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A NON-LINEAR DIFFERENCE EQUATION WITH TWO PARAMETERS. II

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Abstract

The paper discusses solutions of period 4 for the difference equation

$$y_{n+1} = 2k/\{1 + (y_n - m)^2\},$$

where k and m are real parameters, with $k > 0$. For given values of k and m there are at most three solutions with period 4 and equations are set up to determine the elements of these solutions and the stability of each solution. Only real solutions are considered. The procedure that is used to find these solutions allows unstable solutions to be identified as well as stable solutions.

In a previous paper, solutions of period 2 and period 3 were examined for this equation and there was evidence of anomalous behaviour in the way the stability intervals occurred. Some preliminary information about solutions of period 4 was mentioned in the discussion. The present paper provides more complete results, which confirm the anomalous behaviour and give a better idea of how the stability criterion changes for different families of solutions. These results are used to indicate the variety of behaviour that can be found for one-parameter systems by imposing suitable conditions on m and k.

1. Introduction

In a previous paper [2], which will be cited as Paper I, the motivation for studying the equation

$$y_{n+1} = F(y_n) = 2k/\{1 + (y_n - m)^2\} \tag{1.1}$$

was set out. It arose from an iteration formula for solving a cubic equation [4], and equation (1.1), with m real and $k > 0$, was seen to be a standardised form of the general problem. From that point of view, the main concern was to ensure

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that the iteration procedure converged quickly, and an adequate method was developed in Section 7 of Paper I. At the same time it was of interest to have information about periodic solutions of this two-parameter difference equation, since most of the information for problems of this kind is for one-parameter equations [3], although more recently two-parameter equations have attracted attention in the literature [1, 5, 6].

Solutions with minimum period 2 ($C2$ solutions) and with minimum period 3 ($C3$ solutions) were studied in Paper I, and Table 2 of that paper gives critical values of k for a number of values of m. For a given value of m, $C2$ solutions occur for $k > k^*$, where

$$k^* = (1 + m^2)\{-m + \surd(1 + m^2)\}, \tag{1.2}$$

and there is a single $C2$ solution for each $k > k^*$. For $m < 1$, all the $C2$ solutions are stable, but for $m > 1$ the $C2$ solutions are stable for $k^* < k < k_5$, unstable for $k_5 < k < k_6$, and stable again for $k > k_6$, where

$$k_5 = (1 + m^2)\{m - \surd(m^2 - 1)\}, \qquad k_6 = (1 + m^2)\{m + \surd(m^2 - 1)\}. \tag{1.3}$$

When $k = k_5$ or $k = k_6$, the solution is on the border-line for stability, with the stability criterion, S_2, equal to -1. When $m = 1$, almost all the $C2$ solutions are stable. In this case, $k_5 = k_6 = 2$ and there is a $C2$ solution with $S_2 = -1$ when $k = 2$. For all other admissible values of k, the $C2$ solution has $-1 < S_2 < 1$.

It turns out that the $C4$ solutions (solutions with minimum period 4) occur for $m > 1$ and $k_5 < k < k_6$, that is, in the region where the $C2$ solutions are unstable. For m slightly greater than 1 there is only one $C4$ solution for each admissible k and these solutions are stable. This is the situation up to about $m_1 = 1.2$. (More precise values for m_1 and other critical values are given later.) The next critical value, m_2, is roughly 2.31 and for $m_1 < m < m_2$, there is a single $C4$ solution for each k, with two sub-intervals for k in which the solutions are stable. One of these has k_5 as its lower limit while the other has k_6 as its upper limit. For $m > m_2$, there are three $C4$ solutions for some values of k, and it is possible to have two more sub-intervals for k where stable $C4$ solutions occur. In Paper I it was noted that for $m = 3$, superstable $C4$ solutions could be identified for four different values of k, and in fact each of these superstable solutions occurs in a separate "window of stability" whose extent is determined in the course of this paper. This type of behaviour occurs for $m > m_3$, where m_3 is approximately 2.33. For $m_2 < m < m_3$, there is an intermediate situation where stable $C4$ solutions occur in three discrete k-intervals. Essentially these sub-divisions arise because the stability criterion for the relevant solutions goes from $+1$ to a minimum value, and then back to $+1$ (as k increases, for a given value of m). If the minimum is less than -1, two intervals of stability occur; otherwise there is a single interval of stability. Note that counting superstable solutions can be misleading. The number of superstable solutions in a stability interval can vary from 0 to 2, depending on the level of the minimum.

If we take b_1, b_2, b_3, b_4 to be the elements of a $C4$ solution, then it is a matter of choice which element we label as b_1 and it is better initially to use a symmetric function of the b_i such as the sum of the four elements, α. Section 2 sets up various equations involving α and other symmetrical functions and in Section 3 these equations are combined to give a cubic equation for α (for given values of m and k). Once α has been determined, other symmetrical functions follow and we can evaluate the elements of the solution and the stability criterion (Section 4). Some details of the numerical calculations are given in Section 5. It was found that the usual method of solution broke down for certain values of m and k, where the equation for α has a double root, and this situation is considered in Section 6, together with some other special cases.

In Section 7, the results are re-examined in a different way. In discussing equation (1.1) we need information about the solutions for a two-dimensional array of points and this information was built up by seeing how the solutions changed along lines m = constant. In effect, this gives solutions for a family of one-parameter problems, with k as the parameter for each particular value of m. However we can also look at other families of one-parameter problems by, say, specifying k as a function of m. Some examples are given in Section 7 and this brings out a greater variety of behaviour then before. It is clear that the "period-doubling path to chaos" is only one pattern out of a number of possibilities for one-parameter problems.

2. Preliminary equations

If equation (1.1) is iterated four times, to obtain y_{n+4} as a function of y_n, equating y_{n+4} to y_n gives a polynomial equation of degree 17. The roots of this equation include the elements of the $C4$ solutions but they also include the $C2$ solution and the equilibrium solutions, since $y_{n+4} = y_n$ holds for these degenerate cases also. The degenerate cases account for a factor of degree 5 in the polynomial equation and the remaining factor, of degree 12, must provide the elements of the $C4$ solutions. If (b_1, b_2, b_3, b_4) is a typical $C4$ solution, then

$$h(y_n) = (y_n - b_1)(y_n - b_2)(y_n + b_3)(y_n - b_4) \tag{2.1}$$

contributes a factor of degree 4 to the polynomial equation. We see that at most there can be three factors of this type in the polynomial of degree 12, so at most there should be three $C4$ solutions for each pair (m, k).

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For a $C4$ solution the elements b_i must be positive and distinct, with

$$2k = (1 + m^2)b_2 - 2mb_1b_2 + b_1^2b_2 \tag{2.2}$$

and with three similar equations obtained by cyclic permutation of the subscripts. Since it does not matter which element we take as b_1, it is convenient to work with functions of the b_i which have cyclic symmetry, for example

$$\sum b_1^2b_2 = b_1^2b_2 + b_2^2b_3 + b_3^2b_4 + b_4^2b_1,$$

where Σ is used for cyclic summation over the subscripts 1, 2, 3 and 4. In particular we can use

$$\alpha = \sum b_1, \quad \beta_1 = \sum b_1b_2, \quad \beta_2 = b_1b_3 + b_2b_4,$$
$$\gamma = \sum b_1b_2b_3, \quad \delta = b_1b_2b_3b_4.$$

With this notation,

$$h(y) = y^4 - \alpha y^3 + \beta y^2 - \gamma y + \delta, \tag{2.3}$$

where $\beta = \beta_1 + \beta_2$, and if we can determine α, β_1, β_2, γ, δ the elements b_i are the roots of $h(y) = 0$. Our procedure is to express γ and δ in terms of α, β_1 and β_2, then β_1 and β_2 in terms of α and finally to establish an equation for α (given m and k). As noted in Section 1, the equation for α turns out to be a cubic, in line with our expectation that there are at most three $C4$ solutions for any pair (m, k).

