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RESEARCH ARTICLE

# Abelian supplements in almost simple groups

Mauro Costantini[1], Andrea Lucchini[2] and Daniele Nemmi[3]

[1]Università di Padova, Dipartimento di Matematica "Tullio Levi-Civita", Via Trieste 63, 35121 Padova, Italy;
E-mail: costantini@math.unipd.it.
[2]Università di Padova, Dipartimento di Matematica "Tullio Levi-Civita", Via Trieste 63, 35121 Padova, Italy;
E-mail: lucchini@math.unipd.it (corresponding author).
[3]Università di Padova, Dipartimento di Matematica "Tullio Levi-Civita", Via Trieste 63, 35121 Padova, Italy;
E-mail: dnemmi@math.unipd.it.

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**Abstract**
Let $G$ be an almost simple group with socle $G_0$. In this paper we prove that whenever $G/G_0$ is abelian, then there exists an abelian subgroup $A$ of $G$ such that $G = AG_0$. We propose a few applications of this structural property of almost simple groups.

## 1. Introduction

The main result of this paper is the following consequence of the classification of the finite simple groups.

**Theorem 1.** *Let $G$ be an almost simple group with socle $G_0$. If $G/G_0$ is abelian, then $G$ contains an abelian subgroup $A$ such that $G = AG_0$.*

Notice that in general it is not true that if $N$ is a normal subgroup of a finite group $G$ and $G/N$ is abelian, then $N$ has an abelian supplement in $G$. For example, if $G$ is a finite $p$-group and $N$ is the Frattini subgroup of $G$, then $G/N$ is abelian, but $G$ is the unique supplement of $N$, so the statement fails whenever $G$ is not abelian. However, Theorem 1 has also some consequences beyond almost simple groups. In fact, we will prove the following corollary as well, on groups with $F(G) = 1$, where $F(G)$ is the Fitting subgroup of $G$.

**Corollary 2.** *Let $G$ be a finite group and suppose that $F(G) = 1$. Let $N = \mathrm{soc}(G)$. If $a, b$ are two elements of $G$ and $[a, b] \in N$, then there exist $n, m \in N$ such that $[an, bm] = 1$.*

We now describe an application of the previous corollary that was our original motivation to look for results in this direction. Let $G$ be a finite noncyclic group and denote by $d(G)$ the smallest cardinality of a generating set of $G$. The *rank graph* $\Gamma(G)$ associated to $G$ is the graph whose vertices are the elements of $G$ and where $x$ and $y$ are adjacent vertices if there exists a generating set $X$ of $G$ of cardinality $d(G)$ such that $\{x, y\}$ is a subset of $X$. Moreover, we denote by $\Delta(G)$ the subgraph of $\Gamma(G)$ induced by its non-isolated vertices. When $d(G) = 2$, the graph $\Gamma(G)$ is known with the name of *generating graph*. It

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***Table 1.*** *The proof of Theorem 1 in the various cases. Notice that* $\mathrm{Alt}_6 \cong \mathrm{PSL}_2(9)$ *has been considered in the linear one.*

| | $G_0$ | Reference |
|---|---|---|
| alternating | $\mathrm{Alt}_n$, $G_0 \neq \mathrm{Alt}_6$ | Corollary 7 |
| classical | $A_{n-1}(q) = \mathrm{PSL}_n(q)$ | Theorem 19 |
| | ${}^2A_{n-1}(q) = \mathrm{PSU}_n(q)$ | Theorem 19 |
| | $B_n(q), C_n(q)$ | Theorem 21 |
| | $D_n(q)$ | Theorem 25, 26 |
| | ${}^2D_n(q)$ | Theorem 24 |
| exceptional | $E_6(q)$ | Theorem 22 |
| | ${}^2E_6(q)$ | Theorem 23 |
| | $E_7(q)$ | Theorem 21 |
| | ${}^3D_4(q), E_8(q), F_4(q), G_2(q)$, ${}^2B_2(2^r), {}^2G_2(3^r), {}^2F_4(2^r)'$ | Corollary 7 |
| sporadic | all | Corollary 7 |

was defined by Liebeck and Shalev in [14], and it has been widely studied by several authors; as survey references, we recommend [5] and [15]. Many strong structural results about $\Gamma(G)$ are known in the case where $G$ is simple, and this reflects the rich group theoretic structure of these groups. For example, if $G$ is a nonabelian simple group, then the only isolated vertex of $\Gamma(G)$ is the identity [13] and the graph $\Delta(G)$ is connected with diameter two [3], and if $|G|$ is sufficiently large, it admits a Hamiltonian cycle [4] (it is conjectured that the condition on $|G|$ can be removed). Moreover, in recent years, there has been considerable interest in attempting to classify the groups $G$ for which $\Gamma(G)$ shares the strong properties of the generating graphs of simple groups. Recently, a remarkable result has been proved – that the identity is the unique isolated vertex of $\Gamma(G)$ if and only if all proper quotients of $G$ are cyclic [6]. An open question is whether $\Delta(G)$ is connected for any finite group $G$ with $d(G) = 2$. The answer is known to be positive if $G$ is soluble [9] (and in this case, the diameter of $\Delta(G)$ is at most three [16]), if $G$ is a direct product of finite simple groups [10] (but examples in which the diameter is arbitrarily large can be exhibited) or if $G$ is a group whose proper quotients are all cyclic [6]. However, only partial results are known for arbitrary finite groups. Clearly, the same question can be asked in the more general case when $d(G) \geqslant 2$. In a recent preprint [18], Corollary 2 plays a crucial role in the proof of the following result.

**Theorem 3** [18]. *If* $d(G) \geqslant 3$, *then* $\Delta(G)$ *is connected.*

When $d(G) = 2$, the techniques used to prove Theorem 3 encounter some obstacles, but they can suggest a starting point for the case of the generating graph as well.

The proof of Theorem 1 strongly depends on the classification of the finite simple groups. It is articulated in various cases which are proved separately along the paper. Table 1 contains, for every non-abelian simple group $G_0$, the location of the corresponding proof. The statement is clearly true if $G/G_0$ is cyclic: indeed, in this case, $\langle g \rangle$ is an abelian supplement of $G_0$ in $G$ for every $g \in G$ such that $G = \langle G_0, g \rangle$. This implies in particular that Theorem 1 is true if $G_0$ is an alternating group (with the possible exception of $\mathrm{Alt}_6$) or a sporadic simple group (see Corollary 7), so we may restrict our attention to the case when $G_0$ is a simple group of Lie type. To explore the different possibilities that can arise when $G_0$ is a simple group of Lie type, a detailed knowledge of the automorphism group of $G_0$ is needed. Recall in particular that if $\alpha \in \mathrm{Aut}(G_0)$, then there exist inner, diagonal, field and graph automorphisms, $g, \delta, \phi, \rho$ such that $\alpha = g\delta\phi\rho$. The easiest case is when all the diagonal automorphisms of $G$ are inner. In this case, $\mathrm{Aut}(G_0)$ splits over $G_0$ so, in the assumption of Theorem 1, $G_0$ certainly admits an abelian complement in $G$. Clearly, the same argument can be applied whenever $\mathrm{Aut}(G_0)$ splits over $G_0$. The simple groups with this property have been classified in [17] (see Theorem 9). Unfortunately, in many cases, $\mathrm{Aut}(G_0)$ does not split over $G_0$. In these cases, the proof of Theorem 1 requires harder work that goes by a case-by-case inspection. Roughly speaking, denoting by $d$ the index of $\mathrm{Inn}(G_0)$ in the group

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$\mathrm{Inndiag}(G_0)$ of the inner-diagonal automorphisms of $G_0$, the larger $d$ is, the more situations arise in which the proof of Theorem 1 requires greater care. Already when $G_0 = \mathrm{PSL}_2(q)$, although $d \leqslant 2$ in this case, the proof is not immediate. We discuss this case in Theorem 10, and we suggest the reader to pay particular attention to the proof of this theorem, which is, up to some extent, representative of the type of arguments needed in the general case. When $G_0$ is a linear or a unitary group, we give an explicit construction of an abelian supplement of $G_0$ in $G$. This requires patient and tiring work to cover the different possibilities, but it can be easily followed even by the reader less familiar with the properties of simple groups since it is essentially based on elementary considerations of linear algebra. As a by-product of our proof, a description of maximal abelian subgroups of $\mathrm{Out}(G_0)$ is obtained.

The analysis of the remaining simple groups of Lie type is somewhat facilitated by the fact that $d$ is at most 4, although a more detailed description of $\mathrm{Aut}(G_0)$ and its action on root subgroups is needed. The arguments for the different families of simple groups of Lie type are similar, but each family has its own peculiarities, so a case-by-case analysis is unavoidable.

We conclude this introduction by giving an outline of the structure of the paper. We begin with Section 2 in which we set the stage with some notation and preliminary results. Then, invoking the classification of the finite simple groups, we roughly classify the possibilities for $G$ in Theorem 1. We notice already in Section 2 that the alternating and sporadic groups can be easily ruled out. Thus, in the following sections, we look at the different possibilities for the simple groups of Lie type. In Sections 3, we deal with linear and unitary groups. After that, in Section 4, we give more details on groups of Lie type, viewed as Chevalley groups, which will be the framework in which we deal with the remaining cases.

- Section 5: groups of type $C_n(q)$, $B_n(q)$ and $E_7(q)$;
- Sections 6 and 7: groups of type $E_6(q)$ and ${}^2E_6(q)$;
- Section 8: groups of type ${}^2D_n(q)$;
- Sections 10 and 11: groups of type $D_n(q)$.

Finally, in Section 12, we conclude with the proof of Corollary 2.

## 2. Notation and preliminary results

In this section, we will present the main strategy for the proof of Theorem 1 and prove some preliminary results which will also establish the main theorem for some families of almost simple groups. We fix the notation we will use throughout all the paper.

As usual, if $X$ is a subgroup of group $Y$, we will denote by $C_X(Y)$ and $N_X(Y)$ the centraliser and the normaliser of $Y$ in $X$, respectively. Moreover, if $x_1, x_2 \in X$, then $[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$.

If $X$ is a matrix, we denote by ${}^tX$ the transpose of $X$.

For a finite group $H$, let

$$\upsilon : \mathrm{Aut}(H) \longrightarrow \mathrm{Out}(H) \cong \mathrm{Aut}(H)/\mathrm{Inn}(H)$$

be the canonical projection. If $G_0$ is a finite non-abelian simple group, we identify $G_0 \cong \mathrm{Inn}(G_0)$, and from now on, $\upsilon$ will usually denote the above map for $H = G_0$.

Recall that a subgroup $H$ of a finite group $G$ is said to be a supplement for a normal subgroup $N$ of $G$ if $HN = G$. The following definition will provide the language we will use in the proof of our main result.

**Definition 4.** Let $H$ be a finite group. If $T$ is an abelian subgroup of $\mathrm{Out}(H)$, we say that $\tilde{T} \leqslant \mathrm{Aut}(H)$ is a $T$-abelian supplement if $\tilde{T}$ is abelian and surjects onto $T$ in the quotient by $\mathrm{Inn}(H)$. An almost simple group $G$ with socle $G_0$ is said to be abelian supplemented if $T = G/G_0$ is abelian and there is a $T$-abelian supplement in $\mathrm{Aut}(G_0)$.

Notice in particular that if $T$ and $\tilde{T}$ are as in the previous definition and $\upsilon(G) = T$, then $G = \tilde{T}G_0$ with $\tilde{T}$ abelian. Therefore, proving Theorem 1 is equivalent to proving that for every non-abelian simple

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***Table 2.*** *Index $d$ of $G_0$ in* $\mathrm{Inndiag}(G_0)$.

| | $G_0$ | $d$ |
|---|---|---|
| untwisted | $A_{n-1}(q) = \mathrm{PSL}_n(q)$ | $(n, q-1)$ |
| | $B_n(q), C_n(q)$ | $(2, q-1)$ |
| | $D_n(q)$ | $(4, q^n-1)$ |
| | $E_6(q)$ | $(3, q-1)$ |
| | $E_7(q)$ | $(2, q-1)$ |
| | $E_8(q), F_4(q), G_2(q)$ | 1 |
| twisted | ${}^2A_{n-1}(q) = \mathrm{PSU}_n(q)$ | $(n, q+1)$ |
| | ${}^2D_n(q)$ | $(4, q^n+1)$ |
| | ${}^2E_6(q)$ | $(3, q+1)$ |
| | ${}^2B_2(2^r), {}^3D_4(q), {}^2G_2(3^r), {}^2F_4(2^r)$ | 1 |

group $G_0$ and every abelian $T \leqslant \mathrm{Out}(G_0)$, there exists a $T$-abelian supplement. The strategy of the proof of Theorem 1 is in fact the following: given $G_0$, we analyse all the abelian subgroups $T$ of $\mathrm{Out}(G_0)$ and, by the classification of the finite simple groups, prove that there exists a $T$-abelian supplement in a case-by-case inspection. Actually, it is not necessary to check each abelian subgroup of $\mathrm{Out}(G_0)$, but only the maximal abelian ones, as it is shown by the following lemma.

**Lemma 5.** *Let $T \leqslant S \leqslant \mathrm{Out}(G_0)$ with $T$ and $S$ abelian. If there exists an $S$-abelian supplement, then there exists a $T$-abelian supplement as well.*

*Proof.* Let $\tilde{S}$ be an $S$-abelian supplement. Let $\tilde{T}$ be the preimage of $T$ by the map $\upsilon|_{\tilde{S}}$. Then $\tilde{T} \leqslant \tilde{S}$, and so it is abelian; moreover, $\upsilon(\tilde{T}) = \upsilon|_{\tilde{S}}(\tilde{T}) = T$, and so $\tilde{T}$ is a $T$-abelian supplement. □

In particular, whenever $\mathrm{Out}(G_0)$ is abelian, to prove Theorem 1, it is enough to check that there exists an $\mathrm{Out}(G_0)$-abelian supplement.

Now we will establish some results that give sufficient conditions on $\mathrm{Out}(G_0)$ and an abelian subgroup $T$ for the existence of a $T$-abelian supplement.

**Lemma 6.** *Let $T$ be a cyclic subgroup of $\mathrm{Out}(H)$, for a finite group $H$. Then there exists a $T$-abelian supplement.*

*Proof.* Let $T = \langle t \rangle$ and let $\tilde{t} \in \mathrm{Aut}(H)$ be a preimage of $t$ under $\upsilon$. Then $\tilde{T} = \langle \tilde{t} \rangle$ is a $T$-abelian supplement. □

The previous lemma, together with Lemma 5, shows the following.

**Corollary 7.** *If $G_0$ is a finite non-abelian simple group and $\mathrm{Out}(G_0)$ is cyclic, then every almost simple group $G$ with socle $G_0$ is abelian supplemented. This holds for*

- *$G_0 = \mathrm{Alt}_n$, $n \geqslant 5$, with $n \neq 6$;*
- *$G_0 = {}^3D_4(q), E_8(q), F_4(q), G_2(q), {}^2B_2(2^r), {}^2G_2(3^r), {}^2F_4(2^r)'$;*
- *$G_0$ is a sporadic simple group.*

Noticing that $\mathrm{Alt}_6 \cong \mathrm{PSL}_2(9)$, this corollary reduces our investigation to the groups of Lie type.

In what follows, $G_0 = {}^sL(q)$ is a simple group of Lie type and $q = p^m$, where $p$ is a prime. The list of finite simple groups of Lie type and a full explanation of the notation ${}^sL(q)$ may be found in Section 4. We denote by $d$ the index of $G_0$ in $\mathrm{Inndiag}(G_0)$, the subgroup of $\mathrm{Aut}(G_0)$ generated by the inner and diagonal automorphisms of $G_0$ (see Section 4 or [7] for further details). We give the values of $d$ in Table 2 to provide a quick reference to look up, since such values play a central role in the proofs.

The Tits group ${}^2F_4(2)'$, also considered as a group of Lie type, does not appear in Table 2. It is well known that $\mathrm{Aut}({}^2F_4(2)') = {}^2F_4(2)$, and the extension does not split. We are now able to state another fundamental ingredient for the proof of Theorem 1.

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**Lemma 8.** *If* $\mathrm{Aut}(H)$ *splits over* $\mathrm{Inn}(H)$*, then there exists a* $T$*-abelian supplement for every abelian* $T \leqslant \mathrm{Out}(H)$.

*Proof.* Let $K$ be a complement of $\mathrm{Inn}(H)$ in $\mathrm{Aut}(H)$. Then $K \cong \mathrm{Out}(H)$ and the subgroup of $K$ corresponding to $T$ is a $T$-abelian supplement. □

In [17], Lucchini, Menegazzo and Morigi gave a complete classification of all simple groups of Lie type $G_0$ for which $\mathrm{Aut}(G_0)$ splits over $G_0$. Their main result is the following.

