summarizes the connections between the lift of a rel pseudo-Anosov map $\tilde{\phi}$ to the universal Abelian cover $\tilde{M}$ and a corresponding twisted skew product τ.

PROPOSITION 6.1. *Assume that ϕ is a rel pseudo-Anosov map on the compact surface M, $\tilde{M}$ is the universal Abelian cover of M, $\tilde{\phi}$ is a lift of ϕ to $\tilde{M}$ and τ is a twisted skew product with base shift (Σ, σ) that corresponds to $\tilde{\phi}$:*

(a) *$(\tilde{M}, \tilde{\phi})$ is transitive (topologically mixing) if and only if τ is transitive (topologically mixing);*

(b) *τ has the ftp;*

(c) $\mathrm{rot}_F(\tilde{\phi}) = \mathrm{rot}_F(\tau)$ *and these sets have dimension* $k = \mathrm{rank}(\mathbb{Z}^d/F)$.

Proof. The proof of (a) was indicated in §6.3. To prove (b), note that if $\Gamma \subset H_1(M)$ is co-finite and $\phi_*(\Gamma) = \Gamma$, then $\tilde{\phi}$ on the universal Abelian cover descends to a $\tilde{\phi}_\Gamma$ on the quotient cover $\tilde{M}_\Gamma = \tilde{M}/\Gamma$. The map $\tilde{\phi}_\Gamma$ is a lift of ϕ to the compact surface $\tilde{M}_\Gamma$ and, thus, is itself pseudo-Anosov and so $\tilde{\phi}_\Gamma$ is transitive. The semiconjugacy $\tilde{\alpha}$ from $(\Sigma \times \mathbb{Z}^d, \tau)$ to $(\tilde{M}, \tilde{\phi})$ descends to a semiconjugacy $\tilde{\alpha}_\Gamma$ from $(\Sigma \times (\mathbb{Z}^d/\Gamma), \tau_\Gamma)$ to $(\tilde{M}_\Gamma, \tilde{\phi}_\Gamma)$ and so τ_Γ is transitive as remarked at the end of §6.1.

To prove (c) assume that the Fried quotient is non-trivial. Let h_F be the height function of the projection τ_F of τ to the Fried quotient and B be the cocycle defined by (3.3) on the Fried cover of ϕ and $\alpha : \Sigma \to M$ be the semiconjugacy from the base shift to the map ϕ on the surface. Now if $\alpha(s) = x$, then by construction, $\tilde{\alpha}(s, 0) := \tilde{x}$ is a lift of x. Thus using (3.3) and the fact that $\tilde{\alpha}_F$ is a semiconjugacy,

$$B(x, n) = \tilde{\beta}_F(\tilde{\phi}_F^n(\tilde{\alpha}_F(s, 0))) - \tilde{\beta}_F(\tilde{\alpha}_F(s, 0)) = \tilde{\beta}_F \circ \tilde{\alpha}_F(\tau_F^n(s, 0)) - \tilde{\beta}_F \circ \tilde{\alpha}_F(s, 0).$$

Since, by definition, $h_F(s, n) = \pi_2(\tau_F^n(s, 0)) - \pi_2(s, 0)$ and so by the Fried quotient version of (6.5) we have that for any $s \in \Sigma$ and $n \in \mathbb{N}$,

$$|h_F(s, n) - B(\alpha(s), n)| \leq 2\sqrt{k}. \tag{6.6}$$

From (6.6) and the definitions of the rotation vectors it then follows directly that for $s \in \Sigma$, $\mathrm{rot}(s, \tau_F)$ exists if and only if $\mathrm{rot}(\alpha(s), \tilde{\phi}_F)$ does, and if they exist they are equal, and thus the first statement in (c) follows. The assertion regarding the dimension is proved in Remark 7.7 below. □

7. *Transitivity of untwisted skew products*

The main goal of this section is to prove the following theorem which gives conditions for transitivity of an untwisted skew product using the rotation set. It is ultimately based on the criterion for transitivity using the displacement set provided by Coudene's theorem 7.2 below, however the rotation set is more tractable under the iterations and translations of maps required here.

THEOREM 7.1. *Assume that τ is an untwisted skew product with group factor $G = \mathbb{Z}^d$ and the base shift (Σ, σ) a transitive subshift of finite type. The following are equivalent:*

(a) *τ is transitive;*

(b) *τ has the ftp and $0 \in \mathrm{Int}(\mathrm{rot}(\tau))$;*

(c) *τ has the ftp and its periodic points are dense in $\Sigma \times \mathbb{Z}^d$.*

7.1. *Coudene's transitivity theorem.* The next theorem gives conditions on the displacement set which ensure a untwisted skew product is transitive. While the theorem given here is stated in more general terms than in [**Cou04**, Theorem 9], the method of proof indicated there works for the version given here. We need the generalization to countable state Markov shifts in the main induction argument below. In covering space language the analog of this theorem was proved in [**BGH93**] for the case of $G = \mathbb{Z}$ and in [**Par03**] for $G = \mathbb{Z}^2$.

THEOREM 7.2. (Coudene) *Let $\tau : \Sigma \times G \to \Sigma \times G$ be an untwisted skew product with base shift Σ a transitive countable Markov shift and group component G a finitely generated Abelian group. The untwisted skew product τ is transitive if and only if $\langle D(\tau)\rangle_+ = G$.*

Remark 7.3. The analog of Theorem 7.2 for lifted measures is quite different in character. Let τ be an untwisted skew product over the subshift of finite type (Σ, σ) with group component $\mathbb{Z}^d$ and height function h and let μ be an ergodic, shift-invariant Gibbs measure with $\int h \, d\mu = 0$. Rees [**Ree81**] and then Guivarc'h [**Gui89**] showed that the lift of μ to a τ-invariant (infinite) measure is ergodic if and only if $d = 1$ or 2.

7.2. *The finite lifting property and transitivity.* For untwisted skew products we first obtain various conditions equivalent to the ftp using the displacement set $D(\tau)$. Note that for untwisted skew products by definition the twisting isomorphism $\Psi = \mathrm{Id}$ and so the quotient τ_Γ is defined for all subgroups $\Gamma \subset G$.

LEMMA 7.4. *Assume that τ is an untwisted skew product with group factor $G = \mathbb{Z}^d$ and base shift (Σ, σ) a transitive countable Markov shift. The following are equivalent:*

(a) *τ has the ftp;*

(b) *$\langle D(\tau)\rangle = \mathbb{Z}^d$;*

(c) *for every co-finite Γ, $\langle D(\tau_\Gamma)\rangle_+ = \langle D(\tau_\Gamma)\rangle = \mathbb{Z}^d / \Gamma$.*

Proof. Since the group factor of τ_Γ is $\mathbb{Z}^d / \Gamma$, Theorem 7.2 says that τ_Γ is transitive if and only if $\langle D(\tau_\Gamma)\rangle_+ = \mathbb{Z}^d / \Gamma$. If Γ is co-finite, as remarked in §3.1, $\langle D(\tau_\Gamma)\rangle_+ = \langle D(\tau_\Gamma)\rangle$ in the finite group $\mathbb{Z}^d / \Gamma$. Thus we have the equivalence of (a) and (c). Now assume that (b) holds: as noted in (5.4), $\langle D(\tau_\Gamma)\rangle = \langle D(\tau)\rangle / (\langle D(\tau)\rangle \cap \Gamma) = \mathbb{Z}^d / \Gamma$, and so (c) follows.

Now we prove that (b) not holding implies (c) not holding. If (b) is false, then $\hat{H} := \langle D(\tau)\rangle$ is a proper subgroup of $\mathbb{Z}^d$. It follows easily from the fundamental fact (3.2) that there is always a co-finite H with $\hat{H} \subset H$ and H is *not* pure, and thus $|\mathbb{Z}^d / H| > 1$. Again using (5.4) we have $\langle D(\tau_H)\rangle = \langle D(\tau)\rangle / (\langle D(\tau)\rangle \cap H) = \hat{H} / \hat{H} \ncong \mathbb{Z}^d / H$, finishing the proof. □

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Using the ftp we also get new conditions for transitivity again for untwisted skew products.

THEOREM 7.5. *Assume that* τ *is an untwisted skew product with group factor* $G = \mathbb{Z}^d$ *and the base shift* (Σ, σ) *a transitive countable state Markov shift. The following are equivalent:*

(a) τ *is transitive;*

(b) $D(\tau) = \langle D(\tau)\rangle = \langle D(\tau)\rangle_+ = \mathbb{Z}^d$*;*

(c) τ *has the ftp and there is a rank-d subgroup* $H \subset \langle D(\tau)\rangle_+$*;*

(d) *there exists a finite set* $\{d_i\} \subset D(\tau)$ *so that* $\langle\{d_i\}\rangle = \mathbb{Z}^d$ *and* $0 = \sum a_i d_i$ *with the* a_i *positive integers.*

Proof. If τ is transitive, then for every $\mathbf{n} \in \mathbb{Z}^d$ and state s_0 for Σ, treating τ as a countable state Markov shift, there is an allowable transition $(s_0, 0) \to (s_0, \mathbf{n})$. This implies that σ has a period point with displacement $\mathbf{n}$. Thus, (a) implies that $D(\tau) = \mathbb{Z}^d$, and since obviously $D(\tau) \subset \langle D(\tau)\rangle_+ \subset \langle D(\tau)\rangle \subset \mathbb{Z}^d$, the other equalities in (b) follow. Conversely, Theorem 7.2 shows that (b) implies (a).

Now assume that (d) holds. By hypothesis $\langle\{d_i\}\rangle = \mathbb{Z}^d$ and thus for any $\mathbf{n} \in \mathbb{Z}^d$ there are integers b_i with $\mathbf{n} = \sum b_i d_i$. Since the given $a_i > 0$ in (d), we may find $k > 0$ with $ka_i + b_i > 0$ for all i. Thus, $\mathbf{n} = \sum (ka_i + b_i) d_i$ and so $\mathbb{Z}^d = \langle\{d_i\}\rangle_+ \subset \langle D(\tau)\rangle_+$, implying that $\mathbb{Z}^d = \langle D(\tau)\rangle_+$, and so τ is transitive by Theorem 7.2, showing that (d) implies (a).