A number of equations can be obtained directly from equation (2.2) and the three similar equations. Adding the equations gives

$$(1 + m^2)\alpha - 2m\beta_1 + \sum b_1^2b_2 = 8k. \tag{2.4}$$

Multiplying equation (2.2) by b_3 and summing cyclically gives

$$(1 + m^2)\beta_1 - 2m\gamma + \sum b_1^2b_2b_3 = 2k\alpha \tag{2.5}$$

and in the same way we can multiply equation (2.2) by b_4, b_2, b_3b_4, $b_2b_3b_4$, $b_3b_4^2$, $b_4^2 - b_2b_3$ and then sum cyclically in each case. This gives

$$2(1 + m^2)\beta_2 - 2m\gamma + \sum b_1b_2^2b_3 = 2k\alpha, \tag{2.6}$$

$$(1 + m^2)\sum b_1^2 - 2m\sum b_1b_2^2 + \sum b_1^2b_2^2 = 2k\alpha, \tag{2.7}$$

$$(1 + m^2)\gamma - 8m\delta + \alpha\delta = 2k\beta_1, \tag{2.8}$$

$$(1 + m^2)\sum b_1^2b_2b_3 - 2m\alpha\delta + \beta_1\delta = 2k\gamma, \tag{2.9}$$

$$(1 + m^2)\sum b_1b_2b_3^2 - 2m\alpha\delta + \beta_1\delta = 2k\sum b_1b_2^2, \tag{2.10}$$

$$(1 + m^2)\sum\left(b_1^2b_3 - b_2^2b_3\right) - 2m\sum\left(b_1^2b_2b_3 - b_1b_2^2b_3\right) = 2k\left(-\beta_1 + \sum b_1^2\right). \tag{2.11}$$

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We can also make use of identities such as

$$\sum b_1^2 = \alpha^2 - 2\beta, \tag{2.12}$$

$$\sum\left(b_1^2 b_2 + b_1 b_2^2\right) = \alpha\beta_1 - 2\gamma, \tag{2.13}$$

$$\sum b_1^2 b_3 = \alpha\beta_2 - \gamma = \sum b_1 b_3^2, \tag{2.14}$$

$$\sum b_1^2 b_2^2 + 2\sum b_1 b_2^2 b_3 = \beta_1^2 - 4\delta, \tag{2.15}$$

$$\sum b_1 b_2^2 b_3 = \alpha\gamma - 4\delta - \beta_1\beta_2, \tag{2.16}$$

$$\sum\left(b_1^2 b_2 b_3 + b_1 b_2 b_3^2\right) = \beta_1\beta_2, \tag{2.17}$$

$$\sum\left(b_1^2 b_2^2 b_3 + b_1 b_2^2 b_3^2\right) = \beta_1\gamma - 2\alpha\delta. \tag{2.18}$$

Another way of using (2.2) is to multiply by b_4 and to subtract the corresponding equation with the subscripts moved forward (cyclically) by two. This gives

$$2k(b_4 - b_2) = b_2 b_4 (b_3 - b_1)(2m - b_1 - b_3). \tag{2.19}$$

Similarly,

$$2k(b_1 - b_3) = b_3 b_1 (b_4 - b_2)(2m - b_2 - b_4), \tag{2.20}$$

and we can multiply the two equations together, cancel out a non-zero factor $(b_1 - b_3)(b_2 - b_4)$ and end with

$$-4k^2 = \delta\left(4m^2 - 2m\alpha + \beta_1\right). \tag{2.21}$$

3. Combination of equations in Section 2

In equation (2.11), we can substitute for $\Sigma b_1^2 b_3$, $\Sigma b_2^2 b_3$, $\Sigma b_1^2 b_2 b_3$, $\Sigma b_1 b_2^2 b_3$ and Σb_1^2 from equations (2.14), (2.4), (2.5), (2.6) and (2.12). This gives

$$\gamma(1 + m^2) = 6k\beta_1 + \{(\alpha - 4m)(1 + m^2) + 4k\}\beta_2 + \alpha(1 + m^2)^2 - 8k(1 + m^2) - 2k\alpha^2, \tag{3.1}$$

or if we put $2k = p(1 + m^2)$

$$\gamma = 3p\beta_1 + (\alpha - 4m + 2p)\beta_2 + (\alpha - 4p)(1 + m^2) - p\alpha^2. \tag{3.2}$$

In the same way, we can eliminate Σb_1^2, $\Sigma b_1^2 b_2$, $\Sigma b_1 b_2^2$, $\Sigma b_1^2 b_2^2$ and $\Sigma b_1 b_2^2 b_3$ between equations (2.4), (2.6), (2.7), (2.12), (2.13) and (2.15). This gives

$$\beta_1^2 = 4\delta + (2 + 2m\alpha - 2m^2)\beta_1 - (2 + 2m^2)\beta_2 + (1 + m^2)\{p(3\alpha - 8m) + 2m\alpha - \alpha^2\}. \tag{3.3}$$

A similar equation, obtained by eliminating $\Sigma b_1 b_2^2 b_3$ between equations (2.6) and (2.16), is

$$\beta_1\beta_2 = (\alpha - 2m)\gamma - 4\delta + (1 + m^2)(2\beta_2 - p\alpha). \tag{3.4}$$

The summations on the left-hand side of equation (2.18) do not occur in any of the other equations, but we can write

$$\sum b_1^2 b_2^2 b_3 = \sum\{(b_2 b_3)(b_1^2 b_2)\} = \sum b_2 b_3\{2k - (1+m^2)b_2 + 2m b_1 b_2\}$$
$$= 2k\beta_1 - (1+m^2)\sum b_1^2 b_2 + 2m \sum b_1 b_2^2 b_3$$

and the summations on the right-hand side are available from equations (2.4) and (2.6). In the same way,

$$\sum b_1 b_2^2 b_3^2 = \sum\{(b_1 b_3)(b_2^2 b_3)\} = \sum b_1 b_3\{2k - (1+m^2)b_3 + 2m b_2 b_3\}$$
$$= 4k\beta_2 - (1+m^2)\sum b_1 b_3^2 + 2m \sum b_1 b_2 b_3^2.$$

Now $\Sigma b_1 b_3^2$ can be replaced from equation (2.14) and we can combine equations (2.17), (2.5) and (3.4) to give

$$\sum b_1 b_2 b_3^2 = (\alpha - 4m)\gamma - 4\delta + (1+m^2)(\beta_1 + 2\beta_2 - 2p\alpha). \tag{3.5}$$

With this information available for the two summations in equation (2.18) we can make use of this equation and obtain

$$(4m^2 - 2m\alpha + \beta_1)\gamma$$
$$= \alpha\delta + (1+m^2)\{2p\beta_1 + (2p-\alpha)\beta_2 - 2mp\alpha\} + (\alpha - 4p)(1+m^2)^2. \tag{3.6}$$

This is after using equation (2.8) to simplify the terms involving δ.

If we compare equations (2.21) and (3.6), we note that the co-factor of δ in the first of these equations is the same as the co-factor of γ in the second. This suggests that we combine them, using equation (2.8) in the process. From equation (2.8),

$$\begin{aligned}(4m^2 - 2m\alpha + \beta_1)&\{(1+m^2)\gamma + \delta(\alpha - 8m)\}\\ &= (4m^2 - 2m\alpha + \beta_1)\{p(1+m^2)\beta_1\}\\ &= (1+m^2)\{p(4m^2 - 2m\alpha)\beta_1 + p\beta_1^2\}.\end{aligned} \tag{3.7}$$

If we use equations (3.6) and (2.21) on the left-hand side and equation (3.3) to replace β_1^2 on the right-hand side, the resulting equation is

$$(\alpha - 4p)(1+m^2)\{\delta - (1+m^2)(\beta_2 - 1 - m^2 - p\alpha + 4mp)\} = 0. \tag{3.8}$$

Hence, provided $\alpha \neq 4p$, we come out with

$$\delta = (1+m^2)\{\beta_2 - 1 - m^2 - p(\alpha - 4m)\}. \tag{3.9}$$

Strictly we should add another proviso, that $\alpha \neq 8m$, since we multiplied equation (2.21) by $\alpha - 8m$ in the argument above.

