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LARGE DEVIATIONS FOR RANDOMLY CONNECTED NEURAL NETWORKS: II. STATE-DEPENDENT INTERACTIONS

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Abstract

We continue the analysis of large deviations for randomly connected neural networks used as models of the brain. The originality of the model relies on the fact that the directed impact of one particle onto another depends on the state of both particles, and they have random Gaussian amplitude with mean and variance scaling as the inverse of the network size. Similarly to the spatially extended case (see Cabana and Touboul (2018)), we show that under sufficient regularity assumptions, the empirical measure satisfies a large deviations principle with a good rate function achieving its minimum at a unique probability measure, implying, in particular, its convergence in both averaged and quenched cases, as well as a propagation of a chaos property (in the averaged case only). The class of model we consider notably includes a stochastic version of the Kuramoto model with random connections.

Keywords: Large deviation; convergence of measure; neural network; disordered system

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1. Introduction

We pursue the study of randomly connected neural networks inspired from neurobiology. In the companion paper [5], we studied spatially extended neural networks with space-dependent delays and random interactions with mean and variance depending on cell locations, and scaling as the inverse of the network size. In that model, as well as in all previous works dealing with similar interaction coefficient scalings, the fact that the interaction between two particles may depend on the state of both particles was neglected. However, it is now accepted that the interaction between two neurons depends on the state of both the pre-synaptic and post-synaptic cell; see [8] and [9]. This type of state-dependent interaction is much more general and actually ubiquitous in the life sciences, e.g. in models of collective animal behavior [6] or natural coupled oscillators such as those described by the canonical Kuramoto model; see [11], [12], [19], and [20]. In this paper we address the dynamics of networks with state-dependent interactions and random coupling amplitudes in a general setting. In detail, we consider the interaction of N agents described by a real state variable $(X_t^{i,N})_{i=1,\dots,N} \in \mathbb{R}^N$ and satisfying a stochastic differential equation (SDE) of the type of [5, Equation (1)], i.e.

$$\mathrm{d}X_t^{i,N} = \bigg(f(r_i, t, X_t^{i,N}) + \sum_{j=1}^{N} J_{ij} b(X_t^{i,N}, X_t^{j,N})\bigg)\,\mathrm{d}t + \lambda\,\mathrm{d}W_t^i, \tag{1.1}$$

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where f describes the intrinsic dynamics of the particle, J_{ij} models the random interaction amplitude, $b(x, y)$ is the typical impact of a particle with state y on a particle with state x, and each particle is subject to independent Brownian fluctuations (W_t^i); see [5] for details of this equation.

Following the general methodology introduced in [1]–[3] and [10], and also used in the companion paper [5], we will show that using large deviations techniques the empirical measure of system (1.1), averaged over the disorder parameters, satisfies a large deviations principle (LDP), with an explicit good rate function that has a unique minimum implying convergence of the network equations towards a non-Markovian complex mean-field equation. Taking into account general interactions gives rise to a number of specific difficulties compared to previous works. In particular, the dependence on the state of the particle induces complex interdependences between processes that prevent us from isolating the exponential martingale terms as carried out when $b(x, y) = S(y)$. We handle this issue using specific estimates that will lead us to restrict the time horizon.

The paper is organized as follows. We start by introducing the mathematical setting and main results in Section 2. The proofs are found in the following sections. In Section 3 we establish a partial LDP for the averaged empirical measure, which relies on the identification of the good rate function as well as on exponential tightness and upper bounds on closed sets for the sequence of empirical measures. (In the companion paper [5] we proved tightness and upper bounds only for compact sets to avoid any constraint on the time horizon.) In Section 4 we demonstrate that the good rate function admits a unique minimum Q and prove the averaged and quenched convergence of the empirical measure towards Q using the methodology introduced in [5]. We conclude with a discussion on a few perspectives as well as some open research directions.

2. Mathematical setting and statement of the results

The general mathematical setting was introduced in [5, Section 2]. The aim of this second paper is to deal with complex interactions of the form $b(x, y)$ which depend on the state of both particles.

In order to expedite the analysis, we neglect spatial aspects already addressed in the first paper and, in particular, consider the following:

(i) the synaptic coefficients identically distributed with law $\mathcal{N}(\bar{J}/N, \sigma^2/N)$,

(ii) diffusion coefficients independent of space, and

(iii) no interaction delay.

Formally, this amounts to assuming, in the general framework of [5, Section 2], that $J(r, r') \equiv \bar{J} \in \mathbb{R}$, $\sigma(r, r') \equiv \sigma \in \mathbb{R}_+^*$, $\lambda(r) \equiv \lambda > 0$, and $\tau(r, r') \equiv 0$. Therefore, the initial conditions are real variables ($C_\tau = \mathbb{R}$) and the trajectories belong to $\mathcal{C} = \mathcal{C}([0, T], \mathbb{R})$.

Our results hold under the condition that the horizon of time T is such that

$$\frac{2\sigma^2 \|b\|_\infty^2 T}{\lambda^2} < 1. \tag{2.1}$$

Compared to the results of the companion paper [5], this condition entails stronger results on $Q^N(\hat{\mu}_N \in \cdot)$: an *exponential* tightness (Theorem 2.1) and an upper bound for *closed sets* (and not restricted to compact sets, Theorem 2.2).

We summarize these results in the next theorems.

Theorem 2.1. *For small enough T for inequality (2.1) to hold, there exists a unique double-layer probability distribution $Q \in \mathcal{M}_1^+(\mathcal{C} \times D)$ such that*

$$Q^N(\hat{\mu}_N \in \cdot) \xrightarrow{\mathcal{L}} \delta_Q(\cdot) \in \mathcal{M}_1^+(\mathcal{M}_1^+(\mathcal{C} \times D))$$

exponentially fast, where '$\xrightarrow{\mathcal{L}}$' denotes weak convergence of probability measures for processes.

