A CRITERION FOR NORMALCY

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1. Introduction

Gavrilov [2] has shown that a holomorphic function $f(z)$ in the unit disc $|z|<1$ is normal, in the sense of Lehto and Virtanen [5, p. 86], if and only if $f(z)$ does not possess a sequence of ρ-points in the sense of Lange [4]. Gavrilov has also obtained an analagous result for meromorphic functions by introducing the property that a meromorphic function in the unit disc have a sequence of P-points. He has shown that a meromorphic function in the unit disc is normal if and only if it does not possess a sequence of P-points. In the same paper it was shown that if $\{z_n\}$ is a sequence of ρ-points for the function $f(z)$ holomorphic in the unit disc, then $\{z_n\}$ is also a sequence of P-points. Moreover if $\{z_n\}$ is a sequence of P-points for the holomorphic function $f(z)$, then there is a subsequence of $\{z_n\}$ which is a sequence of ρ-points for the function $f(z)$. Thus for holomorphic functions there is a strong relationship between sequences of ρ-points and sequences of P-points. In this paper we extend the concept of a function possessing a sequence of ρ-points so as to be applicable to meromorphic as well as holomorphic functions in the unit disc. It is shown that a sequence $\{z_n\}$ of points of the unit disc is a sequence of ρ-points for a meromorphic function $f(z)$ if and only if $\{z_n\}$ is a sequence of P-points for $f(z)$. From this equivalence and from Gavrilov's criterion for normalcy quoted above, there follows a new criterion for normalcy. A function $f(z)$ meromorphic in the unit disc is normal if and only if it does not possess a sequence of ρ-points. In a subsequent paper this criterion for normalcy will be exploited in studying the distribution of values of meromorphic functions.

Let z and z' be two points of the unit disc. We will denote by $\rho(z, z')$ the hyperbolic non-Euclidean distance between z and z'. For any two points a and a' on the Riemann sphere, we will denote the chordal

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distance between a and a' by $\chi(a, a')$. A family F of functions defined in the unit disc is said to be equicontinuous at a point z_0 of the unit disc if for each positive number r there is a positive number s such that

$$\chi(f(z_0),\ f(z)) < r, \text{ for } \rho(z_0, z) < s \text{ and } f \text{ in } F.$$

2. Preliminaries

Definition 1. Let $f(z)$ be a meromorphic function in the unit disc. A sequence of points $\{z_n\}$ of the unit disc is called a sequence of P-points for the function $f(z)$ if for each $r > 0$ and each subsequence $\{z_{n_k}\}$ the function $f(z)$ assumes every value, with at most two exceptions, infinitely often in the union of the discs

$$D_k = \{z : \rho(z, z_{n_k}) < r\},\quad k = 1, 2, \cdots.$$

Theorem 1. *Let $f(z)$ be a meromorphic function in the unit disc. A sequence of points $\{z_n\}$ of the unit disc is a sequence of P-points for the function $f(z)$ if and only if there is a sequence of points $\{w_n\}$ of the unit disc and a positive number r such that*

(1) $\rho(z_n, w_n) \to 0$, *for* $n \to \infty$ *and* $\chi(f(z_n), f(w_n)) > r$, *for* $n = 1, 2, \cdots$.

Proof. Suppose $\{z_n\}$ is a sequence for which there is no corresponding sequence $\{w_n\}$ satisfying (1). Then for any positive number r, one can find a sequence of indices

$$n(1) < n(2) < \cdots < n(k) < \cdots,$$

such that for all sufficiently large k,

$$\chi(f(z_{n(k)}),\ f(z)) < r, \text{ for } \rho(z_{n(k)}, z) < 1/k.$$

If in particular we let r be any positive number which is smaller than the diameter of the Riemann sphere, then it can be shown that the subsequence $\{z_{n(k)}\}$ associated with r has itself a subsequence which is not a sequence of P-points. Namely, any subsequence of $\{z_{n(k)}\}$ whose images under $f(z)$ converge on the Riemann sphere cannot be a sequence of P-points. Thus $\{z_n\}$ has a subsequence which is not a sequence of P-points. But from the definition it is clear that any subsequence of a sequence of P-points is also a sequence of P-points. Hence $\{z_n\}$ is not a sequence of P-points.