If we substitute for γ and δ in equation (2.8), using equations (3.2) and (3.9), we can cancel a common factor $2 + 2m^2$ and get

$$p\beta_1 + (\alpha + p - 6m)\beta_2 = p(\alpha^2 - 6m\alpha + 2 + 18m^2) - 4m(1+m^2). \tag{3.10}$$

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Similarly, equations (3.3) and (3.4) can be rewritten as

$$\beta_1^2 = (2 + 2m\alpha - 2m^2)\beta_1 + 2(1 + m^2)\beta_2 + (1 + m^2)\{p(8m - \alpha) + 2m\alpha - \alpha^2\} - 4(1 + m^2)^2, \quad (3.11)$$

$$\beta_1\beta_2 = (3\alpha - 6m)p\beta_1 + \{(2\alpha - 4m)p + \alpha^2 - 6m\alpha + 6m^2 - 2\}\beta_2 + p\{-\alpha^3 + 2m\alpha^2 - (\alpha + 8m)(1 + m^2)\} + (1 + m^2)(\alpha^2 - 2m\alpha + 4 + 4m^2). \quad (3.12)$$

Equations (3.10), (3.11) and (3.12) provide three relationships between β_1 and β_2, so in theory it should be possible to eliminate them and get an equation for α. In practice, it is more convenient to look for further linear relationships between β_1 and β_2.

If we insert δ from equation (3.9) into equation (2.21), the only non-linear term in β_1 and β_2 is $\beta_1\beta_2$ and this can be replaced by linear terms from equation (3.12). This leads to a linear equation

$$\{p(2m - 2\alpha) + 1 + m^2\}\beta_1 + \{p(4m - 2\alpha) - \alpha^2 + 8m\alpha - 10m^2 + 2\}\beta_2 = (1 + m^2)(p^2 + \alpha^2 + 4) + p\{-\alpha^3 + 4m\alpha^2 - (1 + 13m^2)\alpha - 8m + 8m^3\}. \quad (3.13)$$

We can obtain a slightly neater alternative if we use equation (3.10) to replace $p\beta_1(2m - 2\alpha)$. This leaves

$$(1 + m^2)\beta_1 + (2mp + \alpha^2 - 6m\alpha + 2m^2 + 2)\beta_2 = (1 + m^2)(p^2 + \alpha^2 + 4 + 8m^2 - 8m\alpha) + p\{\alpha^3 - 10m\alpha^2 + (3 + 35m^2)\alpha - 12m - 28m^3\}. \quad (3.14)$$

There are several ways of obtaining a third linear relationship between β_1 and β_2. Subtracting equation (2.10) from equation (2.9) gives

$$(1 + m^2)\left\{\sum b_1^2 b_2 b_3 - \sum b_1 b_2 b_3^2\right\} = 2k\left\{\gamma - \sum b_1 b_2^2\right\}$$

and we can use equations (2.5), (3.5), (2.13) and (2.4) to replace the summations. This brings in terms in γ and δ but equation (3.2) and (3.9) allow them to be expressed in terms of β_1 and β_2. The resulting linear equation is fairly cumbersome but by using equations (3.10) and (3.14) it reduces to

$$3p^2\beta_1 + \{2p^2 + (\alpha - 4m)p - \alpha^2 + 10m\alpha - 2 - 26m^2\}\beta_2 = p^2\alpha^2 + (1 + m^2)(2p^2 - \alpha^2 + 10m\alpha - 4 - 28m^2) + p\{-\alpha^3 + 14m\alpha^2 - (3 + 67m^2)\alpha + 16m + 112m^3\}. \quad (3.15)$$

In general, equations (3.10), (3.14) and (3.15) are linearly independent and lead to a cubic equation for α.

If we eliminate β_1 from equations (3.10) and (3.14), we get an equation of the form

$$L_1\beta_2 = N_1, \tag{3.16}$$

where

$$L_1 = 2mp^2 + p(\alpha^2 - 6m\alpha + 1 + m^2) - (\alpha - 6m)(1 + m^2), \tag{3.17}$$

$$\begin{aligned} N_1 = {} & p^3(1 + m^2) + p^2\{\alpha^3 - 10m\alpha^2 + (3 + 35m^2)\alpha - 12m - 28m^3\} \\ & + p(1 + m^2)(2 - 10m^2 - 2m\alpha) + 4m(1 + m^2)^2. \end{aligned} \tag{3.18}$$

In the same way, if we eliminate β_1 from equations (3.10) and (3.15) we get

$$L_2\beta_2 = N_2, \tag{3.19}$$

where

$$L_2 = p^2 + p(2\alpha - 14m) + (\alpha^2 - 10m\alpha + 2 + 26m^2), \tag{3.20}$$

$$\begin{aligned} N_2 = {} & p^2(2\alpha^2 - 18m\alpha + 4 + 52m^2) \\ & + p\{\alpha^3 - 14m\alpha^2 + (3 + 67m^2)\alpha - 28 - 124m^3\} \\ & + (1 + m^2)(\alpha^2 - 10m\alpha + 4 + 28m^2). \end{aligned} \tag{3.21}$$

For equations (3.16) and (3.19) to be consistent, the condition is that

$$G(\alpha, m, p) \equiv L_2N_1 - L_1N_2 = 0. \tag{3.22}$$

From the expressions for L_1, L_2, N_1 and N_2,

$$G(\alpha, m, p) = g_0\alpha^3 + g_1\alpha^2 + g_2\alpha + g_3, \tag{3.23}$$

with

$$\begin{aligned} g_0 & = p^4 - 6mp^3 + (3 + 11m^2)p^2 - 6mp(1 + m^2) + (1 + m^2)^2 \\ & = (p^2 - 3mp + 1 + m^2)^2 + p^2, \end{aligned} \tag{3.24}$$

$$\begin{aligned} g_1 = {} & -14mp^4 + (1 + 77m^2)p^3 - (38m + 134m^3)p^2 \\ & + 72m^2(1 + m^2)p - 12m(1 + m^2)^2, \end{aligned} \tag{3.25}$$

$$\begin{aligned} g_2 = {} & (5 + 73m^2)p^4 - (42m + 362m^3)p^3 + (11 + 226m^2 + 567m^4)p^2 \\ & - m(1 + m^2)(28 + 292m^2)p + (4 + 48m^2)(1 + m^2)^2, \end{aligned} \tag{3.26}$$

$$\begin{aligned} g_3 = {} & p^5(1 + m^2) - mp^4(34 + 146m^2) + m^2p^3(188 + 604m^2) \\ & - mp^2(52 + 496m^2 + 828m^4) + m^2p(1 + m^2)(112 + 400m^2) \\ & - 16m(1 + 4m^2)(1 + m^2)^2. \end{aligned} \tag{3.27}$$

From equation (3.24), g_0 is positive for $p > 0$ and indeed it is straightforward to show that g_0 has a minimum value $16/25$. Hence $G(\alpha, m, p) = 0$ gives a cubic equation for α for any pair (m, p). Also, for $m \leqslant 0$ and $p > 0$, g_0, g_1, g_2 and g_3

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are all positive and in this case $G = 0$ cannot have a positive root. This rules out a $C4$ solution since we need $b_i > 0$ for a periodic solution. (From equation (1.1), we have $0 < y_1 \leqslant 2k$ for all values of y_0 and indeed $0 < y_n \leqslant 2k$ for $n \geqslant 1$. This means that in looking for equilibrium solutions or periodic solutions, we can take $0 \leqslant y_0 \leqslant 2k$ and regard the iteration as a mapping of an interval into itself.)