The existence of Q and the exponential convergence results follow from:

(i) the exponential tightness of the sequence $Q^N(\hat{\mu}_N \in \cdot)$,

(ii) a partial LDP for the empirical measure relying on an upper bound for closed sets, and

(iii) a characterization of the set of minima of the good rate function.

Theorem 2.2. (Partial LDP.) *For small enough T for inequality (2.1) to hold, we have:*

(i) *for any real number $M \in \mathbb{R}$, there exists a compact subset K_M such that for any integer N,*

$$\frac{1}{N} \log Q^N(\hat{\mu}_N \notin K_M) \leq -M.$$

(ii) *there exists a good rate function $H\colon \mathcal{M}_1^+(\mathcal{C} \times D)$ such that for any closed subset F of $\mathcal{M}_1^+(\mathcal{C} \times D)$,*

$$\limsup_{N\to\infty} \frac{1}{N} \log Q^N(\hat{\mu}_N \in F) \leq -\inf_F H.$$

The proof can be found in Section 3.

Theorem 2.3. (Minima of the rate function.) *The good rate function H achieves its minimal value at a unique probability measure $Q \in \mathcal{M}_1^+(\mathcal{C} \times D)$ satisfying*

$$Q \simeq P, \qquad \frac{\mathrm{d}Q}{\mathrm{d}P}(x, r) = \mathcal{E}\biggl[\exp\biggl\{\frac{1}{\lambda}\int_0^T G_t^Q(x)\,\mathrm{d}W_t(x, r) - \frac{1}{2\lambda^2}\int_0^T (G_t^Q(x))^2\,\mathrm{d}t\biggr\}\biggr],$$

where $(W_t(\cdot, r))_{t\in[0,T]}$ is a P_r-Brownian motion, and $G^Q(x)$ is under $(\tilde{\Omega}, \tilde{\mathcal{F}}, \mathcal{P})$, a Gaussian process with mean

$$\mathcal{E}[G_t^Q(x)] = \int_{\mathcal{C}\times D} \bar{J} b(x_t, y_t)\,\mathrm{d}Q(y, r')$$

and covariance

$$\mathcal{E}[G_t^Q(x) G_s^Q(x)] = \int_{\mathcal{C}\times D} \sigma^2 b(x_t, y_t) b(x_s, y_s)\,\mathrm{d}Q(y, r').$$

The proof of this theorem is the topic of Section 4. Combining both results, the general result of Sznitman [21, Lemma 3.1] leads to the next theorem.

Theorem 2.4. (Propagation of chaos.) *For small enough T for inequality (2.1) to hold, Q^N is Q-chaotic in the sense that for any $m \in \mathbb{N}^*$, any collection of bounded continuous functions $\varphi_1, \dots, \varphi_m\colon \mathcal{C} \times D \to \mathbb{R}$, and any set of nonzero distinct integers $k_1, \dots, k_m$, we have*

$$\lim_{N\to\infty} \int_{(\mathcal{C}\times D)^N} \prod_{j=1}^m \varphi_j(x^{k_j}, r_{k_j})\,\mathrm{d}Q^N(\boldsymbol{x}, \boldsymbol{r}) = \prod_{j=1}^m \int_{\mathcal{C}\times D} \varphi_j(x, r)\,\mathrm{d}Q(x, r).$$

Our results partially extend to the quenched case as stated in the next theorem.

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Theorem 2.5. (Quenched results.) *For small enough T for inequality (2.1) to hold, we have the quenched upper bound*

$$\limsup_{N\to\infty} \frac{1}{N} \log Q_{\boldsymbol{r}}^N(J)(\hat{\mu}_N \in F) \le -\inf_F H, \qquad \mathcal{P}\text{-a.s. for all closed } F \subset \mathcal{M}_1^+(\mathcal{C} \times D),$$

where H is the good rate function introduced in Theorem 2.2 and we abbreviate $\mathcal{P}$-almost surely to $\mathcal{P}$-a.s. In particular, for almost every realization of $\boldsymbol{r}$ and J, $Q_{\boldsymbol{r}}^N(J)(\hat{\mu}_N \in \cdot)$ is exponentially tight and converges in law toward δ_Q exponentially fast. Eventually, this implies $\mathcal{P}$-almost sure convergence of the empirical measure to Q.

3. The LDP

In this section we prove the existence of a partial LDP for the averaged empirical measure. We start by constructing the appropriate good rate function before obtaining an upper bound and an exponential tightness result. Many points of the proof proceed as in the companion paper [5] or as in earlier works; see [1], [4], and [10]. To avoid repetition we will often refer to these contributions and focus our attention on the new difficulties arising in this state-dependent interactions setting.

3.1. Construction of the good rate function

For $\mu \in \mathcal{M}_1^+(\mathcal{C} \times D)$, we define the two following functions respectively on $[0, T]^2 \times \mathcal{C}$ and $[0, T] \times \mathcal{C}$:

$$K_\mu(s, t, x) := \frac{\sigma^2}{\lambda^2} \int_{\mathcal{C}\times D} b(x_t, y_t) b(x_s, y_s)\, \mathrm{d}\mu(y, r'),$$

$$m_\mu(t, x) := \frac{\bar{J}}{\lambda} \int_{\mathcal{C}\times D} b(x_t, y_t)\, \mathrm{d}\mu(y, r').$$

Both functions are well defined as $(y, r) \to b(x_t, y_t)b(x_s, y_s)$ and $(y, r) \to b(x_t, y_t)$ are continuous for the uniform norm on $\mathcal{C} \times D$, and μ is a Borel measure. They are bounded, i.e.

$$|K_\mu(s, t, x)| \le \frac{\sigma^2 \|b\|_\infty^2}{\lambda^2} \quad \text{and} \quad |m_\mu(t, x)| \le \frac{\bar{J}\|b\|_\infty}{\lambda}.$$

They are also continuous by the dominated convergence theorem.