Now assume that (b) holds and so $D(\tau) = \mathbb{Z}^d$. Let $d_1, \ldots, d_d$ be a basis for $\mathbb{Z}^d$ and for $i = 1, \ldots, d$, let $d_{d+i} = -d_i$. Thus, $\langle\{d_i\}\rangle = \mathbb{Z}^d$ and $\sum d_i = 0$ as required for (d).

Now assume that (c) holds. As noted in (5.4), if $\pi_H : \mathbb{Z}^d \to \mathbb{Z}^d/H$ is the projection, then $\pi_H(\langle D(\tau)\rangle_+) = \langle D(\tau_H)\rangle_+$ and $\pi_H(\langle D(\tau)\rangle) = \langle D(\tau_H)\rangle$. Since H has rank d, it is co-finite and so $\langle D(\tau_H)\rangle = \langle D(\tau_H)\rangle_+$, thus $\pi_H(\langle D(\tau)\rangle) = \pi_H(\langle D(\tau)\rangle_+)$. Now since we are assuming that τ has the ftp, by Lemma 7.4 we have $\langle D(\tau)\rangle = \mathbb{Z}^d$, and so $\pi_H(\mathbb{Z}^d) = \pi_H(\langle D(\tau)\rangle_+)$. Thus, for each $\mathbf{n} \in \mathbb{Z}^d$ there is a $\mathbf{m} \in \langle D(\tau)\rangle_+$ with $\mathbf{n} + H = \mathbf{m} + H$. This means that for some $h \in H$, $\mathbf{n} = \mathbf{m} + h$. Thus, since $h \in H \subset \langle D(\tau)\rangle_+$, we have that $\mathbf{n} \in \langle D(\tau)\rangle_+$. Since $\mathbf{n}$ was arbitrary, $\langle D(\tau)\rangle_+ = \mathbb{Z}^d$ and so by Theorem 7.2 again, τ is transitive. Thus, (c) implies (a).

As remarked in §4.1, for any subgroup $\Gamma \subset \mathbb{Z}^d$, τ is semiconjugate to τ_Γ, and so if τ is transitive, then so is τ_Γ. Thus, (a) coupled with (b) implies (c) using $\mathbb{Z}^d$ itself as the rank d-subgroup H, finishing the proof. □

7.3. *Rotation sets over subshifts of finite type.* A few of the fundamental properties of the rotation set over a transitive subshift of finite type are needed in the following. Basic results were obtained by Fried in [**Fri82b**] and [**Fri82a**] in the context of suspension flows, and much more detailed results were obtained by Ziemian in [**Zie95**] for the more general case of the Birkhoff averages of any bounded vector-valued function (see also [**MT91**]).

Recall that for a subshift of finite type (Σ, σ), a *simple block* is an allowable block that starts and ends with the same symbol and contains no other symbol more than once. Simple blocks are also sometimes called elementary blocks or minimal loops. A *simple* *periodic point* is a periodic point of σ constructed by infinite concatenation of a simple block. Let

$$D_{\text{simp}}(\tau) = \{h(p, n) \mid p \text{ is a simple periodic point with period } n\}.$$

It follows immediately from the definition of simple blocks that any sequence in Σ can be written as the concatenation of such blocks. In particular, a periodic point p of period n can be constructed by concatenating simple blocks $p_1, p_2, \ldots, p_k$ of periods $n_1, n_2, \ldots, n_k$ with $n = \sum p_j$. Thus, $h(p, n) = \sum h(p_j, n_j)$ which implies that $\langle D(\tau)\rangle_+ \subset \langle D_{\text{simp}}(\tau)\rangle_+$. Since the other implication is trivial,

$$\langle D(\tau)\rangle_+ = \langle D_{\text{simp}}(\tau)\rangle_+ \text{ and similarly, } \langle D(\tau)\rangle = \langle D_{\text{simp}}(\tau)\rangle. \tag{7.1}$$

For a linear transformation $L : \mathbb{Z}^d \to \mathbb{Z}$, let $\hat{L}$ denote its linear extension to $\hat{L} : \mathbb{R}^d \to \mathbb{R}$.

THEOREM 7.6. *Assume that τ is an untwisted skew product with base shift a transitive subshift of finite type and group factor $\mathbb{Z}^d$.*

(a) *The rotation set* $\text{rot}(\tau)$ *is equal to the convex hull of the rotation vectors of the simple periodic points.*

(b) *Assuming that* $0 \in \text{rot}(\tau)$, *the rotation set* $\text{rot}(\tau)$ *is d-dimensional if and only if $\langle D(\tau)\rangle$ is a rank d-subgroup of $\mathbb{Z}^d$.*

(c) *If τ has the ftp and* $0 \in \text{rot}(\tau)$, *then the rotation set* $\text{rot}(\tau)$ *is d-dimensional.*

Proof. For a proof of (a), see [**Zie95**, Theorem 3.4] and compare with [**Fri82b**, Lemma 3] and [**MT91**, Proposition 3.2].

We now prove the equivalence in (b). From (a) we know that $\text{rot}(\tau)$ is a convex hull in $\mathbb{R}^d$ with extreme points in $\mathbb{Q}^d$, and by assumption, $0 \in \text{rot}(\tau)$. This implies that $\dim(\text{rot}(\tau)) < d$ if and only if $\text{rot}(\tau)$ is in the kernel of some linear $\hat{L}$ which is the extension of a linear, onto $L : \mathbb{Z}^d \to \mathbb{Z}$. Now for each simple periodic point p_i of period n_i, $n_i \,\text{rot}(p_i) = h(p_i, n_i)$, and from (7.1), $\langle D(\tau)\rangle = \langle D_{\text{simp}}(\tau)\rangle$. This implies that $\text{rot}(\tau) \subset \ker(\hat{L})$ if and only if $\langle D(\tau)\rangle \subset \ker(L)$. However $\langle D(\tau)\rangle \subset \ker(L)$ for some linear, onto $L : \mathbb{Z}^d \to \mathbb{Z}$ if and only if $\langle D(\tau)\rangle$ has rank less than d. This proves (b).

If τ has the ftp, by Lemma 7.4 $\langle D(\tau)\rangle = \mathbb{Z}^d$, and so (c) follows from (b). □

Remark 7.7. We now give the proof of the assertion regarding the dimension in Theorem 6.1(c). Assume that $\tilde{\phi}$ and τ correspond and $\mathbb{Z}^d/F = \mathbb{Z}^k$ for $k > 0$ (if $k = 0$ there is nothing to prove). In addition, let $\tilde{\alpha}_F$ be the projection of the semiconjugacy between $\tilde{\phi}$ and τ to one between $\tilde{\phi}_F$ and τ_F

If s is a periodic point for the base shift, then $\text{rot}(s, \tau_F) = \mathbf{p}/q \in \mathbb{Q}^k$, and so if $\tilde{x}_F = \tilde{\alpha}_F(s, 0)$, then $\text{rot}(\tilde{x}_F, \tilde{\phi}_F) = \mathbf{p}/q$ also, and thus by (5.6), $0 \in \text{rot}(\delta_{-\mathbf{p}}\, \tilde{\phi}_F^q)$ $= \text{rot}(T_{-\mathbf{p}}\, (\tau_F)^q)$. Now $\delta_{-\mathbf{p}}\, \tilde{\phi}_F^q$ is a lift to $\tilde{M}_F$ of the pseudo-Anosov map ϕ^q, and so $T_{-\mathbf{p}}\, (\tau_F)^q$ has the ftp. Thus, by Theorem 7.6(c), $\text{rot}(T_{-\mathbf{p}}\, (\tau_F)^q)$ has dimension k and so using (3.4), so does $\text{rot}(\tau_F) = \text{rot}(\tilde{\phi}_F)$.

The next lemma describes another property of the rotation set over a subshift of finite type: if there is a large enough displacement in any ‘direction’ in the cover, then there is a point with rotation vector in that direction. One consequence is a useful condition for deciding whether zero is in the interior of a rotation set. The author learned this argument from David Fried in 1983, and a version of it was used in [**BGH93**].

LEMMA 7.8. *Assume that τ is a untwisted skew product with group component $\mathbb{Z}^d$, height function h and base shift the transitive subshift of finite type (Σ, σ). For any onto linear functional $L : \mathbb{Z}^d \to \mathbb{Z}$, there exists a $C > 0$ so that if there is a point $s \in \Sigma$ and a positive integer n with $L(h(s, n)) > C$, then there is a period-n' point s' with $\hat{L}(\mathrm{rot}(s')) > 0$, and if there are s and n with $L(h(s, n)) < -C$, then there is a s'' with $\hat{L}(\mathrm{rot}(s'')) < 0$.*

Thus, $0 \in \mathrm{Int}(\mathrm{rot}(\tau))$ if and only if for every linear, onto $L : \mathbb{Z}^d \to \mathbb{Z}$,

$$\sup\{L(h(s, m)) \mid s \in \Sigma, m \in \mathbb{N}\} = \infty. \tag{7.2}$$

Proof. Since (Σ, σ) is transitive, for each pair of symbols i, j there is an allowable transition $i \to j$ for σ. Define $d_{i,j} \in \mathbb{Z}^d$ as the group coordinate of the lift of these transitions to transitions for τ, so $(i, 0) \to (j, d_{i,j})$. Given L, let $C_1 = \max_{i,j}\{|L(d_{i,j})|\}$ and $C = \max\{2C_1, 1\}$. Assume now that for some $s \in \Sigma$ and $n > 0$ we have $L(h(s, n)) > C$. The first $n + 1$-symbols $s_0 s_1 \ldots s_n$ in the given sequence s give $s_0 \to s_n$. Using the transitivity of (Σ, σ), we have $s_n \to s_0$. The concatenation of these two allowable transitions is $s_0 \to s_n \to s_0$ which lifts to

$$(s_0, 0) \to (s_n, h(s, n)) \to (s_0, h(s, n) + d_{s_n,s_0}).$$

Thus, if s' is constructed by infinite concatenation of the block corresponding to $s_0 \to s_n \to s_0$ and n' is the sum of n and the length of the transition $s_n \to s_0$, then s' is a periodic point of period n' and $L(h(s', n')) = L(h(s, n)) + L(d_{s_n,s_0}) > C/2$. Since $\mathrm{rot}(s', n') = h(s', n')/n'$ and $\hat{L}$ is linear we have $\hat{L}(\mathrm{rot}(s')) > C/(2n')$. The construction of s'' is similar.

