Journal of the Inst. of Math. Jussieu (2012) **11**(1), 189–220
doi:10.1017/S1474748010000290

C^*-PSEUDO-MULTIPLICATIVE UNITARIES, HOPF C^*-BIMODULES AND THEIR FOURIER ALGEBRAS

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(Received 10 February 2010; accepted 3 September 2010)

Abstract We introduce C^*-pseudo-multiplicative unitaries and concrete Hopf C^*-bimodules for the study of quantum groupoids in the setting of C^*-algebras. These unitaries and Hopf C^*-bimodules generalize multiplicative unitaries and Hopf C^*-algebras and are analogues of the pseudo-multiplicative unitaries and Hopf–von Neumann-bimodules studied by Enock, Lesieur and Vallin. To each C^*-pseudo-multiplicative unitary, we associate two Fourier algebras with a duality pairing and in the regular case two Hopf C^*-bimodules. The theory is illustrated by examples related to locally compact Hausdorff groupoids. In particular, we obtain a continuous Fourier algebra for a locally compact Hausdorff groupoid.

Keywords: quantum groupoid; groupoid; duality

AMS 2010 *Mathematics subject classification:* Primary 46L55

1. Introduction

Multiplicative unitaries, which were first systematically studied by Baaj and Skandalis [**3**], are fundamental to the theory of quantum groups in the setting of operator algebras and to generalizations of Pontrjagin duality [**28**]. First, one can associate to every locally compact quantum group a multiplicative unitary [**13, 14, 17**]. Out of this unitary, one can construct two Hopf C^*-algebras, where one coincides with the initial quantum group, while the other is the generalized Pontrjagin dual of the quantum group. The duality manifests itself by a pairing on dense Fourier subalgebras of the two Hopf C^*-algebras. These Hopf C^*-algebras can be completed to Hopf–von Neumann algebras and are reduced in the sense that they correspond to the regular representations of the quantum group and of its dual, respectively.

Much of the theory of quantum groups has been generalized for quantum groupoids in a variety of settings, for example, for finite quantum groupoids in the setting of finite-dimensional C^*-algebras by Böhm, Szlachányi, Nikshych and others [**5–7, 18**] and for measurable quantum groupoids in the setting of von Neumann algebras by Enock, Lesieur and Vallin [**9–11, 16**]. Fundamental for the second theory are the Hopf–von Neumann bimodules and pseudo-multiplicative unitaries introduced by Vallin [**32, 33**].

In this article, we introduce generalizations of multiplicative unitaries and Hopf C^*-algebras that are suited for the study of locally compact quantum groupoids in the setting of C^*-algebras, and extend some of the results on multiplicative unitaries that were obtained by Baaj and Skandalis in [**3**]. In particular, we associate to every regular C^*-pseudo-multiplicative unitary two Hopf C^*-bimodules and two Fourier algebras with a duality pairing.

Our concepts are related to their von Neumann-algebraic counterparts as follows. In the theory of quantum groups, one can use the multiplicative unitary to pass between the setting of von Neumann algebras and the setting of C^*-algebras and thus obtains a bijective correspondence between measurable and locally compact quantum groups. This correspondence breaks down for quantum groupoids—already for ordinary spaces, considered as groupoids consisting entirely of units, a measure does not determine a topology. In particular, one cannot expect to pass from a measurable quantum groupoid in the setting of von Neumann algebras to a locally compact quantum groupoid in the setting of C^*-algebras in a canonical way. The reverse passage, however, is possible, at least on the level of the unitaries and the Hopf bimodules.

Fundamental to our approach is the framework of modules, relative tensor products and fibre products in the setting of C^*-algebras introduced in [**25**]. That article also explains in detail how the theory developed here can be reformulated in the setting of von Neumann algebras, where we recover Vallin's notions of a pseudo-multiplicative unitary and a Hopf–von Neumann bimodule, and how to pass from the level of C^*-algebras to the setting of von Neumann algebras by means of various functors.

The theory presented here overcomes several restrictions of our former generalizations of multiplicative unitaries and Hopf C^*-algebras [**27**] (see also [**26**]). It was applied already in [**31**] to the definition and study of compact C^*-quantum groupoids, and in [**30**] to the study of reduced crossed products for coactions of Hopf C^*-bimodules on C^*-algebras and to an extension of the Baaj–Skandalis duality theorem. In [**29**], we furthermore associate to every C^*-pseudo-multiplicative unitary a C^*-tensor category of (co)representations and two universal Hopf C^*-bimodules that are related to the reduced Hopf C^*-bimodules studied here similarly like the universal to the reduced C^*-algebra of a group or groupoid.

Organization

This article is organized as follows. We start with preliminaries, summarizing notation, terminology and some background on Hilbert C^*-modules.

In § 2, we recall the notion of a multiplicative unitary and define C^*-pseudo-multiplicative unitaries. This definition involves C^*-modules over C^*-bases and their relative tensor product, which were introduced in [**25**] and which we briefly recall. As an example, we construct the C^*-pseudo-multiplicative unitary of a locally compact Hausdorff groupoid. We shall come back to this example frequently.

In § 3, we associate to every well behaved C^*-pseudo-multiplicative unitary two Hopf C^*-bimodules. These Hopf C^*-bimodules are generalized Hopf C^*-algebras, where the target of the comultiplication is no longer a tensor product but a fibre product that is taken relative to an underlying C^*-base. Inside these Hopf C^*-bimodules, we identify dense convolution subalgebras which can be considered as generalized Fourier algebras, and construct a dual pairing on these subalgebras. To illustrate the theory, we apply all constructions to the unitary associated to a groupoid G, where one recovers the reduced groupoid C^*-algebra of G on one side and the function algebra of G on the other side.

In § 4, we show that every C^*-pseudo-multiplicative unitary satisfying a certain regularity condition is well behaved. This condition is satisfied, for example, by the unitaries associated to groupoids and by the unitaries associated to compact quantum groupoids. Furthermore, we collect some results on proper and étale C^*-pseudo-multiplicative unitaries.

Terminology and notation

Given a subset Y of a normed space X, we denote by $[Y] \subset X$ the closed linear span of Y. We call a linear map ϕ between normed spaces *contractive* or a *linear contraction* if $\|\phi\| \leqslant 1$.

All sesquilinear maps like inner products of Hilbert spaces are assumed to be conjugate-linear in the first component and linear in the second one. Let H, K be Hilbert spaces. We denote by X' the commutant of a subset $X \subseteq \mathcal{L}(H)$. Given a C^*-subalgebra $A \subseteq \mathcal{L}(H)$ and a $*$-homomorphism $\pi\colon A \to \mathcal{L}(K)$, we put

$$\mathcal{L}^{\pi}(H, K) := \{T \in \mathcal{L}(H, K) \mid Ta = \pi(a)T \text{ for all } a \in A\}. \tag{1.1}$$

We shall use some theory of groupoids; for background, see [**22**] or [**20**]. Given a groupoid G, we denote its unit space by G^0, its range map by r, its source map by s, and let $G_r \times_r G = \{(x, y) \in G \times G \mid r(x) = r(y)\}$, $G_s \times_r G = \{(x, y) \in G \times G \mid s(x) = r(y)\}$ and $G^u = r^{-1}(u)$, $G_u = s^{-1}(u)$ for each $u \in G^0$.

We shall make extensive use of (right) Hilbert C^*-modules and the internal tensor product; a standard reference is [**15**]. Let A and B be C^*-algebras. Given Hilbert C^*-modules E and F over B, we denote by $\mathcal{L}_B(E, F)$ the space of all adjointable operators from E to F. Let E and F be C^*-modules over A and B, respectively, and let $\pi\colon A \to \mathcal{L}_B(F)$ be a $*$-homomorphism. Recall that the internal tensor product $E \otimes_\pi F$ is the Hilbert C^*-module over B which is the closed linear span of elements $\eta \otimes_\pi \xi$, where $\eta \in E$ and $\xi \in F$ are arbitrary and $\langle \eta \otimes_\pi \xi | \eta' \otimes_\pi \xi'\rangle = \langle \xi | \pi(\langle \eta|\eta'\rangle)\xi'\rangle$ and $(\eta \otimes_\pi \xi)b = \eta \otimes_\pi \xi b$ for all $\eta, \eta' \in E$, $\xi, \xi' \in F$, $b \in B$ [**15**, § 4]. We denote the internal tensor product by '$\otimes$' and drop the index π if the representation is understood; thus, for example, $E \otimes F = E \otimes_\pi F = E \otimes_\pi F$.

We also define a *flipped internal tensor product* $F\,{}_\pi\!\otimes E$ as follows. We equip the algebraic tensor product $F \odot E$ with the structure maps $\langle \xi \odot \eta | \xi' \odot \eta'\rangle := \langle \xi | \pi(\langle \eta|\eta'\rangle)\xi'\rangle$, $(\xi \odot \eta)b := \xi b \odot \eta$, form the separated completion, and obtain a Hilbert C^*-module $F\,{}_\pi\!\otimes E$ over B which is the closed linear span of elements $\xi\,{}_\pi\!\otimes \eta$, where $\eta \in E$ and $\xi \in F$ are arbitrary and $\langle \xi\,{}_\pi\!\otimes \eta | \xi'\,{}_\pi\!\otimes \eta'\rangle = \langle \xi | \pi(\langle \eta|\eta'\rangle)\xi'\rangle$ and $(\xi\,{}_\pi\!\otimes \eta)b = \xi b\,{}_\pi\!\otimes \eta$ for all $\eta, \eta' \in E$, $\xi, \xi' \in F$, $b \in B$. As above, we drop the index π and simply write '$\otimes$' instead of '${}_\pi\!\otimes$' if the representation π is understood. Evidently, the usual and the flipped internal tensor product are related by a unitary map

$$\Sigma\colon F \otimes E \xrightarrow{\cong} E \otimes F,$$
$$\eta \otimes \xi \mapsto \xi \otimes \eta.$$

https://doi.org/10.1017/S1474748010000290 Published online by Cambridge University Press

For each $\xi \in E$, the maps $F \to E \oslash F$ and $F \to F \owedge E$ given by $\eta \mapsto \xi \oslash \eta$ and $\eta \mapsto \eta \owedge \xi$, respectively, are adjointable, and the adjoints are given by $\xi' \oslash \eta \mapsto \pi(\langle\xi|\xi'\rangle)\eta$ and $\eta \owedge \xi' \mapsto \pi(\langle\xi|\xi'\rangle)\eta$, respectively.

Finally, let E_1, E_2 be Hilbert C^*-modules over A, let F_1, F_2 be Hilbert C^*-modules over B with representations $\pi_i \colon A \to \mathcal{L}_B(F_i)$ $(i = 1, 2)$, and let $S \in \mathcal{L}_A(E_1, E_2)$, $T \in \mathcal{L}_B(F_1, F_2)$ such that $T\pi_1(a) = \pi_2(a)T$ for all $a \in A$. Then there exists a unique operator $S \oslash T \in \mathcal{L}_B(E_1 \oslash F_1, E_2 \oslash F_2)$ such that $(S \oslash T)(\eta \oslash \xi) = S\eta \oslash T\xi$ for all $\eta \in E_1$, $\xi \in F_1$, and $(S \oslash T)^* = S^* \oslash T^*$ [**8**, Proposition 1.34].

2. C^*-pseudo-multiplicative unitaries

Recall that a multiplicative unitary on a Hilbert space H is a unitary $V \colon H \otimes H \to H \otimes H$ that satisfies the *pentagon equation* $V_{12}V_{13}V_{23} = V_{23}V_{12}$ (see [**3**]). Here, V_{12}, V_{13}, V_{23} are operators on $H \otimes H \otimes H$ defined by $V_{12} = V \otimes \mathrm{id}$, $V_{23} = \mathrm{id} \otimes V$, $V_{13} = (\Sigma \otimes \mathrm{id})V_{23}(\Sigma \otimes \mathrm{id}) = (\mathrm{id} \otimes \Sigma)V_{12}(\mathrm{id} \otimes \Sigma)$, where $\Sigma \in \mathcal{L}(H \otimes H)$ denotes the flip $\eta \otimes \xi \mapsto \xi \otimes \eta$. If G is a locally compact group with left Haar measure λ, then the formula

$$(Vf)(x, y) = f(x, x^{-1}y) \tag{2.1}$$

defines a linear bijection of $C_c(G \times G)$ which extends to a unitary on $L^2(G \times G, \lambda \otimes \lambda) \cong L^2(G, \lambda) \otimes L^2(G, \lambda)$. This unitary is multiplicative, and the pentagon equation amounts to associativity of the multiplication in G.

We shall generalize the notion of a multiplicative unitary so that it covers the example above if we replace the group G by a locally compact Hausdorff groupoid G. In that case, formula (2.1) defines a linear bijection from $C_c(G_s \times_r G)$ to $C_c(G_r \times_r G)$. If G is finite, that bijection is a unitary from $l^2(G_s \times_r G)$ to $l^2(G_r \times_r G)$, and these two Hilbert spaces can be identified with tensor products of $l^2(G)$ with $l^2(G)$ relative to the algebra $C(G^0)$. For a general groupoid, the algebraic tensor product of modules has to be replaced by a refined version. In the measurable setting, the appropriate substitute is the tensor product of Hilbert modules relative to a von Neumann algebra also known as Connes fusion (see [**33**]). To take the topology of G into account, we shall work in the setting of C^*-algebras and use the relative tensor product of C^*-modules over C^*-bases introduced in [**25**].

2.1. The relative tensor product of C^*-modules over C^*-bases

Fundamental to the definition of a C^*-pseudo-multiplicative unitary is the relative tensor product of C^*-modules over C^*-bases introduced in [**25**]. We briefly recall this construction; for further details, see [**25**]. An example will appear in § 2.3.

C^*-modules over C^*-bases

A *C^*-base* is a triple $(\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ consisting of a Hilbert space $\mathfrak{K}$ and two commuting non-degenerate C^*-algebras $\mathfrak{B}, \mathfrak{B}^\dagger \subseteq \mathcal{L}(\mathfrak{K})$. A C^*-base should be thought of as a C^*-algebraic counterpart to pairs consisting of a von Neumann algebra and its commutant. As an example, one can associate to every faithful KMS-state μ on a C^*-algebra B the C^*-base

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$(H_\mu, B, B^{\mathrm{op}})$, where H_μ is the GNS-space for μ and B and B^{op} act on $H_\mu = H_{\mu^{\mathrm{op}}}$ via the GNS-representations [**25**, Example 2.2]. The *opposite* of a C^*-base $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ is the C^*-base $\mathfrak{b}^\dagger := (\mathfrak{K}, \mathfrak{B}^\dagger, \mathfrak{B})$.

Let $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be a C^*-base. A C^**–*$\mathfrak{b}$*-module* is a pair $H_\alpha = (H, \alpha)$, where H is a Hilbert space and $\alpha \subseteq \mathcal{L}(\mathfrak{K}, H)$ is a closed subspace satisfying $[\alpha\mathfrak{K}] = H$, $[\alpha\mathfrak{B}] = \alpha$, and $[\alpha^*\alpha] = \mathfrak{B} \subseteq \mathcal{L}(\mathfrak{K})$. If H_α is a C^*–$\mathfrak{b}$-module, then α is a Hilbert C^*-module over $\mathfrak{B}$ with inner product $(\xi, \xi') \mapsto \xi^*\xi'$ and there exist isomorphisms

$$\alpha \otimes \mathfrak{K} \to H,\ \xi \otimes \zeta \mapsto \xi\zeta, \qquad \mathfrak{K} \otimes \alpha \to H,\ \zeta \otimes \xi \mapsto \xi\zeta, \tag{2.2}$$

and a non-degenerate representation

$$\rho_\alpha \colon \mathfrak{B}^\dagger \to \mathcal{L}(H), \quad \rho_\alpha(b^\dagger)(\xi\zeta) = \xi b^\dagger \zeta \quad \text{for all } b^\dagger \in \mathfrak{B}^\dagger,\ \xi \in \alpha,\ \zeta \in \mathfrak{K}.$$

A *morphism* between C^*–$\mathfrak{b}$-modules H_α and K_β is an operator $T \in \mathcal{L}(H, K)$ satisfying $T\alpha \subseteq \beta$ and $T^*\beta \subseteq \alpha$. We denote the set of all morphisms by $\mathcal{L}(H_\alpha, K_\beta)$. If $T \in \mathcal{L}(H_\alpha, K_\beta)$, then $T\rho_\alpha(b^\dagger) = \rho_\beta(b^\dagger)T$ for all $b^\dagger \in \mathfrak{B}^\dagger$, and left multiplication by T defines an operator in $\mathcal{L}_{\mathfrak{B}}(\alpha, \beta)$ which we again denote by T.