Since K_μ has a covariance structure, we can define a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \gamma)$ and a family of stochastic processes $(G^\mu(x))_{x\in\mathcal{C},\, \mu\in\mathcal{M}_1^+(\mathcal{C}\times D)}$ such that $G^\mu(x)$ is a centered Gaussian process with covariance $K_\mu(\cdot, \cdot, x)$ under the measure γ. We denote by $\mathcal{E}_\gamma$ the expectation under γ.

Remark 3.1. For the sake of measurability under Borel measures of $\mathcal{M}_1^+(\mathcal{C} \times D)$, it is convenient to choose a family $(G_t^\mu(x))_{\mu,x}$ regular in x, which can be achieved by constructing the process explicitly as in [10, Remark 2.14].

We recall that for any Gaussian process $(G_t)_{t\in[0,T]}$ of $(\hat{\Omega}, \hat{\mathcal{F}}, \gamma)$, and any $t \in [0, T]$,

$$\Lambda_t(G) := \exp\left\{-\frac{1}{2}\int_0^t G_s^2\, \mathrm{d}s\right\}\left(\mathcal{E}_\gamma\left[\exp\left\{-\frac{1}{2}\int_0^t G_s^2\, \mathrm{d}u\right\}\right]\right)^{-1}.$$

We also define for any $\nu \in \mathcal{M}_1^+(\mathcal{C} \times D)$, $(x, r) \in \mathcal{C} \times D$, and $t \in [0, T]$,

$$L_t^\nu(x, r) := \int_0^t G_s^\nu(x)(\mathrm{d}W_s(x, r) - m_\nu(s, x)\, \mathrm{d}s), \qquad V_t^\nu(x, r) := W_t(x, r) - \int_0^t m_\nu(s, x)\, \mathrm{d}s.$$

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Moreover, we introduce

$$\mathrm{d}\gamma_{\widetilde{K}^t_{\nu,x}}(\omega) := \Lambda_t(G^\nu(\omega, x))\,\mathrm{d}\gamma(\omega) \quad \text{for all } \omega \in \hat{\Omega},$$

for any $t \in [0, T]$, $x \in \mathcal{C}$, $\nu \in \mathcal{M}_1^+(\mathcal{C} \times D)$. Neveu [15] proved that $\gamma_{\widetilde{K}^t_{\nu,x}}$ is a probability measure, under which $G^\nu(x)$ is a centered Gaussian process with covariance

$$\widetilde{K}^t_{\nu,x}(s, u) := \mathcal{E}_\gamma[G^\nu_u(x) G^\nu_s(x) \Lambda_t(G^\nu(x))].$$

Moreover, for any fixed $N \in \mathbb{N}^*$ and for all $(\boldsymbol{x}, \boldsymbol{r}) \in (\mathcal{C} \times D)^N$, let

$$X^N_i(\boldsymbol{x}, \boldsymbol{r}) := \int_0^T G^{i,N}_t(\boldsymbol{x})\,\mathrm{d}W_t(x^i, r_i) - \frac{1}{2}\int_0^T G^{i,N}_t(\boldsymbol{x})^2\,\mathrm{d}t,$$

where $G^{i,N}_t(\boldsymbol{x}) := (1/\lambda)\sum_{j=1}^N J_{ij} b(x^i_t, x^j_t)$. As proved in [5], we have the following good properties.

Proposition 3.1. *There exists $C_T > 0$ such that for any $\nu \in \mathcal{M}_1^+(\mathcal{C} \times D)$, $x \in \mathcal{C}$, $t \in [0, T]$,*

$$\sup_{0 \le s,u \le t} \widetilde{K}^t_{\nu,x}(s, u) \le C_T, \qquad \Lambda_t(G^\nu(x)) \le C_T, \tag{3.1}$$

$$\mathcal{E}_\gamma\biggl[\exp\biggl\{-\frac{1}{2}\int_0^T G^\nu_t(x)^2\,\mathrm{d}t\biggr\}\biggr] = \exp\biggl\{-\frac{1}{2}\int_0^T \widetilde{K}^t_{\nu,x}(t, t)\,\mathrm{d}t\biggr\}.$$

Moreover, if $(G_t)_{0 \le t \le T}$ and $(G'_t)_{0 \le t \le T}$ are two centered Gaussian processes of $(\hat{\Omega}, \hat{\mathcal{F}}, \gamma)$ with uniformly bounded covariance, then there exists $\tilde{C}_T > 0$ such that for all $t \in [0, T]$,

$$|\Lambda_t(G) - \Lambda_t(G')| \le \tilde{C}_T \biggl\{\int_0^t \mathcal{E}_\gamma[(G_s - G'_s)^2]^{1/2}\,\mathrm{d}s + \int_0^t |G_s^2 - {G'_s}^2|\,\mathrm{d}s\biggr\}. \tag{3.2}$$

Lemma 3.1. *It holds that*

$$\frac{\mathrm{d}Q^N}{\mathrm{d}P^{\otimes N}}(\boldsymbol{x}, \boldsymbol{r}) = \exp\{N\bar{\Gamma}(\hat{\mu}_N)\},$$

where

$$\bar{\Gamma}(\hat{\mu}_N) := \frac{1}{N}\sum_{i=1}^N \log \mathcal{E}_\gamma\biggl[\exp\biggl\{\int_0^T (G^{\hat{\mu}_N}_t(x^i) + m_{\hat{\mu}_N}(t, x^i))\,\mathrm{d}W_t(x^i, r_i) - \frac{1}{2}\int_0^T (G^{\hat{\mu}_N}_t(x^i) + m_{\hat{\mu}_N}(t, x^i))^2\,\mathrm{d}t\biggr\}\biggr].$$

As in [5], this lemma suggests that a version of Varadhan's lemma and an LDP might hold. In the next lemma we properly define the associated Varadhan functional.