**Theorem 9.** *Let* $G_0 = {}^sL_n(q)$ *be a simple group of Lie type,* $q = p^m$*. Then* $\mathrm{Aut}(G_0)$ *splits over* $G_0$ *if and only if one of the following conditions holds:*

(1) $G_0$ *is untwisted, not of type* $D_n(q)$*, and* $(\frac{q-1}{d}, d, m) = 1$*;*
(2) $G_0 = D_n(q)$ *and* $(\frac{q^n-1}{d}, d, m) = 1$*;*
(3) $G_0$ *is twisted, not of type* ${}^2D_n(q)$ *or* ${}^2F_4(2)'$*, and* $(\frac{q+1}{d}, d, m) = 1$*;*
(4) $G_0 = {}^2D_n(q)$*, and either* $n$ *is odd or* $p = 2$*.*

We are now ready to begin the investigation of the various types of almost simple groups, starting with the ones with linear socle.

## 3. Linear amd unitary groups

In this section, we prove Theorem 1 in the linear case. We begin with the easiest case $n = 2$, which is better understood on its own and gives us an explicit model for the more general setting. Then we deal with the case $n \geqslant 3$. More specifically, we prove some technical lemmas and analyse all the different types of maximal abelian subgroups $T$ of the outer automorphism group, showing the existence of $T$-abelian supplements in each case. Finally, the main result of this section is contained in Theorem 19.

We start by recalling the structure of the automorphism group of $\mathrm{PSL}_n(q)$ (a more detailed description can be found in [23, 3.3.4]). The group $\mathrm{PGL}_n(q)$ acts as a group of automorphisms of $\mathrm{PSL}_n(q)$, and the corresponding quotient group $\mathrm{PGL}_n(q)/\mathrm{PSL}_n(q)$ is a cyclic group of order $d = (n, q-1)$, called the group of diagonal outer automorphisms. This group is generated by the element $\delta$ corresponding to the automorphism of $\mathrm{GL}_n(q)$ induced by the conjugation with the diagonal matrix $\mathrm{diag}(\lambda, 1 \dots, 1)$, being $\lambda$ a generator of the multiplicative group $\mathbb{F}_q^\times$. The automorphism group of the field $\mathbb{F}_q$ of order $q = p^m$ is a cyclic group of order $m$ generated by the Frobenius automorphism $x \mapsto x^p$. This induces an automorphism $\phi$ of $\mathrm{GL}_n(q)$ by mapping each matrix entry to its $p$-th power. We denote by $\Gamma\mathrm{L}_n(q)$ the semidirect product of $\mathrm{GL}_n(q)$ with this group of field automorphisms, and correspondingly the extension of $\mathrm{PGL}_n(q)$ by the induced group of field automorphisms is denoted by $\mathrm{P\Gamma L}_n(q)$. The duality automorphism of $\mathrm{GL}_n(q)$ is the map that takes a matrix to the transpose of its inverse. For $n = 2$, this duality map is an inner automorphism of $\mathrm{SL}_2(q)$. For $n > 2$, the duality automorphism induces an automorphism $\gamma$ of $\mathrm{PSL}_n(q)$ of order 2 that spans $\mathrm{Aut}(\mathrm{PSL}_n(q))/\mathrm{P\Gamma L}_n(q)$. We shall identify field and graph automorphisms with their corresponding images in $\mathrm{Out}(\mathrm{PSL}_n(q))$. They generate a subgroup $\langle \phi, \gamma \rangle$ which is isomorphic to the direct product of a cyclic group of order $m$ with a cyclic group of order 2. It can be easily seen that $\delta^\phi = \delta^p$ and $\delta^\gamma = \delta^{-1}$.

We identify the general unitary group $\mathrm{GU}_n(q)$ as the subgroup of the unitary matrices of $\mathrm{GL}_n(q^2)$. Let $G_0 := \mathrm{PSU}_n(q)$. The quotient $\mathrm{PGU}_n(q)/\mathrm{PSU}_n(q)$ is cyclic of order $d = (n, q+1)$, generated by the automorphism $\delta$ induced by the conjugation with the diagonal matrix $\mathrm{diag}(\lambda, 1, \dots, 1)$, denoting by $\lambda$ and element of the field $\mathbb{F}_{q^2}$ of order $q+1$. The outer automorphism group of $G_0$ is described in [23, 3.6.3]. We have

$$\mathrm{Out}(G_0) = \langle \delta \rangle \rtimes \langle \phi \rangle,$$

where $\phi$ is the field automorphism which raises the coefficients of every matrix to the power $p$. In particular, we have $|\phi| = 2m$ and $\delta^\phi = \delta^p$.

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**Theorem 10.** *Let $G$ be a finite almost simple group with socle $G_0 = \mathrm{PSL}_2(q)$. Then $G$ contains an abelian subgroup $H$ such that $G = HG_0$.*

*Proof.* Let $Z := \mathrm{Z}(\mathrm{GL}_2(q))$. We can suppose that $q$ is odd; otherwise, $d = 1$ and $\mathrm{Aut}(G_0)$ splits over $G_0$. In this case, $\mathrm{Out}(G_0) = \langle\delta\rangle \times \langle\phi\rangle$, with $|\delta| = 2$, $|\phi| = m$ and $[\delta, \phi] = 1$. So $\mathrm{Out}(G_0)$ is abelian, and by Lemma 5, it is enough to prove the case $G = \mathrm{Aut}(G_0)$.

Let

$$A := \begin{pmatrix} 0 & -\lambda \\ 1 & 0 \end{pmatrix}. \qquad B := \begin{pmatrix} \lambda^{\frac{p-1}{2}} & 0 \\ 0 & 1 \end{pmatrix}.$$

We have

$$A^{\phi B} = \begin{pmatrix} \lambda^{\frac{1-p}{2}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\lambda^{p} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda^{\frac{p-1}{2}} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -\lambda^{\frac{p+1}{2}} \\ \lambda^{\frac{p-1}{2}} & 0 \end{pmatrix} = \lambda^{\frac{p-1}{2}} A.$$

Therefore,

$$[A, \phi B] \in Z,$$

$\upsilon(AZ) = \delta$, and $\upsilon(\phi BZ)$ can be $\phi\delta$ or $\phi$. In any case,

$$\upsilon(\langle AZ, \phi BZ\rangle) = \mathrm{Out}(G_0),$$

and therefore,

$$\langle A, \phi B\rangle Z/Z$$

is an $\mathrm{Out}(G_0)$-abelian supplement. □

From now on, $G_0 \in \{\mathrm{PSL}_n(q), \mathrm{PSU}_n(q)\}$ with $n \geqslant 3$, so $d = (n, q + e)$, with $e = -1$ in the linear case, $e = 1$ in the unitary case. We write also $Z$ for the center of $\mathrm{GL}_n(q)$ or $\mathrm{GU}_n(q)$, respectively. If $G_0 = \mathrm{PSL}_n(q)$, then $\mathrm{Out}(G_0) = \langle\delta\rangle \rtimes \langle\phi, \gamma\rangle$, with $|\phi| = m$, $|\gamma| = 2$, $[\phi, \gamma] = 1$, $\delta^{\phi} = \delta^{p}$ and $\delta^{\gamma} = \delta^{-1}$. If $G_0 = \mathrm{PSU}_n(q)$, then $\mathrm{Out}(G_0) = \langle\delta\rangle \rtimes \langle\phi\rangle$, with $|\phi| = 2m$ and $\delta^{\phi} = \delta^{p}$.

The following lemma shows that in order to prove Theorem 1 in the linear and unitary cases when $n \geqslant 3$, it is sufficient to investigate only two cases, which we will deal with in Propositions 15 and 18, respectively.

**Lemma 11.** *To prove Theorem 1 in the linear and unitary case when $n \geqslant 3$, we can reduce our investigation to finding abelian supplements for the abelian subgroups $T \leqslant \mathrm{Out}(G_0)$ of the following form:*

(1) $T = \left\langle \delta^{k}, \phi^{s}\gamma^{\varepsilon}\delta^{j} \right\rangle$ *with $\varepsilon \in \{0, 1\}$, $k \mid d$ and $k \neq d$;*
(2) $T = \left\langle \delta^{d/2}, \phi^{s}\delta^{j}, \gamma\delta^{k} \right\rangle$ *with $d$ even.*

*Notice that for unitary groups, case (2) does not occur, and in case (1), $\varepsilon = 0$.*

*Proof.* Proving Theorem 1 for an almost simple group $G_0$ is equivalent to finding a $T$-abelian supplement for every abelian $T \leqslant \mathrm{Out}(G_0)$. By Corollary 7, we can assume that $T$ is not cyclic. Let $\pi\colon \mathrm{Out}(G_0) \to \mathrm{Out}(G_0)/\langle\delta\rangle = \langle\phi, \gamma\rangle$. If $\pi(T) = \langle\phi^{s}\gamma^{\varepsilon}\rangle$ is cyclic, then $T$ is of the form $T = \left\langle \delta^{k}, \phi^{s}\gamma^{\varepsilon}\delta^{j} \right\rangle$ with $k \mid d$, and we are in case (1). If $\pi(T)$ is not cyclic, then $\pi(T) = \langle\phi^{s}, \gamma\rangle$. Suppose $d$ is odd. Then $T$ is 2-generated and of the form $T = \left\langle \phi^{s}\delta^{j}, \gamma\delta^{k} \right\rangle$. Since $d$ is odd, $\gamma\delta^{k}$ is conjugate to $\gamma$ in $\langle\delta, \gamma\rangle$, so up to conjugation, we can assume $k = 0$ and therefore $\delta^{j} = 1$, since $[\phi^{s}\delta^{j}, \gamma] = 1$; therefore, $\tilde{T} := T$ is a $T$-abelian supplement. Suppose $d$ is even. If $T$ is 2-generated, it is of the form $T = \left\langle \phi^{s}\delta^{j}, \gamma\delta^{k} \right\rangle$, and since $\delta^{d/2} \in \mathrm{Z}(\mathrm{Out}(G_0))$, $T$ is contained in an abelian subgroup of $\mathrm{Out}(G_0)$ of the form $\left\langle \delta^{d/2}, \phi^{s}\delta^{j}, \gamma\delta^{k} \right\rangle$,

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and we are in case (2). If it is 3-generated, it is of the form $T = \langle \delta^l, \phi^s\delta^j, \gamma\delta^k \rangle$, and in order to be abelian, we should have $[\delta^l, \gamma\delta^k] = 1$; therefore, $l = d/2$, and again we are in case (2). $\square$

In the sequel, we will use a lot the following special matrices defined from some integers $w, l, c \in \mathbb{Z}$ with $w \geqslant 2$ (we choose to define such matrices only for $w \geqslant 2$ to avoid ambiguity in the definition and behaviour when $w = 1$; this choice is irrelevant in the proofs). Let $\mathbb{F}$ be the algebraic closure of the field with $p$ elements and $\lambda$ be an element of $\mathbb{F}$ of order $q + e$, with $e \in \{-1, 1\}$. Then we define

$$A_{w,l} := \begin{pmatrix} 0 & 0 & \dots & 0 & (-1)^{w-1}\lambda^l \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \in \mathrm{GL}_w(\mathbb{F})$$

and

$$X_{w,c} := \begin{pmatrix} \lambda^{c(w-1)} & 0 & \cdots & 0 & 0 \\ 0 & \lambda^{c(w-2)} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda^c & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \in \mathrm{GL}_w(\mathbb{F}).$$

Notice that

$$\det A_{w,l} = \lambda^l.$$

**Remark 12.** Notice that $A_{w,l}$ and $X_{w,c}$ can be viewed as elements of $\mathrm{GL}_n(q)$ if $e = -1$ and as elements of $\mathrm{GU}_n(q)$ if $e = 1$.

We now introduce a technical lemma which is the key ingredient of the proofs in this section.

**Lemma 13.** *Let $w, l, c \in \mathbb{Z}$ be integers with $w \geqslant 2$ and*

$$A := A_{w,l}(\lambda) \qquad X := X_{w,c}(\lambda).$$

*If*

$$cw \equiv lp^s(-1)^\varepsilon - l \mod q + e,$$

*then we have*

$$A^{\phi^s\gamma^\varepsilon X} = \lambda^c A.$$

*Proof.* First, notice that

$$A^\gamma = \begin{pmatrix} 0 & 0 & \dots & 0 & (-1)^{w-1}\lambda^{-l} \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix},$$

and therefore,

$$A^{\phi^s\gamma^\varepsilon} = \begin{pmatrix} 0 & 0 & \dots & 0 & (-1)^{w-1}\lambda^{lp^s(-1)^\varepsilon} \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

The statement follows by computing the action of $A^{\phi^s\gamma^\varepsilon X}$ and $\lambda^c A$ on the canonical basis. $\square$

We show now the existence of $T$-abelian supplements for $T$ of type (1). We start with the following lemma.

**Lemma 14.** *Let $n, m \geqslant 1$, and $d = (n, m)$. Then there exists an integer $y$ such that $yn \equiv d \bmod m$ and $(y, d) = 1$. In particular, we find an integer $y$ such that $yn \equiv d \bmod q + e$ and $(y, d) = 1$.*

*Proof.* Since $\left(\frac{n}{d}, \frac{m}{d}\right) = 1$, there exists $\overline{y} \in \mathbb{Z}$ such that $\overline{y}\frac{n}{d} \equiv 1 \bmod \frac{m}{d}$. Now let

$$d = p_1^{\alpha_1} \dots p_{\tilde{l}}^{\alpha_{\tilde{l}}} p_{\tilde{l}+1}^{\alpha_{\tilde{l}+1}} \dots p_l^{\alpha_l}$$

be its prime factorisation, where we have ordered the primes in a way such that $p_i$ divides $\overline{y}$ if and only if $1 \leqslant i \leqslant \tilde{l}$.

Let

$$y = \overline{y} + p_{\tilde{l}+1} \cdots p_l \frac{m}{d}.$$

For every $p_i$, we have that $p_i$ does not divide $y$ because if $1 \leqslant i \leqslant \tilde{l}$, then $\overline{y}$ is divisible by $p_i$ while $p_{\tilde{l}+1} \cdots p_l \frac{m}{d}$ is not (since $(\overline{y}, \frac{m}{d}) = 1$), and if $\tilde{l} < i \leqslant l$, $p_i$ divides $p_{\tilde{l}+1} \cdots p_l \frac{m}{d}$ but not $\overline{y}$. Therefore, $(y, d) = 1$; moreover, $y \equiv \overline{y} \bmod \frac{m}{d}$ and $yn \equiv d \bmod m$. $\square$

**Proposition 15.** *Let $T = \left\langle \delta^k, \phi^s\gamma^\varepsilon\delta^j \right\rangle$ be abelian with $k \mid d$ and $k \neq d$. Then there exists a $T$-abelian supplement.*

*Proof.* By Lemma 14, there exists $y \in \mathbb{Z}$ such that $yn \equiv d \bmod q + e$ and $(y, d) = 1$. Since $T$ is abelian,

$$\delta^k = (\delta^k)^{\phi^s\gamma^\varepsilon\delta^j} = \delta^{kp^s(-1)^\varepsilon},$$

which means $d \mid k((-1)^\varepsilon p^s - 1)$ or, equivalently,

$$t \mid (-1)^\varepsilon p^s - 1, \qquad t := d/k.$$

First, suppose $t = n$, so $t = d = n$ and $k = 1$. In this case, $T = \langle \delta, \phi^s\gamma^\varepsilon \rangle$ and

$$\tilde{T} := \left\langle A_{n,1}, \phi^s\gamma^\varepsilon X_{n, \frac{(-1)^\varepsilon p^s - 1}{n}} \right\rangle Z/Z$$

is a $T$-abelian supplement since $\upsilon(A_{n,1}Z) = \delta$ and by applying Lemma 13 with

$$(w, l, c) = \left(n, 1, \frac{(-1)^\varepsilon p^s - 1}{n}\right),$$

we have

$$\left[A_{n,1}, \phi^s\gamma^\varepsilon X_{n, \frac{(-1)^\varepsilon p^s - 1}{n}}\right] \in Z.$$

So in the sequel, we can suppose $t \neq n$.

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**Step 1.** We construct matrices $A, X$ such that $\det A = \lambda^k$ and $[A, \phi^s\gamma^\varepsilon X] \in Z$.