To prove the last statement of the lemma, first note that $0 \in \mathrm{Int}(\mathrm{rot}(\tau))$ if and only if for every linear, onto $L : \mathbb{Z}^d \to \mathbb{Z}$ there exists an $r \in \mathrm{rot}(\tau)$ with $\hat{L}(r) > 0$. Assume now that (7.2) holds for every linear, onto $L : \mathbb{Z}^d \to \mathbb{Z}$. Thus, in particular, if $C(L)$ is the constant depending on L given by the first paragraph of the theorem, there is a point s and an $m > 0$ with $L(h(s, m)) > C(L)$ and so there is a point s' with $\hat{L}(\mathrm{rot}(s')) > 0$, and so $0 \in \mathrm{Int}(\mathrm{rot}(\tau))$. Now conversely, assume that there exists a linear, onto L such that for all s, there is a constant K with $\sup_{m \in \mathbb{N}} L(h(s, m)) < K$. Then certainly for any s for which the rotation vector exists, we have $\hat{L}(\mathrm{rot}(s)) \leq 0$, and so $0 \notin \mathrm{Int}(\mathrm{rot}(\tau))$. □

Remark 7.9. Note that the lemma does not say that there always exists a point s as in the first paragraph of the statement. A trivial example is when Σ is the full two-shift and $h \equiv 0$.

7.4. *Proof of Theorem 7.1.* Before giving the proof of the theorem stated at the beginning of this section we need a small fact about convex polytopes. If $\{x_1, \ldots, x_k\}$ is a finite set of points in $\mathbb{Q}^d$ and zero is in the interior of their convex hull, then there are positive integers b_j with $0 = \sum_{j=1}^{k} b_j x_j$. The proof is an exercise.

Proof of Theorem 7.1. If τ is transitive, then certainly any quotient is transitive and so τ has the ftp. To show that (a) implies (c) we must show that transitivity of a countable Markov shift implies it has dense periodic points. This is standard: given any allowable

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block $s_0 \dots s_n$, we must find a periodic point which contains that block. Since the shift is transitive, there is an allowable transition $s_n \to s_0$. The concatenated transitions $s_0 \to s_n \to s_0$ repeated infinitely often yield the required periodic point. If $0 \notin \mathrm{Int}(\mathrm{rot}(\tau))$, then it follows from the second statement of Lemma 7.8 that τ is not transitive, and so (a) implies (b).

Now assume that (c) holds. Since τ has a periodic point, certainly $0 \in \mathrm{rot}(\tau)$ and since τ has the ftp, by Theorem 7.6(c), $\mathrm{rot}(\tau)$ is d-dimensional. Thus, if $0 \in \mathrm{Fr}(\mathrm{rot}(\tau))$ there is a linear, onto $L : \mathbb{Z}^d \to \mathbb{Z}$ with $\hat{L}(r) \geq 0$ for all $r \in \mathrm{rot}(\tau)$ and using Theorem 7.6(a), there is a point $s \in \Sigma$ with $\hat{L}(\mathrm{rot}(s)) > 0$. Thus, we may find an $m > 0$ with $L(h(s, m)) > 2C$, where C is the constant associated with L from Lemma 7.8. Now, since by hypothesis periodic points of τ are dense, treating τ as a Markov shift we see that there must be a periodic orbit t' of τ that begins with the block $(s_0, 0), (s_1, \mathbf{n}_1), \dots, (s_m, \mathbf{n}_m)$ with $n_j = \mathbf{n}_{j-1} + h(s_{j-1}, s_j)$ and $L(\mathbf{n}_M) > 2C$. Since t' is periodic, it must continue with a block $(s_m, \mathbf{n}_m), (s'_{m+1}, \mathbf{n}_{m+1}), \dots, (s'_{m+k}, \mathbf{n}_{m+k}), (s_0, 0)$ for some $k > 0$. Thus, if $\hat{s}$ is any allowable sequence for σ beginning with $s_m, s'_{m+1}, \dots, s'_{m+k}, s_0$, we have that $L(h(\hat{s}, k+1)) < -C$. Thus, by Lemma 7.8, there is a point s' with $\hat{L}(\mathrm{rot}\, s') < 0$, a contradiction.

Now assume that (b) holds. Let $\{p_i\}$ be the finite set of simple periodic points of σ and let $d_j = h(p_j, k_j)$ where k_j is the period of p_j. Lemma 7.4 coupled with the fact that $\langle D_{\mathrm{simp}}(\tau)\rangle = \langle D(\tau)\rangle$ yields that $\langle\{d_j\}\rangle = \mathbb{Z}^d$. Since each $\mathrm{rot}(p_j) = d_j/k_j \in \mathbb{Q}^d$, by the small fact on convex polytopes given above, $0 = \sum b_j \,\mathrm{rot}(p_j)$ with all of the b_j positive integers. Thus, letting $a_j = b_j(\prod k_i)/k_j$, we have $0 = \sum a_j d_j$ with all a_j positive integers, and so by Theorem 7.5, τ is transitive. □

8. *Examples*

We collect some simple examples which illustrate why various hypotheses are required. In all of these examples the base shift is the full two-shift and the skew product is untwisted. The simple blocks of the full two-shift written as one-step transitions are $1 \to 1$, $0 \to 0$, $0 \to 1 \to 0$ and $1 \to 0 \to 1$. We compute $\mathrm{rot}(\tau)$, $\langle D(\tau)\rangle$ and $\langle D(\tau)\rangle_+$ using the simple periodic points as per (7.1) and Theorem 7.6(a).

(a) For η_1, let the group component be $\mathbb{Z}$ and the height function $f_1 \equiv 1$. Then $\mathrm{rot}(\eta_1) = \{1\}$, $\langle D(\eta_1)\rangle = \mathbb{Z}$ and $\langle D(\eta_1)\rangle_+ = \{1, 2, \dots\}$. So η_2 has the ftp, is not transitive and its rotation set lacks interior and has no periodic points.

(b) For η_2, let the group component be $\mathbb{Z}$ and the height function f_2 be given by $f_2(00) = -2$, $f_2(11) = 2$, $f_2(01) = f_2(10) = 0$. Then $\mathrm{rot}(\eta_2) = [-2, 2]$ and $\langle D(\eta_2)\rangle = \langle D(\eta_2)\rangle_+ = 2\mathbb{Z}$, and so η_2 is not transitive and does not have the ftp. Thus, $0 \in \mathrm{Int}(\mathrm{rot}(\eta_2))$ does not suffice to imply transitivity. Note also that under η_2, $\Sigma \times \mathbb{Z}$ splits into a pair of transitive subsystems, $\Sigma \times (2\mathbb{Z})$ and $\Sigma \times (2\mathbb{Z} + 1)$. Thus, in particular, the periodic points of η_2 are dense, but η_2 is not transitive.

(c) For η_3, let the group component be $\mathbb{Z}$ and the height function f_3 be given by $f_3(00) = -1$, $f_3(11) = 1$, $f_3(01) = f_3(10) = 0$. Then $\mathrm{rot}(\eta_3) = [-1, 1]$ and $\langle D(\eta_3)\rangle = \langle D(\eta_3)\rangle_+ = \mathbb{Z}$, and so η_3 is transitive. However, note that $\eta_3^2 = \eta_2$ from the previous example. Thus, η_3 is transitive, but η_3^2 is not.

(d) For η_4, let the group component be $\mathbb{Z}^2$ and the height function f_4 be given by $f_4(00) = (1, 0) = f_4(01)$ and $f_4(11) = (0, 1) = f_4(10)$. Then $\text{rot}(\eta_4)$ is the line segment connecting (1, 0) and (0, 1) and has no interior. Also, $\langle D(\eta_4)\rangle = \mathbb{Z}^2$ and so η_4 has the ftp. Finally, $\langle D(\eta_4)\rangle_+ = \{(a, b) \mid a \geq 0, b \geq 0\} - \{(0, 0)\}$, and so η_4 is not transitive. This illustrates that the ftp alone does not imply that the rotation set has interior.

9. *Lemmas*

9.1. *Subgroup lemma.* The following elementary, technical lemma will be essential in the proof of the main induction step given in Lemma 10.1.

LEMMA 9.1. *If $\Gamma \subset \mathbb{Z}^d$ is a rank d-subgroup and $w \in \mathbb{Z}^d$ is a given element, then there exist elements $g_1, g_2, \ldots, g_{2d} \in \Gamma$ and a rank d-subgroup H so that*

$$H \subset \langle g_1 + w, g_2 + w, \ldots, g_{2d} + w\rangle_+.$$

Proof. To begin with assume that we are in the special case where the given w is actually an element of Γ. Let $\hat{g}_1, \ldots, \hat{g}_d$ be a basis for Γ, and for $1 \leq j \leq d$, let $g_j = \hat{g}_j - w$, and for $d < j \leq 2d$, let $g_j = -\hat{g}_j - w$. Then clearly $\langle g_1 + w, g_2 + w, \ldots, g_{2d} + w\rangle_+ = \Gamma$, so in this special case the required H is just Γ itself. Henceforth, we assume that $w \notin \Gamma$.

The strategy of the rest of the proof is to show that we may choose the elements $g_1, g_2, \ldots, g_{2d} \in \Gamma$ so that if $k_j = g_j + w$, then there are positive integers c_j with

$$\sum_{j=1}^{2d} c_j k_j = 0.$$

Thus, for any j_0,

$$-c_{j_0} k_{j_0} = \sum_{j \neq j_0} c_j k_j,$$

and so, $-c_{j_0} k_{j_0} \in \langle\{k_j\}\rangle_+$. Now obviously, $c_{j_0} k_{j_0} \in \langle\{k_j\}\rangle_+$, and so

$$H := \langle c_1 k_1, \ldots, c_d k_d\rangle \subset \langle\{k_j\}\rangle_+.$$

The last step in the proof is to show that the elements $\{c_1 k_1, \ldots, c_d k_d\}$ are linearly independent, yielding that H has rank d.