Proposition 3.2. *Let*

$$X^\mu(x, r) := \int_0^T (G^\mu_t(x) + m_\mu(t, x))\,\mathrm{d}W_t(x, r) - \frac{1}{2}\int_0^T (G^\mu_t(x) + m_\mu(t, x))^2\,\mathrm{d}t.$$

The map

$$\Gamma := \mu \in \mathcal{M}_1^+(\mathcal{C} \times D) \to \begin{cases} \displaystyle\int_{\mathcal{C} \times D} \log \mathcal{E}_\gamma[\exp\{X^\mu(x, r)\}]\,\mathrm{d}\mu(x, r) & \textit{if } I(\mu \mid P) < \infty, \\ +\infty & \textit{otherwise} \end{cases}$$

is well defined in $\mathbb{R} \cup \{+\infty\}$, and satisfies:

(i) $\Gamma \le I(\cdot \mid P)$,

(ii) *If $2\sigma^2\|b\|^2_\infty T/\lambda^2 < 1$, there exists $\iota \in (0, 1)$, $e \ge 0$, $|\Gamma(\mu)| \le \iota I(\mu \mid P) + e$.*

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The proof differs slightly from that of [5, Proposition 3] as the dependence of the Gaussian process G^{μ} in x prevents us from extracting it from the integral over P_r. We thus reproduce the important lines of the proof, and rely on Hölder's inequality to deal with this new problem.

Proof of Proposition 3.2. We suppose that $I(\mu \mid P) < +\infty$ and $\mu \ll P$ as the result is otherwise trivial. As $W(\cdot, r)$ is a P_r-Brownian motion, Girsanov's theorem ensures that the stochastic integral $\int_0^T (G_t^{\mu}(x) + m_{\mu}(t,x))\, \mathrm{d}W_t(x,r)$ is well defined γ-a.s. under μ.

(i) Following the proof of [5, Proposition 3], we obtain, for any $\alpha \geq 1$,

$$\alpha \int_{\mathcal{C}\times D} \log(\mathcal{E}_{\gamma}[\exp\{X^{\mu}(x,r)\}] \vee M^{-1})\, \mathrm{d}\mu(x,r) \leq I(\mu \mid P) + \log\biggl\{M^{-\alpha} + \mathcal{E}_{\gamma}\biggl[\int_D \int_{\mathcal{C}} \exp\{\alpha X^{\mu}(x,r)\}\, \mathrm{d}P_r(x)\, \mathrm{d}\pi(r)\biggr]\biggr\}, \tag{3.3}$$

$$\alpha \int_{\mathcal{C}\times D} (\log \mathcal{E}_{\gamma}[\exp\{X^{\mu}(x,r)\}])^{-}\, \mathrm{d}\mu(x,r) \leq I(\mu \mid P) + \alpha C_T + \log\biggl\{\mathcal{E}_{\gamma}\biggl[\int_D \int_{\mathcal{C}} \exp\{\alpha X^{\mu}(x,r)\}\, \mathrm{d}P_r(x)\, \mathrm{d}\pi(r)\biggr]\biggr\}, \tag{3.4}$$

with the right-hand side of these two inequalities being possibly infinite. Moreover, $W(\cdot, r)$ being a P_r-Brownian motion, the martingale property yields, for $\alpha = 1$,

$$\mathcal{E}_{\gamma}\biggl[\int_D \int_{\mathcal{C}} \exp\{\alpha X^{\mu}(x,r)\}\, \mathrm{d}P_r(x)\, \mathrm{d}\pi(r)\biggr] = 1,$$

so that the result follows by sending $M \to +\infty$.

(ii) Let $\alpha > 1$. Using the martingale property and Hölder's inequality, we have

$$\int_{\mathcal{C}\times D} \mathcal{E}_{\gamma}[\exp\{\alpha X^{\mu}(x,r)\}]\, \mathrm{d}P(x,r) \leq \biggl\{\int_{\mathcal{C}\times D} \mathcal{E}_{\gamma}\biggl[\exp\biggl\{\frac{\alpha^2(\alpha+1)}{2}\int_0^T (G_t^{\mu}(x) + m_{\mu}(t,x))^2\, \mathrm{d}t\biggr\}\biggr]\, \mathrm{d}P(x,r)\biggr\}^{(\alpha-1)/\alpha}.$$

Under the short-time hypothesis $2\sigma^2\|b\|_{\infty}^2 T/\lambda^2 < 1$, we can proceed as in [5, Proposition 9] to prove finiteness of the right-hand side for small enough $\alpha - 1$, as we are able to rely on the following identity, valid for $\zeta \sim \mathcal{N}(\alpha, \beta)$ with $\beta < 1$:

$$\mathbb{E}\biggl[\exp\biggl\{\frac{1}{2}\zeta^2\biggr\}\biggr] = \frac{1}{\sqrt{1-\beta}} \exp\biggl\{\frac{\alpha^2}{2(1-\beta)}\biggr\} = \exp\biggl\{\frac{1}{2}\biggl(\frac{\alpha^2}{1-\beta} - \log(1-\beta)\biggr)\biggr\}.$$

Hence, by the Jensen and Fubini inequalities, it follows that there exists a constant C_T uniform in $x \in \mathcal{C}$ such that

$$\mathcal{E}_{\gamma}\biggl[\exp\biggl\{\frac{\alpha^2(\alpha+1)T}{2}\int_0^T (G_t^{\mu}(x) + m_{\mu}(t,x))^2 \frac{\mathrm{d}t}{T}\biggr\}\biggr] \leq \mathrm{e}^{C_T},$$

which implies that

$$\int_{\mathcal{C}\times D} \mathcal{E}_{\gamma}[\exp\{\alpha X^{\mu}(x,r)\}]\, \mathrm{d}P(x,r) \leq \exp\{(\alpha-1)C_T\}. \tag{3.5}$$