Let $\mathfrak{b}_1, \dots, \mathfrak{b}_n$ be C^*-bases, where $\mathfrak{b}_i = (\mathfrak{K}_i, \mathfrak{B}_i, \mathfrak{B}_i^\dagger)$ for each i. A C^**–*$(\mathfrak{b}_1, \dots, \mathfrak{b}_n)$*-module* is a tuple $(H, \alpha_1, \dots, \alpha_n)$, where H is a Hilbert space and (H, α_i) is a C^*–$\mathfrak{b}_i$-module for each i such that $[\rho_{\alpha_i}(\mathfrak{B}_i^\dagger)\alpha_j] = \alpha_j$ whenever $i \neq j$. In the case $n = 2$, we abbreviate ${}_\alpha H_\beta := (H, \alpha, \beta)$. We note that if $(H, \alpha_1, \dots, \alpha_n)$ is a C^*–$(\mathfrak{b}_1, \dots, \mathfrak{b}_n)$-module, then $[\rho_{\alpha_i}(\mathfrak{B}_i^\dagger), \rho_{\alpha_j}(\mathfrak{B}_j^\dagger)] = 0$ whenever $i \neq j$. The set of *morphisms* between C^*–$(\mathfrak{b}_1, \dots, \mathfrak{b}_n)$-modules $\mathcal{H} = (H, \alpha_1, \dots, \alpha_n)$, $\mathcal{K} = (K, \gamma_1, \dots, \gamma_n)$ is the set

$$\mathcal{L}(\mathcal{H}, \mathcal{K}) := \bigcap_{i=1}^{n} \mathcal{L}(H_{\alpha_i}, K_{\gamma_i}) \subseteq \mathcal{L}(H, K).$$

The relative tensor product

Let $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be a C^*-base, let H_β be a C^*–$\mathfrak{b}$-module, and let K_γ be a C^*–$\mathfrak{b}^\dagger$-module. The *relative tensor product* of H_β and K_γ is the Hilbert space

$$H \mathbin{{}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma} K := \beta \otimes \mathfrak{K} \otimes \gamma.$$

It is spanned by elements $\xi \otimes \zeta \otimes \eta$, where $\xi \in \beta$, $\zeta \in \mathfrak{K}$, $\eta \in \gamma$, and

$$\langle \xi \otimes \zeta \otimes \eta | \xi' \otimes \zeta' \otimes \eta' \rangle = \langle \zeta | \xi^*\xi'\eta^*\eta'\zeta' \rangle = \langle \zeta | \eta^*\eta'\xi^*\xi'\zeta' \rangle$$

for all $\xi, \xi' \in \beta$, $\zeta, \zeta' \in \mathfrak{K}$, $\eta, \eta' \in \gamma$. The formula $\xi \otimes \zeta \otimes \eta \mapsto \eta \otimes \zeta \otimes \xi$ obviously defines a unitary *flip*

$$\Sigma \colon H \mathbin{{}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma} K \to K \mathbin{{}_\gamma\underset{\mathfrak{b}^\dagger}{\otimes}{}_\beta} H.$$

Using the unitaries in (2.2) on H_β and K_γ, respectively, we shall make the following identifications without further notice:

$$H \mathbin{{}_{\rho_\beta}\otimes} \gamma \cong H \mathbin{{}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma} K \cong \beta \otimes_{\rho_\gamma} K, \qquad \xi\zeta \otimes \eta \equiv \xi \otimes \zeta \otimes \eta \equiv \xi \otimes \eta\zeta.$$

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For all $S \in \rho_\beta(\mathfrak{B}^\dagger)'$ and $T \in \rho_\gamma(\mathfrak{B})'$, we have operators

$$S \otimes \mathrm{id} \in \mathcal{L}(H_{\rho_\beta} \otimes \gamma) = \mathcal{L}(H {}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K), \qquad \mathrm{id} \otimes T \in \mathcal{L}(\beta \otimes_{\rho_\gamma} K) = \mathcal{L}(H {}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K).$$

If $S \in \mathcal{L}(H_\beta)$ or $T \in \mathcal{L}(K_\gamma)$, then $(S \otimes \mathrm{id})(\xi \otimes \eta\zeta) = S\xi \otimes \eta\zeta$ or $(\mathrm{id} \otimes T)(\xi\zeta \otimes \eta) = \xi\zeta \otimes T\eta$, respectively, for all $\xi \in \beta$, $\zeta \in \mathfrak{K}$, $\eta \in \gamma$, so that we can define

$$S \underset{\mathfrak{b}}{\otimes} T := (S \otimes \mathrm{id})(\mathrm{id} \otimes T) = (\mathrm{id} \otimes T)(S \otimes \mathrm{id}) \in \mathcal{L}(H {}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K)$$

for all (S, T) in $\mathcal{L}(H_\beta) \times \rho_\gamma(\mathfrak{B})'$ or $\rho_\beta(\mathfrak{B}^\dagger)' \times \mathcal{L}(K_\gamma)$.

For each $\xi \in \beta$ and $\eta \in \gamma$, there exist bounded linear operators

$$|\xi\rangle_1 \colon K \to H {}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K,\ \omega \mapsto \xi \otimes \omega, \qquad |\eta\rangle_2 \colon H \to H {}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K,\ \omega \mapsto \omega \otimes \eta,$$

whose adjoints $\langle\xi|_1 := |\xi\rangle_1^*$ and $\langle\eta|_2 := |\eta\rangle_2^*$ are given by $\xi' \otimes \omega \mapsto \rho_\gamma(\xi^*\xi')\omega$ and $\omega \otimes \eta' \mapsto \rho_\beta(\eta^*\eta')\omega$, respectively. We put

$$|\beta\rangle_1 := \{|\xi\rangle_1 \mid \xi \in \beta\} \subseteq \mathcal{L}(K, H {}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K)$$

and similarly define $\langle\beta|_1$, $|\gamma\rangle_2$, $\langle\gamma|_2$.

Let $\mathcal{H} = (H, \alpha_1, \dots, \alpha_m, \beta)$ be a C^*–$(\mathfrak{a}_1, \dots, \mathfrak{a}_m, \mathfrak{b})$-module and let $\mathcal{K} = (K, \gamma, \delta_1, \dots, \delta_n)$ be a C^*–$(\mathfrak{b}^\dagger, \mathfrak{c}_1, \dots, \mathfrak{c}_n)$-module, where $\mathfrak{a}_i = (\mathfrak{H}_i, \mathfrak{A}_i, \mathfrak{A}_i^\dagger)$ and $\mathfrak{c}_j = (\mathfrak{L}_j, \mathfrak{C}_j, \mathfrak{C}_j^\dagger)$ for all i, j. We put

$$\alpha_i \triangleleft \gamma := [|\gamma\rangle_2 \alpha_i] \subseteq \mathcal{L}(\mathfrak{H}_i, H {}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K), \qquad \beta \triangleright \delta_j := [|\beta\rangle_1 \delta_j] \subseteq \mathcal{L}(\mathfrak{L}_j, H {}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K)$$

for all i, j. Then

$$(H {}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K, \alpha_1 \triangleleft \gamma, \dots, \alpha_m \triangleleft \gamma, \beta \triangleright \delta_1, \dots, \beta \triangleright \delta_n)$$

is a C^*–$(\mathfrak{a}_1, \dots, \mathfrak{a}_m, \mathfrak{c}_1, \dots, \mathfrak{c}_n)$-module, called the *relative tensor product* of $\mathcal{H}$ and $\mathcal{K}$ and denoted by $\mathcal{H} \underset{\mathfrak{b}}{\otimes} \mathcal{K}$. For all i, j and $a^\dagger \in \mathfrak{A}_i^\dagger$, $c^\dagger \in \mathfrak{C}_j^\dagger$,

$$\rho_{(\alpha_i \triangleleft \gamma)}(a^\dagger) = \rho_{\alpha_i}(a^\dagger) \underset{\mathfrak{b}}{\otimes} \mathrm{id}, \qquad \rho_{(\beta \triangleright \delta_j)}(c^\dagger) = \mathrm{id} \underset{\mathfrak{b}}{\otimes} \rho_{\delta_j}(c^\dagger).$$

The relative tensor product is functorial, unital and associative in the following sense. Let $\tilde{\mathcal{H}} = (\tilde{H}, \tilde{\alpha}_1, \dots, \tilde{\alpha}_m, \tilde{\beta})$ be a C^*–$(\mathfrak{a}_1, \dots, \mathfrak{a}_m, \mathfrak{b})$-module, $\tilde{\mathcal{K}} = (\tilde{K}, \tilde{\gamma}, \tilde{\delta}_1, \dots, \tilde{\delta}_n)$ a C^*–$(\mathfrak{b}^\dagger, \mathfrak{c}_1, \dots, \mathfrak{c}_n)$-module, and $S \in \mathcal{L}(\mathcal{H}, \tilde{\mathcal{H}})$, $T \in \mathcal{L}(\mathcal{K}, \tilde{\mathcal{K}})$. Then there exists a unique operator

$$S \underset{\mathfrak{b}}{\otimes} T \in \mathcal{L}(\mathcal{H} \underset{\mathfrak{b}}{\otimes} \mathcal{K}, \tilde{\mathcal{H}} \underset{\mathfrak{b}}{\otimes} \tilde{\mathcal{K}})$$

satisfying

$$(S \underset{\mathfrak{b}}{\otimes} T)(\xi \otimes \zeta \otimes \eta) = S\xi \otimes \zeta \otimes T\eta$$

https://doi.org/10.1017/S1474748010000290 Published online by Cambridge University Press

for all $\xi \in \beta$, $\zeta \in \mathfrak{K}$, $\eta \in \gamma$. Next, the triple $\mathcal{U} := (\mathfrak{K}, \mathfrak{B}^\dagger, \mathfrak{B})$ is a C^*–$(\mathfrak{b}^\dagger, \mathfrak{b})$-module and the maps

$$l_{\mathcal{H}} \colon H \,{}_{\beta}\!\underset{\mathfrak{b}}{\otimes}_{\mathfrak{B}^\dagger} \mathfrak{K} \to H,\ \xi \obslash \zeta \oslash b^\dagger \mapsto \xi b^\dagger \zeta, \qquad r_{\mathcal{K}} \colon \mathfrak{K}_{\mathfrak{B}} \underset{\mathfrak{b}}{\otimes}_{\gamma} K \to K,\ b \obslash \zeta \oslash \eta \mapsto \eta b \zeta \tag{2.3}$$

are isomorphisms of C^*–$(\mathfrak{a}_1, \dots, \mathfrak{a}_m, \mathfrak{b})$-modules and C^*–$(\mathfrak{b}^\dagger, \mathfrak{c}_1, \dots, \mathfrak{c}_n)$-modules

$$\mathcal{H} \underset{\mathfrak{b}}{\otimes} \mathcal{U} \to \mathcal{H} \quad \text{and} \quad \mathcal{U} \underset{\mathfrak{b}}{\otimes} \mathcal{K} \to \mathcal{K},$$

respectively, natural in $\mathcal{H}$ and $\mathcal{K}$. Finally, let $\mathfrak{d}$, $\mathfrak{e}_1, \dots, \mathfrak{e}_l$ be C^*-bases, $\hat{\mathcal{K}} = (K, \gamma, \delta_1, \dots, \delta_n, \epsilon)$ a C^*–$(\mathfrak{b}^\dagger, \mathfrak{c}_1, \dots, \mathfrak{c}_n, \mathfrak{d})$-module and $\mathcal{L} = (L, \phi, \psi_1, \dots, \psi_l)$ a C^*–$(\mathfrak{d}^\dagger, \mathfrak{e}_1, \dots, \mathfrak{e}_l)$-module. Then there exists a canonical isomorphism

$$a_{\mathcal{H}, \hat{\mathcal{K}}, \mathcal{L}} \colon (H \,{}_{\beta}\!\underset{\mathfrak{b}}{\otimes}_{\gamma} K) \,{}_{\beta \triangleright \epsilon}\!\underset{\mathfrak{d}}{\otimes}_{\phi} L \to \beta \obslash_{\rho_\gamma} K \,{}_{\rho_\epsilon}\!\oslash \phi \to H \,{}_{\beta}\!\underset{\mathfrak{b}}{\otimes}_{\gamma \triangleleft \phi} (K \,{}_{\epsilon}\!\underset{\mathfrak{d}}{\otimes}_{\phi} L),$$

which is an isomorphism of C^*–$(\mathfrak{a}_1, \dots, \mathfrak{a}_m, \mathfrak{c}_1, \dots, \mathfrak{c}_n, \mathfrak{e}_1, \dots, \mathfrak{e}_l)$-modules

$$(\mathcal{H} \underset{\mathfrak{b}}{\otimes} \hat{\mathcal{K}}) \underset{\mathfrak{d}}{\otimes} \mathcal{L} \to \mathcal{H} \underset{\mathfrak{b}}{\otimes} (\hat{\mathcal{K}} \underset{\mathfrak{d}}{\otimes} \mathcal{L}).$$

We identify the Hilbert spaces above and denote them by

$$H \,{}_{\beta}\!\underset{\mathfrak{b}}{\otimes}_{\gamma} K \,{}_{\epsilon}\!\underset{\mathfrak{d}}{\otimes}_{\phi} L.$$

2.2. The definition of C^*-pseudo-multiplicative unitaries

Let $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be a C^*-base, $(H, \hat{\beta}, \alpha, \beta)$ a C^*–$(\mathfrak{b}^\dagger, \mathfrak{b}, \mathfrak{b}^\dagger)$-module, and

$$V \colon H \,{}_{\hat{\beta}}\!\underset{\mathfrak{b}^\dagger}{\otimes}_{\alpha} H \to H \,{}_{\alpha}\!\underset{\mathfrak{b}}{\otimes}_{\beta} H$$

a unitary satisfying

$$V(\alpha \triangleleft \alpha) = \alpha \triangleright \alpha, \qquad V(\hat{\beta} \triangleright \beta) = \hat{\beta} \triangleleft \beta, \qquad V(\hat{\beta} \triangleright \hat{\beta}) = \alpha \triangleright \hat{\beta}, \qquad V(\beta \triangleleft \alpha) = \beta \triangleleft \beta \tag{2.4}$$

in $\mathcal{L}(\mathfrak{K}, H \,{}_{\alpha}\!\underset{\mathfrak{b}}{\otimes}_{\beta} H)$. Then all operators in the following diagram are well defined:

$$\begin{array}{ccccc}
H \,{}_{\hat{\beta}}\!\underset{\mathfrak{b}^\dagger}{\otimes}_{\alpha} H \,{}_{\hat{\beta}}\!\underset{\mathfrak{b}^\dagger}{\otimes}_{\alpha} H & \xrightarrow{V_{12}} & H \,{}_{\alpha}\!\underset{\mathfrak{b}}{\otimes}_{\beta} H \,{}_{\hat{\beta}}\!\underset{\mathfrak{b}^\dagger}{\otimes}_{\alpha} H & \xrightarrow{V_{23}} & H \,{}_{\alpha}\!\underset{\mathfrak{b}}{\otimes}_{\beta} H \,{}_{\alpha}\!\underset{\mathfrak{b}}{\otimes}_{\beta} H \\
\downarrow V_{23} & & & & \uparrow V_{12} \\
H \,{}_{\hat{\beta}}\!\underset{\mathfrak{b}^\dagger}{\otimes}_{(\alpha \triangleright \alpha)} (H \,{}_{\alpha}\!\underset{\mathfrak{b}}{\otimes}_{\beta} H) & & & & (H \,{}_{\hat{\beta}}\!\underset{\mathfrak{b}^\dagger}{\otimes}_{\alpha} H) \,{}_{(\alpha \triangleleft \alpha)}\!\underset{\mathfrak{b}}{\otimes}_{\beta} H \\
\downarrow \Sigma_{23} & & & & \uparrow \Sigma'_{23} \\
H \,{}_{\hat{\beta}}\!\underset{\mathfrak{b}^\dagger}{\otimes}_{\alpha} H \,{}_{\beta}\!\underset{\mathfrak{b}^\dagger}{\otimes}_{\alpha} H & & \xrightarrow{\qquad V_{12} \qquad} & & (H \,{}_{\alpha}\!\underset{\mathfrak{b}}{\otimes}_{\beta} H) \,{}_{(\hat{\beta} \triangleleft \beta)}\!\underset{\mathfrak{b}^\dagger}{\otimes}_{\alpha} H
\end{array} \tag{2.5}$$

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where we adopted the leg notation [**3**] and wrote

$$V_{12} \text{ for } V \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id},\ V \underset{\mathfrak{b}}{\otimes} \mathrm{id}, \qquad V_{23} \text{ for } V \underset{\mathfrak{b}}{\otimes} \mathrm{id},\ V \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id}, \qquad \Sigma_{23} \text{ for } \mathrm{id} \underset{\mathfrak{b}^\dagger}{\otimes} \Sigma,$$

and where Σ'_{23} denotes the isomorphism

$$\begin{aligned}(H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H) {}_{(\hat\beta \lhd \beta)}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} H &\cong (H {}_{\rho_\alpha}\otimes \beta) {}_{\rho_{(\hat\beta\lhd\beta)}}\otimes \alpha \xrightarrow{\cong} (H {}_{\rho_{\hat\beta}}\otimes \alpha) {}_{\rho_{(\alpha\lhd\alpha)}}\otimes \beta \\ &\cong (H {}_{\hat\beta}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} H) {}_{(\alpha\lhd\alpha)}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H\end{aligned}$$

given by $(\zeta \otimes \xi) \otimes \eta \mapsto (\zeta \otimes \eta) \otimes \xi$. We furthermore write V_{13} for

$$\Sigma'_{23}(V \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id})\Sigma_{23}.$$

Definition 2.1. A *C^*-pseudo-multiplicative unitary* is a tuple $(\mathfrak{b}, H, \hat\beta, \alpha, \beta, V)$ consisting of a C^*-base $\mathfrak{b}$, a C^*–$(\mathfrak{b}^\dagger, \mathfrak{b}, \mathfrak{b}^\dagger)$-module $(H, \hat\beta, \alpha, \beta)$, and a unitary

$$V : H {}_{\hat\beta}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} H \to H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H$$

such that (2.4) holds and diagram (2.5) commutes. We frequently call just V a C^*-pseudo-multiplicative unitary.

Remarks and examples 2.2.

- If $\mathfrak{b}$ is the trivial C^*-base $(\mathbb{C}, \mathbb{C}, \mathbb{C})$, then

$$H {}_{\hat\beta}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} H \cong H \otimes H \cong H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H,$$

  and V is a multiplicative unitary.

- If we consider $\rho_{\hat\beta}$ and ρ_β as representations $\rho_{\hat\beta}, \rho_\beta : \mathfrak{B} \to \mathcal{L}(H_\alpha) \cong \mathcal{L}_{\mathfrak{B}}(\alpha)$, then the map $\alpha {}_{\rho_{\hat\beta}}\otimes \alpha \cong \alpha \lhd \alpha \to \alpha \rhd \alpha \cong \alpha \otimes_{\rho_\beta} \alpha$ given by $\omega \mapsto V\omega$ is a pseudo-multiplicative unitary on C^*-modules in the sense of [**27**].

- Assume that $\mathfrak{b} = \mathfrak{b}^\dagger$; then $\mathfrak{B} = \mathfrak{B}^\dagger$ is commutative. If $\hat\beta = \alpha$, then the pseudo-multiplicative unitary in (ii) is a pseudo-multiplicative unitary in the sense of O'uchi [**19**]. If additionally $\hat\beta = \alpha = \beta$, then the unitary in (ii) is a continuous field of multiplicative unitaries in the sense of Blanchard [**4**].