4. Stability criterion and solution for cyclic elements

For given values of m and k, we have $p = 2k/(1 + m^2)$ and g_0, g_1, g_2, g_3 can be calculated. The only problem is that we may be losing some accuracy by taking the difference between large terms. For example, when $m = 3$

$$g_2 = 662p^4 - 9900p^3 + 47972p^2 - 79680p + 43600$$

and we want to use values of p between 0 and 6. For some of the calculations a change of origin was made to alleviate this problem. When g_0 to g_3 have been found, the next step is to solve the cubic equation for α and this was done by using the iteration procedure developed in Section 7 of Paper I. For each real positive α, equation (3.16) or equation (3.19) gives β_2 (provided L_1 and L_2 are not too close to zero) and equation (3.14) then gives β_1. Equations (3.2) and (3.9) give γ and δ, while β is simply $\beta_1 + \beta_2$. Thus we can form $h(y)$, as given by equation (2.3), and solve $h(y) = 0$ to obtain b_1, b_2, b_3 and b_4. Some comments on this are given below. It was also convenient to evaluate the stability criterion, S, for each $C4$ solution after α, β, γ and δ had been determined. The stability criterion is

$$S = F'(b_1)F'(b_2)F'(b_3)F'(b_4), \tag{4.1}$$

and S must be between -1 and $+1$ for local stability. From equations (1.1) and (2.2)

$$F'(b_1) = \frac{-4k(b_1 - m)}{\{1 + (b_1 - m)^2\}^2} = \frac{4k(m - b_1)}{(2k/b_2)^2} = \frac{b_2^2(m - b_1)}{k}. \tag{4.2}$$

It follows that

$$\begin{aligned} S &= (b_1 b_2 b_3 b_4)^2 (m - b_1)(m - b_2)(m - b_3)(m - b_4)/k^4 \\ &= (\delta^2/k^4)h(m). \end{aligned} \tag{4.3}$$

Thus S can be evaluated without knowing b_1, b_2, b_3, b_4 explicitly and indeed S can be found and used even for cases where the solution for the b_i is complex. (This occurs when the transition from $S > 1$ to $S < 1$ marks the transition from complex to real solutions.)

In solving for the cyclic elements, we note from equations (2.19) and (2.20) that $2m - b_1 - b_3$ and $2m - b_2 - b_4$ must have opposite signs. (From equation (2.21) their product, $4m^2 - 2m\alpha + \beta_1$, is non-zero and negative.) If we take $b_1 + b_3 > 2m$, then we must have

$$b_1 + b_3 > 2m > b_2 + b_4. \tag{4.4}$$

Now $b_1 + b_3$ and $b_2 + b_4$ are the roots of the quadratic

$$0 = (X - b_1 - b_3)(X - b_2 - b_4) = X^2 - \alpha X + \beta_1, \tag{4.5}$$

and for real, distinct roots we must have $\alpha^2 > 4\beta_1$. If this condition is satisfied we can take $b_1 + b_3$ as the larger root of equation (4.5), with $b_2 + b_4$ as the smaller root. This is simply a matter of how we label the elements and in the same way we can assume that b_1 is the larger of the pair b_1 and b_3. From equation (2.20), $b_1 > b_3$ then implies that $b_4 > b_2$. If we write

$$X_1 = b_1 + b_3, \quad X_2 = b_2 + b_4, \quad Y_1 = b_1 b_3, \quad Y_2 = b_2 b_4, \tag{4.6}$$

then a knowledge of X_1 and Y_1 is enough to give b_1 and b_3, and a knowledge of X_2 and Y_2 is enough to give b_2 and b_4. We can say that Y_1 and Y_2 are the roots of the quadratic

$$0 = (Y - b_1 b_3)(Y - b_2 b_4) = Y^2 - \beta_2 Y + \delta \tag{4.7}$$

but unfortunately we do not know whether $b_1 b_3$ or $b_2 b_4$ is the larger root. (Again, $\beta_2^2 \geqslant 4\delta$ is a necessary condition for real roots.) However we can get around this difficulty by using equations (2.19) and (2.20) to obtain

$$\begin{aligned} 2kb_1b_3(b_4 - b_2)^2(2m - b_2 - b_4) \\ = 2kb_2b_4(b_1 - b_3)^2(b_1 + b_3 - 2m). \end{aligned} \tag{4.8}$$

This gives

$$Y_1\{(2m - X_2)X_2^2\} - Y_2\{X_1 - 2m)X_1^2\} = 4\delta(4m - \alpha). \tag{4.9}$$

Also, from equation (2.14),

$$\begin{aligned} \alpha\beta_2 - \gamma &= b_1b_3(b_1 + b_3) + b_2b_4(b_2 + b_4) \\ &= Y_1X_1 + Y_2X_2. \end{aligned} \tag{4.10}$$

From these two equations

$$\begin{aligned} Y_1\{X_1^3(X_1 - 2m) + X_2^3(2m - X_2)\} \\ = 4\delta(4m - \alpha)X_2 + (\alpha\beta_2 - \gamma)X_1^2(X_1 - 2m), \end{aligned} \tag{4.11}$$

with a similar equation for Y_2. The equation $Y_1 + Y_2 = \beta_2$ might have been used as an alternative to equation (4.9) but instead it was incorporated in a check, which was to evaluate $Y_1^2 - \beta_2 Y_1 + \delta$ and $Y_2^2 - \beta_2 Y_2 + \delta$ and ensure that they were close to zero. (From equation (4.7), these expressions should be exactly zero.)

To find b_1 and b_3 we have to solve the quadratic

$$0 = (Z - b_1)(Z - b_3) = Z^2 - X_1 Z + Y_1$$

and for real distinct roots we must have

$$(b_1 - b_3)^2 = X_1^2 - 4Y_1 > 0. \tag{4.12}$$

If this condition is satisfied, we can take

$$b_1 - b_3 = \surd(X_1^2 - 4Y_1), \tag{4.13}$$

and in the same way we can take

$$b_4 - b_2 = \surd(X_2^2 - 4Y_2), \tag{4.14}$$

provided $X_2^2 > 4Y_2$. Thus the solution for b_1, b_2, b_3, and b_4 can be obtained without solving $h(y) = 0$ as a quartic equation. The knowledge of β_1 and β_2 allows us to compute the b_i via linear and quadratic equations.

The procedure given above was adequate in most cases although some modification was made for a few values of m and k where special difficulties arose. These difficulties are probably easier to explain in relation to the general run of the numerical results, which are discussed in the next section.

5. Numerical results

The computational work was carried out on a Univac 1100 computer, using double precision for all variables. A number of numerical checks were included and at the end the quantities

$$e_i = b_{i+1} - F(b_i) \qquad (i = 1, 2, 3),$$
$$e_4 = b_1 - F(b_4) \tag{5.1}$$

were calculated. The e's represent deviations from an exact fit to equation (1.1) and provide an obvious check on the accuracy of the solutions for b_1, b_2, b_3, b_4. The normal practice was to evaluate β_2 from equation (3.16) and from equation (3.19), then to use the mean of the two estimates in the later calculations unless some discrepancy was apparent. It became clear that L_1 and L_2 could be zero for some values of m and k, so a lower limit of 0.8 was imposed on the modulus of these quantities. When $|L_1|$ and $|L_2|$ were both less than 0.8, β_2 was obtained from a quadratic equation

$$(6m - \alpha - p)\beta_2^2 - v_1\beta_2 + v_2 = 0, \tag{5.2}$$

where

$$v_1 = \alpha^3 - (p + 12m)\alpha^2 + (2 + 46m^2 + 6mp)\alpha - 48m^3 - 20m^2p - 4p, \tag{5.3}$$

$$v_2 = p\{\alpha^4 - 16m\alpha^3 + (2 + 94m^2)\alpha^2 - (20m + 244m^3)\alpha - 4 + 24m^2 + 220m^4\} + (1 + m^2)\{\alpha^3 - 12m\alpha^2 + (4 + 48m^2 - p^2)\alpha - 8m - 56m^3 + 12mp^2\}. \tag{5.4}$$

This quadratic comes from equation (3.12) for $\beta_1\beta_2$. If we multiply equation (3.12) by p, we can replace $p\beta_1$ on the left-hand side from equation (3.10) and on the right-hand side we can replace $3p^2\beta_1 + 2p^2\beta_2$ from equation (3.15). This eliminates β_1 and leaves the quadratic equation for β_2.