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Inequalities (3.3)–(3.5) ensure that, under the condition $2\sigma^2\|b\|_\infty^2 T/\lambda^2 < 1$ and for $\alpha > 1$,

$$|\Gamma(\mu)| \leq \iota I(\mu \mid P) + e$$

with $\iota := 1/\alpha$ and $e := (2\alpha - 1)C_T$. □

Define

$$H(\mu) := \begin{cases} I(\mu \mid P) - \Gamma(\mu) & \text{if } I(\mu \mid P) < \infty, \\ \infty & \text{otherwise;} \end{cases}$$

for any $\nu \in \mathcal{M}_1^+(\mathcal{C} \times D)$,

$$\Gamma_\nu := \mu \in \mathcal{M}_1^+(\mathcal{C} \times D) \to \begin{cases} \int_{\mathcal{C}\times D} \log \mathcal{E}_\gamma[\exp\{X^\nu(x, r)\}]\, \mathrm{d}\mu(x, r) & \text{if } I(\mu \mid P) < \infty, \\ +\infty & \text{otherwise,} \end{cases}$$

$$H_\nu \colon \mu \to \begin{cases} I(\mu \mid P) - \Gamma_\nu(\mu) & \text{if } I(\mu \mid P) < +\infty, \\ +\infty & \text{otherwise,} \end{cases}$$

as well as the following probability measure on $\mathcal{C} \times D$:

$$\mathrm{d}Q_\nu(x, r) := \exp\{\bar{\Gamma}_\nu(\delta_{(x,r)})\}\, \mathrm{d}P(x, r) := \mathcal{E}_\gamma[\exp\{X^\nu(x, r)\}]\, \mathrm{d}P(x, r). \tag{3.6}$$

As in [5, Theorem 6], we have the relatively intuitive result of the next theorem.

Theorem 3.1. *It holds that H_ν and $I(\cdot \mid Q_\nu)$ are equal on $\mathcal{M}_1^+(\mathcal{C} \times D)$. In particular, H_ν is a good rate function that attains its unique minimum at Q_ν.*

We introduce the Wasserstein distance on $\mathcal{M}_1^+(\mathcal{C} \times D)$, compatible with the weak topology, i.e.

$$d_T^V(\mu, \nu) := \inf_\xi \bigg\{ \int_{(\mathcal{C}\times D)^2} [\|x - y\|_{\infty,T}^2 + \|r - r'\|_{\mathbb{R}^d}^2]\, \mathrm{d}\xi((x, r), (y, r')) \bigg\}^{1/2},$$

where the infimum is taken on the laws $\xi \in \mathcal{M}_1^+((\mathcal{C} \times D)^2)$ with marginals μ and ν. Moreover, we denote for any $t \in [0, T]$, and any $(x, r), (y, r') \in \mathcal{C} \times D$,

$$d_t((x, r), (y, r')) := (\|x - y\|_{\infty,t}^2 + \|r - r'\|_{\mathbb{R}^d}^2)^{1/2},$$

where we recall that $\|x - y\|_{\infty,t} := \sup_{0\leq s\leq t} |x_s - y_s|^2$, and also

$$d_t^V(\mu, \nu) := \inf_\xi \bigg\{ \int_{(\mathcal{C}\times D)^2} d_t((x, r), (y, r'))^2\, \mathrm{d}\xi((x, r), (y, r')) \bigg\}^{1/2}.$$

The metric d_T^V controls the regularity of the mean and variance structure of the Gaussian interactions and, in the long run (see Theorem 3.2), of the error between H and its approximation H_ν, as we show in the next proposition.

Proposition 3.3. *There exists $C_T > 0$ such that for any $\mu, \nu \in \mathcal{M}_1^+(\mathcal{C} \times D)$, $x \in C$, $t \in [0, T]$, and $u, s \in [0, t]$,*

$$\begin{aligned} &|m_\mu(t, x) - m_\nu(t, x)| + |K_\mu(t, s, x) - K_\nu(t, s, x)| + |\widetilde{K}^t_{\mu,x}(s, u) - \widetilde{K}^t_{\nu,x}(s, u)| \\ &\quad \leq C_T d_T^V(\mu, \nu). \end{aligned} \tag{3.7}$$

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Proof. First, observe that for any $\xi \in \mathcal{M}_1^+((\mathcal{C} \times D)^2)$ with marginals μ and ν,

$$\begin{aligned}|m_\mu(t,x) - m_\nu(t,x)| &= \left|\frac{\bar{J}}{\lambda}\int_{\mathcal{C}\times D} b(x_t, y_t)\,\mathrm{d}(\mu-\nu)(y,r')\right| \\ &\le \frac{\bar{J}}{\lambda}\int_{(\mathcal{C}\times D)^2} |b(x_t,y_t) - b(x_t,z_t)|\,\mathrm{d}\xi((y,r'),(z,\tilde{r}')) \\ &\le \frac{\bar{J}K_b}{\lambda}\left\{\int_{(\mathcal{C}\times D)^2} \|y-z\|_{\infty,t}^2\,\mathrm{d}\xi((y,r'),(z,\tilde{r}'))\right\}^{1/2},\end{aligned}$$

where the second inequality follows from the Cauchy–Schwarz inequality, so that $|m_\mu(t,x) - m_\nu(t,x)| \le C_T d_T^V(\mu,\nu)$.

Fix now $\xi \in \mathcal{M}_1^+((\mathcal{C} \times D)^2)$ with marginals μ and ν. Letting (G, G') be a γ-bi-dimensional centered Gaussian processes with covariance, i.e.

$$K_\xi(s,t,x) := \frac{\sigma^2}{\lambda^2}\int_{(\mathcal{C}\times D)^2}\begin{pmatrix} b(x_s,y_s)b(x_t,y_t) & b(x_s,y_s)b(x_t,z_t) \\ b(x_s,z_s)b(x_t,y_t) & b(x_s,z_s)b(x_t,z_t)\end{pmatrix}\mathrm{d}\xi((y,r'),(z,\tilde{r}')), \tag{3.8}$$

we easily obtain (see [5, Proof of Proposition 5]) the inequalities

$$\begin{aligned}&|K_\mu(t,s,x) - K_\nu(t,s,x)| \\ &\qquad \le C_T\{\mathcal{E}_\gamma[(G_t - G_t')^2]^{1/2} + \mathcal{E}_\gamma[(G_s - G_s')^2]^{1/2}\}, \\ &|\widetilde{K}_{\mu,x}^t(s,u) - \widetilde{K}_{\nu,x}^t(s,u)| \\ &\qquad \le C_T\left\{\left(\int_0^t \mathcal{E}_\gamma[(G_v - G_v')^2]\,\mathrm{d}v\right)^{1/2} + \mathcal{E}_\gamma[(G_s - G_s')^2]^{1/2} + \mathcal{E}_\gamma[(G_u - G_u')^2]^{1/2}\right\},\end{aligned}$$

where the last inequality follows from (3.1) and (3.2).