Since $t \mid n$ and $t \neq 1, n$, we have that both $t \geqslant 2$ and $n - t \geqslant 2$, and so we can define

$$A := \begin{pmatrix} A_{t,y} & 0 \\ 0 & A_{n-t,k-y} \end{pmatrix}.$$

First, notice that

$$\det A = \det A_{t,y} \det A_{n-t,k-y} = \lambda^y \lambda^{k-y} = \lambda^k;$$

therefore $\upsilon(AZ) = \delta^k$.

Let $r := ((-1)^\varepsilon p^s - 1)/t$ and define

$$X := \begin{pmatrix} X_{t,yr} & 0 \\ 0 & X_{n-t,yr} \end{pmatrix}.$$

We have that

$$y(p^s(-1)^\varepsilon - 1) = yrt \mod q + e,$$

so applying Lemma 13 with $(w, l, c) = (t, y, yr)$, we get

$$A_{t,y}^{\phi^s\gamma^\varepsilon X_{t,yr}} = \lambda^{yr} A_{t,y}.$$

Moreover, recalling that $kt = d \equiv ny \mod q + e$, we have that

$$(k - y)(p^s(-1)^\varepsilon - 1) = (k - y)rt = krt - yrt = yr(n - t) \mod q + e,$$

so applying Lemma 13 with $(w, l, c) = (n - t, k - y, yr)$, we get

$$A_{n-t,k-y}^{\phi^s\gamma^\varepsilon X_{n-t,yr}} = \lambda^{yr} A_{n-t,k-y}.$$

Therefore, we have

$$A^{\phi^s\gamma^\varepsilon X} = \lambda^{yr} A$$

or, equivalently,

$$[A, \phi^s\gamma^\varepsilon X] \in Z.$$

**Step 2.** We find a matrix $C$ such that $[A, C] = 1$ with $\det C = \lambda$. Moreover, $C \in \mathrm{GL}_n(q)$ if $e = -1$, $C \in \mathrm{GU}_n(q)$ if $e = 1$.

Recall that $(y, t) = 1$, so there exist $a, b \in \mathbb{Z}$ such that $ay + bt = 1$. Let

$$C_0 := \lambda^b A_{t,y}^a.$$

We have that $[A_{t,y}, C_0] = 1$ and

$$\det C_0 = \det {A_{t,y}}^a \lambda^{bt} = \lambda^{ay+bt} = \lambda.$$

Let

$$C := \begin{pmatrix} C_0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We have $[A, C] = 1$ and $\det C = \det C_0 = \lambda$.

**Step 3.** We complete the proof by constructing a $T$-abelian supplement.

Let $u \in \mathbb{Z}$ such that $\upsilon(XZ) = \delta^u$. Combining Steps 1 and 2, we get

$$[A, \phi^s \gamma^\varepsilon X C^{j-u}] \in Z,$$

with $\upsilon(AZ) = \delta^k$ and $\upsilon(XC^{j-u}Z) = \delta^u \delta^{j-u} = \delta^j$. Therefore,

$$\tilde{T} := \langle A, \phi^s \gamma^\varepsilon X C^{j-u} \rangle Z/Z$$

is a $T$-abelian supplement. □

To continue our investigation, we need a few more lemmas.

**Lemma 16.** *Let $A, B \in \mathrm{GL}_n(q)$. Then*

$$[\phi^s A, \gamma B] = 1$$

*if and only if*

$$B = A^T B^{\phi^s} A.$$

*Proof.* Easy computation. □

**Lemma 17.** *Assume that we are in linear case, so $e = -1$ and $\lambda$ is an element of order $q-1$ in $\mathbb{F}_q$. If $\alpha, \beta \in \mathbb{Z}$ are such that*

$$\beta \equiv 2\alpha + p^s \beta \mod q-1,$$

*then*

$$[\phi^s X_{w,\alpha}, \gamma X_{w,\beta}] = 1.$$

*Proof.* Since $X_{w,\alpha}, X_{w,\beta}$ are diagonal, for Lemma 16, we just need to check that

$$X_{w,\beta} = X_{w,\alpha} X_{w,\beta}^{p^s} X_{w,\alpha}.$$

By inspecting the coefficients on the diagonal, for every $1 \leqslant i \leqslant w$, we have

$$\lambda^{\beta(w-i)} = \lambda^{(2\alpha + p^s\beta)(w-i)} = \lambda^{\alpha(w-i)} \lambda^{p^s\beta(w-i)} \lambda^{\alpha(w-i)},$$

by the hypothesis on $\alpha$ and $\beta$. □

We show now the existence of $T$-abelian supplements for $T$ of type (2). Remember that this case occurs only when $T$ is linear.

**Proposition 18.** *Let $d$ be even and suppose $T$ is abelian of the form $T = \langle \delta^{d/2}, \phi^s \delta^j, \gamma \delta^k \rangle$. Then we can find a $T$-abelian supplement.*

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*Proof.* As in the previous case, this proof is articulated in different steps.

**Step 1.** We find an integer $y \in \mathbb{Z}$ such that $yn \equiv d \bmod q-1$ and $y$ is odd.

This follows from Lemma 14 and the fact that $d = (n, q-1)$ is assumed to be even.

**Step 2.** We construct matrices $A, X_\phi, X_\gamma \in \mathrm{GL}_n(q)$ such that $\det A = \lambda^{d/2}$ and $\hat{T}_1 := \langle A, \phi^s X_\phi, \gamma X_\gamma \rangle Z/Z$ is abelian.

Since $n-2 \geqslant 2$, we can define

$$A := \begin{pmatrix} A_{2,y} & 0 \\ 0 & A_{n-2,d/2-y} \end{pmatrix},$$

so that $\upsilon(AZ) = \delta^{d/2}$ since

$$\det A = \det A_{2,y} \det A_{n-2,d/2-y} = \lambda^y \lambda^{d/2-y} = \lambda^{d/2}.$$

Let $r := (p^s-1)/2$. Considering the automorphisms $\phi^s$ and $\gamma$, let us now argue as in Step 1 of Proposition 15 and construct

$$X_\phi := \begin{pmatrix} X_{2,yr} & 0 \\ 0 & X_{n-2,yr} \end{pmatrix},$$

so that

$$A^{\phi^s X_\phi} = \lambda^{yr} A,$$

and

$$X_\gamma := \begin{pmatrix} X_{2,-y} & 0 \\ 0 & X_{n-2,-y} \end{pmatrix},$$

so that

$$A^{\gamma X_\gamma} = \lambda^{-y} A.$$

Since

$$-y \equiv 2yr - yp^s \mod q-1,$$

by Lemma 17, we have

$$\left[\phi^s X_{2,yr}, \gamma X_{2,-y}\right] = \left[\phi^s X_{n-2,yr}, \gamma X_{n-2,-y}\right] = 1,$$

and therefore,

$$\left[\phi^s X_\phi, \gamma X_\gamma\right] = 1.$$

From this, we obtain that

$$\hat{T}_1 := \langle A, \phi^s X_\phi, \gamma X_\gamma \rangle Z/Z$$

is abelian.

**Step 3.** We construct matrices $X'_\phi, X'_\gamma \in \mathrm{GL}_n(q)$ such that $\det X'_\gamma = \lambda^y \det X_\gamma$ and $\hat{T}_2 := \left\langle A, \phi^s X'_\phi, \gamma X'_\gamma \right\rangle Z/Z$ is abelian.

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Let us define

$$C_\phi := \begin{pmatrix} A_{2,y}^{-r} & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_n(q), \quad C_\gamma := \begin{pmatrix} A_{2,y} & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_n(q),$$

and

$$X'_\phi := X_\phi C_\phi, \quad X'_\gamma := X_\gamma C_\gamma.$$

Since $C_\phi, C_\gamma \in C_{\mathrm{GL}_n(q)}(A)$, we have

$$A^{\phi^s X'_\phi} = \lambda^{yr} A, \qquad A^{\gamma X'_\gamma} = \lambda^{-y} A.$$

A straightforward computation shows also that

$$\left[\phi^s X'_\phi, \gamma X'_\gamma\right] = 1.$$

Therefore,

$$\det X'_\gamma = \det X_\gamma \det C_\gamma = \det X_\gamma \det A_{2,y} = \lambda^y \det X_\gamma,$$

and

$$\hat{T}_2 := \left\langle A, \phi^s X'_\phi, \gamma X'_\gamma \right\rangle Z/Z$$

is abelian.

**Step 4.** We complete the proof by constructing a $T$-abelian supplement.

Let $\upsilon(X_\gamma Z) = \delta^u$ for some $u \in \mathbb{Z}$, so $\upsilon(X'_\gamma Z) = \delta^{y+u}$. Given that $y$ is odd, one of $u-k$ or $y+u-k$ is even. Since $\gamma\delta^x$ is conjugate to $\gamma\delta^y$ in $\langle \delta, \gamma \rangle$ if $y-x$ is even, one of $\gamma\delta^u$ or $\gamma\delta^{u+y}$ is conjugate to $\gamma\delta^k$.

Let

$$\hat{T} := \begin{cases} \hat{T}_1 & \text{if } u-k \text{ is even} \\ \hat{T}_2 & \text{if } u+y-k \text{ is even} \end{cases},$$

so that there exists a matrix $R \in \mathrm{GL}_n(q)$ such that $\upsilon(\hat{T}^R) = \left\langle \delta^{d/2}, \phi^s\delta^l, \gamma\delta^k \right\rangle$ for some $l \in \mathbb{Z}$. Notice that this group being abelian means $2l \equiv -k(p^s-1) \bmod d$. In the same way, since $T = \left\langle \delta^{d/2}, \phi^s\delta^j, \gamma\delta^k \right\rangle$ is abelian, it means that $2j \equiv -k(p^s-1) \bmod d$, but then $2l \equiv 2j \bmod d$, which means $l \equiv j \bmod d/2$ and $\upsilon(\hat{T}^R) = T$. Therefore, $\tilde{T} := \hat{T}^R$ is a $T$-abelian supplement. □

We have therefore proved Theorem 1 in the linear and unitary cases.

**Theorem 19.** *Let $G$ be an almost simple group with socle $G_0 \in \{\mathrm{PSL}_n(q), \mathrm{PSU}_n(q)\}$. If $G/G_0$ is abelian, then $G$ contains an abelian subgroup $H$ such that $G = HG_0$.*

## 4. Notation for groups of Lie type

By Theorem 9 and Table 2, to prove Theorem 1, we are left to deal with the following cases:

$$B_n(q), C_n(q), D_n(q), E_7(q),\ q = p^m, p \neq 2,$$

$$^2D_n(q),\ q = p^m, p \neq 2, n \text{ even},$$

$$E_6(q),\ q = p^m, q \equiv 1 \bmod 3,$$

and

$$^2E_6(q),\ q = p^m, q \equiv -1 \bmod 3.$$

We give a brief introduction of the tools that we are going to use.

For the definitions and automorphisms of simple groups of Lie type, we refer to [7] (see also [20]). The automorphism groups of the finite simple groups of Lie type (untwisted and twisted) have been determined by Steinberg in [19] and by Griess, Lyons in [12]. We denote by $\mathbb{F}_q$ the field with $q = p^m$ elements, where $p$ is a prime. We briefly recall that the Chevalley group (or untwisted group of Lie type) $L(q)$, viewed as a group of automorphisms of a Lie algebra $L_{\mathbb{F}_q}$ over $\mathbb{F}_q$, obtained from a complex finite dimensional simple Lie algebra $L$, is the group generated by certain automorphisms $x_\alpha(t)$, where $t$ runs over $\mathbb{F}_q$ and $\alpha$ runs over the root system $\Phi$ associated to $L$. The finite untwisted groups of Lie type $L(q)$ are

$$A_n(q), n \geqslant 1, B_n(q), C_n(q), n \geqslant 2, D_n(q), n \geqslant 4, E_6(q), E_7(q), E_8(q), F_4(q), G_2(q).$$

It is well known that $L(q)$ is simple, except in the case $L(q) = A_1(2), A_1(3), B_2(2), G_2(2)$ ([7, Theorem 11.1.2]). The groups $A_1(2), A_1(3)$ are soluble. The group $B_2(2)$ is isomorphic to $S_6$. The derived group of $G_2(2)$ is isomorphic to $\mathrm{PSU}_3(3)$.

For every $\alpha \in \Phi$, $t \in \mathbb{F}_q^\times$, one defines $n_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$, $n_\alpha = n_\alpha(1)$ and the subgroup $N = \langle n_\alpha(t) \mid \alpha \in \Phi, t \in \mathbb{F}_q^\times\rangle$ of $L(q)$.

Let $\Delta = \{\alpha_1, \dots, \alpha_n\}$ be a system of simple roots of $\Phi$. We shall use the numbering and the description of the simple roots in terms of the canonical basis $(e_1, \dots, e_r)$ of an appropriate $\mathbb{R}^r$ as in [1], Planches I-IX. We denote by $Q$ the root lattice, by $P$ the weight lattice and by $W$ the Weyl group; $s_i$ is the simple reflection associated to $\alpha_i$, $\{\omega_1, \dots, \omega_n\}$ are the fundamental weights, $w_0$ is the longest element of $W$, and $A = (a_{ij})$ is the Cartan matrix (hence, $\alpha_i = \sum_j a_{ij}\omega_j$).

Let $\mathrm{Hom}(Q, \mathbb{F}_q^\times)$ be the group of $\mathbb{F}_q$-characters of $Q$ (i.e., group homomorphisms from $Q$ to $\mathbb{F}_q^\times$). For any $\chi \in \mathrm{Hom}(Q, \mathbb{F}_q^\times)$, one defines the automorphism $h(\chi)$ of $L_{\mathbb{F}_q}$ ([7, p. 98]). Let $\hat{H} = \{h(\chi) \mid \chi \in \mathrm{Hom}(Q, \mathbb{F}_q^\times)\}$. The map $\chi \mapsto h(\chi)$ is an isomorphism of $\mathrm{Hom}(Q, \mathbb{F}_q^\times)$ onto $\hat{H}$. We have $\hat{H} \leqslant N_{\mathrm{Aut}L_{\mathbb{F}_q}}(L(q))$. The automorphism of $L(q)$ induced by $h(\chi)$ maps $x_\alpha(t)$ to $h(\chi)x_\alpha(t)h(\chi)^{-1} = x_\alpha(\chi(\alpha)t)$ ([7, p. 100]). Let $H = \hat{H} \cap L(q)$. Then $h(\chi)$ lies in $H$ if and only if $\chi$ can be extended to an $\mathbb{F}_q$-character of $P$. The number $d$ in Table 2 relative to the untwisted case is the order of $\hat{H}/H$. We denote by $\mathrm{Inndiag}(L(q))$ the group $L(q)\hat{H}$, the group of *inner-diagonal* automorphisms of $L(q)$. Any automorphism $\phi$ of $\mathbb{F}_q$ induces a *field automorphism*, still denoted by $\phi$, of $L(q)$, which is defined by $x_\alpha(t)^\phi = x_\alpha(t^\phi)$. In particular, the automorphism $x \mapsto x^p$ of $\mathbb{F}_q$ induces the field automorphism denoted by $g \mapsto g^{[p]}$ of $L(q)$. Note that $h(\chi)^{[p]} = h(\chi)^p = h(p\chi)$ for every $\mathbb{F}_q$-character $\chi$ of $Q$.

We recall that a symmetry of the Dynkin diagram of $L(q)$ is a permutation $\rho$ of the nodes of the diagram, such that the number of bonds joining nodes $i, j$ is the same as the number of bonds joining nodes $\rho(i)$, $\rho(j)$, for any $i \neq j$. A nontrivial symmetry $\rho$ of the Dynkin diagram can be extended to a map of the space $E = \mathbb{R}P$ into itself (an isometry if $L = A_n$, $D_n$, $E_6$), still denoted by $\rho$. This map yields an outer automorphism, again denoted by $\rho$, of $L(q)$; $\rho$ is said to be a *graph automorphism* of $L(q)$ (see [7, p. 200–210] for the detailed description). It is defined as follows.

(a) If $L(q) = A_n(q), n \geqslant 2, D_n(q), E_6(q)$

$$x_\alpha(t)^\rho = x_{\rho(\alpha)}(\gamma_\alpha t),$$

where $\alpha \in \Phi$, $t \in \mathbb{F}_q$, $\gamma_\alpha \in \mathbb{Z}$; the $\gamma_\alpha$ can be chosen so that $\gamma_\alpha = 1$ if $\alpha \in \pm\Delta$.