Equation (5.2) gives two values for β_2 and we can use equation (3.14) to determine the corresponding values for β_1. Each pair was then substituted in equation (3.11) and the pair which gave better agreement was taken as the appropriate pair in the subsequent calculations. (Occasionally other information was taken into account in deciding which pair to use.) Once β_1 and β_2 had been found the solution for γ, δ, S, b_i, e_i followed as before.

Although no real $C4$ solutions were expected for $m < 1$, some runs were made with m between 0 and 1 and with k between 0.25 and 3.0. In each case there was a single real value for α but the solution for the elements b_i was complex, with $S > 1$. It was clear that, for a given value of m, S had a minimum around $k = 2.0$. The minimum value decreased as m increased, with a minimum of 1.8, 1.4 and 1.1 recorded for $m = 0.9$, 0.95 and 0.99, respectively. For $m = 1.0$ and $k = 2.0$, a solution with $S = 1$ was recorded and for $m > 1$ real solutions appeared for some values of k.

As mentioned in Section 1, there is a $C2$ solution which has $S_2 = -1$ for $m = 1$ and $k = 2$. Its elements are $b_1 = 2 + \sqrt{2}$ and $b_2 = 2 - \sqrt{2}$ and we can think of it as the limiting case of a $C4$ solution which has $b_1 = b_3$, $b_2 = b_4$ and $S = (-1)^2 = 1$. The computed $C4$ solution for $m = 1$ and $k = 2$ gave values of $\alpha, \beta, \gamma, \delta$, $(b_1 - b_3)^2$ and $(b_2 - b_4)^2$ which agreed with this interpretation (to 12 decimal places), although the solution for the b_i was recorded as complex. Presumably $(b_1 - b_3)^2$ and $(b_2 - b_4)^2$ came out as very small negative numbers and the programme stopped (as instructed) when that happened.

This difficulty appeared in a slightly different form for $m > 1$ and k close to k_5 or k_6. For example, for k slightly greater than k_5 the computations gave a real solution with errors e_i that were larger than usual, rising to 10^{-6} or 10^{-5} in some instances. This appeared to be because $b_1 - b_3$ and $b_2 - b_4$ were close to zero and the calculated values of $(b_1 - b_3)^2$ and $(b_2 - b_4)^2$ had errors of about the same size as the calculated values, say about 10^{-12}. When the square root was taken to obtain $b_1 - b_3$ and $b_4 - b_2$, the square root had an error of order 10^{-6}.

For $m > 1$, real $C4$ solutions were obtained for $k_5 < k < k_6$. The cubic equation for α always has at least one real root, and for $m < m_2 = 2.305976$ there was only one real root. We can call this the α_0 root, and refer to the corresponding solutions for b_i as the α_0 family or main family of solutions. For this family, S was greater than 1 for $k < k_5$ and for $k > k_6$ and the solution for the b_i was then complex. For $k_5 < k < k_6$, S decreased rapidly from $+1$ to a minimum and then increased again more slowly, with $S \to 1$ as $k \to k_6$. For $1 < m < m_1 = 1.202517$ the minimum was greater than -1 so there was a single interval of stability (k_5, k_6). For $m > m_1$ the minimum was less than -1 and there were two intervals of stability (k_5, k_5^*) and (k_6^*, k_6), where k_5^* and k_6^* are the values of k for which $S = -1$. Some values for $k^*, k_5, k_5^*, k_6^*, k_6$ are given in Table 1 and it will be seen that the length of the (k_5, k_5^*) interval is tending to decrease as m increases, but the (k_6^*, k_6) interval is getting larger.

TABLE 1. Critical values of k (for a given m)

		Main family of $C4$ solutions				Additional $C4$ solutions			
m	k^*	k_5	k_5^*	k_6^*	k_6	k_7	k_7^*	k_8^*	k_8
1.00	0.8284	2.0000	–	–	2.0000	–	–	–	–
1.25	0.8989	1.2812	1.6125	2.7762	5.1250	–	–	–	–
1.50	0.9840	1.2414	1.3593	5.1146	8.5086	–	–	–	–
2.00	1.1803	1.3397	1.3918	11.2151	18.6603	–	–	–	–
2.50	1.3962	1.5132	1.5468	20.5265	34.7368	1.879595	1.888580	3.12443	3.44141
3.00	1.6228	1.7157	1.7409	33.9905	58.2842	1.917941	1.919330	5.27415	6.10187
3.50	1.8557	1.9331	1.9533	52.4775	90.8169	2.078314	2.078917	8.00027	9.57957
4.00	2.0928	2.1593	2.1762	76.8405	133.8407	2.274070	2.274430	11.52553	14.11337

$m_1 = 1.202517, \quad m_2 = 2.305976, \quad m_3 = 2.334496.$

For $m > m_2$, three real solutions occurred over an interval $k_7 < k < k_8$, where k_7 and k_8 were determined numerically for various values of m. One of these solutions was the α_0 solution, which was unstable in (k_7, k_8), and two additional families of solutions were recognisable. For one of these additional families, S was greater than 1 in (k_7, k_8) and we can refer to these solutions as the unstable additional family. For this family, $S \to 1$ as $k \to k_7$ and as $k \to k_8$, with S rising to a maximum at some intermediate value of k. For the other additional family, $S = 1$ at $k = k_7$, with S decreasing to a minimum at an intermediate value of k and approaching 1 again as $k \to k_8$. We can refer to this as the "stable" family of additional solutions since it has $|S| < 1$ for some values of k. For $m > m_3 = 2.334496$, the minimum value of S was less than -1 and hence there were two intervals of stability, say (k_7, k_7^*) and (k_8^*, k_8), where k_7^* and k_8^* are the values of k at which $S = -1$ for this family of solutions. For $m_2 < m < m_3$ the minimum S

was greater than −1 and the interval of stability was then (k_7, k_8). Some numerical values are given in Table 1 and for comparison the corresponding critical values for $C3$ solutions are listed in Table 2. (The notation is that $C3$ solutions occur for $K_1 \leqslant k \leqslant K_2$ and one of the solutions is stable for $K_1 < k < K_1^*$ and for $K_2^* < k < K_2$.)

At $k = k_7$ and $k = k_8$, the two additional solutions coalesce to give a single solution, with $S = 1$.

TABLE 2. Critical values of k (for a given m)
$C3$ solutions

m	K_1	K_1^*	K_2^*	K_2
$\sqrt{3}$	2.3094	–	–	2.3094
2.0	1.6667	1.7503	2.9784	5.0000
2.5	1.6850	1.7014	5.6889	10.3984
3.0	1.8350	1.8432	9.2799	18.1650
3.5	2.0256	2.0308	14.1180	28.8911
4.0	2.2352	2.2391	20.4424	43.0981

In determining the critical values for m and k, it was often useful to improve the solutions for the elements b_i by a simple but effective device. If we have a value b_1 which is an approximation to the exact value b_1^* for an element for the $C4$ solution then we can iterate equation (1.1) four times to obtain, say b_5. Now, to the first order in $b_1 - b_1^*$, the iteration should give

$$b_5 - b_1^* = S(b_1 - b_1^*), \tag{5.5}$$

and it follows that

$$(1 - S)b_1^* = b_5 - Sb_1. \tag{5.6}$$

In general, this gives an improved estimate of b_1^*, and we can use this improved estimate to re-calculate b_2, b_3, b_4, $\alpha, \beta, \gamma, \delta$ and S. The procedure can then be repeated and the solution errors e_i calculated to monitor the process. One interesting point is that the $C4$ solution need not be stable; a large value of $|S|$ becomes an advantage rather than a nuisance. Clearly, equation (5.6) does not help when $S = 1$ but in practice the equation was used with $1 - S$ of order 10^{-6}, without any difficulty arising. (Occasionally there were indications that the iterations were settling into periodic solutions of small amplitude for S.)