- Assume that $\mathfrak{b}$ is the C^*-base associated to a faithful proper KMS-weight μ on a C^*-algebra B (see [**25**, Example 2.9]). Then μ extends to a normal, semifinite and faithful weight $\tilde\mu$ on $[\![\mathfrak{B}]\!]$, and with respect to the canonical isomorphisms

$$H {}_{\hat\beta}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} H \cong H {}_{\rho_{\hat\beta}}\underset{\tilde\mu^{\mathrm{op}}}{\bar\otimes}{}_{\rho_\alpha} H$$

  and

$$H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H \cong H {}_{\rho_\alpha}\underset{\tilde\mu}{\bar\otimes}{}_{\rho_\beta} H$$

  (see [**25**, Corollary 2.21]), V is a pseudo-multiplicative unitary on Hilbert spaces in the sense of Vallin [**33**].

- In [**31**], a C^*-pseudo-multiplicative unitary is associated to every compact C^*-quantum groupoid.
- The *opposite* of a C^*-pseudo-multiplicative unitary $(\mathfrak{b}, H, \hat{\beta}, \alpha, \beta, V)$ is the tuple $(\mathfrak{b}, H, \beta, \alpha, \hat{\beta}, V^{\mathrm{op}})$, where V^{op} denotes the composition

$$\Sigma V^* \Sigma \colon H \mathbin{{}_{\beta}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}} H \xrightarrow{\Sigma} H \mathbin{{}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}} H \xrightarrow{V^*} H \mathbin{{}_{\hat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}} H \xrightarrow{\Sigma} H \mathbin{{}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\hat{\beta}}} H.$$

  A tedious but straightforward calculation shows that this is a C^*-pseudo-multiplicative unitary.

2.3. The C^*-pseudo-multiplicative unitary of a groupoid

Let G be a locally compact, Hausdorff, second countable groupoid with left Haar system λ and associated right Haar system λ^{-1}, and let μ be a measure on G^0 with full support. We associate to this data a C^*-pseudo-multiplicative unitary such that the underlying pseudo-multiplicative unitary and the associated unitary on C^*-modules are the ones introduced by Vallin [**33**] and O'uchi [**19, 27**], respectively. We focus on the aspects that are new in the present setting.

Define measures ν, ν^{-1} on G by

$$\int_G f \, \mathrm{d}\nu := \int_{G^0} \int_{G^u} f(x) \, \mathrm{d}\lambda^u(x) \, \mathrm{d}\mu(u), \qquad \int_G f \, \mathrm{d}\nu^{-1} = \int_{G^0} \int_{G_u} f(x) \, \mathrm{d}\lambda_u^{-1}(x) \, \mathrm{d}\mu(u)$$

for all $f \in C_c(G)$. Thus, $\nu^{-1} = i_*\nu$, where $i \colon G \to G$ is given by $x \mapsto x^{-1}$. We assume that μ is quasi-invariant in the sense that ν and ν^{-1} are equivalent, and denote by $D := \mathrm{d}\nu/\mathrm{d}\nu^{-1}$ the Radon–Nikodym derivative.

We identify functions in $C_b(G^0)$ and $C_b(G)$ with multiplication operators on the Hilbert spaces $L^2(G^0, \mu)$ and $L^2(G, \nu)$, respectively, and let

$$\mathfrak{K} := L^2(G^0, \mu), \quad \mathfrak{B} = \mathfrak{B}^\dagger := C_0(G^0) \subseteq \mathcal{L}(\mathfrak{K}), \quad \mathfrak{b} := (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger), \quad H := L^2(G, \nu).$$

Pulling functions on G^0 back to G along r or s, we obtain representations

$$r^* \colon C_0(G^0) \to C_b(G) \hookrightarrow \mathcal{L}(H), \qquad s^* \colon C_0(G^0) \to C_b(G) \hookrightarrow \mathcal{L}(H).$$

We define Hilbert C^*-modules $L^2(G, \lambda)$ and $L^2(G, \lambda^{-1})$ over $C_0(G^0)$ as the respective completions of the pre-C^*-module $C_c(G)$, the structure maps being given by

$$\langle \xi' | \xi \rangle(u) = \int_{G^u} \overline{\xi'(x)} \xi(x) \, \mathrm{d}\lambda^u(x), \quad \xi f = r^*(f)\xi \quad \text{in the case of } L^2(G, \lambda),$$
$$\langle \xi' | \xi \rangle(u) = \int_{G_u} \overline{\xi'(x)} \xi(x) \, \mathrm{d}\lambda_u^{-1}(x), \quad \xi f = s^*(f)\xi \quad \text{in the case of } L^2(G, \lambda^{-1}),$$

respectively, for all $\xi, \xi' \in C_c(G)$, $u \in G^0$, $f \in C_0(G^0)$.

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Lemma 2.3. *There exist embeddings* $j\colon L^2(G,\lambda) \to \mathcal{L}(\mathfrak{K}, H)$ *and* $\hat{\jmath}\colon L^2(G,\lambda^{-1}) \to \mathcal{L}(\mathfrak{K}, H)$ *such that for all* $\xi \in C_c(G)$, $\zeta \in C_c(G^0)$,

$$(j(\xi)\zeta)(x) = \xi(x)\zeta(r(x)), \qquad (\hat{\jmath}(\xi)\zeta)(x) = \xi(x)D^{-1/2}(x)\zeta(s(x)).$$

Proof. Let $E := L^2(G,\lambda)$, $\hat{E} := L^2(G,\lambda^{-1})$, and $\xi, \xi' \in C_c(G)$, $\zeta, \zeta' \in C_c(G^0)$. Then

$$\langle j(\xi')\zeta' | j(\xi)\zeta\rangle = \int_{G^0}\int_{G^u} \overline{\xi'(x)\zeta'(r(x))}\xi(x)\zeta(r(x))\,\mathrm{d}\lambda^u(x)\,\mathrm{d}\mu(u) = \langle \zeta' | \langle \xi' | \xi\rangle_E \zeta\rangle,$$

$$\begin{aligned}\langle \hat{\jmath}(\xi')\zeta' | \hat{\jmath}(\xi)\zeta\rangle &= \int_G \overline{\xi'(x)\zeta'(s(x))}\xi(x)\zeta(s(x))\,\underbrace{D^{-1}(x)\,\mathrm{d}\nu(x)}_{=\mathrm{d}\nu^{-1}(x)} \\ &= \int_{G^0}\int_{G_u} \overline{\zeta'(u)\xi'(x)}\xi(x)\zeta(u)\,\mathrm{d}\lambda_u^{-1}(x)\,\mathrm{d}\mu(u) \\ &= \langle \zeta' | \langle \xi' | \xi\rangle_{\hat{E}} \zeta\rangle.\end{aligned}$$

□

Let $\alpha := \beta := j(L^2(G,\lambda))$ and $\hat{\beta} := \hat{\jmath}(L^2(G,\lambda^{-1}))$. Easy calculations lead us to the following lemma.

Lemma 2.4. $(H, \hat{\beta}, \alpha, \beta)$ *is a* C^**–*$(\mathfrak{b}^\dagger, \mathfrak{b}, \mathfrak{b}^\dagger)$*-module,* $\rho_\alpha = \rho_\beta = r^*$ *and* $\rho_{\hat{\beta}} = s^*$*, and* j *and* $\hat{\jmath}$ *are unitary maps of Hilbert* C^**-modules over* $C_0(G^0) \cong \mathfrak{B}$.

We define measures $\nu^2_{s,r}$ on $G_s \times_r G$ and $\nu^2_{r,r}$ on $G_r \times_r G$ by

$$\begin{aligned}\int_{G_s\times_r G} f\,\mathrm{d}\nu^2_{s,r} &:= \int_{G^0}\int_{G^u}\int_{G^{s(x)}} f(x,y)\,\mathrm{d}\lambda^{s(x)}(y)\,\mathrm{d}\lambda^u(x)\,\mathrm{d}\mu(u), \\ \int_{G_r\times_r G} g\,\mathrm{d}\nu^2_{r,r} &:= \int_{G^0}\int_{G^u}\int_{G^u} g(x,y)\,\mathrm{d}\lambda^u(y)\,\mathrm{d}\lambda^u(x)\,\mathrm{d}\mu(u)\end{aligned}$$

for all $f \in C_c(G_s \times_r G)$, $g \in C_c(G_r \times_r G)$. Routine calculations show that there exist isomorphisms

$$\Phi_{\hat{\beta},\alpha}\colon H\, {}_{\hat{\beta}}\!\underset{\mathfrak{b}^\dagger}{\otimes}\!{}_\alpha\, H \to L^2(G_s \times_r G, \nu^2_{s,r})$$

and

$$\Phi_{\alpha,\beta}\colon H\, {}_{\alpha}\!\underset{\mathfrak{b}}{\otimes}\!{}_\beta\, H \to L^2(G_r \times_r G, \nu^2_{r,r})$$

such that for all $\eta, \xi \in C_c(G)$ and $\zeta \in C_c(G^0)$,

$$\begin{aligned}\Phi_{\hat{\beta},\alpha}(\hat{\jmath}(\eta) \otimes \zeta \otimes j(\xi))(x,y) &= \eta(x)D^{-1/2}(x)\zeta(s(x))\xi(y), \\ \Phi_{\alpha,\beta}(j(\eta) \otimes \zeta \otimes j(\xi))(x,y) &= \eta(x)\zeta(r(x))\xi(y).\end{aligned}$$

We shall use these isomorphisms to identify the spaces above without further notice.

Theorem 2.5. *There exists a* C^**-pseudo-multiplicative unitary* $(\mathfrak{b}, H, \hat{\beta}, \alpha, \beta, V)$ *such that* $(V\omega)(x,y) = \omega(x, x^{-1}y)$ *for all* $\omega \in C_c(G_s \times_r G)$ *and* $(x,y) \in G_r \times_r G$.

https://doi.org/10.1017/S1474748010000290 Published online by Cambridge University Press

Proof. Straightforward calculations show that $(H, \hat{\beta}, \alpha, \beta)$ is a C^*–$(\mathfrak{b}^\dagger, \mathfrak{b}, \mathfrak{b}^\dagger)$-module. Using left-invariance of λ, one finds that the bijection $V_0 \colon C_c(G_s \times_r G) \to C_c(G_r \times_r G)$ given by $(V_0\omega)(x, y) = \omega(x, x^{-1}y)$ for all $\omega \in C_c(G_s \times_r G)$ and $(x, y) \in G_r \times_r G$ extends to a unitary

$$V \colon H \underset{\mathfrak{b}^\dagger}{{}_{\hat{\beta}}\otimes_{\alpha}} H \cong L^2(G_s \times_r G) \to L^2(G_r \times_r G) \cong H \underset{\mathfrak{b}}{{}_{\alpha}\otimes_{\beta}} H.$$

We claim that $V(\hat{\beta} \triangleright \hat{\beta}) = \alpha \triangleright \hat{\beta}$. For each $\xi, \xi' \in C_c(G)$, $\zeta \in C_c(G^0)$, and $(x, y) \in G_s \times_r G$,

$$\begin{aligned}(V|\hat{\jmath}(\xi)\rangle_1 \hat{\jmath}(\xi')\zeta)(x, y) &= (|\hat{\jmath}(\xi)\rangle_1 \hat{\jmath}(\xi')\zeta)(x, x^{-1}y) \\ &= \xi(x)\xi'(x^{-1}y)D^{-1/2}(x)D^{-1/2}(x^{-1}y)\zeta(s(y)), \\ (|j(\xi)\rangle_1 \hat{\jmath}(\xi')\zeta)(x, y) &= \xi(x)\xi'(y)D^{-1/2}(y)\zeta(s(y)).\end{aligned}$$

Using standard approximation arguments and the fact that $D(x)D(x^{-1}y) = D(y)$ for $\nu^2_{r,r}$-almost all $(x, y) \in G_r \times_r G$ (see [**12**] or [**20**, p. 89]), we find that $V(\hat{\beta} \triangleright \hat{\beta}) = [T(C_c(G_r \times_r G))] = \alpha \triangleright \hat{\beta}$, where for each $\omega \in C_c(G_r \times_r G)$,

$$(T(\omega)\zeta)(x, y) = \omega(x, y)D^{-1/2}(y)\zeta(s(y)) \quad \text{for all } \zeta \in C_c(G^0),\ (x, y) \in G_r \times_r G.$$

Similar calculations show that the remaining relations in (2.4) hold. Tedious but straightforward calculations show that diagram (2.5) commutes (see also [**33**]). Therefore, V is a C^*-pseudo-multiplicative unitary. □

3. The legs of a C^*-pseudo multiplicative unitary

To every regular multiplicative unitary V on a Hilbert space H, Baaj and Skandalis associate two Hopf C^*-algebras $(\hat{A}_V, \hat{\Delta}_V)$ and (A_V, Δ_V) as follows [**3**]. The C^*-algebras $\hat{A}_V$ and A_V are the norm closures of the subspaces $\hat{A}^0_V$ and A^0_V of $\mathcal{L}(H)$ given by

$$\hat{A}^0_V := \{(\mathrm{id} \,\bar{\otimes}\, \omega)(V) \mid \omega \in \mathcal{L}(H)_*\}, \qquad A^0_V := \{(\upsilon \,\bar{\otimes}\, \mathrm{id})(V) \mid \upsilon \in \mathcal{L}(H)_*\}, \tag{3.1}$$

and the $*$-homomorphisms $\hat{\Delta}_V \colon \hat{A}_V \to M(\hat{A}_V \otimes \hat{A}_V) \subseteq \mathcal{L}(H \otimes H)$ and $\Delta_V \colon A_V \to M(A_V \otimes A_V) \subseteq \mathcal{L}(H \otimes H)$ are given by

$$\hat{\Delta}_V \colon \hat{a} \mapsto V^*(1 \otimes \hat{a})V, \qquad \Delta_V \colon a \mapsto V(a \otimes 1)V^*, \tag{3.2}$$

respectively. Applied to the multiplicative unitary of a locally compact group G, this construction yields the C^*-algebras $C_0(G)$ and $C^*_{\mathrm{r}}(G)$, and $\hat{\Delta} \colon C_0(G) \to M(C_0(G) \otimes C_0(G)) \cong C_b(G \times G)$ and $\Delta \colon C^*_{\mathrm{r}}(G) \to M(C^*_{\mathrm{r}}(G) \otimes C^*_{\mathrm{r}}(G))$ are given by

$$\hat{\Delta}(f)(x, y) = f(xy) \quad \text{for all } f \in C_0(G), \qquad \Delta(U_x) = U_x \otimes U_x \quad \text{for all } x \in G, \tag{3.3}$$

where $U \colon G \to M(C^*_{\mathrm{r}}(G))$, $x \mapsto U_x$, is the canonical embedding.

To adapt these constructions to C^*-pseudo-multiplicative unitaries, we have to generalize the notion of a Hopf C^*-algebra and identify the targets of the comultiplications

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$\hat{\Delta}_V$ and Δ_V. For the C^*-pseudo-multiplicative unitary of a groupoid G, we expect to obtain the C^*-algebras $\hat{A}_V = C_0(G)$ and $A_V = C^*_r(G)$ with $*$-homomorphisms $\hat{\Delta}$ and Δ given by the same formulae as in (3.3). Then the target of $\hat{\Delta}$ would be $M(C_0(G_s \times_r G))$, and $C_0(G_s \times_r G)$ can be identified with the relative tensor product

$$C_0(G)\,{}_{s^*}\underset{C_0(G^0)}{\otimes}{}_{r^*}\,C_0(G)$$

of $C_0(G^0)$-algebras [**4**]. But the target of Δ cannot be described in a similar way, and in general, we need to replace the balanced tensor product by a fibre product relative to some base. In the setting of von Neumann algebras, the targets of the comultiplications can be described using Sauvageot's fibre product [**24, 32**]. The appropriate construction in the setting of C^*-algebras is given below.

3.1. The fibre product and Hopf C^*-bimodules

Fundamental to the notion of a Hopf C^*-bimodule is the fibre product of C^*-algebras over C^*-bases introduced in [**25**]. We briefly recall this construction and subsequently introduce Hopf C^*-bimodules; for additional motivation and details, see [**25**]. Two examples can be found in § 3.4.

Let $\mathfrak{b}_1, \dots, \mathfrak{b}_n$ be C^*-bases, where $\mathfrak{b}_i = (\mathfrak{K}_i, \mathfrak{B}_i, \mathfrak{B}_i^\dagger)$ for each i. A *(non-degenerate) C^*–$(\mathfrak{b}_1, \dots, \mathfrak{b}_n)$-algebra* consists of a C^*–$(\mathfrak{b}_1, \dots, \mathfrak{b}_n)$-module $(H, \alpha_1, \dots, \alpha_n)$ and a (non-degenerate) C^*-algebra $A \subseteq \mathcal{L}(H)$ such that $\rho_{\alpha_i}(\mathfrak{B}_i^\dagger)A$ is contained in A for each i. We are interested in the cases $n = 1, 2$ and abbreviate $A_H^\alpha := (H_\alpha, A)$, $A_H^{\alpha,\beta} := ({}_\alpha H_\beta, A)$.

Let $\mathcal{A} = (\mathcal{H}, A)$ and $\mathcal{C} = (\mathcal{K}, C)$ be C^*–$(\mathfrak{b}_1, \dots, \mathfrak{b}_n)$-algebras, where

$$\mathcal{H} = (H, \alpha_1, \dots, \alpha_n) \quad \text{and} \quad \mathcal{K} = (K, \gamma_1, \dots, \gamma_n).$$

A *morphism* from $\mathcal{A}$ to $\mathcal{C}$ is a $*$-homomorphism $\pi\colon A \to C$ satisfying $[\mathcal{L}^\pi(\mathcal{H}, \mathcal{K})\alpha_i] = \gamma_i$ for each i, where $\mathcal{L}^\pi(\mathcal{H}, \mathcal{K}) = \mathcal{L}^\pi(H, K) \cap \mathcal{L}(\mathcal{H}, \mathcal{K})$. One easily verifies that every morphism π between C^*–$\mathfrak{b}$-algebras A_H^α and C_K^γ satisfies $\pi(\rho_\alpha(b^\dagger)) = \rho_\gamma(b^\dagger)$ for all $b^\dagger \in \mathfrak{B}^\dagger$.