(b) If $L(q) = B_2(q), F_4(q)$ and $q = 2^m$

$$x_\alpha(t)^\rho = x_{\rho(\alpha)}(t^{\lambda(\rho(\alpha))}),$$

where $\alpha \in \Phi$, $t \in \mathbb{F}_q$, $\lambda(\alpha) = 1$ if $\alpha$ is short, $\lambda(\alpha) = 2$ if $\alpha$ is long. Note that $\rho^2$ is the field automorphism $x_\alpha(t) \mapsto x_\alpha(t^2)$, so $\rho$ has order $2m$.

(c) If $L(q) = G_2(q)$ and $q = 3^m$

$$x_\alpha(t)^\rho = x_{\rho(\alpha)}(t^{\lambda(\rho(\alpha))}),$$

where $\alpha \in \Phi$, $t \in \mathbb{F}_q$, $\lambda(\alpha) = 1$ if $\alpha$ is short, $\lambda(\alpha) = 3$ if $\alpha$ is long. Note that $\rho^2$ is the field automorphism $x_\alpha(t) \mapsto x_\alpha(t^3)$, so $\rho$ has order $2m$.

Graph and field automorphisms commute; the subgroup $R$ they generate (denoted by $\Phi_K\Gamma_K$ in [11, Theorem 2.5.12]) normalises $\mathrm{Inndiag}(L(q))$. We have

$$L(q) \trianglelefteq L(q)\hat{H} = \mathrm{Inndiag}(L(q)) \trianglelefteq \mathrm{Inndiag}(L(q)) : R = \mathrm{Aut}(L(q)).$$

*We shall identify field and graph automorphisms with their corresponding images in* $\mathrm{Out}(L(q))$. The action of $\mathrm{Out}(L(q))/\mathrm{Outdiag}(L(q))$ on $\mathrm{Outdiag}(L(q))$ is described in [2, §1.7.2] and [11, Theorem 2.5.12].

We have $H \triangleleft N$ and $N/H \cong W$. For $w \in W$, we denote by $\dot{w}$ a representative of $w$ in $N$; for each $i = 1, \dots, n$, $n_{\alpha_i}$ is a representative of $s_i$ in $N$. For short, we denote $n_{\alpha_i}$ by $n_i$. Note that $n_i$ lies in $L(p)$, so that it is fixed by field automorphisms of $L(q)$.

Next, we consider the finite twisted groups. These are defined as certain subgroups of appropriate untwisted groups $L(q^s)$ over the field $\mathbb{F}_{q^s}$ with $q^s$ elements, $q = p^m$ as usual (the list may be found in [7, p. 251]):

$${}^2A_n(q), n \geqslant 2, {}^2D_n(q), n \geqslant 4, {}^3D_4(q), {}^2E_6(q), {}^2B_2(2^{2m+1}), {}^2F_4(2^{2m+1}), {}^2G_2(3^{2m+1}).$$

Note that for the types $A_n$, $D_n$, $E_6$, we have used the notation ${}^sL(q)$ instead of ${}^sL(q^s)$ (used in [7, p. 251]) to stick with the notation in [17]. They are all simple, except for the groups ${}^2A_2(2)$, ${}^2B_2(2)$, ${}^2F_4(2)$ and ${}^2G_2(3)$ ([7, Theorem 14.4.1]). The groups ${}^2A_2(2)$, ${}^2B_2(2)$ are soluble. The derived subgroup of ${}^2G_2(3)$ is isomorphic to the simple group $\mathrm{PSL}_2(8)$. The derived subgroup ${}^2F_4(2)'$ of ${}^2F_4(2)$ has index 2 in ${}^2F_4(2)$, and it is a simple group called the Tits group. For the simple groups $G_0$ of type ${}^3D_4(q)$, ${}^2B_2(2^{2m+1})$, ${}^2F_4(2^{2m+1})$, ${}^2G_2(3^{2m+1})$, ${}^2F_4(2)'$, the group $\mathrm{Out}(G_0)$ is cyclic ([11, Theorem 2.5.12, 12]). Therefore, every almost simple group $G$ with socle $G_0$ is abelian supplemented (Corollary 7). Moreover, we have ${}^2A_n(q) \cong \mathrm{PSU}_n(q)$, so we are left to deal with ${}^2D_n(q)$ and ${}^2E_6(q)$. We observe that if $n$ is odd or $p = 2$, then $\mathrm{Aut}({}^2D_n(q))$ splits over ${}^2D_n(q)$ (Theorem 9), so that there exists a $T$-abelian supplement for every abelian $T \leqslant \mathrm{Out}({}^2D_n(q))$ (Lemma 8). In view of this discussion, we shall deal with the remaining cases. Below, we give a short description of these groups

So, let us assume that $L$ is of type $D_n$ or $E_6$, and $\tau$ is an order 2 symmetry of the Dynkin diagram. The twisted group ${}^2L(q)$ is a certain subgroup of the Chevalley group $L(q^2)$ ([7, Definition 13.4.2]). Let $E$ be the real vector space spanned by the roots (or the weights). Then $\tau$ induces an automorphism (in fact an isometry), still denoted by $\tau$, of $E$ fixing both $Q$ and $P$. Let $\chi$ be an $\mathbb{F}_{q^2}$-character of $Q$ (or $P$). We say that $\chi$ is *self-conjugate* if $\chi(\tau(x)) = \chi(x)^q$ for every $x$ in $Q$ (or $P$). Let $\hat{H}^1 = \{h(\chi) \mid \chi : Q \to \mathbb{F}_{q^2}^\times$ is a self-conjugate character of $Q\}$. We have $\hat{H}^1 \leqslant N_{\mathrm{Aut}L_{\mathbb{F}_{q^2}}}({}^2L(q))$. Let $H^1 = \hat{H}^1 \cap {}^2L(q)$. Then $h(\chi)$ lies in $H^1$ if and only if $\chi$ can be extended to a self-conjugate $\mathbb{F}_{q^2}$-character of $P$. The number $d$ in Table 2 relative to the twisted case is the order of $\hat{H}^1/H^1$. We denote by $\mathrm{Inndiag}({}^2L(q))$ the group ${}^2L(q)\hat{H}^1$ of *inner-diagonal* automorphisms of ${}^2L(q)$. Any automorphism $\phi$ of $\mathbb{F}_{q^2}$ induces the field automorphism $\phi$ of $L(q^2)$, which leaves ${}^2L(q)$ invariant and therefore induces an automorphism of ${}^2L(q)$ (also called a *field automorphism*). If $R^1$ is the group of field automorphisms of ${}^2L(q)$, we have

$${}^2L(q) \trianglelefteq {}^2L(q)\hat{H}^1 = \mathrm{Inndiag}({}^2L(q)) \trianglelefteq \mathrm{Inndiag}({}^2L(q)) : R^1 = \mathrm{Aut}({}^2L(q)).$$

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In general, we have $(1-z)P \leqslant Q$ for every $z \in W$. For Coxeter elements, equality holds:

**Lemma 20.** *Let $\alpha_1, \dots, \alpha_n$ be the simple roots (in any fixed order), $\omega_1, \dots, \omega_n$ the corresponding fundamental weights. Then*

$$(1 - s_1 \cdots s_n)\omega_i = \alpha_i + z_1\alpha_1 + \cdots + z_{i-1}\alpha_{i-1}$$

*with $z_1, \dots, z_{i-1} \in \mathbb{Z}$. In particular, $(1 - s_1 \cdots s_n)P = Q$.*

*Proof.* We have $s_i(\omega_j) = \omega_j - \delta_{ij}\alpha_i$ for every $i, j$. For $i = 1$, we have $s_1 \cdots s_n\omega_1 = s_1\omega_1 = \omega_1 - \alpha_1$; hence, $(1 - s_1 \cdots s_n)\omega_1 = \alpha_1$. Let $1 < i \leqslant n$. Then $s_1 \cdots s_{i-1}(\alpha_i) = \alpha_i + z_1\alpha_1 + \cdots + z_{i-1}\alpha_{i-1}$, with $z_u \in \mathbb{Z}$ for $u = 1, \dots, i-1$. Then

$$\begin{aligned}(1 - s_1 \cdots s_n)\omega_i &= \omega_i - s_1 \cdots s_i\omega_i = \omega_i - s_1 \cdots s_{i-1}(\omega_i - \alpha_i)\\ &= \omega_i - (\omega_i - s_1 \cdots s_{i-1}\alpha_i) = s_1 \cdots s_{i-1}\alpha_i = \alpha_i + z_1\alpha_1 + \cdots + z_{i-1}\alpha_{i-1}.\end{aligned}$$

□

Let $\chi$ be a character of $Q$, $w$ in $W$. We define the character $w\chi$ in the following way. For $x \in Q$, we put $w\chi(x) := \chi(w^{-1}x)$ (i.e., $w\chi = \chi \circ w^{-1}$). We also define $\tau\chi$, where $\tau$ is a graph automorphism, by $(\tau\chi)x := \chi(\tau^{-1}x)$ for $x \in Q$ (hence, $\tau\chi = \chi \circ \tau^{-1}$). Note that for $w \in W$, we have ([7, Theorem 7.2.2])

$$\dot{w}h(\chi)\dot{w}^{-1} = h(w\chi).$$

Since we are assuming $\Phi$ of type $D_n$ or $E_6$, there is a Coxeter element $w$ in $W$ fixed by $\tau$. We may choose a representative $\dot{w}$ of $w$ in $N$ over the prime field and fixed by $\tau$. Let $F = \phi^j$ or $\phi^j\tau$, for some integer $j$. Then $F$ fixes $\dot{w}$ and acts on $\hat{H}$; hence, it induces an automorphism $g$ of $\mathrm{Hom}(Q, \mathbb{F}_q^\times)$ given by $F(h(\chi)) = h(g(\chi))$. Let $\chi : Q \to \mathbb{F}_q^\times$ be a fixed character, $x = \dot{w}h(\chi)$. We shall look for an element $y = h(\chi') \in \hat{H}$ such that $[x, Fy] = 1$; that is,

$$xFy = Fyx \iff y^{-1}F^{-1}xFy = x \iff y^{-1}F(x)y = x$$

so that

$$h(\chi')^{-1}\dot{w}h(g(\chi))h(\chi') = \dot{w}h(\chi).$$

We have $h(\chi')^{-1}\dot{w} = \dot{w}\dot{w}^{-1}h(\chi')^{-1}\dot{w} = \dot{w}h(w^{-1}\chi')^{-1} = \dot{w}h(-w^{-1}\chi')$; hence,

$$\begin{aligned}\dot{w}h(-w^{-1}\chi')h(g(\chi))h(\chi') &= \dot{w}h(\chi),\\ h(-w^{-1}\chi')h(g(\chi))h(\chi') &= h(\chi),\end{aligned}$$

and finally,

$$h((1 - w^{-1})\chi') = h((1 - g)\chi), \quad (1 - w^{-1})\chi' = (1 - g)\chi,$$
$$\chi' \circ (1 - w) = (1 - g)\chi.$$

We shall be interested in the following cases:

$F : s \mapsto s^{[p^i]}$, then

$$\chi' \circ (1 - w) = (1 - p^i)\chi,$$

$F : s \mapsto s^{[p]\tau}$, then

$$\chi' \circ (1 - w) = (1 - p\tau)\chi.$$

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By Lemma 20, we have $(1-w)^{-1}Q = P$. Let $\Delta = |P/Q| = \det A$. Then $\Delta P \leqslant Q$. Note that if $\Phi = D_n$ with $n$ even, then $2P \leqslant Q$ since $P/Q \cong C_2 \times C_2$ (the inverses of the Cartan matrices may be explicitly found in [22]). We put $\Delta_1 = |P/Q|$ unless $\Phi = D_n$, $n$ even, in which case we put $\Delta_1 = 2$. Then

$$\Delta_1(1-w)^{-1}Q \leqslant Q,$$

and we may define the character

$$\zeta_\chi = \chi \circ \Delta_1(1-w)^{-1} : Q \to \mathbb{F}_q^\times$$

and $h(\zeta_\chi) \in \hat{H}$.

We start with the cases $B_n(q), C_n(q), E_7(q)$.

## 5. $C_n(q)$, $B_n(q)$, $n \geqslant 2$, $E_7(q)$

Here, $L$ is of type $C_n$, $B_n$ or $E_7$, $G_0 = L(q)$, $q = p^m$, $d = (q-1, 2)$, and we assume that $\mathrm{Aut}(G_0)$ does not split over $G_0$, so $(\frac{q-1}{d}, d, m) \neq 1$. Therefore, $d = 2$ and $p$ is odd:

$$\mathrm{Out}(G_0) = \langle\delta\rangle \times \langle\phi\rangle,$$

$|\delta| = 2$, $|\phi| = m$. We fix an $\mathbb{F}_q$-character $\chi$ of $Q$ which cannot be extended to a character of $P$, so that $h(\chi)$ induces $\delta$ in $\mathrm{Out}(G_0)$. We look for an $\mathbb{F}_q$-character $\chi'$ so that $[\dot{w}h(\chi), \phi h(\chi')] = 1$; that is,

$$\chi' \circ (1-w) = (1-p)\chi.$$

We have $\Delta_1 = 2$, so $\zeta_\chi = \chi \circ 2(1-w)^{-1}$. We take

$$\chi' = \frac{1-p}{2}\,\zeta_\chi$$

so $h(\chi') = h(\zeta_\chi)^{\frac{1-p}{2}}$. Therefore,

$$\tilde{T} = \langle \dot{w}h(\chi), \phi h(\chi')\rangle$$

is an $\mathrm{Out}(G_0)$-abelian supplement (arguing as in the $\mathrm{PSL}_2(q)$ case).

We have proved the following.

**Theorem 21.** *Let $G$ be a finite almost simple group with socle $G_0 = C_n(q), B_n(q)$ or $E_7(q)$. Then $G$ contains an abelian subgroup $A$ such that $G = AG_0$.*

## 6. $E_6(q)$

Here, $L$ is of type $E_6$, $G_0 = L(q)$, $q = p^m$, $d = (q-1, 3)$, and we assume that $\mathrm{Aut}(G_0)$ does not split over $G_0$, so $(\frac{q-1}{d}, d, m) \neq 1$. Therefore, $d = 3$ and $p \neq 3$:

$$\mathrm{Out}(G_0) = \langle\delta\rangle \rtimes \langle\phi, \tau\rangle,$$

where $|\delta| = 3$, $|\phi| = m$, $\delta^\phi = \delta^p$, $\delta^\tau = \delta^{-1}$ and $[\phi, \tau] = 1$. We fix an $\mathbb{F}_q$-character $\chi$ of $Q$ which can not be extended to a character of $P$.

Let $\pi\colon \mathrm{Out}(G_0) \to \mathrm{Out}(G_0)/\langle\delta\rangle = \langle\phi, \tau\rangle$. Let $T$ be a noncyclic abelian subgroup of $\mathrm{Out}(G_0)$. If $\pi(T)$ is not cyclic, then $\pi(T) = \langle\phi^s, \tau\rangle$. Therefore, $T = \langle\phi^s\delta^i, \tau\delta^k\rangle$. But $\tau\delta^k$ is conjugate to $\tau$ under $\langle\delta\rangle$; hence, we may assume $T = \langle\phi^s\delta^i, \tau\rangle$, so $T = \langle\phi^s, \tau\rangle \leqslant \langle\phi, \tau\rangle$ and $\tilde{T} = \langle\phi^s, \tau\rangle$ is a $T$-abelian supplement.

We are left with the case where $\pi(T)$ is cyclic; that is, $\pi(T) = \langle\phi^s\tau^\epsilon\rangle$. Then $T = \langle\delta, \phi^s\tau^\varepsilon\rangle$. Since $p \neq 3$, we have $p \equiv 1$ or $-1 \bmod 3$.

Let $p \equiv 1 \bmod 3$. Then $[\delta, \phi] = 1$, and we get $\varepsilon = 0$, $T \leqslant \langle \delta, \phi \rangle$, so by Lemma 5, it is enough to consider the case

$$p \equiv 1 \mod 3\,,\; T = \langle \delta, \phi \rangle \qquad \text{(case 1)}.$$

Let $p \equiv -1 \bmod 3$. Then $\delta^{\phi} = \delta^{-1}$. If $\varepsilon = 1$, $T = \langle \delta, \phi^s \tau \rangle$. Since $[\delta, \phi^s \tau] = 1$, $s$ must be odd. Therefore, $T \leqslant \langle \delta, \phi\tau, \phi^2 \rangle = \langle \delta, \phi\tau \rangle$. If $\varepsilon = 0$, $T = \langle \delta, \phi^s \rangle$, so $s$ is even, and again, $T \leqslant \langle \delta, \phi^2 \rangle < \langle \delta, \phi\tau \rangle$. Therefore, it is enough to consider

$$p \equiv -1 \mod 3\,,\; T = \langle \delta, \phi\tau \rangle \qquad \text{(case 2)}.$$

Summarising, we only have to deal with cases 1, 2.