6. Special cases

In the derivation of equation (3.9) from equation (3.8), it was assumed that $\alpha - 4p$ was non-zero and it was noted also that a proviso $\alpha \neq 8m$ was needed. From the numerical work, solutions with $\alpha = 4p$ and solutions with $\alpha = 8m$ occur, although there was no indication that anything unusual happened in either case. To check this, it was assumed that $\alpha = 4p$ and the various equations were developed on this assumption. For $p \neq 2m$, the basic equation for α was replaced by an equation linking p and m, which proved to be the same as $G(4p, m, p) = 0$ [equation (3.22)].

In the same way, assuming $\alpha = 8m \neq 4p$ led to an equation between p and m which was simply $G(8m, m, p) = 0$. The other equations used in the solution were also unaffected.

For the special case $\alpha = 4p = 8m$, equation (2.2) is replaced by

$$(2m - b_2)(1 + m^2) = b_1^2 b_2 - 2mb_1b_2, \tag{6.1}$$

and several of the equations are simpler. For example,

$$\sum b_1^2 b_2 = \sum b_1 b_2^2 = \gamma = 2m\beta_1, \tag{6.2}$$

$$\beta_1\beta_2 = (6m^2 - 2)\beta_1 + 32m^2(1 + m^2), \tag{6.3}$$

$$2\delta = (1 + 3m^2)\beta_1 + (1 + m^2)\beta_2 - 24m^2(1 + m^2), \tag{6.4}$$

$$\begin{aligned}\beta_1^2 &= 4\delta + (2 + 14m^2)\beta_1 - (2 + 2m^2)\beta_2 - 16m^2(1 + m^2)\\ &= (4 + 20m^2)\beta_1 - 64m^2(1 + m^2).\end{aligned} \tag{6.5}$$

Equation (6.5) gives either $\beta_1 = 16m^2$ or $\beta_1 = 4 + 4m^2$.

For $\beta_1 = 16m^2$, we get $\beta_2 = 8m^2$, $\gamma = 32m^3$ and $\delta = 16m^4$ and the corresponding solution is an equilibrium solution with $b_1 = b_2 = b_3 = b_4 = 2m$. In similar fashion, $\beta_1 = 4 + 4m^2$ leads to

$$\beta_2 = 14m^2 - 2, \qquad \beta = 2 + 18m^2,$$

$$\gamma = 8m(1 + m^2), \qquad \delta = (1 + m^2)^2$$

and the equation $h(X) = 0$ for the cyclic elements becomes

$$\begin{aligned}0 &= X^4 - 8mX^3 + (2 + 18m^2)X^2 - 8m(1 + m^2)X + (1 + m^2)^2\\ &= (X^2 - 4mX + 1 + m^2)^2.\end{aligned} \tag{6.6}$$

For $m^2 > 1/3$, the solution is real but degenerate, with

$$b_1 = b_3 = 2m + \surd(3m^2 - 1), \quad b_2 = b_4 = 2m - \surd(3m^2 - 1).$$

Thus the condition $\alpha = 4p = 8m$ does not lead to a non-degenerate solution of period 4.

For the numerical work the special cases $\alpha = 4p$ and $\alpha = 8m$ were of no consequence but, as mentioned in Section 5, the computational technique had to be altered when L_1 and L_2 were close to zero. This suggested that it would be worth examining points where $L_1 = 0$ and $L_2 = 0$ and finding what happens to the $C4$ solutions at these points. From equations (3.17) and (3.20), we are concerned with points at which

$$p\alpha^2 - (6mp + 1 + m^2)\alpha + 2mp^2 + (p + 6m)(1 + m^2) = 0, \tag{6.7}$$

$$\alpha^2 + (2p - 10m)\alpha + p^2 - 14mp + 26m^2 + 2 = 0. \tag{6.8}$$

Eliminating α gives an equation $Q(m, p) = 0$, where

$$Q(m, p) = p^6 - 12mp^5 + (8 + 64m^2)p^4 - (46m + 110m^3)p^3 + (8 + 80m^2 + 72m^4)p^2 - 20m(1 + m^2)^2 p + 2(1 + m^2)^3. \tag{6.9}$$

The equation for α can be written as

$$(2p^2 - 4mp + 1 + m^2)\alpha = -p^3 + 16mp^2 - (1 + 25m^2)p + 6m(1 + m^2). \tag{6.10}$$

For a finite $C4$ solution, β_2 must be finite and this implies that L_1 cannot be zero without having N_1 zero also (from equation (3.16)). In the same way, N_2 should be zero when L_2 is zero. It can be verified that the condition $Q(m, p) = 0$ also arises from taking $L_1 = 0$, $L_2 = 0$, $N_2 = 0$ simultaneously and from the simultaneous equations $L_1 = 0$, $L_2 = 0$, $N_1 = 0$, $N_2 = 0$. The numerical work agrees with this.

TABLE 3. Values of m and k for which $L_1 = 0$, $L_2 = 0$

m	Values of k			
1.25	1.175541,	1.444044		
1.50	1.272026,	2.712829		
2.00	2.041693,	6.130276		
2.50	3.314790,	11.499324		
3.00	5.166206,	19.357997;	5.719589,	6.101476
3.50	7.699261,	30.215996;	8.329993,	9.425541
4.00	11.022819,	44.577753;	11.829435,	13.886449

Since $G(\alpha, m, p) = L_2N_1 - L_1N_2$, it follows that G and its first order partial derivatives are zero at points where L_1, L_2, N_1 and N_2 are all zero. However, although this is a sufficient condition for a double root it is not a necessary condition. Also, the double root for α need not lead to a real solution for b_1 to b_4. For the double root, α is given by equation (6.10) and, for given values of m and p, there will be another real value of α for which L_1, L_2, N_1 and N_2 are non-zero even although $Q(m, p) = 0$.

Table 3 lists some values of m and k for which $Q(m, p) = 0$, and it will be seen that there are two values of k for $m = 1.25$, 1.50, 2.00 and 2.50, with two additional values of k for larger values of m. The numerical calculations give two critical values for m, namely

$$m_{11} = 1.233603, \quad m_{12} = 2.880208,$$

and for a given value of m the equation $Q(m, p) = 0$ has no real roots for $1 < m < m_{11}$, two real roots for $m_{11} < m < m_{12}$ and four real roots for $m > m_{12}$. Some of these roots occur in (m, k) regions where no real $C4$ solutions are expected (for example, $m = 1.25$, $k = 1.175541$), others where a single real $C4$ solution is expected (for example, $m = 1.50$, $k = 2.712829$) and others where three real $C4$ solutions are expected (for example, $m = 3.00$, $k = 5.719589$). It turns out that the occurrence of these special points does not change the general picture, although they cause some anomalous results in the numerical work.

For example, for $m = 1.25$ and k going from 1.17551 to 1.17558, in steps of 10^{-5}, the equation for α has one real root near 6.78, well away from the value of α given by equation (6.10). For this value of α, L_1 and L_2 are not close to zero and the solution goes through in the usual way. (It comes out as a complex solution for the b_i). The equation for α also has two complex roots, $g \pm hi$, with the imaginary part h of order 10^{-5} or 10^{-6}. If we take h as positive, h should decrease to zero as k goes through its critical value and then increase again, thus providing a double real root when $h = 0$. The double root should give the value of α required by equation (6.10). The numerical work agreed with this reasonably well, although for $k = 1.17554$ the equation for α gave three real roots. Two of these differed by 2×10^{-6}, which is about as close to a double root as we might expect. When these values of α were used in equation (5.2) they gave complex values for β_2 and the solution for the b_i stopped at this point.