Let $\mathfrak{b}$ be a C^*-base, A_H^β a C^*–$\mathfrak{b}$-algebra, and B_K^γ a C^*–$\mathfrak{b}^\dagger$-algebra. The *fibre product* of A_H^β and B_K^γ is the C^*-algebra

$$A\,{}_\beta\underset{\mathfrak{b}}{*}{}_\gamma\,B := \{x \in \mathcal{L}(H\,{}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma\,K) \mid x|\beta\rangle_1, x^*|\beta\rangle_1 \subseteq [|\beta\rangle_1 B] \text{ as subsets of } \mathcal{L}(K, H\,{}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma\,K),$$
$$x|\gamma\rangle_2, x^*|\gamma\rangle_2 \subseteq [|\gamma\rangle_2 A] \text{ as subsets of } \mathcal{L}(H, H\,{}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma\,K)\}.$$

If A and B are unital, so is $A\,{}_\beta\underset{\mathfrak{b}}{*}{}_\gamma\,B$, but otherwise, $A\,{}_\beta\underset{\mathfrak{b}}{*}{}_\gamma\,B$ may be degenerate. Clearly, conjugation by the flip

$$\Sigma\colon H\,{}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma\,K \to K\,{}_\gamma\underset{\mathfrak{b}^\dagger}{\otimes}{}_\beta\,H$$

yields an isomorphism

$$\mathrm{Ad}_\Sigma\colon A\,{}_\beta\underset{\mathfrak{b}}{*}{}_\gamma\,B \to B\,{}_\gamma\underset{\mathfrak{b}^\dagger}{*}{}_\beta\,A.$$

If $\mathfrak{a}$, $\mathfrak{c}$ are C^*-bases, $A_H^{\alpha,\beta}$ is a C^*–$(\mathfrak{a}, \mathfrak{b})$-algebra and $B_K^{\gamma,\delta}$ a C^*–$(\mathfrak{b}^\dagger, \mathfrak{c})$-algebra, then

$$A_H^{\alpha,\beta} \underset{\mathfrak{b}}{*} B_K^{\gamma,\delta} = ({}_\alpha H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K_\delta, A\,{}_\beta\underset{\mathfrak{b}}{*}{}_\gamma\,B)$$

is a C^*–$(\mathfrak{a}, \mathfrak{c})$-algebra, called the *fibre product* of $A_H^{\alpha,\beta}$ and $B_K^{\gamma,\delta}$.

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Let $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ be C^*-bases, ϕ a morphism of C^*–$(\mathfrak{a},\mathfrak{b})$-algebras $\mathcal{A} = A_H^{\alpha,\beta}$ and $\mathcal{C} = C_L^{\kappa,\lambda}$, and ψ a morphism of C^*–$(\mathfrak{b}^\dagger,\mathfrak{c})$-algebras $\mathcal{B} = B_K^{\gamma,\delta}$ and $\mathcal{D} = D_M^{\mu,\nu}$. Then there exists a unique morphism of C^*–$(\mathfrak{a},\mathfrak{c})$-algebras

$$\phi \underset{\mathfrak{b}}{*} \psi \colon \mathcal{A} \underset{\mathfrak{b}}{*} \mathcal{B} \to \mathcal{C} \underset{\mathfrak{b}}{*} \mathcal{D}$$

such that

$$(\phi \underset{\mathfrak{b}}{*} \psi)(x)R = Rx \quad \text{for all } x \in A\,{}_{\beta}\underset{\mathfrak{b}}{*}{}_{\gamma}\, B \text{ and } R \in I_M J_H + J_L I_K,$$

where

$$I_X = \mathcal{L}^{\phi}(H,L) \underset{\mathfrak{b}}{\otimes} \mathrm{id}_X, \qquad J_Y = \mathrm{id}_Y \underset{\mathfrak{b}}{\otimes} \mathcal{L}^{\psi}(K,M)$$

for $X \in \{K,M\}$, $Y \in \{H,L\}$.

The fibre product need not be associative, but whenever it appears as the target of a comultiplication, coassociativity will compensate the non-associativity.

Definition 3.1. A *comultiplication* on a C^*–$(\mathfrak{b}^\dagger,\mathfrak{b})$-algebra $A_H^{\beta,\alpha}$ is a morphism Δ from $A_H^{\beta,\alpha}$ to

$$A_H^{\beta,\alpha} \underset{\mathfrak{b}}{*} A_H^{\beta,\alpha}$$

that is coassociative in the sense that

$$(\Delta \underset{\mathfrak{b}}{*} \mathrm{id}) \circ \Delta = (\mathrm{id} \underset{\mathfrak{b}}{*} \Delta) \circ \Delta$$

as maps from A to

$$\mathcal{L}(H\,{}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}\, H\,{}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}\, H).$$

A *Hopf C^*-bimodule over* $\mathfrak{b}$ is a C^*–$(\mathfrak{b}^\dagger,\mathfrak{b})$-algebra with a comultiplication. A *morphism* of Hopf C^*-bimodules $(A_H^{\beta,\alpha},\Delta_A)$, $(B_K^{\delta,\gamma},\Delta_B)$ over $\mathfrak{b}$ is a morphism π from $A_H^{\beta,\alpha}$ to $B_K^{\delta,\gamma}$ satisfying

$$\Delta_B \circ \pi = (\pi \underset{\mathfrak{b}}{*} \pi) \circ \Delta_A.$$

3.2. The Hopf C^*-bimodules of a C^*-pseudo-multiplicative unitary

Let $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be a C^*-base, $(H, \hat\beta, \alpha, \beta)$ a C^*–$(\mathfrak{b}^\dagger,\mathfrak{b},\mathfrak{b}^\dagger)$-module and

$$V \colon H\,{}_{\hat\beta}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}\, H \to H\,{}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}\, H$$

a C^*-pseudo-multiplicative unitary. We associate to V two algebras and, if V is well behaved, two Hopf C^*-bimodules as follows. Let

$$\hat A_V := [\langle\beta|_2 V|\alpha\rangle_2] \subseteq \mathcal{L}(H), \qquad A_V := [\langle\alpha|_1 V|\hat\beta\rangle_1] \subseteq \mathcal{L}(H), \tag{3.4}$$

where

$$|\alpha\rangle_2, |\hat\beta\rangle_1 \subseteq \mathcal{L}(H, H\,{}_{\hat\beta}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}\, H) \quad \text{and} \quad \langle\beta|_2, \langle\alpha|_1 \subseteq \mathcal{L}(H\,{}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}\, H, H)$$

are defined as in § 2.1.

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Proposition 3.2.

(i) $\hat{A}_{V^{\mathrm{op}}} = A_V^*$, $[\hat{A}_V\hat{A}_V] = \hat{A}_V$, $[\hat{A}_V H] = H = [\hat{A}_V^* H]$, $[\hat{A}_V\beta] = \beta = [\hat{A}_V^*\beta]$, *and* $[\hat{A}_V\rho_{\hat{\beta}}(\mathfrak{B})] = [\rho_{\hat{\beta}}(\mathfrak{B})\hat{A}_V] = \hat{A}_V = [\hat{A}_V\rho_\alpha(\mathfrak{B}^\dagger)] = [\rho_\alpha(\mathfrak{B}^\dagger)\hat{A}_V]$.

(ii) $A_{V^{\mathrm{op}}} = \hat{A}_V^*$, $[A_V A_V] = A_V$, $[A_V H] = H = [A_V^* H]$, $[A_V\hat{\beta}] = \hat{\beta} = [A_V^*\hat{\beta}]$, *and* $[A_V\rho_\beta(\mathfrak{B})] = [\rho_\beta(\mathfrak{B})A_V] = A_V = [A_V\rho_\alpha(\mathfrak{B}^\dagger)] = [\rho_\alpha(\mathfrak{B}^\dagger)A_V]$.

We shall prove some of the equations above using commutative diagrams, where the vertices are labelled by Hilbert spaces, the arrows are labelled by single operators or closed spaces of operators, and the composition is given by the closed linear span of all possible compositions of operators.

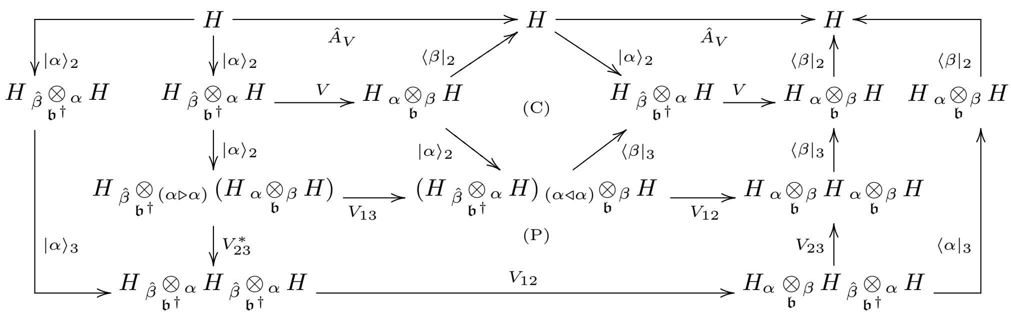
Proof. (i) First,

$$\hat{A}_{V^{\mathrm{op}}} = [\langle\hat{\beta}|_2\Sigma V^*\Sigma|\alpha\rangle_2] = [\langle\hat{\beta}|_1 V^*|\alpha\rangle_1] = A_V^*.$$

Next,

$$[\hat{A}_V\hat{A}_V] = [\langle\beta|_2\langle\alpha|_3 V_{12}|\alpha\rangle_3|\alpha\rangle_2]$$

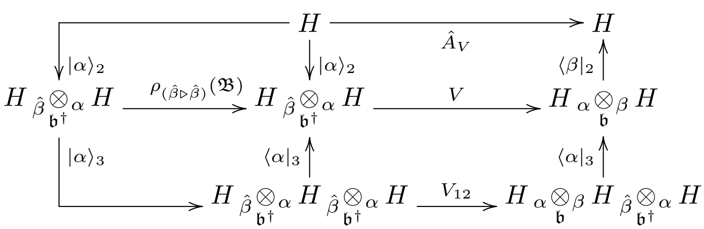
because the diagram below commutes:



Indeed, cell (C) commutes because for all $\xi \in \alpha$, $\eta, \eta' \in \beta$, $\zeta \in H$,

$$|\xi\rangle_2\langle\eta'|_2(\zeta \otimes \eta) = \rho_\alpha(\eta'^*\eta)\zeta \otimes \xi = \rho_{(\alpha\lhd\alpha)}(\eta'^*\eta)(\zeta \otimes \xi) = \langle\eta'|_3|\xi\rangle_2(\zeta \otimes \eta), \tag{3.5}$$

cell (P) is diagram (2.5), and the other cells commute by definition of $\hat{A}_V$ and because of (2.4). Now, $[\langle\beta|_2\langle\alpha|_3 V_{12}|\alpha\rangle_3|\alpha\rangle_2] = \hat{A}_V$ because the following diagram commutes:



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Finally, we prove some of the remaining equations; the other ones follow similarly:

$$\begin{aligned}[]
[\hat{A}_V\beta] &= [\langle\beta|_2V|\alpha\rangle_2\beta] = [\langle\beta|_2|\beta\rangle_2\beta] = [\rho_\alpha(\mathfrak{B}^\dagger)\beta] = \beta,\\
[\hat{A}_V\rho_{\hat{\beta}}(\mathfrak{B})] &= [\langle\beta|_2V|\alpha\rangle_2\rho_{\hat{\beta}}(\mathfrak{B})] = [\langle\beta|_2V|\alpha\mathfrak{B}\rangle_2] = \hat{A}_V,\\
[\rho_{\hat{\beta}}(\mathfrak{B})\hat{A}_V] &= [\rho_{\hat{\beta}}(\mathfrak{B})\langle\beta|_2V|\alpha\rangle_2] = [\langle\beta|_2(\rho_{\hat{\beta}}(\mathfrak{B}) \underset{\mathfrak{b}}{\otimes} \mathrm{id})V|\alpha\rangle_2]\\
&= [\langle\beta|_2V(\mathrm{id} \underset{\mathfrak{b}^\dagger}{\otimes} \rho_\beta(\mathfrak{B}))|\alpha\rangle_2] = [\langle\beta|_2V|\rho_\beta(\mathfrak{B})\alpha\rangle_2] = \hat{A}_V.
\end{aligned}$$

(ii) This follows from (i) after replacing V by V^{op}. □

Define

$$\hat{\Delta}_V\colon \rho_\beta(\mathfrak{B})' \to \mathcal{L}(H \,{}_{\hat{\beta}}\!\underset{\mathfrak{b}^\dagger}{\otimes}\!{}_\alpha\, H) \quad \text{and} \quad \Delta_V\colon \rho_{\hat{\beta}}(\mathfrak{B})' \to \mathcal{L}(H \,{}_\alpha\!\underset{\mathfrak{b}}{\otimes}\!{}_\beta\, H)$$

by

$$\hat{\Delta}_V\colon y \mapsto V^*(\mathrm{id} \underset{\mathfrak{b}}{\otimes} y)V \quad \text{and} \quad \Delta_V\colon z \mapsto V(z \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id})V^*.$$

Evidently,

$$\hat{\Delta}_{V^{\mathrm{op}}} = \mathrm{Ad}_\Sigma \circ \Delta_V \quad \text{and} \quad \Delta_{V^{\mathrm{op}}} = \mathrm{Ad}_\Sigma \circ \hat{\Delta}_V.$$

Moreover, if $\eta \in \beta$ and $\xi \in \alpha$, then $\hat{a} := \langle\eta|_2V|\xi\rangle_2$ lies in $\mathcal{L}(H_\beta) \subseteq \rho_\beta(\mathfrak{B})'$ by Proposition 3.2 and

$$\hat{\Delta}_V(\langle\eta|_2V|\xi\rangle_2) = \hat{\Delta}_V(\hat{a}) = V^*(1 \underset{\mathfrak{b}}{\otimes} \hat{a})V = \langle\eta|_3V_{12}^*V_{23}V_{12}|\xi\rangle_3 = \langle\eta|_3V_{13}V_{23}|\xi\rangle_3. \tag{3.6}$$

Similarly, if $\eta \in \alpha$ and $\xi \in \hat{\beta}$, then $a := \langle\eta|_1V|\xi\rangle_1$ lies in $\rho_{\hat{\beta}}(\mathfrak{B})'$ and

$$\Delta_V(\langle\eta|_1V|\xi\rangle_1) = \Delta_V(a) = V(a \underset{\mathfrak{b}^\dagger}{\otimes} 1)V^* = \langle\eta|_1V_{23}V_{12}V_{23}^*|\xi\rangle_1 = \langle\eta|_1V_{12}V_{13}|\xi\rangle_1. \tag{3.7}$$

Lemma 3.3. *The map $\hat{\Delta}_V$ is a morphism of the C^*–$(\mathfrak{b}, \mathfrak{b}^\dagger)$-algebras*

$$(\rho_\beta(\mathfrak{B})')_H^{\alpha,\hat{\beta}} \quad \textit{and} \quad ((\rho_\beta(\mathfrak{B}) \,{}_{\hat{\beta}}\!\underset{\mathfrak{b}^\dagger}{\otimes}\!{}_\alpha\, \rho_\beta(\mathfrak{B}))')^{(\alpha\triangleleft\alpha),(\hat{\beta}\triangleright\hat{\beta})}_{H \,{}_{\hat{\beta}}\!\underset{\mathfrak{b}^\dagger}{\otimes}\!{}_\alpha\, H},$$

and Δ_V is a morphism of the C^*–$(\mathfrak{b}^\dagger, \mathfrak{b})$-algebras

$$(\rho_{\hat{\beta}}(\mathfrak{B})')_H^{\beta,\alpha} \quad \textit{and} \quad ((\rho_{\hat{\beta}}(\mathfrak{B}) \,{}_\alpha\!\underset{\mathfrak{b}}{\otimes}\!{}_\beta\, \rho_{\hat{\beta}}(\mathfrak{B}))')^{(\beta\triangleleft\beta),(\alpha\triangleright\alpha)}_{H \,{}_\alpha\!\underset{\mathfrak{b}}{\otimes}\!{}_\beta\, H}.$$

Proof. We only prove the assertions concerning $\hat{\Delta}_V$. First,

$$\hat{\Delta}_V(\rho_\beta(\mathfrak{B})') \subseteq (\rho_\beta(\mathfrak{B}) \underset{\mathfrak{b}^\dagger}{\otimes} \rho_\beta(\mathfrak{B}))'$$

because

$$V(\rho_\beta(\mathfrak{B}) \underset{\mathfrak{b}^\dagger}{\otimes} \rho_\beta(\mathfrak{B}))V^* = \rho_\beta(\mathfrak{B}) \underset{\mathfrak{b}}{\otimes} \rho_{\hat{\beta}}(\mathfrak{B}) \subseteq \mathrm{id} \underset{\mathfrak{b}}{\otimes} \rho_\beta(\mathfrak{B})'$$

by (2.4). Next,

$$V^*|\alpha\rangle_1 \subseteq \mathcal{L}^{\hat{\Delta}_V}(H, H\, {}_{\hat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} H)$$

because

$$\hat{\Delta}(y)V^*|\xi\rangle_1 = V^*(\mathrm{id} \underset{\mathfrak{b}}{\otimes} y)|\xi\rangle_1 = V^*|\xi\rangle_1 y$$

for all $y \in \rho_{\hat{\beta}}(\mathfrak{B})'$, $\xi \in \alpha$, and $\alpha \triangleleft \alpha = [V^*|\alpha\rangle_1 \alpha]$ and $\hat{\beta} \triangleright \hat{\beta} = [V^*|\alpha\rangle_1 \hat{\beta}]$ by (2.4). □

Theorem 3.4. *If $\hat{A}_V = \hat{A}_V^*$, then $((\hat{A}_V)_H^{\alpha,\hat{\beta}}, \hat{\Delta}_V)$ is a Hopf C^*-bimodule. Similarly, if $A_V = A_V^*$, then $((A_V)_H^{\beta,\alpha}, \Delta_V)$ is a Hopf C^*-bimodule.*

Proof. We only prove the first assertion; the second one follows by replacing V by V^{op}. Write $\hat{A} = \hat{A}_V$, $\hat{\Delta} = \hat{\Delta}_V$, and assume that $\hat{A} = \hat{A}^*$. By Proposition 3.2, $\hat{\mathcal{A}} := \hat{A}_H^{\alpha,\hat{\beta}}$ is a C^*–$(\mathfrak{b}, \mathfrak{b}^\dagger)$-algebra and $\hat{A} \subseteq \mathcal{L}(H_\beta) \subseteq \rho_\beta(\mathfrak{B})'$. We claim that

$$\hat{\Delta}(\hat{A}) \subseteq \hat{A}\, {}_{\hat{\beta}}\underset{\mathfrak{b}^\dagger}{*}{}_{\alpha} \hat{A}.$$