We consider the Coxeter element $w = s_1 s_4 s_6 s_3 s_2 s_5$, fixed by the graph automorphism $\tau$. In fact, we have

$$\tau(\alpha_1) = \alpha_6, \tau(\alpha_2) = \alpha_2, \tau(\alpha_3) = \alpha_5, \tau(\alpha_4) = \alpha_4, \tau(\alpha_5) = \alpha_3, \tau(\alpha_6) = \alpha_1.$$

We choose a representative $\dot{w}$ of $w$ in $N$ over the prime field and fixed by $\tau$, $\dot{w} = n_1 n_4 n_6 n_3 n_2 n_5$, for instance. Hence, $\tau\dot{w} = \dot{w}\tau$, $\dot{w}\phi = \phi\dot{w}$. Here, $\phi$ is the field automorphism of $G_0$ sending $x$ to $x^{[p]}$. We use the notation $\phi^{-1} x \phi = x^{[p]}$. We have $\Delta_1 = 3$, so $\zeta_\chi = \chi \circ 3(1-w)^{-1}$.

**Case 1**: $p \equiv 1 \bmod 3$, $T = \langle \delta, \phi \rangle$.

We take

$$\chi' = \frac{1-p}{3}\, \zeta_\chi$$

so $h(\chi') = h(\zeta_\chi)^{\frac{1-p}{3}}$. Therefore,

$$\tilde{T} = \langle \dot{w} h(\chi), \phi h(\chi') \rangle$$

is a $T$-abelian supplement.

**Case 2**: $p \equiv -1 \bmod 3$, $T = \langle \delta, \phi\tau \rangle$.

Since $\tau w_0 = -1$, we have

$$(1+\tau)P = (1+\tau)w_0 P = (w_0 + \tau w_0)P = (w_0 - 1)P = (1 - w_0)P \leqslant Q.$$

Hence, by Lemma 20,

$$(1+\tau)(1-w)^{-1} Q = (1+\tau)P \leqslant Q,$$

so $\chi \circ (1+\tau)(1-w)^{-1}$ is an $\mathbb{F}_q$-character of $Q$. We look for an $\mathbb{F}_q$-character $\chi'$ so that $[\dot{w}h(\chi), \phi\tau h(\chi')] = 1$; that is,

$$\chi' \circ (1-w) = (1 - p\tau)\chi.$$

We have $1 - p\tau = 1 + p - p - p\tau = 1 + p - p(1+\tau)$, and we may define

$$\chi' = \frac{1+p}{3}\, \zeta_\chi - p\, \chi \circ (1+\tau)(1-w)^{-1},$$

obtaining a character which satisfies $\chi' \circ (1-w) = (1-p\tau)\chi$. Therefore,

$$\tilde{T} = \langle \dot{w}h(\chi), \phi\tau h(\chi') \rangle$$

is a $T$-abelian supplement.

We have proved the following.

**Theorem 22.** *Let $G$ be a finite almost simple group with socle $G_0 = E_6(q)$. If $G/G_0$ is abelian, then $G$ contains an abelian subgroup $A$ such that $G = AG_0$.*

## 7. ${}^2E_6(q)$

Here, $L$ is of type $E_6$, $G_0 = {}^2E_6(q) \leqslant E_6(q^2)$, $q = p^m$, $d = (q+1, 3)$, and we assume that $\mathrm{Aut}(G_0)$ does not split over $G_0$; that is, $(\frac{q+1}{d}, d, m) \neq 1$. Therefore, $d = 3$ and $q \equiv -1 \bmod 3$, so $p \equiv -1 \bmod 3$ and $m$ is odd:

$$\mathrm{Out}(G_0) = \langle \delta \rangle \rtimes \langle \phi \rangle,$$

where $|\delta| = 3$, $|\phi| = 2m$, $\delta^\phi = \delta^{-1}$.

It is enough to consider the case $T = \langle \delta, \phi^2 \rangle$. We fix a self-conjugate $\mathbb{F}_{q^2}$-character $\chi$ of $Q$ which can not be extended to a self-conjugate $\mathbb{F}_{q^2}$-character of $P$ (so that $h(\chi) \in \hat{H}^1 \setminus H^1$).

We consider the same Coxeter element $w = s_1s_4s_6s_3s_2s_5$ as in the previuos section, and the same representative $\dot{w} = n_1n_4n_6n_3n_2n_5$, which lies in $G_0$.

We look for an element $h(\chi') \in \hat{H}^1$ so that $[\dot{w}h(\chi), \phi^2 h(\chi')] = 1$; that is,

$$\chi' \circ (1-w) = (1-p^2)\chi.$$

We have $\Delta_1 = 3$, so $\zeta_\chi = \chi \circ 3(1-w)^{-1}$. We take

$$\chi' = \frac{1-p^2}{3}\,\zeta_\chi,$$

so $h(\chi') = h(\zeta_\chi)^{\frac{1-p^2}{3}}$.

Note that since $\chi$ is self-conjugate and $\tau w = w\tau$, $\zeta_\chi$ and $\chi'$ are self-conjugate, so $h(\chi')$ lies in $\hat{H}^1$. Therefore,

$$\tilde{T} = \left\langle \dot{w}h(\chi), \phi^2 h(\chi') \right\rangle$$

is a $T$-abelian supplement.

We have proved the following.

**Theorem 23.** *Let $G$ be a finite almost simple group with socle $G_0 = {}^2E_6(q)$. If $G/G_0$ is abelian, then $G$ contains an abelian subgroup $A$ such that $G = AG_0$.*

## 8. ${}^2D_n(q)$, $n$ even

Here, $L$ is of type $D_n$, $n$ even, $G_0 = {}^2D_n(q) \leqslant D_n(q^2)$, $q = p^m$, $d = (q+1, 2)$, and we assume that $\mathrm{Aut}(G_0)$ does not split over $G_0$. Therefore, $d = 2$, $p \neq 2$, and

$$\mathrm{Out}(G_0) = \langle \delta \rangle \times \langle \phi \rangle,$$

where $|\delta| = 2$, $|\phi| = 2m$.

It is enough to consider the case $T = \mathrm{Out}(G_0)$. We fix a self-conjugate $\mathbb{F}_{q^2}$-character $\chi$ of $Q$ which can not be extended to a self-conjugate $\mathbb{F}_{q^2}$-character of $P$ (so that $h(\chi) \in \hat{H}^1 \setminus H^1$).

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We consider the Coxeter element $w = s_1 s_2 \cdots s_{n-1} s_n$, fixed by $\tau$ (which exchanges $\alpha_{n-1}$ and $\alpha_n$), and the representative $\dot{w} = n_1 n_2 \cdots n_{n-1} n_n$, which lies in $G_0$. We look for an element $h(\chi') \in \hat{H}^1$ so that $[\dot{w}h(\chi), \phi h(\chi')] = 1$; that is,

$$\chi' \circ (1 - w) = (1 - p)\chi.$$

We have $\Delta_1 = 2$ (since $n$ is even), so $\zeta_\chi = \chi \circ 2(1 - w)^{-1}$. We take

$$\chi' = \frac{1 - p}{2}\, \zeta_\chi,$$

so $h(\chi') = h(\zeta_\chi)^{\frac{1-p}{2}}$.

Since $\chi$ is self-conjugate and $\tau w = w\tau$, $\zeta_\chi$ and $\chi'$ are self-conjugate, so $h(\chi')$ lies in $\hat{H}^1$. Therefore,

$$\tilde{T} = \langle \dot{w}h(\chi), \phi h(\chi') \rangle$$

is an $\mathrm{Out}(G_0)$-abelian supplement.

We have proved the following.

**Theorem 24.** *Let $G$ be a finite almost simple group with socle $G_0 = {}^2D_n(q)$. Then $G$ contains an abelian subgroup $A$ such that $G = AG_0$.*

## 9. The remaining case

In the next sections, we shall deal with the remaining case: $D_n(q)$, $q = p^m$. We shall use the identifications with classical groups as in [7, Theorem 11.3.2] and [8, 1.11, 1.19]. Here, $\lambda$ is a generator of $\mathbb{F}_q^\times$.

We have $G_0 = P\Omega^+_{2n}(q)$, $\mathrm{Inndiag}(G_0) = P(CO_{2n}(q)^\circ)$, where $CO_{2n}(q)$ if the conformal orthogonal group; that is, the group of orthogonal similitudes of $\mathbb{F}_q^{2n}$; $CO_{2n}(q)^\circ$ is the subgroup of index 2 of $CO_{2n}(q)$ of elements which do not interchange the two families of maximal isotropic subspaces of $\mathbb{F}_q^{2n}$. If $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ is the canonical basis of $\mathbb{F}_q^{2n}$, the bilinear form on $\mathbb{F}_q^{2n}$ corresponds to the matrix

$$K_n = \begin{pmatrix} 0_n & I_n \\ I_n & 0_n \end{pmatrix}.$$

We define the homomorphism $\eta : CO_{2n}(q)^\circ \longrightarrow \mathbb{F}_q^\times$ by

$$\eta(X) = \mu \quad \text{if} \quad {}^tXK_nX = \mu K_n.$$

For $\mu \in \mathbb{F}_q^\times$, let $o_\mu = \left(\begin{smallmatrix} I_n & 0 \\ 0 & \mu I_n \end{smallmatrix}\right)$, so that $\eta(o_\mu) = \mu$.

The graph automorphism $\tau$ of $D_n$ exchanging $\alpha_{n-1}$ and $\alpha_n$ is induced by conjugation with

$$\tau_n = \begin{pmatrix} I_{n-1} & 0 & 0_{n-1} & 0 \\ 0 & 0 & 0 & 1 \\ 0_{n-1} & 0 & I_{n-1} & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \in O_{2n}(q),$$

$\tau_n^2 = 1$, $x^\tau = \tau_n x \tau_n$.

$$x_{\alpha_i}(z)^\tau = x_{\alpha_i}(z), i = 1, \ldots, n-2, \quad x_{\alpha_{n-1}}(z)^\tau = x_{\alpha_n}(z), x_{\alpha_n}(z)^\tau = x_{\alpha_{n-1}}(z).$$

We shall deal with the cases $n$ odd and even separately.

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## 10. $D_n(q)$, $n \geqslant 3$, $n$ odd

Here, $L$ is of type $D_n$, $n$ odd, $G_0 = D_n(q)$, $q = p^m$, $d = (4, q-1)$, and we assume that $\mathrm{Aut}(G_0)$ does not split over $G_0$; hence, $(\frac{q^n-1}{d}, d, m) \neq 1$. In particular, $d = 2$ or 4, and $p$ is odd. Moreover, $m$ is even; hence, 4 divides $q-1$. Therefore, $d = 4$.

$$\mathrm{Out}(G_0) = \langle \delta, \tau, \phi \mid \delta^4 = \tau^2 = 1, \delta^\tau = \delta^{-1}, \phi^m = [\tau, \phi] = 1, \delta^\phi = \delta^p \rangle.$$

In $\Omega_{2n}^+(q)$, we choose

$$\dot{w}_0 = \begin{pmatrix} 0_{n-1} & 0 & I_{n-1} & 0 \\ 0 & 1 & 0 & 0 \\ I_{n-1} & 0 & 0_{n-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

a representative of the longest element $w_0$ of the Weyl group. We have $\dot{w}_0^2 = 1$, $\dot{w}_0\tau_n = \tau_n\dot{w}_0 = K_n$. Let $X \in CO_{2n}(q)^\circ$, $\eta(X) = \mu$ (i.e., ${}^tXK_nX = \mu K_n$). Then ${}^tX = \mu K_n X^{-1} K_n$, so that

$$ {}^tX^{-1} = \eta(X)^{-1}\dot{w}_0\tau_n X \tau_n \dot{w}_0 = \eta(X)^{-1}\dot{w}_0 X^\tau \dot{w}_0. \tag{10.1}$$

We start with $D_3$, exploiting the fact that $D_3 = A_3$. Let $V = \mathbb{F}_q^4$ with canonical basis $\mathcal{B} = (v_1, \ldots, v_4)$ over $\mathbb{F}_q$, $\overline{V} = \overline{\mathbb{F}}_q^4$ with the same basis over $\overline{\mathbb{F}}_q$. Let

$$\sigma : GL(\overline{V}) \to GL(\wedge^2\overline{V}), \quad f \mapsto \wedge^2 f.$$

We choose the basis $\mathcal{B}$ for $\overline{V}$, and the basis $\mathcal{C} = (v_{12}, v_{13}, v_{23}, v_{34}, v_{42}, v_{14})$, where $v_{ij} = v_i \wedge v_j$, for $\wedge^2\overline{V}$. We endow $\wedge^2\overline{V}$ with the symmetric bilinear form with matrix $K_3$ with respect to $\mathcal{C}$. Then $\sigma(GL(\overline{V})) \leqslant CO(\wedge^2\overline{V})^\circ$, $\sigma(GL(V)) \leqslant CO(\wedge^2 V)^\circ$, and, by considering bases, we obtain the homomorphism $\sigma : GL_4(q) \to CO_6(q)^\circ$. We have

$$\sigma : \begin{pmatrix} I_3 & 0 \\ 0 & \mu \end{pmatrix} \mapsto \begin{pmatrix} I_3 & 0_3 \\ 0_3 & \mu I_3 \end{pmatrix} = o_\mu;$$

in particular,

$$\det\begin{pmatrix} I_3 & 0 \\ 0 & \mu \end{pmatrix} = \mu = \eta(o_\mu).$$

Moreover, $\sigma : \mu I_4 \mapsto \mu^2 I_6$. If $X \in GL_4(q)$, $\det X = \mu$, then $X = Y\begin{pmatrix} I_3 & 0 \\ 0 & \mu \end{pmatrix}$ with $Y \in SL_4(q)$, $\sigma(X) = \sigma(Y)o_\mu$ with $\sigma(Y) \in \Omega_6^+(q)$ ([21, Theorem 12.20]); hence,

$$\eta(\sigma(X)) = \mu = \det X. \tag{10.2}$$

From (10.1) and (10.2), we get

$$\sigma({}^tX^{-1}) = {}^t(\sigma(X))^{-1} = (\det X)^{-1}\dot{w}_0\sigma(X)^\tau\dot{w}_0. \tag{10.3}$$

For $X, Y \in GL_4(q)$, $z \in \mathbb{F}_q^\times$, we get

$$\begin{aligned} Y^{-1}X^{[p]}Y = zX \quad &\Rightarrow \quad \sigma(Y)^{-1}\sigma(X)^{[p]}\sigma(Y) = z^2\sigma(X) \\ Y^{-1}({}^tX^{-1})Y = zX \quad &\Rightarrow \quad Z^{-1}\sigma(X)^\tau Z = z^2 \det(X)\sigma(X)\ ,\ Z = \dot{w}_0\sigma(Y) \\ {}^tX^{-1}Y = zY^{[p]}X \quad &\Rightarrow \quad \sigma(X)^\tau Z = z^2 \det(X)Z^{[p]}\sigma(X)\ ,\ Z = \dot{w}_0\sigma(Y) \end{aligned} \tag{10.4}$$

since $\dot{w}_0^{[p]} = \dot{w}_0$.

In Section 3, for a given abelian subgroup $T$ of $\mathrm{Out}(\mathrm{PSL}_4(q))$, we have exhibited a $T$-abelian supplement $\tilde{T}$ by giving matrices in $\mathrm{GL}_4(q)$: the map $\sigma$ allows to solve the problem for $G_0 = P\Omega_6^+(q)$, by giving matrices in $CO_6(q)^\circ$. Now we consider $D_n$, $n$ odd, $n = 1 + 2m$, $n > 3$. The space $\mathbb{F}_q^{2n}$ is the orthogonal direct sum $\mathbb{F}_q^{2n} = U \oplus U^\perp$, where $U = \langle e_1, \ldots, e_{n-3}, f_1, \ldots, f_{n-3}\rangle$, $U^\perp = \langle e_{n-2}, e_{n-1}, e_n, f_{n-2}, f_{n-1}, f_n\rangle$, with $\dim U = 2n-6 = 4(m-1)$. Moreover, $U$ is the direct orthogonal sum of subspaces of dimension 4:

$$U_1 = \langle e_1, e_2, f_1, f_2\rangle, \ldots, U_{m-1} = \langle e_{n-4}, e_{n-3}, f_{n-4}, f_{n-3}\rangle.$$

To define an isometry or more generally an orthogonal similitude of $\mathbb{F}_q^{2n}$, we may give matrices $X_i \in CO_4(q)^\circ$, $\eta(X_i) = \mu$, $i = 1, \ldots, m-1$, $X \in CO_6(q)^\circ$, $\eta(X) = \mu$ and define $Y$ in $\mathrm{GL}_{2n}(q)$ by

$$Y = X_1 \oplus \cdots \oplus X_{m-1} \oplus X.$$

Then $Y \in CO_{2n}(q)^\circ$, with $\eta(Y) = \mu$. If $Y \in CO_{2n}(q)^\circ$ fixes $U^\perp$, then it fixes $U$, and if we write $Y = X \oplus Z$, with $X \in CO_6(q)^\circ$, $Z \in CO_{2n-6}(q)^\circ$, and consider the action of $\phi$ and $\tau$, we get

$$Y^{[p]} = X^{[p]} \oplus Z^{[p]}, \quad Y^\tau = X^\tau \oplus Z$$

since $\tau_n$ acts on the basis $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ just by switching $e_n$ and $f_n$ (here, $Y^\tau = \tau_n Y \tau_n$, $X^\tau = \tau_3 X \tau_3$).