Conversely suppose there is a sequence of points $\{w_n\}$ for which $\rho(z_n, w_n)$

tends to zero while $\chi(f(z_n), f(w_n))$ is bounded away from zero. Let $F = f(g_n(z))$, where $g_n(z) = (z + z_n)/(1 + \bar{z}_n z)$. Then clearly the family F of functions is not equicontinuous at the point 0. Hence [3, p. 244] for each $r > 0$, F is not a normal family in the set $\{z : \rho(0, z) < r\}$, and so by Montel's theorem [3, p. 248] the family F must assume each value, with at most two exceptions, infinitely often in $\{z : \rho(0, z) < r\}$. That is, $f(z)$ assumes each value of the Riemann sphere, with at most two exceptions, infinitely often in the union of the discs

$$\{z : \rho(z_n, z) < r\}, \quad n = 1, 2, \cdots.$$

Since the same argument can be applied for any positive number r and any subsequence of $\{z_n\}$, it follows that $\{z_n\}$ is a sequence of P-points. This concludes the proof.

Lange [4] defined the concept of a sequence of ρ-points for a holomorphic function in the unit disc. We will now define what we mean by a sequence of ρ-points for a meromorphic function in the unit disc. Definition 2 generalizes that given by Lange in that every sequence of ρ-points in the sense of Lange is a sequence of ρ-points in the sense of definition 2. Moreover if a holomorphic function has a sequence $\{z_n\}$ of ρ-points in the sense of definition 2, then $\{z_n\}$ has a subsequence which is a sequence of ρ-points in the sense of Lange. These statements are easily verified by comparing the two definitions.

DEFINITION 2. Let $f(z)$ be a meromorphic function in the unit disc. A sequence of points $\{z_n\}$ of the unit disc is called a sequence of ρ-points for the function $f(z)$ if there are sequences $\{L_n\}$ and $\{r_n\}$, where

(A) $$L_1 > L_2 > \cdots > L_n > \cdots, \quad L_n \to 0, \text{ for } n \to \infty,$$

and

(B) $$r_1 > r_2 > \cdots > r_n > \cdots, \quad r_n \to 0, \text{ for } n \to \infty,$$

and there exists a sequence $\{D_n\}$ of open discs,

$$D_n = \{z : \rho(z_n, z) < r_n\},$$

in the unit disc, having the following property:

(C) in each disc D_n, $n = 1, 2, \cdots$, the function $f(z)$ assumes all values of the Riemann sphere with the possible exception of two sets of values E_n and G_n whose chordal diameters do not exceed L_n.

THEOREM 2. *A sequence of points* $\{z_n\}$ *of the unit disc is a sequence of* ρ*-points for the function* $f(z)$ *meromorphic in the unit disc if and only if for each* $r>0$, *there are sets* $E(r,n)$ *and* $G(r,n)$ *whose chordal diameters do not exceed* r, *and there is an integer* $N(r)$ *such that in each disc* $\{z:\rho(z_n,z)<r\}$, $n>N$, *the function* $f(z)$ *assumes all values of the Riemann sphere with the exception of the sets of values* $E(r,n)$ *and* $G(r,n)$.

Proof. That any sequence of ρ-points satisfies the condition stated in the theorem is obvious. Conversely suppose the sequence $\{z_n\}$ satisfies the condition. Then letting $r=1/m$, $m=1,2,\cdots$, one obtains sets $E(1/m,n)$ and $G(1/m,n)$, $n=1,2,\cdots; m=1,2,\cdots$. Moreover we may choose the integers $N(1/m)$ in such a way that

$$N(1/1)<N(1/2)<\cdots<N(1/m)<\cdots.$$

Now we define

$$r_n=1/m \text{ and } L_n=1/m, \text{ for } N(1/m)<n\leq N(1/(m+1)).$$

For $n\leq N(1/1)$ we define r_n to be 1 and L_n to be the diameter of the Riemann sphere. Having defined the sequences $\{r_n\}$ and $\{L_n\}$ we see from definition 2 that the sequence $\{z_n\}$ is a sequence of ρ-points for $f(z)$. This concludes the proof.