To implement the strategy, as noted in §3.1, if $q = |\mathbb{Z}^d / \Gamma|$, then $qw \in \Gamma$. Let $\overline{g}_1$ be primitive in Γ and in the same direction as w, that is, for some $p > 0$, $p\overline{g}_1 = qw$. Now also from §3.1, $\langle \overline{g}_1\rangle$ is a pure subgroup of Γ and so $\overline{g}_1$ is an element of a basis $\{\overline{g}_1, g_2, \ldots, g_d\}$ of Γ.

Now let $n_0 \in \mathbb{Z}$ be such that

$$n_0 < -p/q < n_0 + 1. \tag{9.1}$$

Note that we have strict inequalities in (9.1) because $w \notin \Gamma$. Now define elements of $\mathbb{Z}^d$,

$$k_1 = (n_0 + 1 + p/q)\overline{g}_1 = (n_0 + 1)\overline{g}_1 + w,$$
$$k_{d+1} = (n_0 + p/q)\overline{g}_1 = n_0\overline{g}_1 + w$$

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and integers

$$A = qn_0 + q + p = q(n_0 + 1 + p/q),$$
$$B = -(qn_0 + p) = -q(n_0 + p/q).$$

Note that $A > 0$, $B > 0$ and a simple calculation shows that

$$Ak_{d+1} + Bk_1 = 0, \tag{9.2}$$

and so $\langle Bk_1\rangle \subset \langle k_1, k_{d+1}\rangle_+$. Thus, letting $g_1 = (n_0+1)\overline{g}_1$ and $g_{d+1} = n_0\overline{g}_1$, this completes the first step of the strategy when $d = 1$, so now assume $d > 1$.

Let

$$k_j = \begin{cases} g_j + w & \text{for } 2 \le j \le d, \\ -g_j + w & \text{for } 2 + d \le j \le 2d. \end{cases}$$

Thus, letting $C = 2p$, another simple calculation for each $2 \le j \le d$ yields

$$Bk_j + Bk_{j+d} + Ck_{d+1} = 0, \tag{9.3}$$

and so summing (9.3) for $j = 2$ to d and adding (9.2) yields

$$Ak_{d+1} + Bk_1 + C(d-1)k_{d+1} + \sum_{j=2}^{d}(Bk_j + Bk_{j+d}) = 0.$$

Thus, as noted in the second paragraph of the proof, this implies

$$H := \langle Bk_1, Bk_2, \ldots, Bk_d\rangle \subset \langle g_1 + w, g_2 + w, \ldots, g_{2d} + w\rangle_+.$$

Finally, to show that $\{Bk_1, Bk_2, \ldots, Bk_d\}$ is linearly independent, it suffices to show that $\{k_1, k_2, \ldots, k_d\}$ is linearly independent over $\mathbb{Q}$. If $\sum a_j k_j = 0$, then using the definition of k_j,

$$0 = (a_1(n_0 + 1 + p/q) + p/q(a_2 + \cdots + a_d))\overline{g}_1 + a_2 g_2 + \cdots + a_d g_d,$$

and so all $a_j = 0$ using the linear independence of $\{\overline{g}_1, g_2, \ldots, g_d\}$. □

9.2. *Nilpotent linear transformations.* In the sufficient conditions for transitivity given in Proposition 10.2 below the twisting matrix is required to satisfy spec $A = \{1\}$. It is often technically convenient to work with the nilpotent matrix $A - \mathrm{id}$ in such cases. A linear transformation $T : \mathbb{Z}^d \to \mathbb{Z}^d$ is said to be *nilpotent of order* J if $T^J = 0$ and $T^{J-1} \neq 0$. The same definition applies to square matrices. A nilpotent transformation of order J generates a chain of kernels,

$$0 = \ker(T^0) \subset \ker(T) \subset \ker(T^2) \subset \cdots \subset \ker(T^J) = \mathbb{Z}^d.$$

It is also easy to check that

$$T(\ker(T^j)) \subset \ker(T^{j-1}) \subset \ker(T^j). \tag{9.4}$$

LEMMA 9.2. *Let $T : \mathbb{Z}^d \to \mathbb{Z}^d$ be a nilpotent homomorphism of order J. Then there is a direct sum decomposition $\mathbb{Z}^d = V_1 \oplus \cdots \oplus V_J$ with each $V_j \neq 0$ so that for all j:*

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(a) $\ker(T^j) = V_1 \oplus \cdots \oplus V_j$;
(b) $T(V_j) \subset V_1 \oplus \cdots \oplus V_{j-1}$;
(c) *if* $p_j : \mathbb{Z}^d \to V_j$ *is the projection, then* $(p_{j-1} \circ T)_{|V_j}$ *is injective for* $j > 1$.

Proof. The proof proceeds by induction on the order of nilpotency J with the case $J = 1$ being trivial. Assume then that the result is true for all nilpotent transformations of order less than J defined on any finite rank-free Abelian group.

For simplicity of notation let $K := \ker(T^{J-1})$ and $\hat{T} = T_{|K}$. By (9.4), $\hat{T}(K) \subset K$, and since $\hat{T}$ has order of nilpotency $J - 1$, by the inductive hypothesis, there is a direct sum decomposition $K = V_1 \oplus \cdots \oplus V_{J-1}$ with each $V_j \neq 0$ so that conditions (a), (b) and (c) hold for all $1 \leq j \leq J - 1$.

Since K is a kernel, one easily sees that it is a pure subgroup of $\mathbb{Z}^d$, and thus for some subgroup $V_J \subset \mathbb{Z}^d$, $\mathbb{Z}^d = K \oplus V_J$. We now check that $\mathbb{Z}^d = V_1 \oplus \cdots \oplus V_J$ satisfies the required conditions (a), (b) and (c) for all $1 \leq j \leq J$.

Since $\ker(T^J) = \mathbb{Z}^d$, condition (a) is obviously satisfied. Now certainly, $V_J \subset \ker(T^J) = \mathbb{Z}^d$, and so by (9.4),

$$T(V_J) \subset T(\ker(T^J)) \subset \ker(T^{J-1}) = V_1 \oplus \cdots \oplus V_{J-1},$$

and so condition (b) is satisfied.

Now we confirm condition (c). We have just shown that $T(V_J) \subset V_1 \oplus \cdots \oplus V_{J-1}$. Thus, if $v \in V_J$ and $p_{J-1} \circ T(v) = 0$, then in fact $T(v) \subset V_1 \oplus \cdots \oplus V_{J-2} = \ker(T^{J-2})$, and so $T^{J-2}(T(v)) = 0$, and so $v \in \ker(T^{J-1}) = K$ and, therefore, $v \in K \cap V_J = 0$. Thus, $\ker(p_{J-1} \circ T_{|V_J}) = \{0\}$, as required. □

9.3. *Integer matrices with spectrum equal to* $\{1\}$. For a linear transformation T or a square, integer matrix A, its *spectrum* is denoted by $\mathrm{spec}(T)$ or $\mathrm{spec}(A)$. We need notation for the block description of a matrix. Given a collection of positive integers $n_1, n_2, \ldots, n_J$ with $\sum n_\alpha = d$, the block description of type $(n_1, n_2, \ldots, n_J)$ of the $d \times d$ matrix A consists of the matrices $B_{\alpha,\beta}$, with $1 \leq \alpha, \beta \leq J$ with the dimension of $B_{\alpha,\beta}$ equal to $n_\alpha \times n_\beta$ and $A = (B_{\alpha,\beta})$, or more explicitly, the (i, j)th entry of $B_{\alpha,\beta}$ is the $(n_1 + \cdots + n_{\alpha-1} + i, n_1 + \cdots + n_{\beta-1} + j)$th entry of A.

The next lemma uses Lemma 9.2 to give a special form for a matrix A representing a linear isomorphism S with $\mathrm{spec}(S) = \{1\}$. The simplest case of a matrix A in the form is lower tridiagonal with all ones on the diagonal and all non-zero entries in the first subdiagonal.

LEMMA 9.3. *If* $S : \mathbb{Z}^d \to \mathbb{Z}^d$ *is an automorphism with* $\mathrm{spec}(S) = \{1\}$, *then there is a collection of numbers* $n_1 \geq n_2 \geq \cdots \geq n_J > 0$ *with* $\sum n_j = d$ *and a basis of* $\mathbb{Z}^d$ *such that with respect to this basis the automorphism* S *is represented by a matrix* A *which when written in block form of type* $(n_1, n_2, \ldots, n_J)$ *has blocks* $B_{\alpha,\beta}$ *satisfying:*
(a) $B_{\alpha,\alpha} = I$ *for* $\alpha = 1, \ldots, J$;
(b) $B_{\alpha,\beta} = 0$ *for* $\alpha < \beta$;
(c) $B_{\alpha,\alpha-1}$ *has rank* n_α *for* $\alpha = 2, \ldots, J$.

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Proof. Let C be the matrix representing S in the standard basis for $\mathbb{Z}^d$. Since by hypothesis $\operatorname{spec}(C) = \{1\}$, we have $\operatorname{spec}(C^T) = \{1\}$ and so $C^T - I$ represents a nilpotent homomorphism which we denote by T. Applying Lemma 9.2 to T, we find a basis for T or equivalently a unimodular matrix E, so that $\bar{A} := E(C^T - I)E^{-1}$ is block factored of type $(n_1, n_2, \ldots, n_J)$, where $n_j = \operatorname{rank}(V_j)$ with the V_j as in Lemma 9.2 and:

(a) $\bar{B}_{\alpha,\alpha} = 0$ for $\alpha = 1, \ldots, J$;

(b) $\bar{B}_{\alpha,\beta} = 0$ for $\alpha > \beta$;

(c) $\bar{B}_{\alpha-1,\alpha}$ has rank n_α for $\alpha = 2, \ldots, J$.

Finally, let $A := \bar{A}^T + I := (E^T)^{-1}(C - I)(E^T) + I = (E^T)^{-1}C(E^T)$, and so A represents S and it is easy to check that it has the required block form using the block form of $\bar{A}$. □

9.4. *Behavior of the Fried quotient under iteration.* For any quotient twisted skew product τ_Γ it follows easily that $(\tau_\Gamma)^k = (\tau^k)_\Gamma$. However, for $k > 0$ letting $F^{(k)} := P(\operatorname{im}(\Psi^k - \mathrm{id}))$, in general one has $F^{(1)} \neq F^{(k)}$. A simple example is $\Psi = -\mathrm{id}$. Thus, the iterate of the Fried quotient and the Fried quotient of the iterate, τ_F^k and $(\tau^k)_{F^{(k)}}$, often act on different spaces and thus are not equal. However, in the special case of $\operatorname{spec}(\Psi) = \{1\}$ we have the following result as a corollary of Lemma 9.3.