We shall proceed as follows. Assume $T$ is an abelian subgroup of $\mathrm{Out}(G_0)$. We consider the analogous subgroup $T$ of $\mathrm{Out}(\mathrm{PSL}_4(q))$. From the $\mathrm{PSL}_4(q)$ case, we have an abelian subgroup of $\mathrm{Aut}(\mathrm{PSL}_4(q))$ given by explicit matrices in $\mathrm{GL}_4(q)$. By using $\sigma$, we obtain corresponding matrices in $CO_6(q)^\circ$ satisfying certain relations. For each such matrix $X$, we define a matrix $X_1 \in CO_4(q)^\circ$ and finally define the matrix $Y = X_1 \oplus \cdots \oplus X_1 \oplus X$ in $CO_{2n}(q)^\circ$ ($m-1$ copies of $X_1$). We shall then obtain a $T$-abelian supplement $\tilde{T}$ in $\mathrm{Aut}(G_0)$.

Let $A, B \in GL_2(q)$ with

$$B^{-1}A^{[p]}B = zA, \quad \det A = \mu,\ z = \mu^{\frac{1}{2}(p-1)}$$

and let $\nu \in \mathbb{F}_q^\times$. Our aim is to define orthogonal similitudes of $\mathbb{F}_q^4$ (with respect to the form given by $K_2$). We put

$$\begin{aligned} a = a(A) &= \begin{pmatrix} A & 0_2 \\ 0_2 & {}^tA^{-1} \end{pmatrix} \begin{pmatrix} I_2 & 0_2 \\ 0_2 & (\det A)I_2 \end{pmatrix} \in CO_4(q)^\circ, \quad \eta(a) = \det A \\ b = b(B, \nu) &= \begin{pmatrix} B & 0_2 \\ 0_2 & {}^tB^{-1} \end{pmatrix} \begin{pmatrix} I_2 & 0_2 \\ 0_2 & \nu I_2 \end{pmatrix} \in CO_4(q)^\circ, \quad \eta(b) = \nu. \end{aligned}$$

From $B^{-1}A^{[p]}B = \mu^{\frac{1}{2}(p-1)}A$, we get

$$b^{-1}a^{[p]}b = \mu^{\frac{1}{2}(p-1)}a, \quad \eta(a) = \det A = \mu, \eta(b) = \nu.$$

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We shall take

$$A = \begin{pmatrix} 0 & -\mu \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \mu^{\frac{1}{2}(p-1)} & 0 \\ 0 & 1 \end{pmatrix}, \quad B^{-1}A^{[p]}B = \mu^{\frac{1}{2}(p-1)}A,$$

$$a = a(A) = a(\mu) = \begin{pmatrix} 0 & -\mu & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & \mu & 0 \end{pmatrix}, \ \eta(a) = \mu, \tag{10.5}$$

$$b = b(B,\nu) = b(\mu,\nu) = \begin{pmatrix} \mu^{\frac{1}{2}(p-1)} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mu^{-\frac{1}{2}(p-1)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \nu & 0 \\ 0 & 0 & 0 & \nu \end{pmatrix}, \ \eta(b) = \nu. \tag{10.6}$$

Then

$$b^{-1}a^{[p]}b = \mu^{\frac{1}{2}(p-1)}a, \quad \eta(a(\mu)) = \mu, \ \eta(b(\mu,\nu)) = \nu.$$

Note that for any $i \in \mathbb{Z}$, we have

$$B^{-1}(A^i)^{[p]}B = \mu^{\frac{1}{2}i(p-1)}A^i, \quad \det A^i = \mu^i,$$

$$b(\det A,\nu)^{-1}a(A^i)^{[p]}\, b(\det A,\nu) = \mu^{\frac{1}{2}i(p-1)}a(A^i), \quad \eta(a(A^i)) = \det A^i = \mu^i, \ \eta(b) = \nu.$$

We shall make use of the explicit matrices in $\mathrm{GL}_4(q)$ from Section 3.

### *10.1.* $p \equiv 1 \bmod 4$

By Lemma 11, in the case when $p \equiv 1 \bmod 4$, the maximal abelian subgroups of $\mathrm{Out}(D_3(q)) \cong \mathrm{Out}(\mathrm{PSL}_4(q))$ are $\langle \delta, \phi \rangle$, $\langle \delta^2, \phi, \tau \rangle$ and $\langle \delta^2, \phi, \tau\delta \rangle$. We are therefore going through such cases.

**Case** $T = \langle \delta, \phi \rangle$. In the $\mathrm{PSL}_4(q)$ case, we took

$$L = \begin{pmatrix} 0 & 0 & 0 & -\lambda \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, M = \begin{pmatrix} \lambda^{\frac{3(p-1)}{4}} & 0 & 0 & 0 \\ 0 & \lambda^{\frac{2(p-1)}{4}} & 0 & 0 \\ 0 & 0 & \lambda^{\frac{p-1}{4}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{T} = \langle L, \phi M \rangle Z(\mathrm{GL}_4(q))/Z(\mathrm{GL}_4(q))$$

We have $M^{-1}L^{[p]}M = \lambda^{\frac{p-1}{4}}L$, hence in $CO_6(q)^\circ$, with $\ell = \sigma(L)$, $m = \sigma(M)$, by (10.2), (10.4),

$$m^{-1}\ell^{[p]}m = \lambda^{\frac{p-1}{2}}\ell,$$

$$\eta(\ell) = \det L = \lambda, \quad \eta(m) = \det M = \lambda^{\frac{3(p-1)}{2}}.$$

We look for $a, b \in CO_4(q)^\circ$ satisfying the same relations using the above procedure. We take $\mu = \lambda$, $\nu = \lambda^{\frac{3(p-1)}{2}}$, $a = a(\lambda)$, $b = b(\lambda, \lambda^{\frac{3(p-1)}{2}})$: if we put $A_1 = a \oplus \cdots \oplus a \oplus \ell$, $B_1 = b \oplus \cdots \oplus b \oplus m$, then

$$A_1, B_1 \in CO_{2n}(q)^\circ, B_1^{-1}A_1^{[p]}B_1 = \lambda^{\frac{p-1}{2}}A_1$$

and

$$\tilde{T} = \langle A_1, \phi B_1 \rangle Z(CO_{2n}(q)^\circ)/Z(CO_{2n}(q)^\circ)$$

is a $T$-abelian supplement.

**Case** $T = \left\langle \delta^2, \phi, \tau \right\rangle$.

In the $\mathrm{PSL}_4(q)$ case for $\left\langle \delta^2, \phi, \gamma \right\rangle$, we took

$$L = \begin{pmatrix} 0 & -\lambda & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda \\ 0 & 0 & 1 & 0 \end{pmatrix}, M = \begin{pmatrix} \lambda^{\frac{p-1}{2}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda^{\frac{p-1}{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, N = \begin{pmatrix} \lambda^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{T} = \langle L, \phi M, \gamma N \rangle Z/Z,$$

with

$$M^{-1} L^{[p]} M = z_1 L \ , \ N^{-1}({}^t L^{-1}) N = z_2 L \ , \ ({}^t M^{-1}) N = z_3 N^{[p]} M,$$

$$z_1 = \lambda^{\frac{1}{2}(p-1)}, z_2 = \lambda^{-1}, z_3 = 1, \det L = \lambda^2, \det M = \lambda^{p-1}, \det N = \lambda^{-2}.$$

Hence, in $CO_6(q)^\circ$, with $\ell = \sigma(L)$, $m = \sigma(M)$, $n = \dot{w}_0 \sigma(N)$,

$$m^{-1} \ell^{[p]} m = \lambda^{p-1} \ell \ , \ n^{-1} \ell^\tau n = \ell \ , \ m^\tau n = \lambda^{p-1} n^{[p]} m,$$

$$\eta(\ell) = \det L = \lambda^2, \eta(m) = \det M = \lambda^{p-1}, \eta(n) = \eta(\dot{w}_0)\eta(\sigma(N)) = \det N = \lambda^{-2}.$$

Recall that $\tau_n$ acts trivially on $U$; hence, we have to define matrices $a, b, c \in CO_4(q)^\circ$ such that

$$b^{-1} a^{[p]} b = \lambda^{p-1} a \ , \ c^{-1} a c = a \ , \ bc = \lambda^{p-1} c^{[p]} b, \text{ that is, } \quad b^{-1} c^{[p]} b = \lambda^{-(p-1)} c,$$

$$\eta(a) = \lambda^2, \eta(b) = \lambda^{p-1}, \eta(c) = \lambda^{-2}.$$

Once we have solved $b^{-1} a^{[p]} b = \lambda^{p-1} a$, we may take $c = a^{-1}$. We take

$$a = a(\lambda^2), b = b(\lambda^2, \lambda^{p-1}), c = a^{-1}.$$

If we put $A_1 = a \oplus \cdots \oplus a \oplus \ell$, $B_1 = b \oplus \cdots \oplus b \oplus m$, $C_1 = c \oplus \cdots \oplus c \oplus n$, then $A_1, B_1, C_1 \in CO_{2n}(q)^\circ$, with

$$B_1^{-1} A_1^{[p]} B_1 = \lambda^{p-1} A_1 \ , \ C_1^{-1} A_1^\tau C_1 = A_1 \ , \ B_1^\tau C_1 = \lambda^{p-1} C_1^{[p]} B_1,$$

$$\eta(A_1) = \lambda^2, \eta(B_1) = \lambda^{p-1}, \eta(C_1) = \lambda^{-2},$$

so that

$$\tilde{T} = \langle A_1, \phi B_1, \tau C_1 \rangle Z/Z$$

is a $T$-abelian supplement.

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**Case** $T = \langle \delta^2, \phi, \tau\delta \rangle$. In the $\mathrm{PSL}_4(q)$ case for $\langle \delta^2, \phi, \gamma\delta \rangle$, we took

$$L = \begin{pmatrix} 0 & -\lambda & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda \\ 0 & 0 & 1 & 0 \end{pmatrix}, M = \begin{pmatrix} \lambda^{\frac{p-1}{2}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda^{\frac{p-1}{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\lambda & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{\frac{1-p}{2}}, N = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{T} = \langle L, \phi M, \gamma N \rangle Z/Z,$$

with

$$M^{-1} L^{[p]} M = z_1 L\ ,\ N^{-1}({}^t L^{-1}) N = z_2 L\ ,\ ({}^t M^{-1}) N = z_3 N^{[p]} M,$$

$$z_1 = \lambda^{\frac{1}{2}(p-1)}, z_2 = \lambda^{-1}, z_3 = 1, \det L = \lambda^2, \det M = \lambda^{\frac{1}{2}(p-1)}, \det N = \lambda^{-1}.$$

Hence, in $CO_6(q)^\circ$, with $\ell = \sigma(L)$, $m = \sigma(M)$, $n = \dot{w}_0 \sigma(N)$,

$$m^{-1} \ell^{[p]} m = \lambda^{p-1} \ell\ ,\ n^{-1} \ell^\tau n = \ell\ ,\ m^\tau n = \lambda^{\frac{1}{2}(p-1)} n^{[p]} m,$$

$$\eta(\ell) = \det L = \lambda^2, \eta(m) = \det M = \lambda^{\frac{1}{2}(p-1)}, \eta(n) = \eta(\dot{w}_0)\eta(\sigma(N)) = \det N = \lambda^{-1}.$$

We have to define matrices $a, b, c \in CO_4(q)^\circ$ such that

$$b^{-1} a^{[p]} b = \lambda^{p-1} a\ ,\ c^{-1} a c = a\ ,\ bc = \lambda^{\frac{1}{2}(p-1)} c^{[p]} b,\ \text{that is,}\quad b^{-1} c^{[p]} b = \lambda^{-\frac{1}{2}(p-1)} c,$$

$$\eta(a) = \lambda^2, \eta(b) = \lambda^{\frac{1}{2}(p-1)}, \eta(c) = \lambda^{-1}.$$

Once we have solved $b^{-1} c^{[p]} b = \lambda^{-\frac{1}{2}(p-1)} c$, we may take $a = c^{-2}$. We take

$$c = a(\lambda^{-1}), b = b(\lambda^{-1}, \lambda^{\frac{1}{2}(p-1)}), a = c^{-2}.$$

If we put $A_1 = a \oplus \cdots \oplus a \oplus \ell$, $B_1 = b \oplus \cdots \oplus b \oplus m$, $C_1 = c \oplus \cdots \oplus c \oplus n$, then $A_1, B_1, C_1 \in CO_{2n}(q)^\circ$, with

$$B_1^{-1} A_1^{[p]} B_1 = \lambda^{p-1} A_1\ ,\ C_1^{-1} A_1^\tau C_1 = A_1\ ,\ B_1^\tau C_1 = \lambda^{\frac{1}{2}(p-1)} C_1^{[p]} B_1,$$

$$\eta(A_1) = \lambda^2, \eta(B_1) = \lambda^{\frac{1}{2}(p-1)}, \eta(C_1) = \lambda^{-1},$$

so that

$$\tilde{T} = \langle A_1, \phi B_1, \tau C_1 \rangle Z/Z$$

is a $T$-abelian supplement.

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### *10.2.* $p \equiv -1 \bmod 4$

By Lemma 11, in the case when $p \equiv -1 \bmod 4$, the maximal abelian subgroups of $\mathrm{Out}(D_3(q)) \cong \mathrm{Out}(\mathrm{PSL}_4(q))$ are $\langle \delta, \phi\tau \rangle$, $\left\langle \delta^2, \phi, \tau \right\rangle$ and $\left\langle \delta^2, \phi\delta, \tau\delta \right\rangle$. We are therefore going through such cases.

**Case** $T = \langle \delta, \phi\tau \rangle$. In the $\mathrm{PSL}_4(q)$ case for $\langle \delta, \phi\gamma \rangle$, we took

$$L = \begin{pmatrix} 0 & 0 & 0 & -\lambda \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, M = \begin{pmatrix} \lambda^{\frac{3(-p-1)}{4}} & 0 & 0 & 0 \\ 0 & \lambda^{\frac{2(-p-1)}{4}} & 0 & 0 \\ 0 & 0 & \lambda^{\frac{-p-1}{4}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{T} = \langle L, \phi\gamma M \rangle Z/Z,$$

with

$$M^{-1}({}^t(L^{[p]})^{-1})M = \lambda^{-\frac{p+1}{4}} L, \quad \det L = \lambda, \det M = \lambda^{-\frac{3}{2}(p+1)}.$$

Hence, in $CO_6(q)^\circ$, with $\ell = \sigma(L)$, $m = \dot{w}_0\sigma(M)$,

$$m^{-1}(\ell^{[p]})^\tau m = \lambda^{\frac{1}{2}(p-1)}\ell,$$

$$\eta(\ell) = \det L = \lambda, \eta(m) = \eta(\dot{w}_0)\eta(\sigma(M)) = \det M = \lambda^{-\frac{3}{2}(p+1)}.$$

We have to define matrices $a, b \in CO_4(q)^\circ$ such that

$$b^{-1}a^{[p]}b = \lambda^{\frac{1}{2}(p-1)}a, \quad \eta(a) = \lambda, \eta(b) = \lambda^{-\frac{3}{2}(p+1)}.$$

We take

$$a = a(\lambda), b = b(\lambda, \lambda^{-\frac{3}{2}(p+1)}).$$

If we put $A_1 = a \oplus \cdots \oplus a \oplus \ell$, $B_1 = b \oplus \cdots \oplus b \oplus m$, then $A_1, B_1 \in CO_{2n}(q)^\circ$, with

$$B_1^{-1}(A_1^{[p]})^\tau B_1 = \lambda^{\frac{1}{2}(p-1)}A_1, \quad \eta(A_1) = \lambda, \eta(B_1) = \lambda^{-\frac{3}{2}(p+1)},$$

so that

$$\tilde{T} = \langle A_1, \phi\tau B_1 \rangle Z/Z$$

is a $T$-abelian supplement.