By (3.6), $\hat{\Delta}(\hat{A}) = [\langle\beta|_3 V_{13} V_{23} |\alpha\rangle_3]$, and the following commutative diagram shows that $[\hat{\Delta}(\hat{A})|\alpha\rangle_2] = [\langle\beta|_3 V_{13}V_{23}|\alpha\rangle_3|\alpha\rangle_2] \subseteq [|\alpha\rangle_2\langle\beta|_2 V|\alpha\rangle_2] = [|\alpha\rangle_2 \hat{A}]$:

$$\begin{array}{ccccccc}
H & \xrightarrow{|\alpha\rangle_2} & H\, {}_{\hat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} H & \xrightarrow{V} & H\, {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H & \xrightarrow{\langle\beta|_2} & H \\
\downarrow{\scriptstyle |\alpha\rangle_2} & & \downarrow{\scriptstyle |\alpha\rangle_3} & & \downarrow{\scriptstyle |\alpha\rangle_3} & & \downarrow{\scriptstyle |\alpha\rangle_2} \\
H\, {}_{\hat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} H & \xrightarrow{V_{23}|\alpha\rangle_3} & H\, {}_{\hat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{(\alpha\triangleright\alpha)} (H\, {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H) & \xrightarrow{V_{13}} & (H\, {}_{\hat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} H)\, {}_{(\alpha\triangleleft\alpha)}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H & \xrightarrow{\langle\beta|_3} & H\, {}_{\hat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} H
\end{array}$$

Similarly, one proves that $[\hat{\Delta}(\hat{A})|\hat{\beta}\rangle_1] = [|\hat{\beta}\rangle_1 \hat{A}]$, and the claim follows. By Lemma 3.3, $\hat{\Delta}$ is a morphism of the C^*–$(\mathfrak{b}, \mathfrak{b}^\dagger)$-algebras $\hat{\mathcal{A}}$ and $\hat{\mathcal{A}} \underset{\mathfrak{b}}{*} \hat{\mathcal{A}}$. It only remains to show that $\hat{\Delta}$ is coassociative. Let $\hat{a} \in \hat{A}$. Then

$$(\hat{\Delta} \underset{\mathfrak{b}^\dagger}{*} \mathrm{id})(\hat{\Delta}(\hat{a})) = V_{12}^*(\mathrm{id} \underset{\mathfrak{b}}{\otimes} \hat{\Delta}(\hat{a}))V_{12} = V_{12}^* V_{23}^*(\mathrm{id} \underset{\mathfrak{b}}{\otimes} \mathrm{id} \underset{\mathfrak{b}}{\otimes} \hat{a})V_{23}V_{12}.$$

Here, we can replace $V_{23}V_{12}$ by $V_{12}^* V_{23} V_{12} = V_{13}V_{23}$. Therefore, $(\hat{\Delta} \underset{\mathfrak{b}^\dagger}{*} \mathrm{id})(\hat{\Delta}(\hat{a}))$ equals

$$V_{23}V_{13}^*(\mathrm{id} \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id} \underset{\mathfrak{b}}{\otimes} \hat{a})V_{13}V_{23} = V_{23}\Sigma_{23}(\hat{\Delta}(\hat{a}) \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id})\Sigma_{23}V_{23} = (\mathrm{id} \underset{\mathfrak{b}^\dagger}{*} \hat{\Delta})(\hat{\Delta}(\hat{a})).$$

□

3.3. The Fourier algebras of a C^*-pseudo-multiplicative unitary

We first introduce certain spaces of maps on C^*-algebras and slice maps on fibre products, and then associate to every Hopf C^*-bimodule several convolution algebras and to every C^*-pseudo-multiplicative unitary two Fourier algebras.

Let $\mathfrak{a} = (\mathfrak{H}, \mathfrak{A}, \mathfrak{A}^\dagger)$ and $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be C^*-bases, H a Hilbert space, H_α a C^*–$\mathfrak{a}$-module, H_β a C^*–$\mathfrak{b}$-module, and $A \subseteq \mathcal{L}(H)$ a closed subspace. We denote by α^∞ the space of all sequences $\eta = (\eta_k)_{k\in\mathbb{N}}$ in α for which the sum $\sum_k \eta_k^* \eta_k$ converges in

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norm, and put $\|\eta\| := \| \sum_k \eta_k^* \eta_k\|^{1/2}$ for each $\eta \in \alpha^\infty$. Similarly, we define β^∞. Standard arguments show that for all $\eta \in \beta^\infty$, $\eta' \in \alpha^\infty$, there exists a bounded linear map

$$\omega_{\eta,\eta'} \colon A \to \mathcal{L}(\mathfrak{H}, \mathfrak{K}), \quad T \mapsto \sum_{k \in \mathbb{N}} \eta_k^* T \eta'_k,$$

where the sum converges in norm and $\|\omega_{\eta,\eta'}\| \leqslant \|\eta\| \, \|\eta'\|$. We put

$$\Omega_{\beta,\alpha}(A) := \{\omega_{\eta,\eta'} \mid \eta \in \beta^\infty, \ \eta' \in \alpha^\infty\} \subseteq L(A, \mathcal{L}(\mathfrak{H}, \mathfrak{K})),$$

where $L(A, \mathcal{L}(\mathfrak{H}, \mathfrak{K}))$ denotes the space of bounded linear maps from A to $\mathcal{L}(\mathfrak{H}, \mathfrak{K})$. If $\beta = \alpha$, we abbreviate $\Omega_\beta(A) := \Omega_{\beta,\alpha}(A)$. It is easy to see that $\Omega_{\beta,\alpha}(A)$ is a subspace of $L(A, \mathcal{L}(\mathfrak{H}, \mathfrak{K}))$ and that the following formula defines a norm on $\Omega_{\beta,\alpha}(A)$:

$$\|\omega\| := \inf\{\|\eta\| \, \|\eta'\| \mid \eta \in \beta^\infty, \ \eta' \in \alpha^\infty, \ \omega = \omega_{\eta,\eta'}\} \quad \text{for all } \omega \in \Omega_{\beta,\alpha}(A).$$

Standard arguments show that $\Omega_{\beta,\alpha}(A)$ is a Banach space. Moreover, if $A = A^*$, then there exists an anti-linear isometry $\Omega_{\beta,\alpha}(A) \to \Omega_{\alpha,\beta}(A)$, $\omega \mapsto \omega^*$, such that $\omega^*(a) = \omega(a^*)^*$ for all $a \in A$ and $(\omega_{\eta,\eta'})^* = \omega_{\eta',\eta}$ for all $\eta \in \beta^\infty$, $\eta' \in \alpha^\infty$.

Proposition 3.5.

(i) *Let π be a morphism of C^*–$\mathfrak{b}$-algebras A_H^α and B_K^γ. Then there exists a linear contraction $\pi^* \colon \Omega_\gamma(B) \to \Omega_\alpha(A)$ given by $\omega \mapsto \omega \circ \pi$.*

(ii) *Let π be a morphism of C^*–$(\mathfrak{a}, \mathfrak{b})$-algebras $A_H^{\alpha,\beta}$ and $B_K^{\gamma,\delta}$. Then there exists a linear contraction $\pi^* \colon \Omega_{\delta,\gamma}(B) \to \Omega_{\beta,\alpha}(A)$ given by $\omega \mapsto \omega \circ \pi$.*

Proof. We only prove (ii). Let $I := \mathcal{L}^\pi({}_\alpha H_\beta, {}_\gamma K_\delta)$ and $\eta \in \delta^\infty$, $\eta' \in \gamma^\infty$. Then there exists a closed separable subspace $I_0 \subseteq I$ such that $\eta_n \in [I_0\beta]$ and $\eta'_n \in [I_0\alpha]$ for all $n \in \mathbb{N}$. We may assume that $I_0 I_0^* I_0 \subseteq I_0$, and then $[I_0 I_0^*]$ is a σ-unital C^*-algebra and has a bounded sequential approximate unit $(u_k)_k$ of the form $u_k = \sum_{l=1}^k T_l T_l^*$, where $(T_l)_l$ is a sequence in I_0 [**15**, Proposition 6.7]. We choose a bijection $i \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and let $\xi_{i(l,n)} := T_l^* \eta_n \in \beta$ and $\xi'_{i(l,n)} := T_l^* \eta'_n \in \alpha$ for all $l, n \in \mathbb{N}$. Then the sum

$$\sum_l \xi_{i(l,n)}^* \xi_{i(l,n)} = \sum_l \eta_n^* T_l T_l^* \eta_n$$

converges to $\eta_n^* \eta_n$ for each $n \in \mathbb{N}$ in norm because $\eta_n \in [I_0\beta]$. Therefore, $\xi \in \beta^\infty$ and $\|\xi\| = \|\eta\|$, and a similar argument shows that $\xi' \in \alpha^\infty$ and $\|\xi'\| = \|\eta'\|$. Finally,

$$\omega_{\xi,\xi'}(a) = \sum_{l,n} \eta_n^* T_l a T_l^* \eta'_n = \sum_{l,n} \eta_n^* \pi(a) T_l T_l^* \eta'_n = \sum_n \eta_n^* \pi(a) \eta'_n = \omega_{\eta,\eta'}(\pi(a))$$

for each $a \in A$, where the sum converges in norm, and hence $\omega_{\eta,\eta'} \circ \pi = \omega_{\xi,\xi'} \in \Omega_{\beta,\alpha}(A)$ and $\|\omega_{\eta,\eta'} \circ \pi\| \leqslant \|\xi\| \, \|\xi'\| = \|\eta\| \, \|\eta'\|$. □

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For each map of the form considered above, we can form a slice map as follows [**25**, Proposition 3.30]. Let H_β be a C^*–$\mathfrak{b}$-module, let K_γ be a C^*–$\mathfrak{b}^\dagger$-module, let $A \subseteq \mathcal{L}(H)$ and $B \subseteq \mathcal{L}(K)$ be closed subspaces, and let

$$A \,{}_\beta \underset{\mathfrak{b}}{\bar{*}}{}_\gamma\, B = \{x \in \mathcal{L}(H \,{}_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma\, K) \mid \langle\beta|_1 x|\beta\rangle_1 \subseteq B,\ \langle\gamma|_2 x|\gamma\rangle_2 \subseteq A\}.$$

Proposition 3.6.

- *There exists a linear contraction*

$$\Omega_\beta(A) \to \Omega_{|\beta\rangle_1}(A \,{}_\beta \underset{\mathfrak{b}}{\bar{*}}{}_\gamma\, B),$$

  $\phi \mapsto \phi * \mathrm{id}$, *such that* $\omega_{\xi,\xi'} * \mathrm{id} = \omega_{\tilde{\xi},\tilde{\xi}'}$ *for all* $\xi, \xi' \in \beta^\infty$, *where* $\tilde{\xi}_n = |\xi_n\rangle_1$ *and* $\tilde{\xi}'_n = |\xi'_n\rangle_1$ *for all* $n \in \mathbb{N}$.

- *There exists a linear contraction*

$$\Omega_\gamma(B) \to \Omega_{|\gamma\rangle_2}(A \,{}_\beta \underset{\mathfrak{b}}{\bar{*}}{}_\gamma\, B),$$

  $\psi \mapsto \mathrm{id} * \psi$, *such that* $\mathrm{id} * \omega_{\eta,\eta'} = \omega_{\tilde{\eta},\tilde{\eta}'}$ *for all* $\eta, \eta' \in \gamma^\infty$, *where* $\tilde{\eta}_n = |\eta_n\rangle_2$ *and* $\tilde{\eta}'_n = |\eta'_n\rangle_2$ *for all* $n \in \mathbb{N}$.

- *We have* $\psi \circ (\phi * \mathrm{id}) = \phi \circ (\mathrm{id} * \psi)$ *for all* $\phi \in \Omega_\beta(A)$ *and* $\psi \in \Omega_\gamma(B)$.

Assume that ${}_\alpha H_\beta$ is a C^*–$(\mathfrak{a}, \mathfrak{b})$-module and that ${}_\gamma K_\delta$ is a C^*–$(\mathfrak{b}^\dagger, \mathfrak{c})$-module. Denote by '$\hat{\otimes}$' the projective tensor product of Banach spaces. Clearly, there exist linear contractions

$$\Omega_\alpha(A)\hat{\otimes}\Omega_\gamma(B) \to \Omega_{(\alpha\triangleleft\gamma)}(A \,{}_\beta \underset{\mathfrak{b}}{\bar{*}}{}_\gamma\, B), \qquad \omega \otimes \omega' \mapsto \omega \boxtimes \omega' := \omega \circ (\mathrm{id} * \omega'),$$
$$\Omega_\beta(A)\hat{\otimes}\Omega_\delta(B) \to \Omega_{(\beta\triangleright\delta)}(A \,{}_\beta \underset{\mathfrak{b}}{\bar{*}}{}_\gamma\, B), \qquad \omega \otimes \omega' \mapsto \omega \boxtimes \omega' := \omega' \circ (\omega * \mathrm{id}).$$

Proposition 3.7. *There exist linear contractions*

$$\Omega_{\alpha,\beta}(A)\hat{\otimes}\Omega_{\gamma,\delta}(B) \to \Omega_{(\alpha\triangleleft\gamma),(\beta\triangleright\delta)}(A \,{}_\beta \underset{\mathfrak{b}}{\bar{*}}{}_\gamma\, B), \qquad \omega \otimes \omega' \mapsto \omega \boxtimes \omega',$$
$$\Omega_{\beta,\alpha}(A)\hat{\otimes}\Omega_{\delta,\gamma}(B) \to \Omega_{(\beta\triangleright\delta),(\alpha\triangleleft\gamma)}(A \,{}_\beta \underset{\mathfrak{b}}{\bar{*}}{}_\gamma\, B), \qquad \omega \otimes \omega' \mapsto \omega \boxtimes \omega',$$

such that for all $\xi \in \alpha^\infty$, $\xi' \in \beta^\infty$, $\eta \in \gamma^\infty$, $\eta' \in \delta^\infty$ *and each bijection* $i \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, *we have* $\omega_{\xi,\xi'} \boxtimes \omega_{\eta,\eta'} = \omega_{\theta,\theta'}$ *and* $\omega_{\xi',\xi} \boxtimes \omega_{\eta',\eta} = \omega_{\theta',\theta}$, *where*

$$\theta_{i(m,n)} = |\eta_n\rangle_2 \xi_m \in \alpha \triangleleft \gamma, \quad \theta'_{i(m,n)} = |\xi'_m\rangle_1 \eta'_n \in \beta \triangleright \delta \quad \text{for all } m, n \in \mathbb{N}.$$

Proof. We only prove the existence of the first contraction. Let ξ, ξ', η, η', i, θ, θ' be as above. Then $\theta \in (\alpha \triangleleft \gamma)^\infty$ and $\|\theta\| \leqslant \|\xi\|\,\|\eta\|$ because

$$\sum_k \theta_k^* \theta_k = \sum_{m,n} \xi_m^* \langle\eta_n|_2|\eta_n\rangle_2 \xi_m = \sum_{m,n} \xi_m^* \rho_\beta(\eta_n^*\eta_n)\xi_m \leqslant \|\eta\|^2 \sum_m \xi_m^*\xi_m \leqslant \|\eta\|^2\|\xi\|^2,$$

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and similarly $\theta' \in (\beta \triangleright \delta)^\infty$ and $\|\theta'\| \leqslant \|\xi'\| \, \|\eta'\|$. Next, we show that $\omega_{\theta,\theta'}$ does not depend on ξ and ξ' but only on $\omega_{\xi,\xi'} \in \Omega_{\alpha,\beta}(A)$. Let $\zeta' \in \mathfrak{K}$ and $x \in A \,{}_{\beta}\overset{*}{\underset{\mathfrak{b}}{}}{}_{\gamma}\, B$. Then

$$\omega_{\theta,\theta'}(x)\zeta' = \sum_{m,n\in\mathbb{N}} \xi_m^* \langle \eta_n|_2 x|\xi'_m\rangle_1 \eta'_n \zeta',$$

where the sum converges in norm. Fix any $n \in \mathbb{N}$. Then we find a sequence $(k_r)_r$ in $\mathbb{N}$ and $\eta''_{r,1}, \dots, \eta''_{r,k_r} \in \gamma$, $\zeta''_{r,1}, \dots, \zeta''_{r,k_r} \in \mathfrak{K}$ such that the sum $\sum_{l=1}^{k_r} \eta''_{r,l}\zeta''_{r,l}$ converges in norm to $\eta'_n\zeta'$ as r tends to infinity. But then

$$\begin{aligned}\sum_m \xi_m^* \langle \eta_n|_2 x|\xi'_m\rangle_1 \eta'_n\zeta' &= \lim_{r\to\infty} \sum_m \sum_{l=1}^{k_r} \xi_m^* \langle \eta_n|_2 x|\xi'_m\rangle_1 \eta''_{r,l}\zeta''_{r,l} \\ &= \lim_{r\to\infty} \sum_{l=1}^{k_r} \sum_m \xi_m^* \langle \eta_n|_2 x|\eta''_{r,l}\rangle_2 \xi'_m \zeta''_{r,l} \\ &= \lim_{r\to\infty} \sum_{l=1}^{k_r} \omega_{\xi,\xi'}(\langle \eta_n|_2 x|\eta''_{r,l}\rangle_2)\zeta''_{r,l}.\end{aligned}$$

Note here that $\langle \eta_n|x|\eta''_{r,l}\rangle_2 \in A$. Therefore, the sum on the left-hand side only depends on $\omega_{\xi,\xi'} \in \Omega_{\alpha,\beta}(A)$ but not on ξ, ξ', and since $n \in \mathbb{N}$ was arbitrary, the same is true for $\omega_{\theta,\theta'}(x)\zeta'$. A similar argument shows that $\omega_{\theta,\theta'}(x)^*\zeta$ depends on $\omega_{\eta,\eta'} \in \Omega_{\gamma,\delta}(B)$ but not on η, η' for each $\zeta \in \mathfrak{K}$. □

Proposition 3.8. *Let $(A_H^{\beta,\alpha}, \Delta)$ be a Hopf C^*-bimodule over $\mathfrak{b}$. Then each of the spaces $\Omega = \Omega_\alpha(A), \Omega_\beta(A), \Omega_{\alpha,\beta}(A), \Omega_{\beta,\alpha}(A)$ is a Banach algebra with respect to the multiplication $\Omega \times \Omega \to \Omega$ given by $(\omega, \omega') \mapsto \omega * \omega' := (\omega \boxtimes \omega') \circ \Delta$.*

Proof. The multiplication is well defined by Propositions 3.5 and 3.7, and associative because Δ is coassociative. □

Now, let $(H, \hat{\beta}, \alpha, \beta)$ be a C^*–$(\mathfrak{b}^\dagger, \mathfrak{b}, \mathfrak{b}^\dagger)$-module, let

$$V \colon H \,{}_{\hat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}\, H \to H \,{}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}\, H$$

be a C^*-pseudo-multiplicative unitary, and let

$$\tilde{\Omega}_{\beta,\alpha} := \Omega_{\beta,\alpha}(\rho_{\hat{\beta}}(\mathfrak{B})'), \qquad \tilde{\Omega}_{\alpha,\hat{\beta}} := \Omega_{\alpha,\hat{\beta}}(\rho_\beta(\mathfrak{B})').$$

Using Lemma 3.3, the inclusions

$$\begin{aligned}(\rho_{\hat{\beta}}(\mathfrak{B}) \,{}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}\, \rho_{\hat{\beta}}(\mathfrak{B}))' &\subseteq \rho_{\hat{\beta}}(\mathfrak{B})' \,{}_{\alpha}\overset{*}{\underset{\mathfrak{b}}{}}{}_{\beta}\, \rho_{\hat{\beta}}(\mathfrak{B})', \\ (\rho_\beta(\mathfrak{B}) \,{}_{\hat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}\, \rho_\beta(\mathfrak{B}))' &\subseteq \rho_\beta(\mathfrak{B}) \,{}_{\hat{\beta}}\overset{*}{\underset{\mathfrak{b}^\dagger}{}}{}_{\alpha}\, \rho_\beta(\mathfrak{B})',\end{aligned}$$

and Proposition 3.7, we define maps

$$\begin{aligned}\tilde{\Omega}_{\beta,\alpha} \times \tilde{\Omega}_{\beta,\alpha} &\to \tilde{\Omega}_{\beta,\alpha}, \qquad (\omega, \omega') \mapsto \omega * \omega' := (\omega \boxtimes \omega') \circ \Delta_V, \\ \tilde{\Omega}_{\alpha,\hat{\beta}} \times \tilde{\Omega}_{\alpha,\hat{\beta}} &\to \tilde{\Omega}_{\alpha,\hat{\beta}}, \qquad (\omega, \omega') \mapsto \omega * \omega' := (\omega \boxtimes \omega') \circ \hat{\Delta}_V.\end{aligned}$$

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Theorem 3.9.