COROLLARY 9.4. *If τ is a twisted skew product with twisting matrix A satisfying* $\operatorname{spec}(A) = \{1\}$, *then the iterate of the Fried quotient is the Fried quotient of the iterate, or $(\tau_{F^{(1)}})^k = (\tau^k)_{F^{(k)}}$, where $F^{(k)} := P(\operatorname{im}(\Psi^k - \mathrm{id}))$.*

Proof. Since $\operatorname{spec}(A) = \{1\}$, we may conjugate A so that it is in the form given by the block factorization of Lemma 9.3. In the proof of that lemma, this block factorization corresponds to an internal direct sum decomposition $\mathbb{Z}^d = W_1 \oplus \cdots \oplus W_J$ as in Lemma 9.2. As a consequence of condition (c) in Lemma 9.3, the purification of $\operatorname{im}(A - \mathrm{id})$ is exactly $F = W_2 \oplus \cdots \oplus W_J$, and so $\mathbb{Z}^d/F$ is naturally identified with W_1.

Denoting the block factorization of A by $B_{\alpha,\beta}$, then for $k > 0$, a simple computation shows that we can obtain such a factorization for A^k by taking the kth power of the block factorization of A. Specifically, if we denote the factorization so obtained for A^k as $B^{(k)}_{\alpha,\beta}$, then subdiagonal blocks satisfy $B^{(k)}_{\alpha+1,\alpha} = k\, B_{\alpha+1,\alpha}$ for $\alpha = 1, \ldots, J-1$. Thus, since $B_{\alpha+1,\alpha}$ has rank n_α, so does $B^{(k)}_{\alpha+1,\alpha}$. This implies that the purification of $\operatorname{im}(A^k - \mathrm{id})$ is $F^{(k)} = W_2 \oplus \cdots \oplus W_J$ for all $k > 0$. Thus, for all $k > 0$, $\mathbb{Z}^d/F^{(k)} = \mathbb{Z}^d/F$ which implies that $(\tau^k)_{F^{(k)}} = (\tau_{F^{(1)}})^k$. □

10. *Transitivity of twisted skew products*

10.1. *The main induction lemma.* When the twisting matrix is a generalized shear, transitivity is proved by treating the twisted skew product as a sequence of untwisted extensions. The main technical step in doing this is given in the next lemma.

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The simplest non-trivial case of the lemma is when τ has group component $\mathbb{Z}^2$, height function $h = (h_1, h_2)$, and twisting matrix

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{10.1}$$

Thus, in coordinates,

$$\tau(s, m, n) = (\sigma(s), m + h_1(s), n + m + h_2(s)).$$

We then define η on $\Sigma \times \mathbb{Z}$ as $\eta(s, m) = (\sigma(s), m + h_1(s))$. As in §4.2 we treat η as a countable state Markov shift on $\Sigma' = \Sigma \times \mathbb{Z}$. After letting $t = (s, m)$ and $g(t) = g(s, m) = m + h_2(s)$, we may write

$$\tau(t, n) = (\eta(t), n + g(t)),$$

and so τ treated as a map on $\Sigma' \times \mathbb{Z}$ is an *untwisted* extension of (Σ', η). In this case the lemma says that if η is transitive and τ has the ftp, then τ is transitive. Note that no separate hypothesis in required in the shearing direction. Roughly speaking, the shear creates a global circulation which is conducive of recurrence.

The lemma allows group components $\mathbb{Z}^k$ and $\mathbb{Z}^\ell$, and it requires that the twisting matrix A be (k, ℓ)-block factored in lower triangular form with identity matrices on the diagonal and the subdiagonal block having full rank.

LEMMA 10.1. *Let $\eta : \Sigma \times \mathbb{Z}^k \to \Sigma \times \mathbb{Z}^k$ be a transitive, untwisted skew product with height function h and base shift (Σ, σ) countable Markov. Let $\Sigma' = \Sigma \times \mathbb{Z}^k$ and assume that $\tau : \Sigma' \times \mathbb{Z}^\ell \to \Sigma' \times \mathbb{Z}^\ell$ is an untwisted skew product given by*

$$\tau(t, n) = (\eta(t), n + g(t)) \tag{10.2}$$

and for $t = (s, m) \in \Sigma'$, the height function g is required to have the form

$$g(t) = g(s, m) = S(m) + f(s), \tag{10.3}$$

with $f : \Sigma \to \mathbb{Z}^\ell$ constant on length-two cylinder sets and $S : \mathbb{Z}^k \to \mathbb{Z}^\ell$ a rank-ℓ homomorphism. If τ has the ftp, then it is transitive.

Proof. Since η is transitive, it certainly has a periodic point, say $t^{(0)} = (s', 0)$ of period n_0. Now since η is untwisted, $t^{(m)} := (s', m)$ is also a periodic orbit of period n_0 for η. We treat $t^{(m)} \in \Sigma'$ and a straightforward computation using (10.2) and (10.3) yields that

$$\begin{aligned} g(t^{(m)}, n_0) &= g(t^{(m)}) + g(\eta(t^{(m)})) + \cdots + g(\eta^{n_0-1}(t^{(m)})) \\ &= g(s', m) + g(\sigma(s'), m + h(s')) + \cdots + g(\sigma^{n_0-1}(s'), m + h(s', n_0 - 1)) \\ &= n_0 S(m) + S(h(s') + \cdots + h(s', n_0 - 1)) + f(s', n_0). \end{aligned}$$

Thus, letting $w = S(h(s') + \cdots + h(s_0, n_0 - 1)) + f(s', n_0)$, we have that $\mathrm{im}(n_0 S) + w \subset D(\tau)$. Since S has rank ℓ by hypothesis, $\mathrm{im}(n_0 S)$ is a rank-ℓ subgroup of $\mathbb{Z}^\ell$. Thus, we may apply Lemma 9.1 to $\mathrm{im}(n_0 S)$ yielding elements $m_1, \ldots, m_{2d} \in \mathbb{Z}^k$ and a rank-ℓ subgroup H, with

$$H \subset \langle n_0 S(m_1) + w, \ldots, n_0 S(m_{2d}) + w \rangle_+ \subset \langle D(\tau) \rangle_+.$$

Thus, by Lemma 7.5, since τ has the ftp by assumption, it is transitive. □

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10.2. *Sufficient conditions for transitivity.* The next proposition uses Lemma 10.1 as the induction step to obtain transitivity in the case when the twisting automorphism is a generalized shear in the form given by Lemma 9.3.

PROPOSITION 10.2. *Assume that $\tau : \Sigma \times \mathbb{Z}^d \to \Sigma \times \mathbb{Z}^d$ is a twisted skew product with height function h and base shift (Σ, σ) countable Markov. If the twisting automorphism Ψ is a generalized shear* ($\text{spec}(\Psi) = \{1\}$)*, the Fried quotient τ_F is transitive, and τ has the ftp, then τ is transitive.*

Proof. Since $\text{spec}(\Psi) = \{1\}$, using Lemma 9.3 we can start by choosing a basis for $\mathbb{Z}^d$ in which Ψ is represented by a matrix A with block factorization $B_{\alpha,\beta}$ for $\alpha, \beta = 1, \ldots, J$ as in that lemma. Further, we adopt the notation of Lemma 9.2 by assuming that the block factorization of A corresponds to the internal direct sum decomposition $\mathbb{Z}^d = W_1 \oplus \cdot \oplus W_J$ with $\text{rank}(W_j) = n_j$. If $p_j : \mathbb{Z}^d \to W_j$ is the projection, let $h_j = p_j \circ h$.

We now define a collection of spaces and untwisted skew products inductively. Let $\Sigma^{(1)} = \Sigma$ and let $\tau_1 : \Sigma^{(1)} \times W_1 \to \Sigma^{(1)} \times W_1$ be defined as $\tau_1(s, m_1) = (\sigma(s), m_1 + h_1(s))$. For $j = 2, \ldots, J$, let $\Sigma^{(j)} = \Sigma^{(j-1)} \times W_{j-1}$ and $\tau_j : \Sigma^{(j)} \times W_j \to \Sigma^{(j)} \times W_j$ is defined using $t^{(j-1)} \in \Sigma^{(j)}$ written as $t^{(j-1)} = (s, m_1, \ldots, m_{j-1})$ by

$$\tau_j(t^{(j-1)}, m_j) = (\tau_{j-1}(t^{(j-1)}), m_j + H_j(t^{(j-1)})),$$

where

$$H_j(t^{(j-1)}) = B_{j,1}m_1 + B_{j,2}m_2 + \cdots + B_{j,j-1}m_{j-1} + h_j(s).$$

Note that $\tau_J = \tau$, the given twisted skew product. Also, if for $j = 1, \ldots, J-1$, we let $\Gamma_j = W_{j+1} \oplus \cdots \oplus W_J$, then $\Psi(\Gamma_j) \subset \Gamma_j$ by the form of A, and τ_j is exactly the quotient map τ_{Γ_j} as defined in §4.1. Thus, since by hypothesis τ has the ftp, each τ_j also has the ftp and the base shift is transitive. As noted in the proof of Corollary 9.4, τ_1 is the Fried quotient of τ and so is transitive by hypothesis. Finally, by the block factorization of A obtained using Lemma 9.3 each $B_{j,j-1}$ is rank n_j, and so each τ_j extends τ_{j-1} as required for Lemma 10.1. Thus, by induction, each τ_j is transitive and so $\tau_J = \tau$ is also transitive. □

10.3. *The main symbolic theorem.* Before we state our main theorem about the transitivity of twisted skew products we recall a few more definitions and facts about an automorphism Ψ of a finitely generated free Abelian group, or equivalently, the matrix that represents it in some basis. For simplicity we just give the terminology and notation for the matrices.