**Case** $T = \left\langle \delta^2, \phi, \tau \right\rangle$. In the $\mathrm{PSL}_4(q)$ case for $\left\langle \delta^2, \phi, \gamma \right\rangle$, we took

$$\tilde{T} = \langle L, \phi M, \gamma N \rangle Z/Z,$$

with the same $L, M, N$ as in the case $p \equiv 1 \bmod 4$, $T = \left\langle \delta^2, \phi, \gamma \right\rangle$. We define $A_1, B_1, C_1 \in CO_{2n}(q)^\circ$ as in this case, and $\tilde{T} = \langle A_1, \phi B_1, \tau C_1 \rangle Z/Z$ is a $T$-abelian supplement.

**Case** $T = \left\langle \delta^2, \phi\delta, \tau\delta \right\rangle$. In the $\mathrm{PSL}_4(q)$ case for $\left\langle \delta^2, \phi\delta, \gamma\delta \right\rangle$, we took

$$\tilde{T} = \langle L, \phi M, \gamma N \rangle Z/Z,$$

with the same $L$, $M$, $N$ as in the case $p \equiv 1 \bmod 4$, $T = \left\langle \delta^2, \phi, \gamma\delta \right\rangle$. Again, we define $A_1, B_1, C_1 \in CO_{2n}(q)^\circ$ in the same way, and $\tilde{T} = \langle A_1, \phi B_1, \tau C_1 \rangle Z/Z$ is a $T$-abelian supplement.

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We have proved the following.

**Theorem 25.** *Let $G$ be an almost simple group with socle $G_0 = D_n(q)$, $n$ odd. If $G/G_0$ is abelian, then there exists an abelian subgroup $A$ such that $G = AG_0$.*

## 11. $D_n(q)$, $n$ even

Here, $L$ is of type $D_n$, $n$ even, $G_0 = D_n(q)$, $q = p^m$, $d = (2, q-1)^2$, and we assume that $\mathrm{Aut}(G_0)$ does not split over $G_0$; hence, $(\frac{q^n-1}{d}, d, m) \neq 1$. In particular, $d \neq 1$; hence, $p$ is odd, $m$ is even and $d = 4$, $\hat{H}/H \cong C_2 \times C_2$.

If $n = 4$, then

$$\mathrm{Out}(G_0) = (\langle \delta_1, \delta_2, \delta_3 \rangle \times \langle \phi \rangle) : S_3,$$

where $S_3 = \langle \rho, \tau \rangle$, $\tau^2 = 1$, $\rho^3 = 1$, $\delta_1\delta_2 = \delta_3$, $\delta_i^2 = \phi^m = [\rho, \phi] = [\tau, \phi] = 1$, $\delta_1^\tau = \delta_2$, $\delta_3^\tau = \delta_3$, $\delta_1^\rho = \delta_2$, $\delta_2^\rho = \delta_3$, $\delta_3^\rho = \delta_1$.

If $n \neq 4$, then

$$\mathrm{Out}(G_0) = (\langle \delta_1, \delta_2, \delta_3 \rangle \times \langle \phi \rangle) : \langle \tau \rangle,$$

where $\tau^2 = 1$, $\delta_1\delta_2 = \delta_3$, $\delta_i^2 = \phi^m = [\tau, \phi] = 1$, $\delta_1^\tau = \delta_2$, $\delta_3^\tau = \delta_3$.

Note that $(\tau\delta_1)^2 = \tau\delta_1\tau\delta_1 = \delta_2\delta_1 = \delta_3$, $(\tau\delta_2)^2 = \delta_3$; hence,

$$\langle \delta_3, \phi, \tau\delta_1 \rangle = \langle \phi, \tau\delta_1 \rangle = \langle \phi, \tau\delta_2 \rangle.$$

We have to consider the following cases. Assume $T$ is an abelian, noncyclic subgroup of $\langle \delta_1, \delta_2, \phi, \tau \rangle$ (which is $\mathrm{Out}(G_0)$ if $n \neq 4$).

Let $D = \langle \delta_1, \delta_2 \rangle$, $\pi : \mathrm{Out}(G_0) \to \mathrm{Out}(G_0)/D = \langle \phi, \tau \rangle$. If $\pi(T)$ is cyclic, then $\pi(T) = \langle \phi^s \tau^\epsilon \rangle$. If $\varepsilon = 0$, $T \leqslant \langle D, \phi^s \rangle$, so

$$T \leqslant \langle \delta_1, \delta_2, \phi \rangle.$$

If $\varepsilon = 1$, $T \leqslant \langle D, \phi^s\tau \rangle$, and $T$ contains an element $\alpha = \phi^s\tau\delta$, $\delta \in D$, $\delta \neq 1$, so either $T = \langle \delta_3, \phi^s\tau \rangle$ or $T = \langle \delta_3, \phi^s\tau\delta_1 \rangle = \langle \delta_3, \phi^s\tau\delta_2 \rangle$. In the first case,

$$T \leqslant \langle \delta_3, \phi, \tau \rangle.$$

In the second case,

$$T \leqslant \langle \phi, \tau\delta_1 \rangle = \langle \phi, \tau\delta_2 \rangle.$$

If $\pi(T)$ is not cyclic, then $\pi(T) = \langle \phi^s, \tau \rangle$. Therefore, either

$$T \leqslant \langle \delta_3, \phi, \tau \rangle$$

or

$$T \leqslant \langle \delta_3, \phi, \tau\delta_1 \rangle = \langle \phi, \tau\delta_1 \rangle = \langle \phi, \tau\delta_2 \rangle.$$

Therefore, if $T \leqslant \langle \delta_1, \delta_2, \phi, \tau \rangle$, by Lemma 5, we only have to deal with the following cases:

$$\begin{aligned} &\text{case 1:} && T = \langle \delta_1, \delta_2, \phi \rangle, \\ &\text{case 2:} && T = \langle \delta_3, \phi, \tau \rangle, \\ &\text{case 3:} && T = \langle \phi, \tau\delta_1 \rangle = \langle \phi, \tau\delta_2 \rangle. \end{aligned}$$

Assume $n = 4$. Let $M = \langle\delta_1, \delta_2, \phi\rangle$, $\zeta : \mathrm{Out}(G_0) \to \mathrm{Out}(G_0)/M = \langle\rho, \tau\rangle$, and $T$ a noncyclic abelian subgroup of $\mathrm{Out}(G_0)$ not contained in $\langle\phi, \rho, \tau\rangle$. Hence, $\zeta(T) = \{1\}$, $\langle\rho^i\tau\rangle$ or $\langle\rho\rangle$. However, $\rho^i\tau$ is conjugate to $\tau$; therefore, we may assume $\zeta(T) = \{1\}$, $\langle\tau\rangle$ or $\langle\rho\rangle$.

If $\zeta(T) = \{1\}$ or $\langle\tau\rangle$, we are in the previous case $T \leqslant \langle\delta_1, \delta_2, \phi, \tau\rangle$. We are left with $\zeta(T) = \langle\rho\rangle$, $T \leqslant \langle\delta_1, \delta_2, \phi, \rho\rangle$, so $T = \langle\phi^s, \phi^t\rho\delta\rangle$, $\delta \in D$, $\delta \neq 1$ since $T$ is abelian and not contained in $\langle\phi, \rho, \tau\rangle$. It follows that $T \leqslant \langle\phi, \rho\delta\rangle$. Moreover, since $\langle\rho\rangle$ acts transitively on $\{\delta_1, \delta_2, \delta_3\}$ and $[\rho, \phi] = 1$, we may assume

$$\text{case 4:} \quad T = \langle\phi, \rho\delta_2\rangle \qquad \text{only for } D_4.$$

We use the same procedure used to deal with the odd $n$ case. It is convenient to start with $G_0 = D_2(q) = P\Omega_4^+(q) \cong \mathrm{PSL}_2(q) \times \mathrm{PSL}_2(q)$.

We have

$$n_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, n_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \text{ in } \Omega_4^+(q).$$

Note that $n_2 = \tau_2 n_1 \tau_2$ and $n_1 n_2 = n_2 n_1$. If

$$g = \begin{pmatrix} f_1 & 0 & 0 & 0 \\ 0 & f_2 & 0 & 0 \\ 0 & 0 & \frac{\mu}{f_1} & 0 \\ 0 & 0 & 0 & \frac{\mu}{f_2} \end{pmatrix}$$

is a diagonal matrix in $CO_4(q)^\circ$, then $\alpha_1(g) = \frac{f_1}{f_2}$, $\alpha_2(g) = \frac{f_1 f_2}{\mu}$. We define $\delta_1$, $\delta_2$, $\delta_3$. Let

$$h_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

in $CO_4(q)^\circ$. Then $\alpha_1(h_1) = \lambda^{-1}$, $\alpha_2(h_1) = 1$. We write for short $h_1 \mapsto h(\chi_1) \in \hat{H}$, where $\chi_1 = (\lambda^{-1}, 1)$ is the $\mathbb{F}_q$-character of $Q$ with $\chi_1(\alpha_1) = \lambda^{-1}$, $\chi_1(\alpha_2) = 1$. We define $\delta_1 := h(\chi_1)G_0$. Moreover, $\chi_2 := \chi_1 \circ \tau = (1, \lambda^{-1})$, $h_2 := h_1^\tau$, $h_2 \mapsto h(\chi_2) \in \hat{H}$, $\delta_2 := h(\chi_2)G_0$; finally, $h_3 := h_1 h_2$, and hence, $h_3 \mapsto h(\chi_3)$, $\chi_3 = \chi_1 + \chi_2 = (\lambda^{-1}, \lambda^{-1})$, $\delta_3 := h(\chi_3)G_0$, so $\delta_3 = \delta_1\delta_2$.

Let

$$x_1 = n_1 h_1 = \begin{pmatrix} 0 & \lambda & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\lambda & 0 \end{pmatrix}, y = \begin{pmatrix} \lambda^{p-1} & 0 & 0 & 0 \\ 0 & \lambda^{\frac{p-1}{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda^{\frac{p-1}{2}} \end{pmatrix}.$$

Then $x_1, y$ are in $CO_4(q)^\circ$ with

$$\eta(x_1) = \eta(h_1) = \lambda\,,\ \eta(y) = \lambda^{p-1}\,,\ y^{-1}x_1^{[p]}y = \lambda^{\frac{p-1}{2}}x_1\,,\ y^\tau = y.$$

We take $x_2 := x_1^\tau = n_2 h_2$, Then

$$\eta(x_2) = \eta(h_2) = \lambda\,,\ y^{-1}x_2^{[p]}y = \lambda^{\frac{p-1}{2}}x_2\,,\ x_1x_2 = x_2x_1.$$

We put $x_3 := x_1x_2$. We have $x_3^\tau = x_3$, $\eta(x_3) = \lambda^2$. Since $n_1, n_2$ are in $\Omega_4^+(q)$, $x_i$ induces $\delta_i$ for $i = 1, 2, 3$. However, we have $y \mapsto h(\chi)$, $\chi = (\lambda^{\frac{p-1}{2}}, \lambda^{\frac{p-1}{2}})$, so $y$ induces $\delta_3$ if $p \equiv -1 \bmod 4$ and the identity if $p \equiv 1 \bmod 4$.

We are in a position to deal with the 3 cases for $D_2$:

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**Case 1**: $T = \langle \delta_1, \delta_2 \rangle \times \langle \phi \rangle$. From the above discussion, we have

$$x_1 x_2 = x_2 x_1, \quad y^{-1} x_1^{[p]} y = \lambda^{\frac{p-1}{2}} x_1, \quad y^{-1} x_2^{[p]} y = \lambda^{\frac{p-1}{2}} x_2;$$

hence,

$$\tilde{T} = \langle x_1, x_2, \phi y \rangle Z/Z$$

is a $T$-abelian supplement.

**Case 2**: $T = \langle \delta_3 \rangle \times \langle \phi \rangle \times \langle \tau \rangle$. We have

$$x_3^\tau = x_3, \quad y^\tau = y, \quad y^{-1} x_3^{[p]} y = \lambda^{p-1} x_3 \quad ;$$

hence,

$$\tilde{T} = \langle x_3, \phi y, \tau \rangle Z/Z$$

is a $T$-abelian supplement.

**Case 3**: $T = \langle \tau\delta_1, \phi \rangle$. We have

$$y^\tau = y, \quad y^{-1} x_1^{[p]} y = \lambda^{\frac{1}{2}(p-1)} x_1;$$

hence,

$$\tilde{T} = \langle \tau x_1, \phi y \rangle Z/Z$$

is a $T$-abelian supplement. Note that $y$ induces the identity if $p \equiv 1 \bmod 4$ and $\delta_3$ if $p \equiv -1 \bmod 4$, but $x_3 = (\tau x_1)^2$.

We now deal with $G_0 = D_n(q)$, $n$ even, $n = 2m$, $n \geqslant 4$. Let $c_i = \alpha_i, i = 1, \ldots, n-2$, and

$$c_{n-1} = \alpha_{n-1} - (\alpha_1 + \alpha_3 + \cdots + \alpha_{n-3}), \quad c_n = \alpha_n - (\alpha_1 + \alpha_3 + \cdots + \alpha_{n-3}).$$

Then $\frac{1}{2}c_{n-1}$, $\frac{1}{2}c_n$ are in $P$, and so $(c_1, \ldots, c_n)$ is a $\mathbb{Z}$-basis of $Q$ and $(c_1, \ldots, c_{n-2}, \frac{1}{2}c_{n-1}, \frac{1}{2}c_n)$ is a $\mathbb{Z}$-basis of $P$. If $\chi : Q \to \mathbb{F}_q^\times$ is a character, then $\chi$ can be extended to a character of $P$ if and only if $\chi(c_{n-1})$ and $\chi(c_n)$ are in $(\mathbb{F}_q^\times)^2$.

We define the characters $\psi_1, \psi_2, \psi_3 : Q \to \mathbb{F}_q^\times$. As usual, $\lambda$ is a generator of $\mathbb{F}_q^\times$. We have

$$\psi_1(\alpha_i) = 1,\ i = 1, \ldots, n-2\ ,\ \psi_1(\alpha_{n-1}) = \lambda\ ,\ \psi_1(\alpha_n) = 1;$$

hence, $\psi_1(c_{n-1}) = \lambda$, $\psi_1(c_n) = 1$. Then we put $\psi_2 = \psi_1 \circ \tau$, so $\psi_2(c_{n-1}) = 1$, $\psi_2(c_n) = \lambda$, and $\psi_3 = \psi_1 + \psi_2$, so $\psi_3(c_{n-1}) = \psi_3(c_n) = \lambda$. Finally $\delta_1 := h(\psi_1)G_0$, $\delta_2 := h(\psi_2)G_0$, $\delta_3 := h(\psi_3)G_0$; hence, $\delta_3 = \delta_1\delta_2$. Each $\delta_i$ induces the corresponding diagonal automorphism of $D_2(q)$ relative to $\alpha_{n-1}$, $\alpha_n$ (denoted above with the same symbols).

Let $U = \langle e_1, \ldots, e_{n-2}, f_1, \ldots, f_{n-2} \rangle$. Then $\mathbb{F}_q^{2n}$ is the orthogonal direct sum $\mathbb{F}_q^{2n} = U \oplus U^\perp$, $U^\perp = \langle e_{n-1}, e_n, f_{n-1}, f_n \rangle$, with $\dim U = 2n - 4 = 4(m-1)$. Moreover, $U$ is the direct orthogonal sum of subspaces of dimension 4:

$$U_1 = \langle e_1, e_2, f_1, f_2 \rangle, \ldots, U_{m-1} = \langle e_{n-3}, e_{n-2}, f_{n-3}, f_{n-2} \rangle.$$

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To define an isometry or more generally an orthogonal similitude of $\mathbb{F}_q^{2n}$, we give matrices $X_i \in CO_4(q)^\circ$, $\eta(X_i) = \mu$, $i = 1, \dots, m-1$, $X \in CO_4(q)^\circ$, $\eta(X) = \mu$ and define $Y$ in $\mathrm{GL}_{2n}(q)$ by

$$Y = X_1 \oplus \cdots \oplus X_{m-1} \oplus X.$$

Then $Y \in CO_{2n}(q)^\circ$, with $\eta(Y) = \mu$. If $Y \in CO_{2n}(q)^\circ$ fixes $U^\perp$, then it fixes $U$, and if we write $Y = X \oplus Z$, with $X \in CO_4(q)^\circ$, $Z \in CO_{2n-4}(q)^\circ$, and consider the action of $\phi$ and $\tau$, we get

$$Y^{[p]} = X^{[p]} \oplus Z^{[p]}, \quad Y^\tau = X^\tau \oplus Z,$$

since $\tau$ acts on the basis $(e_1, \dots, e_n, f_1, \dots, f_n)$ just switching $e_n$ and $f_n$ (here, $Y^\tau = \tau_n Y \tau_n$, $X^\tau = \tau_2 X \tau_2$).