(i) *The maps above turn* $\tilde{\Omega}_{\beta,\alpha}$ *and* $\tilde{\Omega}_{\alpha,\hat{\beta}}$ *into Banach algebras.*

(ii) *There exist contractive homomorphisms* $\hat{\pi}_V : \tilde{\Omega}_{\beta,\alpha} \to \hat{A}_V$ *and* $\pi_V : \tilde{\Omega}_{\alpha,\hat{\beta}} \to A_V$ *such that* $\hat{\pi}_V(\omega_{\xi,\eta}) = \sum_n \langle \xi_n|_2 V|\eta_n\rangle_2$ *and* $\pi_V(\omega_{\eta,\zeta}) = \sum_n \langle \eta_n|_1 V|\zeta_n\rangle_1$ *for all* $\xi \in \beta^\infty$, $\eta \in \alpha^\infty$, $\zeta \in \hat{\beta}^\infty$.

Proof. We only prove the assertions concerning $\tilde{\Omega}_{\beta,\alpha}$.

(i) One only needs to show that the multiplication on $\tilde{\Omega}_{\beta,\alpha}$ is associative. Let $\omega, \omega', \omega'' \in \tilde{\Omega}_{\beta,\alpha}$, where $\omega = \omega_{\eta,\xi}$, $\omega' = \omega_{\eta',\xi'}$, $\omega'' = \omega_{\eta'',\xi''}$ and $\eta, \eta', \eta'' \in \beta^\infty$, $\xi, \xi', \xi'' \in \alpha^\infty$. Then a short calculation shows that for all $x \in \rho_{\hat{\beta}}(\mathfrak{B})'$,

$$((\omega * \omega') * \omega'')(x) = \sum_{k,l,m} \eta_k^* \langle \eta_l'|_2 \langle \eta_m''|_3 V_{12} V_{13} (x \underset{\mathfrak{b}^\dagger}{\otimes} (1 \underset{\mathfrak{b}}{\otimes} 1)) V_{13}^* V_{12}^* |\xi_k\rangle_1 |\xi_l'\rangle_1 \xi_m'',$$

$$(\omega * (\omega' * \omega''))(x) = \sum_{k,l,m} \eta_k^* \langle \eta_l'|_2 \langle \eta_m''|_3 V_{23} V_{12} (x \underset{\mathfrak{b}^\dagger}{\otimes} 1 \underset{\mathfrak{b}^\dagger}{\otimes} 1) V_{12}^* V_{23}^* |\xi_k\rangle_1 |\xi_l'\rangle_1 \xi_m'',$$

and by (2.5), the right-hand sides coincide.

(ii) The map $\hat{\pi}_V$ is well-defined because $\eta^* \hat{\pi}_V(\omega)\xi = \omega(\langle \eta|_1 V|\xi\rangle_1)$ and $\langle \eta|_1 V|\xi\rangle_1 \in A_V \subseteq \rho_{\hat{\beta}}(\mathfrak{B})'$ for all $\eta \in \alpha$, $\xi \in \hat{\beta}$, $\omega \in \tilde{\Omega}_{\beta,\alpha}$, contractive because V is unitary, and a homomorphism because for all $\omega, \omega' \in \tilde{\Omega}_{\beta,\alpha}$, $\eta \in \alpha$ $\xi \in \hat{\beta}$,

$$\begin{aligned}
\eta^* \hat{\pi}_V(\omega) \hat{\pi}_V(\omega')\xi &= (\omega \boxtimes \omega')(\langle \eta|_1 V_{12} V_{13} |\xi\rangle_1) \\
&= (\omega \boxtimes \omega')(\langle \eta|_1 V_{23} V_{12} V_{23}^* |\xi\rangle_1) \\
&= (\omega \boxtimes \omega')(V(\langle \eta|_1 V|\xi\rangle_1 \;{}_{\hat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} \; \mathrm{id})V^*) \\
&= (\omega * \omega')(\langle \eta|_1 V|\xi\rangle_1) \\
&= \eta^* \hat{\pi}_V(\omega * \omega')\xi.
\end{aligned}$$

□

Definition 3.10. We call the algebras $\hat{A}_V^0 := \hat{\pi}_V(\tilde{\Omega}_{\beta,\alpha}) \subseteq \hat{A}_V$ and $A_V^0 := \pi_V(\tilde{\Omega}_{\alpha,\hat{\beta}}) \subseteq A_V$, equipped with the quotient norms coming from the surjections $\hat{\pi}_V$ and π_V, the *Fourier algebra* and the *dual Fourier algebra* of V, respectively.

The pairs $((\hat{A}_V)_H^{\alpha,\hat{\beta}}, \hat{\Delta}_V)$ and $((A_V)_H^{\beta,\alpha}, \Delta_V)$ stand in a generalized Pontrjagin duality which is captured by the following pairing.

Proposition 3.11.

(i) *There exists a bilinear map* $(\cdot|\cdot) : \hat{A}_V^0 \times A_V^0 \to \mathcal{L}(\mathfrak{K})$ *such that* $\omega(\pi_V(\upsilon)) = (\hat{\pi}_V(\omega)|\pi_V(\upsilon)) = \upsilon(\hat{\pi}_V(\omega))$ *for all* $\omega \in \tilde{\Omega}_{\beta,\alpha}$, $\upsilon \in \tilde{\Omega}_{\alpha,\hat{\beta}}$. *This map is non-degenerate in the sense that for each* $\hat{a} \in \hat{A}_V^0$ *and* $a \in A_V^0$, *there exist* $\hat{a}' \in \hat{A}_V^0$ *and* $a' \in A_V^0$ *such that* $(\hat{a}|a') \neq 0$ *and* $(\hat{a}'|a) \neq 0$.

(ii) $(\hat{\pi}_V(\omega)\hat{\pi}_V(\omega')|a) = (\omega \boxtimes \omega')(\Delta_V(a))$ *and* $(\hat{a}|\pi_V(\upsilon)\pi_V(\upsilon')) = (\upsilon \boxtimes \upsilon')(\hat{\Delta}_V(\hat{a}))$ *for all* $\omega, \omega' \in \tilde{\Omega}_{\beta,\alpha}$, $a \in A_V^0$, $\upsilon, \upsilon' \in \tilde{\Omega}_{\alpha,\hat{\beta}}$, $\hat{a} \in \hat{A}_V^0$.

Proof. (i) If $\omega = \omega_{\xi,\xi'}$ and $\upsilon = \omega_{\eta,\eta'}$, where $\xi \in \beta^\infty$, $\xi', \eta \in \alpha^\infty$, $\eta' \in \hat{\beta}^\infty$, then

$$\omega(\pi_V(\upsilon)) = \sum_{m,n} \xi_m^* \langle \eta_n|_1 V|\eta_n'\rangle_1 \xi_m' = \sum_{m,n} \eta_n^* \langle \xi_m|_2 V|\xi_m'\rangle_2 \eta_n' = \upsilon(\hat{\pi}_V(\omega)).$$

(ii) For all ω, ω', a as above,

$$(\hat{\pi}_V(\omega)\hat{\pi}_V(\omega')|a) = (\hat{\pi}_V(\omega * \omega')|a) = (\omega * \omega')(a) = (\omega \boxtimes \omega)(\Delta_V(a)).$$

The second equation follows similarly. □

Part (i) of the preceding result implies the following relation between the Fourier algebras $\hat{A}_V^0$ and A_V^0 and the convolution algebras constructed in Proposition 3.8.

Corollary 3.12. *If $((A_V)_H^{\beta,\alpha}, \hat{\Delta}_V)$ or $((\hat{A}_V)_H^{\alpha,\hat{\beta}}, \Delta_V)$ is a Hopf C^*-bimodule, then there exists an isometric isomorphism of Banach algebras $\hat{\pi}\colon \Omega_{\beta,\alpha}(A_V) \to \hat{A}_V^0$ or $\pi\colon \Omega_{\alpha,\hat{\beta}}(\hat{A}_V) \to A_V^0$, respectively, whose composition with the quotient map $\tilde{\Omega}_{\beta,\alpha} \to \Omega_{\beta,\alpha}(A_V)$ or $\tilde{\Omega}_{\alpha,\hat{\beta}} \to \Omega_{\alpha,\hat{\beta}}(\hat{A}_V)$ is equal to $\hat{\pi}_V$ or π_V, respectively.*

3.4. The legs of the unitary of a groupoid

Let G be a locally compact, Hausdorff, second countable groupoid G as in § 2.3. We keep the notation introduced there and determine the legs of the C^*-pseudo-multiplicative unitary V associated to G. Denote by $m\colon C_0(G) \to \mathcal{L}(H)$ the representation given by multiplication operators, and by $L^1(G,\lambda)$ the completion of $C_c(G)$ with respect to the norm given by $\|f\| := \sup_{u\in G^0} \int_{G^u} |f(u)| \,\mathrm{d}\lambda^u(x)$ for all $f \in C_c(G)$. Then $L^1(G,\lambda)$ is a Banach algebra with respect to the convolution product

$$(f * g)(y) = \int_{G^{r(y)}} g(x) f(x^{-1}y) \,\mathrm{d}\lambda^{r(y)}(x) \quad \text{for all } f, g \in L^1(G,\lambda),\ y \in G,$$

and there exists a contractive algebra homomorphism $L\colon L^1(G,\lambda) \to \mathcal{L}(H)$ such that

$$(L(f)\xi)(y) = \int_{G^{r(y)}} f(x) D^{-1/2}(x) \xi(x^{-1}y) \,\mathrm{d}\lambda^{r(y)}(x) \quad \text{for all } f, \xi \in C_c(G),\ y \in G.$$

Routine arguments show that there exists a unique continuous map

$$L^2(G,\lambda) \times L^2(G,\lambda) \to C_0(G), (\xi,\xi') \mapsto \bar{\xi} * \xi'^*,$$

such that $(\bar{\xi} * \xi'^*)(x) = \int_{G^{r(x)}} \overline{\xi(y)} \xi'(x^{-1}y) \,\mathrm{d}\lambda^{r(x)}(y)$ for all $\xi, \xi' \in C_c(G)$, $x \in G$.

Lemma 3.13. *Let $\hat{a}_{\xi,\xi'} := \langle j(\xi)|_2 V|j(\xi')\rangle_2$ and $a_{\eta,\eta'} := \langle j(\eta)|_1 V|\hat{\jmath}(\eta')\rangle_1 \in A_V^0$, where $\xi, \xi' \in L^2(G,\lambda)$ and $\eta, \eta' \in C_c(G)$. Then $\hat{a}_{\xi,\xi'} = m(\bar{\xi} * \xi'^*)$ and $a_{\eta,\eta'} = L(\bar{\eta}\eta')$.*

Proof. By continuity, we may assume $\xi, \xi' \in C_c(G)$. Then for all $\zeta, \zeta' \in C_c(G)$,

$$\begin{aligned}
\langle \zeta|\hat{a}_{\xi,\xi'}\zeta'\rangle &= \langle \zeta \oslash j(\xi)|V(\zeta' \oslash j(\xi'))\rangle \\
&= \int_G \int_{G^{r(x)}} \overline{\zeta(x)\xi(y)} \zeta'(x) \xi'(x^{-1}y) \,\mathrm{d}\lambda^{r(x)}(y) \,\mathrm{d}\nu(x),
\end{aligned}$$

$$\begin{aligned}
\langle \zeta|a_{\eta,\eta'}\zeta'\rangle &= \langle j(\eta) \oslash \zeta|V(\hat{\jmath}(\eta') \oslash \zeta')\rangle \\
&= \int_G \int_{G^{r(y)}} \overline{\eta(x)\zeta(y)} \eta'(x) D^{-1/2}(x) \zeta'(x^{-1}y) \,\mathrm{d}\lambda^{r(y)}(x) \,\mathrm{d}\nu(y).
\end{aligned}$$

□

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The algebra $\hat{A}_V^0$ can be considered as a continuous Fourier algebra of the locally compact groupoid G. A Fourier algebra for measured groupoids was defined and studied by Renault [**23**], and for measured quantum groupoids by Vallin [**32**].

Remark 3.14. A Fourier algebra $A(G)$ for locally compact groupoids was defined by Paterson in [**21**] as follows. He constructs a Fourier–Stieltjes algebra $B(G) \subseteq C(G)$ and defines $A(G)$ to be the norm-closed subalgebra of $B(G)$ generated by the set $A_{\mathrm{cf}}(G) := \{\hat{a}_{\xi,\xi'} \mid \xi,\ \xi' \in L^2(G,\lambda)\}$. The definition of $B(G)$ in immediately implies that $\|\hat{\pi}_V(\omega_{\xi,\xi'})\|_{B(G)} \leqslant \|\xi\|\,\|\xi'\|$ for all $\xi \in \alpha^\infty$, $\xi' \in \beta^\infty$ with finitely many non-zero components. Therefore, the identity on $A_{\mathrm{cf}}(G)$ extends to a contractive homomorphism from $\hat{A}_V^0$ to $A(G)$.

Remark 3.15. Another Fourier space $\tilde{\mathcal{A}}(G)$ considered in the note following Proposition 13 in [**21**] is defined as follows. For each $\eta \in L^2(G,\lambda)$ and $u \in G^0$, write $\|\xi_n(u)\| := \langle \xi_n | \xi_n \rangle (u)^{1/2}$. Denote by M the set of all pairs (ξ,ξ') of sequences in $L^2(G,\lambda)$ such that the supremum $|(\xi,\xi')|_M := \sup_{u,v\in G^0} \sum_n \|\xi_n(u)\|\,\|\xi'_n(v)\|$ is finite, and denote by $\tilde{\mathcal{A}}(G)$ the completion of the linear span of $A_{\mathrm{cf}}(G)$ with respect to the norm defined by

$$\|\hat{a}\|_{\tilde{\mathcal{A}}(G)} = \inf\left\{ |(\xi,\xi')|_M \;\middle|\; \hat{a} = \sum_n \hat{a}_{\xi_n,\xi'_n} \right\}.$$

The identity on $A_{\mathrm{cf}}(G)$ extends to a linear contraction from $\hat{A}_V^0$ to $\tilde{\mathcal{A}}(G)$ because

$$\|\xi\|^2 = \sup_{u\in G^0} \sum_n \langle \xi_n|\xi_n\rangle(u) = \sup_{u\in G^0} \sum_n \|\xi_n(u)\|^2, \qquad \|\xi'\|^2 = \sup_{v\in G^0} \sum_n \|\xi_n(v)\|^2,$$

for all $\xi,\xi' \in L^2(G,\lambda)^\infty$ and hence $|(\xi,\xi')|_M = \sup_{u,v\in G^0} \sum_n \|\xi_n(u)\|\,\|\xi'_n(v)\| \leqslant \|\xi\|\,\|\xi'\|$.

Recall that the reduced groupoid C^*-algebra $C^*_{\mathrm{r}}(G)$ is the closed linear span of all operators of the $L(g)$, where $g \in L^1(G,\lambda)$ [**22**].

Theorem 3.16. *Let V be the C^*-pseudo-multiplicative unitary of a locally compact groupoid G. Then $((\hat{A}_V)_H^{\beta,\alpha}, \hat{\Delta}_V)$ and $((A_V)_H^{\alpha,\hat{\beta}}, \Delta_V)$ are Hopf C^*-bimodules and*

$$\hat{A}_V = m(C_0(G)),$$
$$(\hat{\Delta}_V(m(f))\omega)(x,y) = f(xy)\omega(x,y)$$

for all $f \in C_0(G)$, $\omega \in H\, {}_{\hat{\beta}}\!\underset{\mathfrak{b}^\dagger}{\otimes}\!{}_{\alpha}\, H$, $(x,y) \in G_s \times_r G$, and

$$A_V = C^*_{\mathrm{r}}(G),$$
$$(\Delta_V(L(g))\omega')(x',y') = \int_{G^{u'}} g(z) D^{-1/2}(z) \omega'(z^{-1}x', z^{-1}y')\,\mathrm{d}\lambda^{u'}(z)$$

for all $g \in C_c(G)$, $\omega' \in H\, {}_{\alpha}\!\underset{\mathfrak{b}}{\otimes}\!{}_{\beta}\, H$, $(x',y') \in G_r \times_r G$, where $u' = r(x') = r(y')$.