The *spectral radius* of A is denoted by $\rho(A)$. Since matrix A is invertible if and only if $\det(A) = \pm 1$, when A is invertible, if $\rho(A) \neq 1$, there must be eigenvalues of modulus both larger and less than one. If $\rho(A) = 1$, the eigenvalues of A must all lie on the unit circle. Now for an eigenvalue λ of A, certainly its minimal polynomial must be a factor of the characteristic polynomial of A. Thus, if $\rho(A) = 1$, all of the algebraic conjugates of any eigenvalue λ must lie on the unit circle. Thus, as a consequence of a theorem of Kronecker (see, for example [**Mar77**, problem 11 on p. 40]), λ must be a root of unity (the author learned this from Peter Sin whom he acknowledges with gratitude). Thus, we see that if A is invertible and $\rho(A) = 1$, then for some integer $N > 0$, $\text{spec}(A^N) = \{1\}$. This allows the following definition.

Definition 10.3. If $A \in \text{SL}(d, \mathbb{Z})$ and has spectral radius equal to one, $\rho(A) = 1$, let $N(A)$ be the least positive integer with $\text{spec}(A^{N(A)}) = \{1\}$.

While many of the preceding results involved skew products with the base shift a countable Markov chain, the main theorem concerns the case where the base shift is a transitive subshift of finite type. This restriction is required in order to use the various properties of the rotation set from §7.3. It is worth remarking, however, that the proof itself uses an induction with untwisted extensions of countable state Markov shifts. It is also worth remarking that since the alternative (b) in the theorem only requires that the twisting matrix A has spectral radius one, we must consider $\tau^{N(A)}$ in order to ensure that the Fried quotient is non-trivial and the rotation set is thus defined. Alternative (b) also contains a condition that is the analog of totally transitive for finite quotients; we say that τ has the *total ftp* if τ^k has the ftp for all $k > 0$.

THEOREM 10.4. *Assume that $\tau : \Sigma \times \mathbb{Z}^d \to \Sigma \times \mathbb{Z}^d$ is a twisted skew product with twisting matrix A, height function h and base shift (Σ, σ) which is a transitive subshift of finite type. The following are equivalent:*
(a) *τ is totally transitive;*
(b) *$\rho(A) = 1$, τ has the total ftp; and $0 \in \text{Int}(\text{rot}_F(\tau^{N(A)}))$;*
(c) *τ has the total ftp and its periodic points are dense in $\Sigma \times \mathbb{Z}^d$;*
(d) *τ is topologically mixing.*

Proof. We first show that (b) implies (a). For simplicity of notation, let $\eta_k = \tau^{kN(A)}$. Using (4.2) the twisting matrix of η_k is $A^{kN(A)}$. By definition $\text{spec}(A^{N(A)}) = \{1\}$, and so for all $k > 0$, spec $A^{kN(A)} = \{1\}$ also. Since by hypothesis τ has the total ftp, each η_k also has the ftp. We now show that the Fried quotient of each η_k is transitive, and then Lemma 10.2 will give that each η_k is transitive.

Corollary 9.4 and (5.6) yield that

$$\text{rot}((\tau^{kN(A)})_{F(kN(A))}) = \text{rot}(((\tau^{N(A)})_{F(N(A))})^k) = k\,\text{rot}((\tau^{N(A)})_{F(N(A))}).$$

Thus, since $0 \in \text{Int}(\text{rot}_F(\tau^{N(A)}))$ by hypothesis, $0 \in \text{Int}(\text{rot}_F(\eta_k))$ for all $k > 0$. As already noted, each η_k has the ftp and so as remarked in §4.3, $(\eta_k)_F$ has the ftp for all $k > 0$. Thus, since each $(\eta_k)_F$ is untwisted, by Theorem 7.1, each $(\eta_k)_F$ is transitive, finishing the proof that each η_k is transitive. Now since $\eta_k = (\tau^k)^{N(A)}$, a power of every τ^k is transitive, and so each τ^k is also transitive, proving (a).

Now, conversely, assume that τ is totally transitive. If $\rho(A) \neq 1$, there must be an eigenvalue λ with $|\lambda| > 1$. Assume first that λ is real and positive and treating A as acting on $\mathbb{R}^d$, let $v_1 \in \mathbb{R}^d$ be an eigenvector corresponding to λ. Extend v_1 to a basis for $\mathbb{R}^d$ and for $w \in \mathbb{R}^d$, let $\Phi(w)$ be its first component with respect to this basis, and so $\Phi(A^k w) = \lambda^k \Phi(w)$.

Since by hypothesis (Σ, σ) is a subshift of finite type, h is bounded and let $|\Phi \circ h| < C$. Now,

$$\tau^k(s, \mathbf{n}) = (\sigma^k(s), A^k\mathbf{n} + A^{k-1}h(s) + \cdots + Ah(\sigma^{k-2}(s)) + h(\sigma^{k-1}(s))), \qquad (10.4)$$

and so

$$|\Phi \circ \pi_2(\tau^k(s, \mathbf{n}))| \geq \lambda^k |\Phi(\mathbf{n})| - (\lambda^{k-1} + \cdots + \lambda + 1)C$$
$$= \lambda^k \left(|\Phi(\mathbf{n})| - \frac{C}{\lambda - 1} \right) + \frac{C}{\lambda - 1}. \tag{10.5}$$

Thus, if $\mathbf{n}_0$ is such that $|\Phi(\mathbf{n}_0)| > C/(\lambda - 1)$, then $|\Phi \circ \pi_2(\tau^k(s, \mathbf{n}_0))| \to \infty$ as $k \to \infty$ and so no point in the open set $\Sigma \times \{\mathbf{n}_0\}$ can have a dense, forward τ orbit, and so τ is not transitive. The case where the eigenvalue λ is complex is similar, but now one uses a Φ that projects onto the two-dimensional subspace associated with λ in the real Jordan form. Thus, τ being transitive implies that $\rho(A) = 1$. Finally, if τ is totally transitive, then certainly any quotient of any power is also transitive and so using Theorem 7.1, τ has the total ftp and $0 \in \mathrm{Int}(\mathrm{rot}_F(\tau^{N(A)}))$, finishing the proof that (a) implies (b).

Now note that the argument just given shows that shows that if $\rho(A) \neq 1$, then the recurrent points of τ cannot be dense in $\Sigma \times \mathbb{Z}^d$. Thus, assuming that (c) holds, we have $\rho(A) = 1$. In addition, if τ has dense periodic points, then so does any power or quotient, so in particular, $(\tau^{N(A)})_F$ has dense periodic points and is untwisted and has the ftp by construction. Thus by Theorem 7.1, $0 \in \mathrm{Int}(\mathrm{rot}_F(\tau^{N(A)}))$ proving that (c) implies (b).

The fact that transitivity implies dense periodic points was noted in the proof of Theorem 7.1, and so (a) implies (c). The equivalence of (a) and (d) for countable state Markov shifts is standard and was noted in §2.2, finishing the proof. □

Remark 10.5. For future reference we note that the argument above based on (10.5) yields that if τ is transitive, then $\rho(A) = 1$. In addition, an analogous argument gives that $|\Phi(\mathbf{n} - \mathbf{n}')| > (2C)/(\lambda - 1)$ implies that for any $s, s' \in \Sigma$,

$$|\Phi \circ \pi_2(\tau^k(s, \mathbf{n})) - \Phi \circ \pi_2(\tau^k(s', \mathbf{n}'))| \to \infty,$$

as $k \to \infty$, where $\pi_2 : \Sigma \times \mathbb{Z}^d \to \mathbb{Z}^d$ is the projection.

11. *Lifted rel pseudo-Anosov maps*

With the help of Theorem 4.2 we now apply Theorem 10.4 on twisted skew products to the study of transitive lifts of rel pseudo-Anosov maps.

11.1. *Necessary and sufficient conditions for total transitivity.*

We recall some definitions about the invariant singular measured foliations, $\mathcal{F}^u$ and $\mathcal{F}^s$, of a rel pseudo-Anosov map. Owing to the presence of a finite number of singularities, there are several ways to define a 'leaf' of the foliation. Fix one foliation, say $\mathcal{F}^u$, and let P be the set comprised of the singular points of $\mathcal{F}^u$ and the boundary components of the surface of M. On the punctured surface $M - P$, $\mathcal{F}^u$ is a non-singular foliation. Points $x \in M - P$ are called *regular points* and the leaf of $\mathcal{F}^u$ containing these points is the leaf of the foliation on $M - P$. Thus, the leaves containing regular points are either immersed lines in M or immersed half-lines which 'begin' at a singularity. In the latter case the singular point is said to be *associated with the leaf*. The leaves which contain non-regular points are called *trivial leaves*. These are of two types: a singular point is consider a trivial leaf as are any leaves which are segments contained in the boundary of M. We also recall the dynamical meaning of the invariant foliations. Given a topological metric d on M,

$$d(\phi^n(x_1), \phi^n(x_2)) \to 0 \tag{11.1}$$

as $n \to \infty$ (respectively, $n \to -\infty$) if and only if x_1 and x_2 are on the same leaf of $\mathcal{F}^s$ ($\mathcal{F}^u$), or their leaves are associated with the same singularity.

In the universal Abelian cover the singular foliations lift to a pair $\tilde{\mathcal{F}}^u$ and $\tilde{\mathcal{F}}^s$. For a point $\tilde{x}$ in $\tilde{M}$ its leaf of $\tilde{\mathcal{F}}^u$ is defined as in the base or, equivalently, project $\tilde{x}$ to $x \in M$, find the leaf of x and then the leaf of $\tilde{x}$ is the lift of this leaf in $\tilde{M}$ which contains $\tilde{x}$. The analog of (11.1) also holds using any equivariant metric $\tilde{d}$.