We remark that

$$\begin{aligned}\mathcal{E}_\gamma[(G_t - G_t')^2] &= \frac{\sigma^2}{\lambda^2}\int_{(\mathcal{C}\times D)^2}(b(x_t,y_t) - b(x_t,z_t))^2\,\mathrm{d}\xi((y,r'),(z,\tilde{r}')) \\ &\le \frac{\sigma^2 K_b^2}{\lambda^2}\int_{(\mathcal{C}\times D)^2} d_T((y,r'),(z,\tilde{r}'))^2\,\mathrm{d}\xi((y,r'),(z,\tilde{r}'))\end{aligned}$$

and taking the infimum over ξ yields the result. □

In the following theorem we control the error between H and H_ν and ensure that the former is a good rate function under the time condition (2.1).

Theorem 3.2. *It holds that:*

(i) *there exists $C_T > 0$ such that for every $\mu, \nu \in \mathcal{M}_1^+(\mathcal{C} \times D)$,*

$$|\Gamma_\nu(\mu) - \Gamma(\mu)| \le C_T(1 + I(\mu \mid P))d_T^V(\mu,\nu).$$

(ii) *If $2\sigma^2\|b\|_\infty^2 T/\lambda^2 < 1$, H is a good rate function.*

Proof. The basic mechanism for the proof is similar to [5, Lemma 4] or [1, Lemmas 3.3 and 3.4]. However, the dependence in x of the Gaussian $G^\mu(x)$ is problematic as we cannot take it out of the integrals on x. In order to deal with this difficulty, we rely on tools from probability theory, such as Fubini's theorem for stochastic integrals, or the Dambis–Dubins–Schwarz (DDS) theorem. We focus our attention on (i), whereas (ii), previously shown without restriction on time in cases where $b(x, y) = S(y)$ (see [4] and [5]), is now valid only under the short-time hypothesis of Proposition 3.2(ii).

As proved in [5], Γ_ν can be expressed as $\Gamma_\nu(\mu) = \Gamma_{1,\nu}(\mu) + \Gamma_{2,\nu}(\mu)$ with

$$\Gamma_{1,\nu}(\mu) := -\frac{1}{2}\int_0^T (\widetilde{K}^t_{\nu,x}(t,t) + m_\nu(t,x)^2)\,\mathrm{d}t\,\mathrm{d}\mu(x,r),$$

and

$$\Gamma_{2,\nu}(\mu) := \begin{cases} \dfrac{1}{2}\displaystyle\int_{\mathcal{C}\times D}\int_{\hat{\Omega}} L^\nu_T(x,r)^2\,\mathrm{d}\gamma_{\widetilde{K}^T_{\nu,x}}\,\mathrm{d}\mu(x,r) & \\ \quad + \displaystyle\int_{\mathcal{C}\times D}\int_0^T m_\nu(t,x)\,\mathrm{d}W_t(x,r)\,\mathrm{d}\mu(x,r) & \text{if } I(\mu \mid P) < \infty, \\ +\infty & \text{otherwise.} \end{cases}$$

The previous decomposition has the effect of splitting the difficulties, i.e. $|\Gamma_\nu(\mu) - \Gamma(\mu)| \le |\Gamma_{1,\nu}(\mu) - \Gamma_1(\mu)| + |\Gamma_{2,\nu}(\mu) - \Gamma_2(\mu)|$. The first term is easily controlled by $C_T d^V_T(\mu,\nu)$ using Proposition 3.3. We now prove that

$$|\Gamma_{2,\nu}(\mu) - \Gamma_2(\mu)| \le C_T(1 + I(\mu \mid P))d^V_T(\mu,\nu).$$

The inequality is trivial when $I(\mu \mid P) = \infty$. We now assume that $I(\mu \mid P) < \infty$ implying that $\mu \ll P$ and the finiteness of $\Gamma(\mu)$ and $\Gamma_\nu(\mu)$. In particular, μ has a Borel-measurable density ρ_μ with respect to P, i.e.

$$\mathrm{d}\mu(x,r) = \rho_\mu(x,r)\,\mathrm{d}P(x,r).$$

Let $\varepsilon > 0$, and let $\xi \in \mathcal{M}^+_1((\mathcal{C}\times D)^2)$ with marginals μ and ν be such that

$$\left\{\int_{(\mathcal{C}\times D)^2} d_T((y,r'),(z,\tilde{r}))^2\,\mathrm{d}\xi((y,r'),(z,\tilde{r}))\right\}^{1/2} \le d^V_T(\mu,\nu) + \varepsilon.$$

Also let $(G(x), G'(x))_{x\in\mathcal{C}}$ be a family of bi-dimensional centered Gaussian process from the probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \gamma)$ with covariance K_ξ defined by (3.8). In the expressions of $\Gamma_{2,\nu}(\mu)$ and $\Gamma_2(\mu)$, we can then replace the triplet (G^μ, G^ν, γ) by (G, G', γ) so that we choose their covariance in terms of K_ξ (see [5, Remark 3]). As proved in Proposition 3.3, we can show that there exists a constant $C_T > 0$ such that for any $t \in [0,T]$, $x \in \mathcal{C}$,

$$\mathcal{E}_\gamma[(G_t(x) - G'_t(x))^2] \le (d^V_T(\mu,\nu) + \varepsilon)^2.$$