Proof. The first assertion will follow from Example 4.3 and Theorem 4.5 in § 4.1. The equations concerning $\hat{A}_V$, A_V and $\hat{\Delta}_V$, Δ_V follow from Lemma 3.13 and straightforward calculations. □

4. Regular, proper and étale C^*-pseudo-multiplicative unitaries

Let

$$V\colon H \underset{\mathfrak{b}^\dagger}{{}_{\hat\beta}\!\otimes_\alpha} H \to H \underset{\mathfrak{b}}{{}_\alpha\!\otimes_\beta} H$$

be a C^*-pseudo-multiplicative unitary as before.

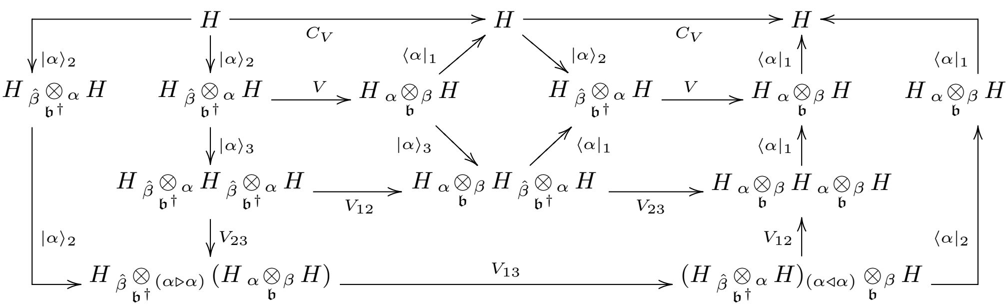
4.1. Regularity

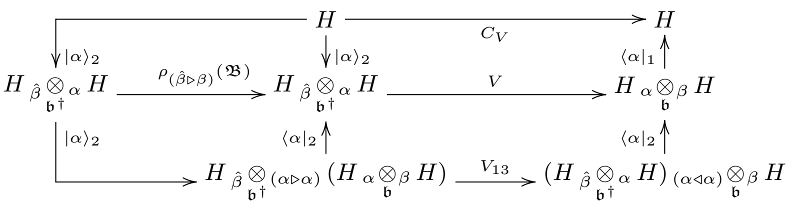
In [**3**], Baaj and Skandalis showed that the pairs $(\hat{A}_V, \hat{\Delta}_V)$ and (A_V, Δ_V) associated to a multiplicative unitary V on a Hilbert space H form Hopf C^*-algebras if the unitary satisfies the regularity condition $[\langle H|_2 V|H\rangle_1] = \mathcal{K}(H)$. This condition was generalized by Baaj in [**1**, **2**] and extended to pseudo-multiplicative unitaries by Enock [**9**]. To adapt it to C^*-pseudo-multiplicative unitaries, we consider the space

$$C_V := [\langle\alpha|_1 V|\alpha\rangle_2] \subseteq \mathcal{L}(H).$$

Proposition 4.1. *$[C_V C_V] = C_V$, $C_{V^{\mathrm{op}}} = C_V^*$, $[C_V\alpha] = \alpha$, and $[C_V\rho_\beta(\mathfrak{B})] = [\rho_\beta(\mathfrak{B})C_V] = C_V = [C_V\rho_{\hat\beta}(\mathfrak{B})] = [\rho_{\hat\beta}(\mathfrak{B})C_V]$.*

Proof. The proof is completely analogous to the proof of Proposition 3.2; for example, the first equation follows from the commutativity of the following two diagrams:





□

Definition 4.2. A C^*-pseudo-multiplicative unitary $(\mathfrak{b}, H, \hat\beta, \alpha, \beta, V)$ is *semi-regular* if $C_V \supseteq [\alpha\alpha^*]$, and *regular* if $C_V = [\alpha\alpha^*]$.

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Examples 4.3.

(i) V is (semi-)regular if and only if V^{op} is (semi-)regular.

(ii) The C^*-pseudo-multiplicative unitary of a locally compact Hausdorff groupoid G (see Theorem 2.5) is regular. To prove this assertion, we use the notation introduced in § 2.3 and calculate that for each $\xi, \xi' \in C_c(G)$, $\zeta \in C_c(G) \subseteq L^2(G, \nu)$, $y \in G$,

$$(\langle j(\xi')|_1 V |j(\xi)\rangle_2 \zeta)(y) = \int_{G^{r(y)}} \overline{\xi'(x)} \zeta(x) \xi(x^{-1}y)\, \mathrm{d}\lambda^{r(y)}(x),$$

$$(j(\xi')j(\xi)^* \zeta)(y) = \xi'(y) \int_{G^{r(y)}} \overline{\xi(x)} \zeta(x)\, \mathrm{d}\lambda^{r(y)}(x).$$

Using standard arguments, we find $[\langle\alpha|_1 V |\alpha\rangle_2] = [S(C_c(G\,{}_r{\times}_r\, G))] = [\alpha\alpha^*]$, where for each $\omega \in C_c(G\,{}_r{\times}_r\, G)$, the operator $S(\omega)$ is given by

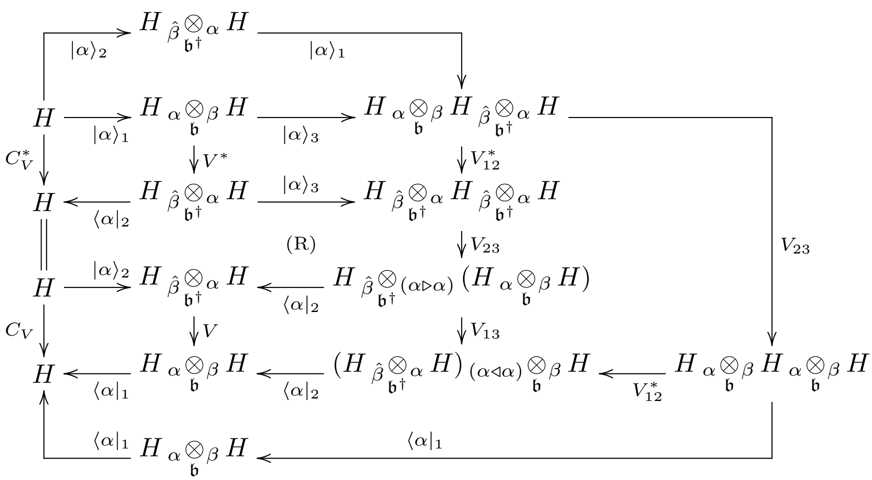
$$(S(\omega)\zeta)(y) = \int_{G^{r(y)}} \omega(x, y)\zeta(x)\, \mathrm{d}\lambda^{r(y)}(x) \quad \text{for all } \zeta \in C_c(G),\ y \in G.$$

(iii) In [**31**], we introduce compact C^*-quantum groupoids and construct for each such quantum groupoid a C^*-pseudo-multiplicative unitary that turns out to be regular.

We now deduce several properties of semi-regular and regular C^*-pseudo-multiplicative unitaries, using commutative diagrams as in § 3.2.

Proposition 4.4. *If V is semi-regular, then C_V is a C^*-algebra.*

Proof. Assume that V is regular. Then the following two diagrams commute, whence $[C_V C_V^*] = [\langle\alpha|_1 \langle\alpha|_1 V_{23} |\alpha\rangle_1 |\alpha\rangle_2] = C_V$:



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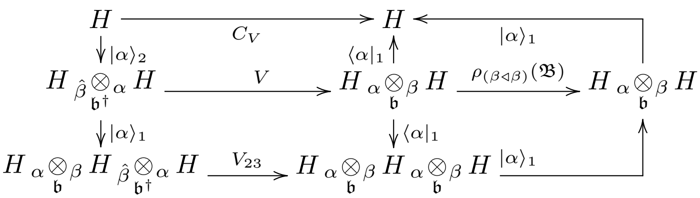


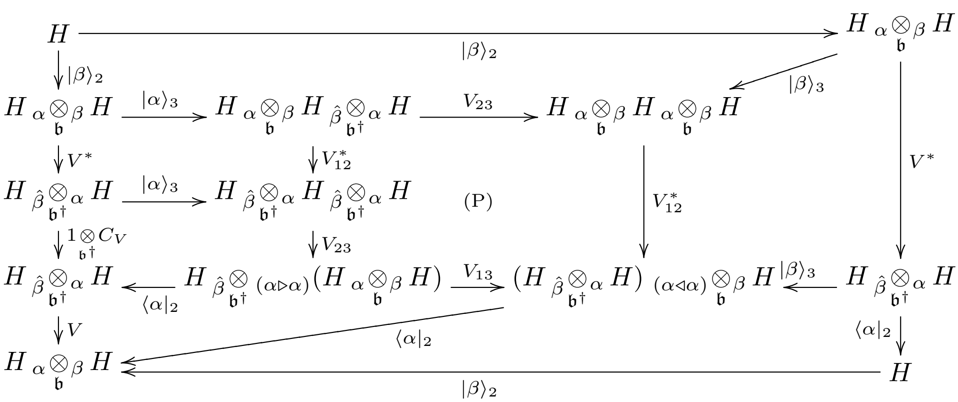
Now, assume that V is semi-regular. Then cell (R) in the first diagram need not commute, but still $[|\alpha\rangle_2\langle\alpha|_2] \subseteq [\langle\alpha|_2 V_{23}|\alpha\rangle_3]$ and hence $[C_V C_V^*] \subseteq [\langle\alpha|_1\langle\alpha|_1 V_{23}|\alpha\rangle_1|\alpha\rangle_2] = C_V$. A similar argument shows that also $[C_V^* C_V] \subseteq C_V$, and from Proposition 3.2 and [**1**, Lemme 3.3], it follows that C_V is a C^*-algebra. □

Theorem 4.5. *If $C_V = C_V^*$, then $((\hat{A}_V)_H^{\alpha,\hat{\beta}}, \hat{\Delta}_V)$ and $((A_V)_H^{\beta,\alpha}, \Delta_V)$ are Hopf C^*-bimodules. In particular, this is the case if V is semi-regular.*

The key step in the proof is the following lemma.

Lemma 4.6. $[V(1 \underset{\mathfrak{b}^\dagger}{\otimes} C_V)V^*|\beta\rangle_2] = [|\beta\rangle_2 \hat{A}_V^*]$.

Proof. The diagram



commutes and shows that

$$[V(1 \underset{\mathfrak{b}^\dagger}{\otimes} C_V)V^*|\beta\rangle_2] = [|\beta\rangle_2\langle\alpha|_2 V^*|\beta\rangle_2] = [|\beta\rangle_2 \hat{A}_V^*].$$

Indeed, cell (P) commutes by (2.5), and the remaining cells by (2.4) or by inspection. □

Proof of Theorem 4.5. By Theorem 3.4, it suffices to show that $\hat{A}_V = \hat{A}_V^*$. But by Proposition 3.2 and Lemma 4.6,

$$\hat{A}_V^* = [\rho_\alpha(\mathfrak{B}^\dagger)\hat{A}_V^*] = [\langle\beta|_2|\beta\rangle_2\hat{A}_V^*] = [\langle\beta|_2 V(1 \underset{\mathfrak{b}^\dagger}{\otimes} C_V)V^*|\beta\rangle_2].$$

Replacing V by V^{op}, we obtain the assertion concerning A_V. □

Remark 4.7. If V is regular, then $[V|\alpha\rangle_2\hat{A}_V] = [|\beta\rangle_2\hat{A}_V]$ and $[V|\hat{\beta}\rangle_1 A_V] = [|\alpha\rangle_1 A_V]$. Indeed, using Lemma 4.6 and the relation $\hat{A}_V = \hat{A}_V^*$ (Theorem 4.5), we find that

$$[V|\alpha\rangle_2\hat{A}_V] = [V|\alpha\rangle_2\langle\alpha|_2 V^*|\beta\rangle_2] = [V(1 \underset{\mathfrak{b}^\dagger}{\otimes} C_V)V^*|\beta\rangle_2] = [|\beta\rangle_2\hat{A}_V],$$

and replacing V by V^{op}, we obtain the second equation.

4.2. Proper and étale C^*-pseudo-multiplicative unitaries

In [**3**], Baaj and Skandalis characterized multiplicative unitaries that correspond to compact or discrete quantum groups by the existence of fixed or cofixed vectors, respectively, and showed that from such vectors, one can construct a Haar state and a counit on the associated legs. We adapt some of their constructions to C^*-pseudo-multiplicative unitaries as follows. Given a C^*–$\mathfrak{b}^{(\dagger)}$-module K_γ, let $M(\gamma) = \{T \in \mathcal{L}(\mathfrak{K}, K) \mid T\mathfrak{B}^{(\dagger)} \subseteq \gamma,\ T^*\gamma \subseteq \mathfrak{B}^{(\dagger)}\}$.

Definition 4.8. A *fixed element* for V is an $\eta \in M(\hat{\beta}) \cap M(\alpha) \subseteq \mathcal{L}(\mathfrak{K}, H)$ satisfying $V|\eta\rangle_1 = |\eta\rangle_1$. A *cofixed element* for V is a $\xi \in M(\alpha) \cap M(\beta) \subseteq \mathcal{L}(\mathfrak{K}, H)$ satisfying $V|\xi\rangle_2 = |\xi\rangle_2$. We denote the set of all fixed/cofixed elements for V by $\mathrm{Fix}(V)/\mathrm{Cofix}(V)$.

Example 4.9. Let V be the C^*-pseudo-multiplicative unitary of a groupoid G (see § 2.3). Identify $M(L^2(G,\lambda))$ in the natural way with the completion of the space

$$\left\{ f \in C(G) \,\middle|\, r\colon \operatorname{supp} f \to G^0 \text{ is proper, } \sup_{u\in G^0} \int_{G^u} |f(x)|^2 \,\mathrm{d}\lambda^u(x) \text{ is finite} \right\}$$

with respect to the norm

$$f \mapsto \sup_{u\in G^0} \left(\int_{G^u} |f(x)|^2 \,\mathrm{d}\lambda^u(x) \right)^{1/2}.$$

As in [**27**, Lemma 7.11], one finds that

(i) $\eta_0 \in M(L^2(G,\lambda))$ is a fixed element if and only if for each $u \in G^0$, $\eta_0|_{G^u\setminus\{u\}} = 0$ almost everywhere with respect to λ^u;

(ii) $\xi_0 \in M(L^2(G,\lambda))$ is a cofixed element if and only if $\xi_0(x) = \xi_0(s(x))$ for all $x \in G$.

Remarks 4.10.

(i) $\mathrm{Fix}(V) = \mathrm{Cofix}(V^{\mathrm{op}})$ and $\mathrm{Cofix}(V) = \mathrm{Fix}(V^{\mathrm{op}})$.

(ii) $\mathrm{Fix}(V)^*\,\mathrm{Fix}(V)$ and $\mathrm{Cofix}(V)^*\,\mathrm{Cofix}(V)$ are contained in $M(\mathfrak{B}) \cap M(\mathfrak{B}^\dagger)$.

(iii) $\rho_\alpha(\mathfrak{B}^\dagger)\,\mathrm{Fix}(V) = \mathrm{Fix}(V)\mathfrak{B}^\dagger \subseteq \hat{\beta}$ and $\rho_{\hat{\beta}}(\mathfrak{B})\,\mathrm{Fix}(V) = \mathrm{Fix}(V)\mathfrak{B} \subseteq \alpha$ because $\mathrm{Fix}(V) \subseteq M(\hat{\beta}) \cap M(\alpha)$, and similarly $\rho_\beta(\mathfrak{B})\,\mathrm{Cofix}(V) \subseteq \alpha$ and $\rho_\alpha(\mathfrak{B}^\dagger)\,\mathrm{Cofix}(V) \subseteq \beta$.

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Lemma 4.11.

(i) $\langle \xi|_2 V|\xi'\rangle_2 = \rho_\alpha(\xi^*\xi') = \rho_{\hat\beta}(\xi^*\xi')$ *for all* $\xi, \xi' \in \mathrm{Cofix}(V)$.

(ii) $\langle \eta|_1 V|\eta'\rangle_1 = \rho_\beta(\eta^*\eta') = \rho_\alpha(\eta^*\eta')$ *for all* $\eta, \eta' \in \mathrm{Fix}(V)$.

Proof. Let $\zeta \in H$ and $\xi, \xi' \in \mathrm{Cofix}(V)$. Then $\langle \xi|_2 V|\xi'\rangle_2\zeta = \langle \xi|_2|\xi'\rangle_2\zeta = \rho_\alpha(\xi^*\xi')\zeta$ and $(\langle \xi|_2 V|\xi'\rangle_2)^*\zeta = \langle \xi'|_2|\xi\rangle_2\zeta = \rho_{\hat\beta}((\xi')^*\xi)\zeta$. The proof of (ii) is similar. □

Proposition 4.12.

(i) $\rho_{\hat\beta}(M(\mathfrak{B}))\,\mathrm{Cofix}(V) \subseteq \mathrm{Cofix}(V)$ *and* $\rho_\beta(\mathfrak{B})\,\mathrm{Fix}(V) \subseteq \mathrm{Fix}(V)$.

(ii) $[\mathrm{Cofix}(V)\,\mathrm{Cofix}(V)^*\,\mathrm{Cofix}(V)] = \mathrm{Cofix}(V)$ *and* $[\mathrm{Fix}(V)\,\mathrm{Fix}(V)^*\,\mathrm{Fix}(V)] = \mathrm{Fix}(V)$.

(iii) $[\mathrm{Cofix}(V)^*\,\mathrm{Cofix}(V)]$ *and* $[\mathrm{Fix}(V)^*\,\mathrm{Fix}(V)]$ *are C^*-subalgebras of* $M(\mathfrak{B}) \cap M(\mathfrak{B}^\dagger)$*; in particular, they are commutative.*

Proof. We only prove the assertions concerning $\mathrm{Cofix}(V)$.