3. A Criterion for Normalcy

We now state our main result.

THEOREM 3. *A function* $f(z)$ *meromorphic in the unit disc is normal if and only if* $f(z)$ *has no sequence of* ρ*-points.*

Proof. This follows immediately from theorem 4 and from Gavrilov's criterion for normalcy which was mentioned in section 1.

THEOREM 4. *A sequence* $\{z_n\}$ *of points of the unit disc is a sequence of* ρ*-points for a function* $f(z)$ *meromorphic in the unit disc if and only if the sequence* $\{z_n\}$ *is a sequence of P-points for the function* $f(z)$.

Proof. From theorem 1 it follows easily that any sequence of ρ-points

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is a sequence of P-points. Conversely suppose $\{z_n\}$ is not a sequence of ρ-points. Then there is a positive number r for which the condition in theorem 2 is not satisfied. Hence there is a subsequence, which we again denote by $\{z_n\}$ such that for each n if D_n is the non-Euclidean disc with center z_n and radius r, then the set of values of the Riemann sphere not assumed by $f(z)$ in D_n cannot be contained in two sets whose chordal diameters do not exceed r. It follows immediately from this that $f(z)$ omits three values $a_n, b_n,$ and c_n in D_n such that

$$\chi(a_n, b_n) \geq r/2, \quad \chi(a_n, c_n) \geq r/2, \quad \text{and} \quad \chi(b_n, c_n) \geq r/2\,; \quad n = 1, 2, \cdots.$$

From $\{z_n\}$ we may choose a subsequence, which we continue to denote by $\{z_n\}$, such that $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ converge respectively to (necessarily distinct) values A, B, and C. Let $f_n(z) = f(g_n(z))$ where $g_n(z)$ is a $1-1$ conformal map of the unit disc onto itself such that $g_n(0) = z_n$. Then for each n, $f_n(z)$ omits a_n, b_n, and c_n in the disc $\{z : \rho(0, z) < r\}$.

We wish to show that $\{f_n(z)\}$ is a normal family of functions. We may assume that one of the values, say A, among A, B, and C is not infinite. Set

$$h_n(z) = [(f_n(z) - c_n)/(f_n(z) - b_n)] \cdot [(a_n - b_n)/(a_n - c_n)].$$

Then $\{h_n(z)\}$ omits 0, 1, and ∞, and so by Montel's theorem $\{h_n(z)\}$ is a normal family of functions in $\{z : \rho(0, z) < r\}$. Solving for $f_n(z)$ in terms of $h_n(z)$, we can verify that $\{f_n(z)\}$ is also a normal family. Hence there is a subsequence, which we continue to denote by $\{f_n(z)\}$, which converges spherically uniformly on $\{z : \rho(0, z) \leq r/2\}$ to a function which is either meromorphic or identically infinite. Since the behavior of $f_n(z)$ in $\{z : \rho(0, z) \leq r/2\}$ is the same as that of $f(z)$ in $\{z : \rho(z_n, z) \leq r/2\}$, for each positive number s, there is an integer N and a positive number R such that

$$\chi(f(z_n), f(z)) < s, \quad \text{for } n > N \text{ and } \rho(z_n, z) < R.$$

Hence by theorem 1 $\{z_n\}$ cannot be a sequence of P-points. We have shown that any sequence which is not a sequence of ρ-points has a subsequence which is not a sequence of P-points. But each subsequence of a sequence of P-points must also be a sequence of P-points, and so any sequence which is not a sequence of ρ-points cannot be a sequence of P-points. This concludes the proof.

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