THEOREM 11.1. *Let $\phi : M \to M$ be rel pseudo-Anosov, $\tilde{M}$ is the universal Abelian cover and $\tilde{\phi}$ is a lift of ϕ to $\tilde{M}$. The following are equivalent:*
(a) *$\tilde{\phi}$ is totally transitive;*
(b) *$\rho(\phi_*) = 1$ and $0 \in \mathrm{Int}(\mathrm{rot}_F(\tilde{\phi}^{N(\phi_*)}))$;*
(c) *the set of periodic points of $\tilde{\phi}$ is dense in $\tilde{M}$;*
(d) *$\tilde{\phi}$ is topologically mixing;*
(e) *there is a periodic regular point $\tilde{x}$ of $\tilde{\phi}$ so that the leaf of the lifted foliation $\tilde{\mathcal{F}}^u$ containing $\tilde{x}$ is dense in $\tilde{M}$.*

Proof. Let τ be a twisted skew product corresponding to $\tilde{\phi}$, and let $\tilde{\alpha}$ be the semiconjugacy given in §6.3. The equivalence of (a), (b), (c) and (d) follows from Proposition 6.1 and Theorem 10.4.

Now assume that (a)–(d) hold and let $\tilde{x} \in \tilde{M}$ be a periodic point for $\tilde{\phi}$ with period n_0. Since the collection of singular points is a discrete set in $\tilde{M}$ and periodic points of $\tilde{\phi}$ are dense, we may assume that $\tilde{x}$ is a regular point. By hypothesis $\tilde{\phi}^{n_0}$ and thus τ^{n_0} have dense forward orbits using Theorem 2.1. Treating τ as a countable state Markov shift, let $\hat{t} = \ldots \hat{t}_{-1}\hat{t}_0\hat{t}_1 \ldots$ be a point whose forward orbit is dense under τ^{n_0}.

Let $t = (s, \mathbf{n}) \in \Sigma \times \mathbb{Z}^d$ be a period-n_0 point of τ with $\tilde{\alpha}(s, \mathbf{n}) = \tilde{x}$. If the periodic block of t is $b = b_0 \ldots b_{n_0-1}$, since τ is transitive we have an allowable $b_0 \to \hat{t}_0$. Call this block c. Now form the sequence $w = \ldots b\, b\, b\, c\, \hat{t}_0\, \hat{t}_1 \ldots$. By construction, $\tilde{\alpha}(o_+(w, \tau^{n_0})) = o_+(\tilde{\alpha}(w), \tilde{\phi}^{n_0})$ is dense in $\tilde{M}$. In addition, $\tau^{-n_0 k}(w) \to t$ as $k \to \infty$ and so $\tilde{\phi}^{-n_0 k}(\tilde{\alpha}(w)) \to \tilde{x}$. Thus, if $\tilde{\mathcal{L}}$ is the leaf of $\tilde{\mathcal{F}}^u$ which contains $\tilde{x}$, since $\tilde{x}$ is a period n_0 point we have $\tilde{\alpha}(w) \in \tilde{\mathcal{L}}$ and so $o_+(\tilde{\alpha}(w), \tilde{\phi}^{n_0}) \subset \tilde{\mathcal{L}}$ as well, and so $\tilde{\mathcal{L}}$ is dense in $\tilde{M}$, finishing the proof that (a)–(d) imply (e).

Now assume that (e) holds and let $\tilde{x}$ be a regular periodic point with period n_0 and its unstable leaf $\tilde{\mathcal{L}}$ dense in $\tilde{M}$. We show that (b) follows by first showing that $\rho(\phi_*) = 1$. Assume to the contrary that $\rho(\phi_*) \neq 1$. Since $\det(\phi_*) = 1$, this implies that ϕ_*^{-1} has an eigenvalue λ with $|\lambda| > 1$. Then, just as in the proof of Theorem 10.4 and using (6.5), there is a linear functional $\Phi : \mathbb{R}^n \to \mathbb{R}$ and a constant $C > 0$ so that $\Phi(\tilde{\beta}(\tilde{y})) > C/(\lambda - 1)$ implies that $\Phi(\tilde{\beta}(\tilde{\phi}^k(\tilde{y}))) \to \infty$ as $k \to -\infty$, where $\tilde{\beta} : \tilde{M} \to \mathbb{R}^d$ is the map constructed in §3.3.

Now since $\tilde{L}$ is an unstable leaf containing the periodic regular point $\tilde{x}$, if I is a small segment in $\tilde{L}$ with $\tilde{x}$ in its interior, then $\bigcap_{j>0} \tilde{\phi}^{-n_0 j}(I) = \tilde{x}$ and $\bigcup_{j\geq 0}(\tilde{\phi}^{n_0 j}(I)) = \tilde{L}$. Since $\tilde{L}$ is dense in $\tilde{M}$ there are certainly $\tilde{y} \in \bigcup_{j\geq 0}(\tilde{\phi}^{n_0 j}(I))$ with $\Phi(\tilde{\beta}(\tilde{y})) > C/(\lambda - 1)$.

Thus, $\Phi(\tilde{\beta}(\tilde{\phi}^n(\tilde{y}))) \to \infty$ as $n \to -\infty$, but $\tilde{\phi}^{-n_0 j_0}(\tilde{y}) \in I$ for some $j_0 > 0$ implies that $\tilde{\phi}^{n_0 j}(\tilde{y}) \to \tilde{x}$ as $j \to -\infty$, a contradiction, finishing the proof that $\rho(\phi_*) = 1$.

As in Definition 10.3, let $N(\phi_*)$ be such that $\mathrm{spec}(\phi_*^{N(\phi_*)}) = \{1\}$. If $\tilde{M}'$ is the Fried cover of $\tilde{\phi}^{N(\phi_*)}$, $\tilde{M}' = \tilde{M}/F^{(N(\phi_*))}$ in the notation of §9.4. Let $\chi : \tilde{M} \to \tilde{M}'$ be the projection and $\tilde{\phi}'$ be the projection of $\tilde{\phi}$ to $\tilde{M}'$. Let $\tilde{x}' = \chi(\tilde{x})$ where $\tilde{x}$ is a period-n_0 regular point with its unstable leaf $\tilde{L}$ dense in $\tilde{M}$. If $\tilde{L}'$ is the unstable leaf in $\tilde{M}'$ which contains $\tilde{x}'$, then certainly $\tilde{L}' = \chi(\tilde{L})$, and so $\tilde{L}'$ is dense in $\tilde{M}'$. Now as above let $I' \subset \tilde{L}'$ be a small segment with $\tilde{x}'$ in its interior and so $\bigcup_{j>0}((\tilde{\phi}')^{n_0 j}(I'))$ is dense in $\tilde{M}'$.

Now let τ' be the Fried quotient of $\tau^{N(\phi_*)}$ and $\tilde{\alpha}'$ the induced semiconjugacy from τ' to $\tilde{\phi}'$. Since $\bigcup_{j>0}((\tilde{\phi}')^{n_0 j}(I'))$ is dense in $\tilde{M}'$, using that fact that $\tilde{\alpha}$ is a quasi-isometry as observed in §6.4, certainly if h' is height function of τ', for every onto linear functional L, the corresponding supremum in (7.2) is infinite. Thus, by Lemma 7.8 and using Lemma 6.1, $0 \in \mathrm{Int}(\mathrm{rot}(\tilde{\phi}')) = \mathrm{Int}(\mathrm{rot}_F(\tilde{\phi}^{N(\phi_*)}))$, completing the proof. □

Remark 11.2. When $G \subset H_1(M)$ with $\phi_*(G) = G$ and $H_1(M)/G$ torsion-free, the analog of Theorem 11.1 for the cover $\tilde{M}_G$ has an almost identical statement and proof. The case where $H_1(M)/G$ has torsion requires a more elaborate statement of condition (b), and we leave it to the interested reader.

Remark 11.3. Rel pseudo-Anosov maps on the torus which act on homology by the skew matrix A in (10.1) are the simplest examples of rel pseudo-Anosov maps which satisfy the hypothesis of the theorem with $\phi_* \neq \mathrm{id}$. Rel pseudo-Anosov maps in this class were studied in [**Doe97, DM97, PW**]. The general notion of rot_F when restricted to this torus shear case was called the shear rotation interval by Doeff, and many of the basic properties as in §7.3 above were proved. A fascinating explicit example of map $\tilde{\psi}$ on the plane which is the lift of a rel pseudo-Anosov map in this class was given by [**CG05**] and analyzed by [**Mac06**]. In the example $\mathrm{rot}_F(\tilde{\psi}) = [0, 1]$, and so by Theorem 11.1, the example is not transitive on the plane. However, for example, the theorem does imply that $\delta_{(-1,0)} \circ \tilde{\psi}^2$ is transitive.

Remark 11.4. When ϕ is pseudo-Anosov rel a non-empty finite set, its foliations have one-prongs and are thus non-orientable. A true pseudo-Anosov map has oriented foliations if and only if $\rho(\phi_*)$ is equal to its dilation $\lambda > 1$ (see, for example, [**BB07**, Lemma 4.3]). Thus, in the situation of Theorem 11.1, where $\rho(\phi_*) = 1$, the invariant foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ are always non-orientable.

Remark 11.5. The characterization of transitive twisted skew products given in Theorem 10.4 allows the twisting matrix to be any $A \in \mathrm{SL}(d, \mathbb{Z})$. For a surface homeomorphism f whose lift is modeled by the skew product, $f_* = A$ has additional structure. When the surface is closed, f_* is symplectic and the addition of boundary components only gives rise to permutations on $H_1(M)$. Thus, the surface dynamics applications do not require the full force of Theorem 10.4.

11.2. *The case when $\phi_* = \mathrm{id}$.* Most of the literature on the dynamics of lifted maps concerns the case of maps isotopic to the identity. In this as well as the more general case where ϕ acts trivially on first homology, condition (b) in Theorem 11.1 is replaced by the simpler condition $0 \in \text{Int}(\text{rot}(\tilde{\phi}))$.

In this case there is a fair amount known about dynamical representatives for elements of the rotation set. For example, it follows from the results of Ziemian [**Zie95**] that for each $r \in \text{Int}(\text{rot}(\tilde{\phi}))$, there is a compact, ϕ-invariant set $Y_r \subset M$, so that $\text{rot}(y, \tilde{\phi}) = r$ for all $y \in Y_r$. In addition, as a consequence of theorem of Jenkinson in [**Jen01**] (see also [**Kwa95**]) for each such r there is a ϕ-invariant, ergodic, fully supported Gibbs probability measure μ_r with $\text{rot}(x, \tilde{\phi}) = r$ for μ_r-almost every point.