Also, for any $t \in [0,T]$,

$$L_t(x,r) := \int_0^t G_s(x)\,\mathrm{d}V^\mu_s(x,r), \qquad L'_t(x,r) := \int_0^t G'_s(x)\,\mathrm{d}V^\nu_s(x,r).$$

Then

$$\begin{aligned} |\Gamma_{2,\nu}(\mu) - \Gamma_2(\mu)| \le{}& \frac{1}{2}\left|\int_{\mathcal{C}\times D} \mathcal{E}_\gamma[L'_T(x,r)^2(\Lambda_T(G'(x)) - \Lambda_T(G(x)))]\,\mathrm{d}\mu(x,r)\right| \\ &+ \frac{1}{2}\left|\int_{\mathcal{C}\times D} \mathcal{E}_\gamma[(L_T(x,r)^2 - L'_T(x,r)^2)\Lambda_T(G(x))]\,\mathrm{d}\mu(x,r)\right| \\ &+ \left|\int_{\mathcal{C}\times D}\int_0^T (m_\nu - m_\mu)(t,x)\,\mathrm{d}W_t(x,r)\,\mathrm{d}\mu(x,r)\right|. \end{aligned}$$

Observe that, by (3.2), we have

$$\begin{aligned}
&\left|\int_{\mathcal{C}\times D}\mathcal{E}_\gamma[L'_T(x,r)^2(\Lambda_T(G'(x))-\Lambda_T(G(x)))]\,\mathrm{d}\mu(x,r)\right|\\
&\quad\le C_T\left\{(d_T^V(\mu,\nu)+\varepsilon)\int_{\mathcal{C}\times D}\mathcal{E}_\gamma[L'_T(x,r)^2]\,\mathrm{d}\mu(x,r)\right.\\
&\qquad\left.+\int_{\mathcal{C}\times D}\int_0^T\mathcal{E}_\gamma\Big[|G_t(x)^2-G'_t(x)^2|L'_T(x,r)^2\Big]\,\mathrm{d}t\,\mathrm{d}\mu(x,r)\right\}\\
&\quad\le C_T(d_T^V(\mu,\nu)+\varepsilon)\int_{\mathcal{C}\times D}\mathcal{E}_\gamma[L'_T(x,r)^2]\,\mathrm{d}\mu(x,r),
\end{aligned}$$

where the second equality follows from the Cauchy–Schwarz inequality, since Isserlis' theorem ensures that

$$\begin{aligned}
&\mathcal{E}_\gamma[(G'_t(x)-G_t(x))^2L'_T(x,r)^2]\\
&\quad=\mathcal{E}_\gamma[(G'_t(x)-G_t(x))^2]\mathcal{E}_\gamma[L'_T(x,r)^2]+2\mathcal{E}_\gamma[(G'_t(x)-G_t(x))L'_T(x,r)]^2\\
&\quad\le 3\mathcal{E}_\gamma[(G'_t(x)-G_t(x))^2]\mathcal{E}_\gamma[L'_T(x,r)^2]\\
&\quad\le 3(d_T(\mu,\nu)+\varepsilon)^2\mathcal{E}_\gamma[L'_T(x,r)^2],
\end{aligned}$$

again using Cauchy–Schwarz for the first inequality and, similarly,

$$\mathcal{E}_\gamma[(G'_t(x)+G_t(x))^2L'_T(x,r)^2]\le C_T\,\mathcal{E}_\gamma[L'_T(x,r)^2].$$

As a consequence, and again using Cauchy–Schwarz,

$$\begin{aligned}
&|\Gamma_{2,\nu}(\mu)-\Gamma_2(\mu)|\\
&\quad\le C_T\Bigg\{\underbrace{\prod_{\varepsilon=\pm1}\left(\int_{\mathcal{C}\times D}\mathcal{E}_\gamma\left[\left(\int_0^T(G_t(x)+\varepsilon G'_t(x))\,\mathrm{d}V_t^\nu(x,r)\right)^2\right]\mathrm{d}\mu(x,r)\right)^{1/2}}_{B_1}\\
&\qquad+(d_T^V(\mu,\nu)+\varepsilon)\underbrace{\int_{\mathcal{C}\times D}\mathcal{E}_\gamma[L'_T(x,r)^2]\,\mathrm{d}\mu(x,r)}_{B_2}\\
&\qquad+\underbrace{\left(\int_{\mathcal{C}\times D}\left|\int_0^T(m_\nu-m_\mu)(t,x)\,\mathrm{d}W_t(x,r)\right|^2\mathrm{d}\mu(x,r)\right)^{1/2}}_{B_3}\\
&\qquad+\underbrace{\prod_{\varepsilon=\pm1}\left(\int_{\mathcal{C}\times D}\mathcal{E}_\gamma\left[\left\{\int_0^T G_t(x)((1+\varepsilon)\,\mathrm{d}W_t(x,r)-(m_\mu(t,x)+\varepsilon m_\nu(t,x))\,\mathrm{d}t)\right\}^2\right]\mathrm{d}\mu(x,r)\right)^{1/2}}_{B_4}\Bigg\}.
\end{aligned}$$

We remark that these four terms can be cast in the form

$$\int_{\mathcal{C}\times D}\mathcal{E}_\gamma\left[\left(\int_0^T H_t(G,G',\mu,\nu)(x)(\alpha\,\mathrm{d}W_t(x,r)-M_t(\mu,\nu)(x)\,\mathrm{d}t)\right)^2\right]\mathrm{d}\mu(x,r)$$

with α equal to 0 or 1. Controlling such terms is the aim of the next technical lemma.