(i) Let $T \in M(\mathfrak{B})$ and $\xi \in \mathrm{Cofix}(V)$. Then $\rho_{\hat\beta}(T)\xi \subseteq M(\beta) \cap M(\alpha)$ because $\rho_{\hat\beta}(\mathfrak{B})\beta \subseteq \beta$ and $\rho_{\hat\beta}(\mathfrak{B})\alpha \subseteq \alpha$. The relation $V(\hat\beta \triangleright \hat\beta) = \alpha \triangleright \hat\beta$ furthermore implies

$$V|\rho_{\hat\beta}(T)\xi\rangle_2 = V\rho_{(\hat\beta\triangleright\hat\beta)}(T)|\xi\rangle_2 = \rho_{(\alpha\triangleright\hat\beta)}(T)V|\xi\rangle_2 = \rho_{(\alpha\triangleright\hat\beta)}(T)|\xi\rangle_2 = |\rho_{\hat\beta}(T)\xi\rangle_2.$$

(ii) Using (i) and the relation $\mathrm{Cofix}(V)^*\,\mathrm{Cofix}(V) \subseteq M(\mathfrak{B}^\dagger)$, we find that

$$[\mathrm{Cofix}(V)\,\mathrm{Cofix}(V)^*\,\mathrm{Cofix}(V)] \subseteq [\mathrm{Cofix}(V)M(\mathfrak{B}^\dagger)] = [\rho_{\hat\beta}(M(\mathfrak{B}))\,\mathrm{Cofix}(V)] \subseteq \mathrm{Cofix}(V).$$

Therefore, $[\mathrm{Cofix}(V)^*\,\mathrm{Cofix}(V)]$ is a C^*-algebra and $\mathrm{Cofix}(V)$ is a Hilbert C^*-module over $[\mathrm{Cofix}(V)^*\,\mathrm{Cofix}(V)]$. Now, [**15**, p. 5] implies that the inclusion above is an equality.

(iii) This follows from (ii) and Remark 4.10 (ii). □

Definition 4.13. The C^*-pseudo-multiplicative unitary V is *étale* if $\eta^*\eta = \mathrm{id}_{\mathfrak{K}}$ for some $\eta \in \mathrm{Fix}(V)$, *proper* if $\xi^*\xi = \mathrm{id}_{\mathfrak{K}}$ for some $\xi \in \mathrm{Cofix}(V)$, and *compact* if it is proper and $\mathfrak{B}$, $\mathfrak{B}^\dagger$ are unital.

Example 4.14. The C^*-pseudo-multiplicative unitary of a groupoid G (§ 2.3) is étale/proper/compact if and only if G is étale/proper/compact. This follows from arguments similar to those in [**27**, Theorem 7.12].

Remarks 4.15.

(i) By Remark 4.10, V is étale/proper if and only if V^{op} is proper/étale.

(ii) If V is proper and $\xi \in \mathrm{Cofix}(V)$, $\xi^*\xi = \mathrm{id}_{\mathfrak{K}}$, then

$$[\rho_{\hat\beta}(\mathfrak{B})\rho_\alpha(\mathfrak{B}^\dagger)] = [\rho_\alpha(\mathfrak{B}^\dagger)\langle \xi|_2 V|\xi\rangle_2\rho_{\hat\beta}(\mathfrak{B})] = [\langle \xi\mathfrak{B}^\dagger|_2 V|\xi\mathfrak{B}\rangle_2] \subseteq [\langle \beta|_2 V|\alpha\rangle_2] = \hat A_V.$$

Similarly, if V is étale, then $[\rho_\beta(\mathfrak{B})\rho_\alpha(\mathfrak{B}^\dagger)] \in A_V$.

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Fixed and cofixed vectors give rise to invariant operator-valued weights and counits on the legs of V as follows.

Definition 4.16. Let $(A_H^{\beta,\alpha}, \Delta)$ be a Hopf C^*-bimodule over $\mathfrak{b}$.

A *bounded left Haar weight* for $(A_H^{\beta,\alpha}, \Delta)$ is a completely positive contraction $\phi\colon A \to \mathfrak{B}$ satisfying $\phi(a\rho_\beta(b)) = \phi(a)b$ and $\phi(\langle \xi|_1 \Delta(a)|\xi'\rangle_1) = \xi^* \rho_\beta(\phi(a))\xi'$ for all $a \in A$, $b \in \mathfrak{B}$, $\xi, \xi' \in \alpha$. We call ϕ *normal* if $\phi \in \Omega_{M(\beta)}(A)$.

Similarly, a *bounded right Haar weight* for $(A_H^{\beta,\alpha}, \Delta)$ is a completely positive contraction $\psi\colon A \to \mathfrak{B}^\dagger$ satisfying $\psi(a\rho_\alpha(b^\dagger)) = \psi(a)b^\dagger$ and $\psi(\langle \eta|_2 \Delta(a)|\eta'\rangle_2) = \eta^* \rho_\alpha(\psi(a))\eta'$ for all $a \in A$, $b^\dagger \in \mathfrak{B}^\dagger$, $\eta, \eta' \in \beta$. We call ψ *normal* if $\psi \in \Omega_{M(\alpha)}(A)$.

A *bounded (left/right) counit* for $(A_H^{\beta,\alpha}, \Delta)$ is a morphism of C^*-$(\mathfrak{b}^\dagger, \mathfrak{b})$-algebras $\epsilon\colon A_H^{\beta,\alpha} \to \mathcal{L}(\mathfrak{K})_{\mathfrak{K}}^{\mathfrak{B}^\dagger, \mathfrak{B}}$ that makes the (left/right one of the) following two diagrams commute:

$$
\begin{array}{ccccc}
A\, {}_\alpha{\underset{\mathfrak{b}}{*}}_\beta\, A & \xleftarrow{\quad\Delta\quad} & & & A \\
\Big\downarrow {\scriptstyle \epsilon \underset{\mathfrak{b}}{*} \mathrm{id}} & & & & \Big\downarrow \\
\mathcal{L}(\mathfrak{K})\, {}_{\mathfrak{B}}{\underset{\mathfrak{b}}{*}}_\beta\, A & \longrightarrow & \mathcal{L}(\mathfrak{K}\, {}_{\mathfrak{B}}{\underset{\mathfrak{b}}{\otimes}}_\beta\, H) & \xrightarrow{\cong} & \mathcal{L}(H)
\end{array}
\qquad
\begin{array}{ccccc}
A & \xrightarrow{\quad\Delta\quad} & & & A\, {}_\alpha{\underset{\mathfrak{b}}{*}}_\beta\, A \\
\Big\downarrow & & & & \Big\downarrow {\scriptstyle \mathrm{id} \underset{\mathfrak{b}}{*} \epsilon} \\
\mathcal{L}(H) & \xleftarrow{\cong} & \mathcal{L}(H\, {}_\alpha{\underset{\mathfrak{b}}{\otimes}}_{\mathfrak{B}^\dagger}\, \mathfrak{K}) & \longleftarrow & A\, {}_\alpha{\underset{\mathfrak{b}}{*}}_{\mathfrak{B}^\dagger}\, \mathcal{L}(\mathfrak{K})
\end{array}
$$

where the isomorphisms

$$\mathcal{L}(\mathfrak{K}\, {}_{\mathfrak{B}}{\underset{\mathfrak{b}}{\otimes}}_\beta\, H) \cong \mathcal{L}(H) \cong \mathcal{L}(H\, {}_\alpha{\underset{\mathfrak{b}}{\otimes}}_{\mathfrak{B}^\dagger}\, \mathfrak{K})$$

are induced by the isomorphisms (2.3).

Remark 4.17. Let $(A_H^{\beta,\alpha}, \Delta)$ be a Hopf C^*-bimodule over $\mathfrak{b}$. Evidently, a completely positive contraction $\phi\colon A \to \mathfrak{B}$ is a normal bounded left Haar weight for $(A_H^{\beta,\alpha}, \Delta)$ if and only if $\phi \in \Omega_{M(\beta)}(A)$ and $(\mathrm{id} * \phi) \circ \Delta = \rho_\beta \circ \phi$. A similar remark applies to right Haar weights.

Theorem 4.18. *Let V be an étale C^*-pseudo-multiplicative unitary.*

(i) *There exists a contractive homomorphism $\hat{\epsilon}\colon \hat{A}_V \to \mathcal{L}(\mathfrak{K})$ such that $\hat{\epsilon}(\langle \eta|_2 V|\xi\rangle_2) = \eta^*\xi$ for all $\eta \in \beta$, $\xi \in \alpha$.*

(ii) *Assume that V is regular and let $D := [\beta^*\alpha]$. Then $D_{\mathfrak{K}}^{\mathfrak{B}, \mathfrak{B}^\dagger}$ is a C^*-$(\mathfrak{b}, \mathfrak{b}^\dagger)$-algebra and $\hat{\epsilon}$ is a morphism from $(\hat{A}_V)_H^{\alpha,\hat{\beta}}$ to $D_{\mathfrak{K}}^{\mathfrak{B}, \mathfrak{B}^\dagger}$ and a bounded counit for $((\hat{A}_V)_H^{\alpha,\hat{\beta}}, \hat{\Delta}_V)$.*

Proof. Choose an $\eta_0 \in \mathrm{Fix}(V)$ with $\eta_0^*\eta_0 = \mathrm{id}_{\mathfrak{K}}$ and define $\hat{\epsilon}\colon \hat{A}_V \to \mathcal{L}(\mathfrak{K})$ by $\hat{a} \mapsto \eta_0^* \hat{a} \eta_0$. Then $\hat{\epsilon}$ is contractive. For all $\xi \in \alpha$, $\eta \in \beta$, $\zeta \in \mathfrak{K}$,

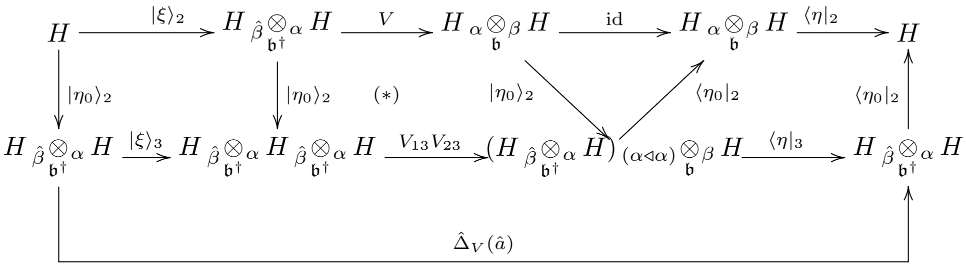
$$\langle \eta|_2 V|\xi\rangle_2 \eta_0 \zeta = \langle \eta|_2 V(\eta_0 \otimes \xi\zeta) = \langle \eta|_2 (\eta_0 \otimes \xi\zeta) = \eta_0(\eta^*\xi)\zeta,$$

and hence $\hat{a}\eta_0 = \eta_0 \hat{\epsilon}(\hat{a})$ and $\hat{\epsilon}(\hat{b}\hat{a}) = \eta_0^* \hat{b}\hat{a}\eta_0 = \eta_0^*\hat{b}\eta_0\hat{\epsilon}(\hat{a}) = \hat{\epsilon}(\hat{b})\hat{\epsilon}(\hat{a})$ for all $\hat{a}, \hat{b} \in \hat{A}_V$.

Assume that V is regular. Then D is a C^*-algebra and $\hat{\epsilon}$ is a morphism because by construction, $\hat{\epsilon}$ is a $*$-homomorphism, $D = \hat{\epsilon}(\hat{A}_V)$, $\eta_0^* \in \mathcal{L}^{\hat{\epsilon}}({}_{\alpha}H_{\hat{\beta}}, {}_{\mathfrak{B}}\mathfrak{K}_{\mathfrak{B}^\dagger})$, and $[\eta_0^*\alpha] \supseteq [\eta_0^*\eta_0\mathfrak{B}] = \mathfrak{B}$ and $[\eta_0^*\hat{\beta}] \supseteq [\eta_0^*\eta_0\mathfrak{B}^\dagger] = \mathfrak{B}^\dagger$. Let $\hat{a} \in \hat{A}_V$. Then

$$\begin{aligned}(\hat{\epsilon} \underset{\mathfrak{b}^\dagger}{*} \mathrm{id})(\hat{\Delta}_V(\hat{a})) &= \langle \eta_0|_1 \hat{\Delta}_V(\hat{a})|\eta_0\rangle_1 \\ &= \langle \eta_0|_1 V^*(1 \underset{\mathfrak{b}}{\otimes} \hat{a})V|\eta_0\rangle_1 \\ &= \langle \eta_0|_1 (1 \underset{\mathfrak{b}}{\otimes} \hat{a})|\eta_0\rangle_1 \\ &= \rho_\beta(\eta_0^*\eta_0)\hat{a} = \hat{a},\end{aligned}$$

and if $\hat{a} = \langle \eta|_2 V|\xi\rangle_2$ for some $\eta \in \beta$, $\xi \in \alpha$, then $(\mathrm{id} * \hat{\epsilon})(\hat{\Delta}_V(\hat{a})) = \langle \eta_0|_2 \hat{\Delta}_V(\hat{a})|\eta_0\rangle_2 = \hat{a}$ because the following diagram commutes:



Indeed, the lower cell commutes by equation (3.6), cell $(*)$ commutes because $V_{23}|\eta_0\rangle_2 = |\eta_0\rangle_2$, and the other cells commute as well. □

Theorem 4.19. *Let V be a proper regular C^*-pseudo-multiplicative unitary. Then there exists a normal bounded left Haar weight ϕ for $((\hat{A}_V)_H^{\alpha,\hat{\beta}}, \hat{\Delta}_V)$.*

Proof. Choose $\xi_0 \in \mathrm{Cofix}(V)$ with $\xi_0^*\xi_0 = \mathrm{id}_{\mathfrak{K}}$. By Proposition 3.2 and Remark 4.10 (i),

$$[\xi_0^*\hat{A}_V\xi_0] = [\xi_0^*\rho_\alpha(\mathfrak{B}^\dagger)\hat{A}_V\rho_\alpha(\mathfrak{B}^\dagger)\xi_0] \subseteq [\beta^*\hat{A}_V\beta] \subseteq \mathfrak{B}^\dagger.$$

Hence, we can define a completely positive map $\phi\colon \hat{A}_V \to \mathfrak{B}^\dagger$ by $\hat{a} \mapsto \xi_0^*\hat{a}\xi_0$, and $\phi \in \Omega_{M(\alpha)}(\hat{A}_V)$. For all $\hat{a} \in \hat{A}_V$,

$$(\mathrm{id} * \phi)(\hat{\Delta}_V(\hat{a})) = \langle \xi_0|_2 V^*(\mathrm{id} \underset{\mathfrak{b}}{\otimes} \hat{a})V|\xi_0\rangle_2 = \langle \xi_0|_2(\mathrm{id} \underset{\mathfrak{b}}{\otimes} \hat{a})|\xi_0\rangle_2 = \rho_\alpha(\xi_0^*\hat{a}\xi_0).$$

□

As an example, consider the C^*-pseudo-multiplicative unitary

$$V\colon H \,{}_{\hat{\beta}}\!\underset{\mathfrak{b}^\dagger}{\otimes}\!{}_{\alpha}\, H \to H \,{}_{\alpha}\!\underset{\mathfrak{b}}{\otimes}\!{}_{\beta}\, H$$

associated to a locally compact, Hausdorff, second countable groupoid G as in § 2.3.

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Proposition 4.20.

(i) *Let G be étale. Then V is étale, $\hat{A}_V \cong C_0(G)$, $\hat{\epsilon}(\hat{A}_V) \cong C_0(G^0)$, and $\hat{\epsilon}$ is given by the restriction of functions on G to functions on G^0.*

(ii) *Let G be proper. Then V^{op} is étale, $A_V = \hat{A}_{V^{\mathrm{op}}} = C^*_{\mathrm{r}}(G)$, and for each $f \in C_c(G)$, the operator $\hat{\epsilon}(L(f)) \in \mathcal{L}(L^2(G^0, \mu))$ is given by*

$$(\hat{\epsilon}(L(f))\zeta)(u) = \int_{G^u} f(x) D^{-1/2}(x) \zeta(s(x))\, \mathrm{d}\lambda^u(x) \quad \text{for all } \zeta \in L^2(G^0, \mu),\ x \in G.$$

Proof. For all $\xi, \xi' \in C_c(G)$, $\zeta \in L^2(G^0, \mu)$ and $u \in G^0$, we have by Lemma 3.13

$$\begin{aligned}(\hat{\epsilon}(m(\bar{\xi} * \xi'^*))\zeta)(u) = (\hat{\epsilon}(\hat{a}_{\xi,\xi'})\zeta)(u) &= (j(\xi)^* j(\xi')\zeta)(u) \\ &= \int_{G^u} \overline{\xi(x)} \xi'(x) \zeta(u)\, \mathrm{d}\lambda^u(x) = (\bar{\xi} * \xi'^*)(u)\zeta(u), \\ (\hat{\epsilon}(L(\bar{\xi}\xi'))\zeta)(u) = (\hat{\epsilon}(a_{\xi,\xi'})\zeta)(u) &= (j(\xi)^* \hat{\jmath}(\xi')\zeta)(u) \\ &= \int_{G^u} \overline{\xi(x)} \xi'(x) D^{-1/2}(x) \zeta(s(x))\, \mathrm{d}\lambda^u(x).\end{aligned}$$

□

Proposition 4.21.

(i) *Let G be proper. Then V is proper, $\hat{A}_V \cong C_0(G)$, and the map $\phi \colon \hat{A}_V \to C_0(G^0)$ given by $(\phi(f))(u) = \int_{G^u} f(x)\, \mathrm{d}\lambda^u(x)$ is a normal bounded left Haar weight for $((\hat{A}_V)^{\alpha,\hat{\beta}}_H, \hat{\Delta}_V)$.*

(ii) *Let G be étale. Then V^{op} is proper and there exists a normal bounded left and right Haar weight ϕ for $((A_V)^{\beta,\alpha}_H, \Delta_V)$ given by $L(f) \mapsto f|_{G^0}$ for all $f \in C_c(G)$.*

Proof. This follows from Theorem 4.19 and calculations similar to those in Proposition 4.20. □

Acknowledgements. This work was supported by the SFB 478 'Geometrische Strukturen in der Mathematik' (funded by the Deutsche Forschungsgemeinschaft (DFG)) and initiated during a stay at the 'Special Programme on Operator Algebras' at the Fields Institute in Toronto, Canada, in July 2007.

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