We shall proceed as follows. Assume $T$ is an abelian subgroup of $\mathrm{Out}(G_0)$, $G_0$ of type $D_n$ ($T \leqslant \langle \delta_1, \delta_2, \phi, \tau\rangle$ if $G_0 = D_4(q)$). We consider the analogous subgroup $T$ of $\mathrm{Out}(D_2(q))$. From the $D_2$ case, we have an abelian subgroup of $\mathrm{Aut}(D_2(q))$ given by explicit matrices in $CO_4(q)^\circ$. For each such matrix $X$, we define matrices $X_1 \in CO_4(q)^\circ$ and $Y = X_1 \oplus \cdots \oplus X_1 \oplus X$ in $CO_{2n}(q)^\circ$ ($m-1$ copies of $X_1$). We shall then obtain a $T$-abelian supplement $\tilde{T}$ in $\mathrm{Aut}(G_0)$.

Recall the matrices $a(\mu)$, $b(\mu, \nu)$ in $CO_4(q)^\circ$ defined in (10.5), (10.6) and the matrices $x_1$, $x_2$, $x_3$, $y \in CO_4(q)^\circ$ defined to deal with $D_2$. We have

$$x_1 x_2 = x_2 x_1, \quad y^{-1} x_1^{[p]} y = \lambda^{\frac{p-1}{2}} x_1, \quad y^{-1} x_2^{[p]} y = \lambda^{\frac{p-1}{2}} x_2,$$

$\eta(x_1) = \eta(x_2) = \lambda$, $\eta(y) = \lambda^{p-1}$, and also

$$x_3^\tau = x_3, \quad y^\tau = y, \quad y^{-1} x_3^{[p]} y = \lambda^{p-1} x_3, \quad \eta(x_3) = \lambda^2.$$

We take $\mu = \lambda$, $\nu = \lambda^{p-1}$; that is,

$$a = a(\lambda) = \begin{pmatrix} 0 & \lambda & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\lambda & 0 \end{pmatrix} (= x_1)\ , \ \eta(a) = \lambda,$$

$$b = b(\lambda, \lambda^{p-1}) = \begin{pmatrix} \lambda^{\frac{1}{2}(p-1)} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda^{\frac{1}{2}(p-1)} & 0 \\ 0 & 0 & 0 & \lambda^{p-1} \end{pmatrix}, \ \eta(b) = \lambda^{p-1},$$

so $b^{-1} a^{[p]} b = \lambda^{\frac{1}{2}(p-1)} a$. We put

$$A_1 = \underbrace{a \oplus \cdots \oplus a}_{m-1} \oplus x_1, A_2 = A_1^\tau = \underbrace{a \oplus \cdots \oplus a}_{m-1} \oplus x_2, B = \underbrace{b \oplus \cdots \oplus b}_{m-1} \oplus y.$$

Then $A_1, A_2, B \in CO_{2n}(q)^\circ$, $\eta(A_1) = \eta(A_2) = \lambda$, $\eta(B) = \lambda^{p-1}$ and

$$A_1 A_2 = A_2 A_1, \quad B^{-1} A_1^{[p]} B = \lambda^{\frac{1}{2}(p-1)} A_1, \quad B^{-1} A_2^{[p]} B = \lambda^{\frac{1}{2}(p-1)} A_2.$$

If, moreover, $A_3 = A_1 A_2$, then

$$A_3^\tau = A_3, \quad B^\tau = B, \quad B^{-1} A_3^{[p]} B = \lambda^{p-1} A_3,$$

and $\eta(A_3) = \lambda^2$. Also, $(\tau A_1)^2 = A_1^\tau A_1 = A_2 A_1 = A_3$.

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For $\gamma = \lambda^{\frac{1}{2}(p-1)}$, we have

$$(\alpha_i(B))_{i=1,\dots,n} = (\underbrace{\gamma, \gamma^{-1}, \dots, \gamma, \gamma^{-1}}_{n-4}, \gamma, \gamma^{-2}, \gamma, \gamma),$$

and

$$c_{n-1}(B) = c_n(B) = \gamma^{2-m} = \lambda^{\frac{1}{2}(p-1)(2-m)},$$

so $B$ induces $\delta_3$ if $m$ is odd and $p \equiv -1 \bmod 4$, and the identity otherwise.

For $\mu \in \mathbb{F}_q^\times$, let

$$h(\mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mu^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \eta(h(\mu)) = \mu.$$

If $H(\mu) = \underbrace{h(\mu) \oplus \cdots \oplus h(\mu)}_{m}$ in $CO_{2n}(q)^\circ$, then

$$(\alpha_i(H(\mu)))_{i=1,\dots,n} = (\underbrace{\mu^{-1}, \mu, \dots, \mu^{-1}, \mu}_{n-2}, \mu^{-1}, 1),$$

$$c_{n-1}(H(\mu)) = \mu^{m-2}, \quad c_n(H(\mu)) = \mu^{m-1}.$$

Note that $A_1$ induces the same diagonal automorphism in $\mathrm{Out}(G_0)$ as $H(\lambda)$ since $A_1 H(\lambda)^{-1} \in N$. Therefore, $A_1$ induces $\delta_1$ if $m$ is odd, and $\delta_2$ if $m$ is even. Hence, $A_2$ induces $\delta_2$ if $m$ is odd, $\delta_1$ if $m$ is even. It follows that $A_3$ induces $\delta_3$.

**Case 1**: $T = \langle \delta_1, \delta_2 \rangle \times \langle \phi \rangle$. In the $D_2(q)$ case, we took

$$\tilde{T} = \langle x_1, x_2, \phi y \rangle Z(CO_4(q)^\circ)/Z(CO_4(q)^\circ).$$

Then

$$\tilde{T} = \langle A_1, A_2, \phi B \rangle Z(CO_{2n}(q)^\circ)/Z(CO_{2n}(q)^\circ)$$

is a $T$-abelian supplement in $\mathrm{Aut}(G_0)$.

**Case 2**: $T = \langle \delta_3 \rangle \times \langle \phi \rangle \times \langle \tau \rangle$. In the $D_2(q)$ case, we took $\tilde{T} = \langle x_3, \phi y, \tau \rangle Z/Z$. Then

$$\tilde{T} = \langle A_3, \phi B, \tau \rangle Z/Z$$

is a $T$-abelian supplement in $\mathrm{Aut}(G_0)$.

**Case 3**: $T = \langle \tau\delta_1, \phi \rangle$. In the $D_2(q)$ case we took $\tilde{T} = \langle \tau x_1, \phi y \rangle Z/Z$. Then

$$\tilde{T} = \langle \tau A_1, \phi B \rangle Z/Z$$

is a $T$-abelian supplement in $\mathrm{Aut}(G_0)$.

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We finally deal with the last case.

**Case 4**: $T = \langle \phi, \rho\delta_2 \rangle$, only for $D_4(q)$. We have defined the matrices $A_1, B$ in $CO_{2n}(q)^\circ$: in the case $n = 4$, they are

$$A_1 = \begin{pmatrix} 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda & 0 \end{pmatrix}, \; B = \begin{pmatrix} \lambda^{\frac{p-1}{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda^{p-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda^{\frac{p-1}{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda^{\frac{p-1}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda^{p-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^{\frac{p-1}{2}} \end{pmatrix}.$$

We have

$$A_1 = n_1 n_3 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\lambda} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\lambda} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix} = n_1 n_3 H(\lambda).$$

In $P(CO_8(q)^\circ) = G_0\hat{H}$, we obtain the elements

$$A_1 \mapsto n_1 n_3 h(\xi_1) \in G_0\hat{H}\ , \ B \mapsto h(\xi) \in \hat{H},$$

where $\xi_1$ is the $\mathbb{F}_q$-character of $Q$

$$\alpha_1 \mapsto \lambda^{-1}, \alpha_2 \mapsto \lambda, \alpha_3 \mapsto \lambda^{-1}, \alpha_4 \mapsto 1.$$

In particular, $c_3 \mapsto 1$, $c_4 \mapsto \lambda$ so that $n_1 n_3 h(\xi_1)$ induces $\delta_2$ in $\operatorname{Out} G_0$, while $\xi$ is the $\mathbb{F}_q$-character of $Q$

$$\alpha_1 \mapsto \lambda^{\frac{p-1}{2}}, \alpha_2 \mapsto \lambda^{1-p}, \alpha_3 \mapsto \lambda^{\frac{p-1}{2}}, \alpha_4 \mapsto \lambda^{\frac{p-1}{2}}.$$

In particular, $c_3 \mapsto 1$, $c_4 \mapsto 1$, so $\xi$ can be extended to a character of $P$; hence, $h(\xi) \in H$. From $B^{-1} A_1^{[p]} B = \lambda^{\frac{1}{2}(p-1)} A_1$, we get $[\phi h(\xi), n_1 n_3 h(\xi_1)] = 1$. Moreover, $h(\xi)^\rho = h(\xi)$; hence,

$$\tilde{T} = \langle \phi h(\xi), \rho n_1 n_3 h(\xi_1) \rangle$$

is a $T$-abelian supplement in $\operatorname{Aut}(G_0)$.

We have proved the following.

**Theorem 26.** *Let $G$ be an almost simple group with socle $G_0 = D_n(q)$, $n$ even. If $G/G_0$ is abelian, then there exists an abelian subgroup $A$ such that $G = AG_0$.*

This completes the proof of Theorem 1.

## 12. Proof of Corollary 2

In the following, we will denote by $F(G)$ and $F^*(G)$, respectively, the Fitting subgroup and the generalized Fitting subgroup of $G$.

*Proof of Corollary 2.* Notice that $F(G) = 1$ implies $N = \operatorname{soc}(G) = F^*(G)$. Let $H = \langle a, b, N \rangle$. If $M$ is a minimal normal subgroup of $H$, then either $M \leqslant N$ or $M \cap N = 1$. However, in the second case, we would have $M \leqslant C_G(N) = C_G(F^*(G)) = Z(F^*(G)) = 1$, a contradiction. This implies $N = \operatorname{soc}(H) = F^*(H)$, and therefore, it is not restrictive to assume $G = \langle a, b, N \rangle$.

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We decompose $N = N_1 \times \cdots \times N_t$ as a product of minimal normal subgroups of $G$. For $1 \leqslant i \leqslant t$, we denote by $\xi_i : G \to \mathrm{Aut}(N_i)$ the map induced by the conjugation action of $G$ on $N_i$. The map $\xi : G \to \prod_{1\leqslant i\leqslant t} \mathrm{Aut}(N_i)$ which sends $g$ to $(g^{\xi_1}, \ldots, g^{\xi_t})$, is an injective homomorphism since $\ker \xi = \bigcap_{1\leqslant i\leqslant t} C_G(N_i) = C_G(N) = 1$. If $t \neq 1$, then by induction, there exist $n_i, m_i \in N_i$ such that $[(an_i)^{\xi_i}, (bm_i)^{\xi_i}] = 1$. But then, setting $n = (n_1, \ldots, n_t)$ and $m = (m_1, \ldots, m_t)$, we have that $[(an)^{\xi}, (bm)^{\xi}] = 1$, and consequently, since $\xi$ is injective, $[an, bm] = 1$.

Hence, it is not restrictive to assume that $N$ is a minimal normal subgroup of $G = \langle a, b, N\rangle$. Write $N = S_1\times\cdots\times S_u$, where $S_1, \ldots, S_u$ are isomorphic non-abelian simple groups, and let $X = N_G(S_1)/C_G(S_1)$. We may identify $G$ with a subgroup of $X \wr \mathrm{Sym}(u)$, so $a = x\sigma$, $b = y\tau$, with $x, y \in X^u$ and $\langle \sigma, \tau\rangle$ is an abelian regular subgroup of $\mathrm{Sym}(u)$. Notice that

$$\frac{X}{S_1} \cong \frac{N_G(S_1)/C_G(S_1)}{S_1C_G(S_1)/C_G(S_1)} \cong \frac{N_G(S_1)}{S_1C_G(S_1)}.$$

Since $S_1C_G(S_1) \geqslant N$, it follows that $X/S_1$ is isomorphic to a section of $G/N$. Since $G/N$ is an abelian group, $X/S_1$ is abelian, and therefore by Theorem 1, there exists an abelian subgroup $Y$ of $X$ such that $X = YS_1$. Then it is not restrictive to assume $\langle a, b\rangle \leqslant Y \wr \langle \sigma, \tau\rangle$. Let $K = \langle a, b\rangle$ and $Z = Y \cap S_1$. The group $KZ^u/Z^u$ is abelian, and we have reduced our problem to finding $n, m \in Z^u$ such that $\langle xn\sigma, ym\tau\rangle$ is abelian. We have

$$\begin{aligned}[][xn\sigma, ym\tau] &= [xn\sigma, \tau][xn\sigma, ym]^{\tau} = [xn, \tau]^{\sigma}[\sigma, \tau][xn, ym]^{\sigma\tau}[\sigma, ym]^{\tau} \\ &= [xn, \tau]^{\sigma}[\sigma, ym]^{\tau} = [x, \tau]^{\sigma}[\sigma, y]^{\tau}[n, \tau]^{\sigma}[\sigma, m]^{\tau}.\end{aligned}$$

Since $[n, \tau]^{\sigma}[\sigma, m]^{\tau} = [n^{\sigma}, \tau][\sigma, m^{\tau}]$, we are looking for $n, m \in Z^u$ such that

$$[x, \tau]^{\sigma}[\sigma, y]^{\tau} = [x\sigma, y\tau] = [\tau, n^{\sigma}][m^{\tau}, \sigma].$$

Notice that $[x\sigma, y\tau] = (z_1, \ldots, z_u) \in Z^u$, with $z_1z_2\cdots z_u = 1$. Let

$$\Lambda := \{(z_1, \ldots, z_u) \in Z^u \mid z_1z_2\cdots z_u = 1\}.$$

In order to conclude our proof, it suffices to prove that for every $(z_1, \ldots, z_u) \in \Lambda$, there exist $\tilde{n}, \tilde{m} \in Z^u$ such that $(z_1, \ldots, z_u) = [\tau, \tilde{n}][\tilde{m}, \sigma]$.

Since $\langle \sigma, \tau\rangle$ is a regular subgroup of $\mathrm{Sym}(u)$, $\sigma = \sigma_1\cdots\sigma_r$ is the product of $r$ disjoint cycles of the same length $s$, with $rs = u$. First, assume $r = 1$. In that case, for every $\lambda \in \Lambda$, there exists $\tilde{m} \in Z^u$ such that $[\tilde{m}, \sigma] = \lambda$, and our conclusion follows by taking $\tilde{n} = 1$. Finally, assume $r \neq 1$. In this case, $\tau = \tau_1\cdots\tau_w$ is the product of $w$ disjoint cycles of the same length, and $\tau$ must permute cyclically the orbits $\Sigma_1, \ldots, \Sigma_r$ of $\sigma$. It is not restrictive to assume that $i \in \Sigma_i$ for $1 \leqslant i \leqslant r$ and that $\tau_1(j) = j + 1$ for $1 \leqslant j \leqslant r-1$. Notice that $[Z^u, \sigma]$ consists of the elements $(k_1, \ldots, k_u) \in Z^u$ with the property that, for any $1 \leqslant i \leqslant r$, $\prod_{\omega\in\Sigma_i} k_\omega = 1$. Given $\lambda \in \Lambda$, we may choose $\tilde{m}$ so that $\lambda[\tilde{m}, \sigma]^{-1} = (v_1, \ldots, v_u) \in Z^u$ with $v_1\cdots v_r = 1$ and $v_j = 1$ if $j > r$. But then we may find $\tilde{n} = (w_1, \ldots, w_r, 1, \ldots, 1)$ so that $[\tau, \tilde{n}] = [\tau_1, \tilde{n}] = (v_1, \ldots, v_u)$, and therefore, $\lambda = [\tau, \tilde{n}][\tilde{m}, \sigma]$. □

**Competing interest.** The authors have no competing interest to declare.

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