Similar results were obtained in other cases where a single real solution for α occurred at almost all neighbouring points. Where there were three real solutions for α at almost all neighbouring points, the double root corresponded to a "cross-over" of two of these solutions. (In Paper I, behaviour of this type was mentioned for the $C3$ solutions at $k = K_0$.) For example, if we take $m = 3.0$ and k close to 5.166206, the main family of solutions has $S \doteqdot -116$, the additional stable family has $S \doteqdot -1.25$ and the additional unstable has $S \doteqdot +10.5$. If we use α_0, α_1 and α_2 for the corresponding values of α, then $\alpha_0 > \alpha_2 > \alpha_1$ for $k = 5.166201$ and $\alpha_0 > \alpha_1 > \alpha_2$ for $k = 5.166211$, with $\alpha_1 = \alpha_2$ at the point where L_1 and L_2 are zero. The values of β_1, β_2, γ, δ and S are appreciably different for the α_1 and α_2 families in this neighbourhood so there is no difficulty in distinguishing them. In the same way, there is a cross-over point for $m = 3.0$ and $k = 5.719589$, where the pattern $\alpha_0 > \alpha_1 > \alpha_2$ changes to $\alpha_1 > \alpha_0 > \alpha_2$ as k increases, with $\alpha_0 = \alpha_1$ at the cross-over point. There is another cross-over point at $m = 3.0$ and $k = 6.101476$, where $\alpha_1 > \alpha_0 > \alpha_2$ changes back to $\alpha_0 > \alpha_1 > \alpha_2$ as k increases. Again, $\alpha_0 = \alpha_1$ at the cross-over point, with the corresponding $C4$ solutions clearly distinguishable.

In contrast, we can compare what happens at $m = 3.0$, $k = k_8 = 6.10187$. At this point, $\alpha_1 = \alpha_2$ and the equation for α again has a double root but as k increases through k_8 we have a transition from three real solutions for α to a single real solution. The α_0 solution continues but α_1 and α_2 become complex for $k > k_8$. At the transition point the two additional solutions merge into a single solution, with $S = 1$ and with L_1 and L_2 non-zero. This is a reminder that $Q(m, p) = 0$ provides a sufficient condition for a double root for α but is not a necessary condition.

In examining the solutions near points where L_1 and L_2 are zero, the stability criterion S was an important item of information, since it helped to discriminate between different families of solutions and was usually enough in itself to classify a solution unequivocally. In addition, it allowed solutions to be improved (as described in Section 5) once a first approximation had been obtained.

7. Discussion of results

The main results of the paper are the critical values of k in Table 1. For a given value of m, the $C4$ solutions first appear at $k = k_5$, as the $C2$ solutions become unstable, and the $C4$ solutions soon become unstable as k increases. In the table, the interval of stability (k_5, k_5^*) decreases as m increases and the ratio $(k_5 - k^*)/(k_5^* - k_5)$ increases from 1.15 at $m = 1.25$ to 3.93 at $m = 4.0$. On the other hand, the interval (k_6^*, k_6) becomes larger as m increases and there are no $C4$ solutions for $k > k_6$. For the additional solutions, there is a similar pattern. The interval of stability (k_7, k_7^*) is becoming smaller as m increases and the interval (k_8^*, k_8) becomes larger. For the values of m which give the additional intervals of stability, there are intervals of stability (K_1, K_1^*) and (K_2^*, K_2) for the $C3$ solutions, with (K_1, K_1^*) between (k_5, k_5^*) and (k_7, k_7^*) and with (K_2^*, K_2) between (k_8^*, k_8) and (k_6^*, k_6). Behaviour of this kind was foreshadowed from the discussion of superstable $C4$ solutions in Paper I, and the new results add corroborative detail.

One advantage of examining a problem with two parameters is that it may be possible to discuss a variety of one-parameter problems by linking the parameters in different ways. For example if we take $2k = m$ in equation (1.1) and impose the condition $m > 0$ (to ensure that we are dealing with positive values of k), the equation for the equilibrium values is

$$0 = Y\{1 + (Y - m)^2\} - m = (Y - m)(Y^2 - mY + 1). \qquad (7.1)$$

For $0 < m < 2$, there is only one real equilibrium value, $Y = m$, but for $m > 2$ there are three equilibrium solutions, say Y_1, Y_2, Y_3, with $0 < Y_1 < Y_2 < Y_3$. We can write $Y_3 = m$,

$$Y_1 = (1/2)\{m - \surd(m^2 - 4)\}, \qquad Y_2 = (1/2)\{m + \surd(m^2 - 4)\}, \quad (7.2)$$

and note that $Y_1 + Y_2 = m$, $Y_1 Y_2 = 1$. From the discussion in Paper I, it is easy to see that Y_1 is a stable equilibrium point, Y_2 is unstable and Y_3 is stable. (For $Y = m$, the stability criterion is zero.). Also, it was shown in Paper I that

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$2k^* - m > 0$, so in this case we have $0 < k < k^*$ for each value of m. This means that there cannot be any $C2$ solutions when $m = 2k$ and presumably no solutions with a longer period. Thus, by taking $m = 2k$, we get a one-parameter problem where the equilibrium solutions are the only periodic solutions.

If we take $k = m(1 + m^2)$, with $m > 0$, the equation for the equilibrium values is

$$0 = (Y - 2m)(Y^2 + 1 + m^2), \tag{7.3}$$

so the only real equilibrium solution is $Y = 2m$. The correspnding stability criterion is $S_1 = -4m^2/(1 + m^2)$, which decreases from 0 to -4 as m increases from 0 to ∞, with $S_1 = -1$ for $m^2 = 1/3$. Thus the equilibrium solution is stable for $0 < m < 1/\sqrt{3}$ and unstable for $m > 1/\sqrt{3}$. From the discussion of $C2$ solutions in Paper I, there is a $C2$ solution for $m > 1/\sqrt{3}$, with elements (b_1, b_2) and stability criterion S_2, where

$$b_{1,2} = 2m \pm \sqrt{(3m^2 - 1)}, \quad S_2 = -2 + \frac{1}{m^2}. \tag{7.4}$$

It follows that the $C2$ solutions are stable for $1/\sqrt{3} < m < 1$ and unstable for $m > 1$. For $m > 1$, there will be at least one $C4$ solution if $k_5 < k < k_6$ and this condition holds, since $k = m(1 + m^2) = (1/2)(k_5 + k_6)$. An inspection of the critical value of k in Table 1 indicates that there is only one $C4$ solution for each m, and that this solution is stable for m slightly greater than 1, but becomes unstable for some value of m between 1.25 and 1.50.

It looks as if the behaviour of the solutions in this case is more in line with the orthodox picture, where stable equilibrium solutions give way to stable solutions of period two as the parameter increases and these in turn give way to stable solutions of period four and so on, with stable solutions of period three appearing last of all. However it can be proved that no $C3$ solutions are possible in this instance, so the process must come to a stop at some intermediate stage. Perhaps the simplest way of showing that no $C3$ solutions can occur is to use the result (from Paper I) that, for a given value of m, $C3$ solutions exist for $K_1 \leqslant k \leqslant K_2$, where $m > \sqrt{3}$ and

$$K_{1,2} = \frac{(m^2 + 1)}{3}\left[m \mp \sqrt{(m^2 - 3)}\right]. \tag{7.5}$$

With $k = m(m^2 + 1)$,

$$k - K_2 = \frac{(m^2 + 1)}{3}\left[2m - \sqrt{(m^2 - 3)}\right] > 0$$

and k is too large for a $C3$ solution to be possible.

Another possibility is to see what happens when k is kept constant and m is allowed to vary from $-\infty$ to $+\infty$. If we take $0 \leqslant y_0 \leqslant 2k$, then equation (1.1) gives a mapping into a fixed interval $[0, 2k]$ for all values of the parameter m.