In contrast, it is a simple consequence of Theorem 11.1 that when $\phi_* = \text{id}$ the rotation number does not exist for the topologically typical point in M. This implies that the topologically generic point is not generic for any ϕ-invariant Borel measure. Here is the argument: assume that $0 \in \text{Int}(\text{rot}(\tilde{\phi}))$. Thus, from Theorem 11.1 there is a dense, G_δ set $X_0 \subset M$ so that $x \in X_0$ implies that for any lift $\tilde{x}$ of x, the orbit $o(\tilde{x}, \tilde{\phi})$ is dense in $\tilde{M}$. Thus, for $x \in X_0$, if $\text{rot}(x, \tilde{\phi})$ exists, it must be zero. However, now pick $\mathbf{p}/q \in \text{Int}(\text{rot}(\tilde{\phi}))$ with $0 \neq \mathbf{p}/q$ and let $\tilde{f} = \delta_{-\mathbf{p},q} \circ \tilde{\phi}^q$. Using (5.6), $0 \in \text{Int}(\text{rot}(\tilde{f}))$ and so again using Theorem 11.1, there is a dense, G_δ set $X_1 \subset M$ so that $x \in X_1$ implies that if $\text{rot}(x, \tilde{f})$ exists, it must be zero and so $\text{rot}(x, \tilde{\phi}) = \mathbf{p}/q$. It then follows that for x in the dense, G_δ-set $X_0 \cap X_1$, $\text{rot}(x, \tilde{\phi})$ cannot exist.

While there are rel pseudo-Anosov maps in every isotopy class, there are some classes, for example the identity class, which cannot contain a true pseudo-Anosov map, i.e. one whose invariant foliations have no interior one-prong singularities. Thurston observed that there are many mapping classes which act trivially on $H_1(M)$ which do contain a true pseudo-Anosov map, or in the language of his classification theorem, are pseudo-Anosov mapping classes [**Thu88**]. The collection of mapping classes which act trivially on $H_1(M)$ is called the Torelli group and its properties have been much studied (see [**Joh83, Far06**] for surveys).

As dynamical systems pseudo-Anosov maps which act trivially on homology are quite interesting, and there are many tools available for their study such as the rotation set and twisted transition matrices [**Fri86, BB**]. They also have useful isotopy stable properties. For example, if g is any homeomorphism which is isotopic to a true pseudo-Anosov ϕ_* with $\phi_* = \text{id}$, then if $\tilde{\phi}$ is a transitive lift to $\tilde{M}$, the corresponding lift $\tilde{g}$ of g will always have 'well-travelled' orbits, i.e. orbits that repeatedly visit every fundamental domain in $\tilde{M}$. This follows from Handel's global shadowing theorem [**Han85**].

11.3. *H_1-transitivity.* We now give the proof of Theorem 1.2 stated in the introduction.

Proof of Theorem 1.2. Fix a lift $\tilde{\phi}$ of ϕ to $\tilde{M}$. Assume that ϕ is H_1-transitive, and so for some $q > 0$, $\mathbf{n} \in \mathbb{Z}^d$, we have that $\eta := \delta_{\mathbf{n}} \circ \tilde{\phi}^q$ is transitive. Now η is a lift of $\tilde{\phi}^q$ and so by Theorem 11.1, $\rho(\phi_*^q) = 1$ and so $\rho(\phi_*) = 1$.

Now, conversely, assume that $\rho(\phi_*) = 1$. By Theorem 6.1(c), $\text{rot}_F(\tilde{\phi}^{N(\phi_*)})$ has an interior. Pick $\mathbf{p}/q \in \text{Int}(\text{rot}_F(\tilde{\phi}^{N(\phi_*)}))$. Now as in the proof of Corollary 9.4, we may write $\mathbb{Z}^d = W_1 \oplus W'$ with W_1 naturally identified with $\mathbb{Z}^d / F^{(kN(\phi_*))}$, for all $k > 0$. Let $\mathbf{m} \in \mathbb{Z}^d$ be $\mathbf{m} = (\mathbf{p}, 0)$ with $0 \in W'$ and so by (3.4), $\eta' := \delta_{-\mathbf{m}} \tilde{\phi}^{qN(\phi_*)}$, has $0 \in \text{Int}(\text{rot}_F(\eta'))$. Now certainly $\rho(\phi_*) = 1$ implies $\rho(\eta') = 1$ and so by Theorem 11.1, η' is transitive.

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The implication (a) implies (c) also follows from Theorem 11.1. Now assume that (c) holds and for the sake of contradiction that $\rho(\phi_*) \neq 1$. Fix a lift $\tilde{\phi}$ and let τ be a twisted skew product that corresponds to it and $\tilde{\alpha}$ the semiconjugacy given in §6.3. Since $\rho(\phi_*) \neq 1$, using Remark 10.5, there is a non-zero linear functional $\Phi : \mathbb{R}^d \to \mathbb{R}$ and a constant $C' > 0$ so $|\Phi(\mathbf{n} - \mathbf{n}')| > C'$ implies that for any $s, s' \in \Sigma$,

$$|\Phi \circ \pi_2(\tau^k(s, \mathbf{n})) - \Phi \circ \pi_2(\tau^k(s', \mathbf{n}'))| \to \infty, \tag{11.2}$$

as $k \to \infty$. However, if $\tilde{\mathcal{L}}$ is a leaf of $\tilde{\mathcal{F}}^u$ that is dense in $\tilde{M}$, certainly there are $(s, \mathbf{n})$ and $(s', \mathbf{n}')$ with $\tilde{\alpha}(s, \mathbf{n}) \in \tilde{\mathcal{L}}$ and $\tilde{\alpha}(s', \mathbf{n}') \in \tilde{\mathcal{L}}$ and $|\Phi(\mathbf{n} - \mathbf{n}')| > C'$. Then by (11.2) and the fact from §6.4 that $\tilde{\alpha}$ is a quasi-isometry from the pseudometric $d_1((s, \mathbf{n}), (s', \mathbf{n}')) = \|\mathbf{n} - \mathbf{n}'\|$ to a lifted metric $\tilde{d}$ on $\tilde{M}$, we have $\tilde{d}(\tilde{\alpha}(\tau^k(s, \mathbf{n})), \tilde{\alpha}(\tau^k(s', \mathbf{n}'))) \to \infty$. Thus, since $\tilde{\alpha}$ is a semiconjugacy, $\tilde{d}(\tilde{\phi}^k(\tilde{\alpha}(s, \mathbf{n})), \tilde{\phi}^k(\tilde{\alpha}(s', \mathbf{n}'))) \to \infty$ as $k \to \infty$ in contradiction to the fact that $\tilde{\alpha}(s, \mathbf{n})$ and $\tilde{\alpha}(s', \mathbf{n}')$ are on the same leaf of the unstable foliation (see (11.1)). □

11.4. *The lifted foliations.* In the invariant foliations associated with a rel pseudo-Anosov map on a compact surface all non-trivial leaves are dense in M. For a H_1-transitive rel pseudo-Anosov map, for the lifted foliations the typical leaf is dense, but there are always non-dense leaves.

PROPOSITION 11.6. *Let $\tilde{\mathcal{F}}^u$ and $\tilde{\mathcal{F}}^s$ be the lifted foliations to the universal Abelian cover $\tilde{M}$ of a rel pseudo-Anosov map $\phi : M \to M$.*

(a) *If the lifted foliation has one dense leaf, there is a dense G_δ-set $Z \subset \tilde{M}$ so that $\tilde{x} \in Z$ implies that the leaf of $\tilde{\mathcal{F}}^u$ containing $\tilde{x}$ is dense in $\tilde{M}$.*
(b) *Each non-trivial leaf of the lifted foliations is unbounded.*
(c) *The lifted foliations always has non-trivial leaves which are not dense.*

Proof. The proof of (a) is a minor alteration of a standard proof: fix a countable base U_n for the topology of $\tilde{M}$. For each n, let A_n be all of the points of $\tilde{M}$ which are contained in a leaf of $\tilde{\mathcal{F}}^u$ which intersects U_n. Since there is a dense leaf by hypothesis, it follows immediately that each A_n is dense. Each A_n is also open as a consequence of our slightly peculiar definition of 'leaf' in §11.1. Now $\cap A_n$ is exactly all of the points of $\tilde{M}$ contained in dense leaves, and by the Baire category theorem, $\cap A_n$ is dense G_δ.

To prove (b) note that if in $\tilde{M}$ a non-trivial leaf was bounded, by letting $\Gamma = N\mathbb{Z}^d$ for N large enough and $\mathbb{Z}^d = H_1(M)$, the rel pseudo-Anosov map $\tilde{\phi}_\Gamma$ on the compact manifold $\tilde{M}/\Gamma$ would possess a non-trivial leaf that was not dense, a contradiction.

The proof of (c) starts with the observation that if $\rho(\phi_*) \neq 1$, then by Theorem 1.2, the lifted foliations have no dense leaves and so (c) certainly follows. So assume now that $\rho(\phi_*) = 1$ and so from Theorem 1.2 we may find a $\tilde{g}$ which is a transitive lift of an iterate $g := \phi^k$. Now in addition assume that $\phi_* = \text{id}$ and so $g_* = \text{id}$ as well. By Theorem 7.6 there is a periodic point x_0 of g such that its lift $\tilde{x}_0$ to $\tilde{M}$ satisfies $\text{rot}(\tilde{x}_0, \tilde{g}) \in \text{Fr}(\text{rot}(\tilde{\phi}))$. Using Lemma 7.8 as in the proof of Theorem 11.1, we see that if the leaf containing $\tilde{x}_0$ (or one of its associated leaves if $\tilde{x}_0$ is a singularity) were dense, $\text{rot}(\tilde{x}_0, \tilde{g})$ would not be a boundary point of $\text{rot}(\tilde{g})$, a contradiction.

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In the more general situation that $\rho(\phi_*) = 1$, use the argument in the previous paragraph to show that the foliations in the Fried quotient of $\phi^{N(\phi_*)}$ have non-dense leaves and so the lifts of these leaves to $\tilde{M}$ are also not dense. □

Remark 11.7. Under a variety of hypotheses which include the case where a rel pseudo-Anosov ϕ is isotopic to the identity on a closed surface, Pollicott and Sharp show in [**PS07**] that the transverse measures on the lifted foliations are ergodic.

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