However, the behaviour of the solution depends on the value of k. To illustrate this we can start with $k = 0.5$, which gives $F(y_n) = 1/\{1 + (y_n - m)^2\}$. In this case there is a single equilibrium solution for each m and the equilibrium solution is stable. To see this, consider the graph of $y = F_0(x) = 1/(1 + x^2)$. This graph is symmetrical about $x = 0$ and the maximum slope occurs at $x = \pm 1/\sqrt{3}$ where the graph has a point of inflection. At the points of inflection the slope has magnitude $(3/8)\sqrt{3} = 0.65$ and it follows that the slope is less than 1 at all points. This means that $|F'(y_n)| < 1$ at all points and hence the line $y_{n+1} = y_n$ cannot intersect the graph of $y_{n+1} = F(y_n)$ in more than one point. Thus we get a single equilibrium point, say Y, with $|F'(Y)| < 1$, which ensures that the equilibrium is stable. This suggests that there should be no $C2$ solutions and this can be checked by noting that $k^* = (m^2 + 1)(-m + \sqrt{(m^2 + 1)})$ has a minimum value which is greater than 0.5. (Indeed $k = 0.5$ was chosen to satisfy this condition is the first place.) We can infer that for $k = 0.5$ there are no periodic solutions other than the equilibrium point.

The minimum value of k^* occurs at $m = 1/\sqrt{3}$, where $k^* = (4/9)\sqrt{3} \doteqdot 0.77$, so if we take $k = 1$ we can be sure there will be $C2$ solutions for some values of m. Now $k^* = 1$ for $m = 0$ and numerical work gives $k^* = 1$ again for $m = m_4 = 1.543689$. Hence the $C2$ solutions can only occur for $0 < m < m_4$. From Paper I, the stability criterion for a $C2$ solution is $S_2 = (1 + m^2)(1 + m^2 - 2km)/k^2$, and this becomes $S_2 = (1 + m^2)(m - 1)^2$ for $k = 1$. With this expression for S_2, S_2 has a minimum value 0 at $m = 1$ and $S_2 \to 1$ as $m \to 0$ or $m \to m_4$. This checks that the $C2$ solutions are stable, so no solutions with longer periods should occur. It was shown in Paper I that for equation (1.1) there could only be one equilibrium solution for $m < \sqrt{3}$ (for any choice of k) and when $k = 1$ the discussion in Paper I can be used to show that there are three distinct equilibrium solutions only for $2 < m < m_5$, where $m_5 = 2.134884$. For $m = 2$ and $m = m_5$ there are double root solutions and for $m < 2$ or $m > m_5$ there is only one real equilibrium solution. The equilibrium solution must be stable for $m < 0$, for $m_4 < m < 2$ and for $m > m_5$, since $0 < k < k^*$ in these intervals, and the equilibrium solution is unstable for $0 < m < m_4$, where $k > k^*$. Where there are three equilibrium values, the smallest and the largest will be stable, with the intermediate one unstable. For $m > 2$ we have $m > 2k$, so Coppel's conditions are satisfied [4, 2], and the iteration converges monotonically to an equilibrium value. Thus for $k = 1$ and m increasing we have an interval where there is a single stable equilibrium solution ($m < 0$), then an interval where there is a stable $C2$ solution ($0 < m < m_4$), then another interval with a single stable equilibrium solution ($m_4 < m < 2$), then an interval with one unstable and two stable equilibrium solutions ($2 < m < m_5$) and finally another interval with a single stable equilibrium solution ($m > m_5$). This is a long way from the standard picture of period-doubling sequences and much closer to the "remerging Feigenbaum trees" described by Bier and Bountis [1].

If we take a slightly larger value for k, say $k = 1.3$, chosen so that k is greater than the minimum value of k_5, we can have stable $C1$ solutions giving way to stable $C2$ solutions as m increases, then an interval where the $C2$ solutions are unstable but stable $C4$ solutions occur. As m increases still further, the $C4$ solutions become complex but there are stable $C2$ solutions (over a shorter interval than before) before these $C2$ solutions disappear and are replaced by $C1$ solutions.

For $k = 1.35$, the pattern is similar except that there are now two m-intervals where stable $C4$ solutions occur. Between these two intervals there is a gap where $C4$ solutions exist but are unstable and presumably stable $C8$ solutions occupy at least part of this gap.

If we move to an appreciably larger value of k, say $k = 5$, and keep k fixed while m increases from $-\infty$ to $+\infty$, the changes become more complex, and cannot be discussed in full with the equations available in this paper and in Paper I. We can introduce the notation that

$$M_i = (\text{value of } m \text{ for which } k_i = 5) \qquad (i = 5, 6, 7, 8),$$
$$M_i^* = (\text{value of } m \text{ for which } k_i^* = 5) \qquad (i = 5, 6, 7, 8),$$
$$M_-^*, M_+^* = (\text{values of } m \text{ for which } k^* = 5),$$
$$M_j = (\text{value of } m \text{ for which } K_j = 5) \qquad (j = 1, 2),$$
$$M_j^* = (\text{value of } m \text{ for which } K_j^* = 5) \qquad (j = 1, 2),$$

and we can take $M_-^* < 0$, $M_+^* > 0$ to distinguish these two values. The various M and M^* values are now critical values of m in the sense that there will be

(i) stable $C1$ solutions for $m < M_-^*$ and $m > M_+^*$,
(ii) stable $C2$ solutions in (M_-^*, M_6) and (M_5, M_+^*),
(iii) stable $C4$ solutions in (M_6, M_6^*), (M_8, M_8^*), (M_7^*, M_7) and (M_5^*, M_5),
(iv) stable $C3$ solutions in (M_2, M_2^*) and (M_1^*, M_1).

Numerically, these critical values are:

$$M_-^* = -1.020498, \quad M_6 = 1.239384, \quad M_6^* = 1.488224, \quad M_2 = 2.00,$$
$$M_2^* = 2.383399, \quad M_8 = 2.807304, \quad M_8^* = 2.941606, \quad M_7^* = 9.798271,$$
$$M_7 = 9.798392, \quad M_1^* = 9.817682, \quad M_1 = 9.819772, \quad M_5^* = 9.860592,$$
$$M_5 = 9.873000, \quad M_+^* = 9.924557.$$

There are obviously large gaps in the information available but it looks as if we have period-doubling at first (as m increases) and solutions of different period appear in the usual sequence until we have stable $C3$ solutions in (M_2, M_2^*).

Then there is a large interval in which solutions of all periods are present, mostly unstable but with occasional small intervals of stability. Close to $m = 10$ there is a smaller interval in which the $C3$ solutions are stable and then the whole process reverses itself very rapidly as m approaches $2k$. The $C3$ solutions disappear at $m = M_1$, the $C4$ solutions at $m = M_5$ and the $C2$ solutions at $m = M_+^*$, leaving the equilibrium solutions as the only periodic solutions for $m > M_+^*$.

Similar behaviour occurs for $k = 10$, where the usual process of bifurcation reverses itself very rapidly as $m \to 2k = 20$. Indeed some preliminary runs indicate that this complete reversal takes place for smaller values of k, down to about $k = 1.675$. For $k = 1.674$ there are $C3$ solutions, with a single interval of stability and no $C3$ solutions outside this interval. Values of the stability criterion down to -0.997 were obtained but they appeared to remain just above -1. For $k = 1.675$, the situation was almost the same except that the minimum value for the stability criterion was about -1.025 and in consequence there were two separate intervals of stability for these $C3$ solutions. The two intervals of stability were of comparable length. It is only as k becomes larger that the first interval of stability for the $C3$k solutions becomes much larger than the second one.

There is obviously plenty of scope for examining other relationships between m and k, but the examples above show how wide a range of behaviour is possible for one-parameter mappings.

In concluding, I should like to thank Mr W. A. Coppel for drawing my attention to the paper by Bier and Bountis [1], which encouraged me to extend the work on solutions with k constant. The investigation was carried out as a Visiting Fellow in the Research School of Physical Sciences, Australian National University and I am grateful to the Department of Theoretical Physics for the facilities it has provided.

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