Lemma 3.2. *Let $\mu \in \mathcal{M}_1^+(\mathcal{C} \times D)$ with $\mu \ll P$ and let the filtration $(\mathcal{F}_t^{\boldsymbol{x}})_{t\in[0,T]}$ on $\mathcal{C}$ be $\mathcal{F}_t^{\boldsymbol{x}} := \sigma(x_s, 0 \le s \le t)$ (the σ-algebra on $\mathcal{C}$ generated by the coordinate process up to time t). Also let:*

- *$x \in \mathcal{C} \to (M_t(x))_{t\in[0,T]}$ be a bounded time-continuous process progressively measurable for the filtration $(\mathcal{F}_t^{\boldsymbol{x}})_{t\in[0,T]}$ and continuous in x,*
- *$(x, \omega) \in \mathcal{C} \times \hat{\Omega} \to (H_t(x, \omega))_{t\in[0,T]}$ be a progressively measurable process for the filtration $(\mathcal{F}_t^{\boldsymbol{x}} \otimes \hat{\mathcal{F}})_{t\in[0,T]}$ such that $(H_t(x, \cdot), t \in [0, T])_{x\in\mathcal{C}}$ is a continuous family of γ-Gaussian processes (possibly deterministic) with uniformly bounded covariance,*

and define

$$A(\mu) := \int_{\mathcal{C}\times D}\int_{\hat{\Omega}}\bigg(\int_0^T H_t(x,\omega)(\alpha\,\mathrm{d}W_t(x,r) - M_t(x)\,\mathrm{d}t)\bigg)^2\,\mathrm{d}\gamma(\omega)\,\mathrm{d}\mu(x,r)$$

with $\alpha \in \{0, 1\}$. Then, there exists a constant $C_T > 0$ independent of μ such that

$$A(\mu) \le C_T\Big\{\alpha(I(\mu \mid P) + 1) + \sup_{x\in\mathcal{C},\, t\in[0,T]} M_t^2(x)\Big\}\sup_{x\in\mathcal{C},\, t\in[0,T]}\mathcal{E}_\gamma[H_t^2(x)]$$

with the right-hand side being possibly infinite.

Proof. As $(a + b)^2 \le 2a^2 + 2b^2$ for all $a, b \in \mathbb{R}$,

$$\begin{aligned} A(\mu,\nu) &\le 2\int_{\mathcal{C}\times D}\int_{\hat{\Omega}}\bigg\{\alpha\underbrace{\bigg(\int_0^T H_t(x,\omega)\,\mathrm{d}W_t(x,r)\bigg)}_{N_T(x,\omega,r)}^2 \\ &\qquad + \bigg(\int_0^T H_t(x,\omega)M_t(x)\,\mathrm{d}t\bigg)^2\bigg\}\,\mathrm{d}\gamma(\omega)\,\mathrm{d}\mu(x,r) \\ &\le 2\alpha\int_{\hat{\Omega}}\int_{\mathcal{C}\times D} N_T^2(x,\omega,r)\,\mathrm{d}\mu(x,r)\,\mathrm{d}\gamma(\omega) \\ &\qquad + 2T\int_{\mathcal{C}\times D}\int_0^T M_t^2(x)\mathcal{E}_\gamma[H_t^2(x)]\,\mathrm{d}t\,\mathrm{d}\mu(x,r) \end{aligned}$$

using Cauchy–Schwarz and Fubini in the second inequality.

Define the Radon–Nikodym density $\rho_\mu(x, r) := (\mathrm{d}\mu/\mathrm{d}P)(x, r)$ and note that for every $r \in D$, $(N_t(, \cdot, \cdot, r))$ is, γ-a.s., a well-defined P_r-martingale. Itô calculus yields, γ-a.s., the indistinguishable equality

$$N_T^2(x,\omega,r) = 2\int_0^T H_t(x,\omega)N_t(x,\omega,r)\,\mathrm{d}W_t(x,r) + \int_0^T H_t^2(x,\omega)\,\mathrm{d}t$$

under P_r so that, γ-a.s.,

$$\begin{aligned} &\int_{\mathcal{C}\times D} N_T^2(x,\omega,r)\rho_\mu(x,r)\,\mathrm{d}P(x,r) \\ &\quad = 2\int_{\mathcal{C}\times D}\int_0^T H_t(x,\omega)N_t(x,\omega,r)\,\mathrm{d}W_t(x,r)\rho_\mu(x,r)\,\mathrm{d}P(x,r) \\ &\qquad + \int_{\mathcal{C}\times D}\int_0^T H_t^2(x,\omega)\,\mathrm{d}t\rho_\mu(x,r)\,\mathrm{d}P(x,r). \end{aligned}$$

Relying again on Fubini's theorem,

$$\begin{aligned} A(\mu,\nu) \le 4\alpha \int_{\mathcal{C}\times D} \mathcal{E}_\gamma\bigg[\int_0^T H_t(x)N_t(x,r)\,\mathrm{d}W_t(x,r)\bigg]\rho_\mu(x,r)\,\mathrm{d}P(x,r) \\ + 2\int_{\mathcal{C}\times D}\int_0^T \mathcal{E}_\gamma[\alpha H_t^2(x)]\,\mathrm{d}t\,\mathrm{d}\mu(x,r) \\ + 2T\int_{\mathcal{C}\times D}\int_0^T M_t^2(x)\mathcal{E}_\gamma[H_t^2(x)]\,\mathrm{d}t\,\mathrm{d}\mu(x,r). \end{aligned} \tag{3.9}$$

Under the favorable assumptions of the lemma, the last two terms on the right-hand side of (3.9) are easily controlled taking the supremum of their integrand on $\mathcal{C}\times[0,T]$. In order to control the first term, we rely on the stochastic Fubini theorem (see [17, Theorem IV.65]) to show that, P-a.s.,

$$\tilde{N}_T(x,r) := \int_0^T \mathcal{E}_\gamma[H_t(x)N_t(x,r)]\,\mathrm{d}W_t(x,r) = \mathcal{E}_\gamma\bigg[\int_0^T H_t(x)N_t(x,r)\,\mathrm{d}W_t(x,r)\bigg].$$

In doing so, we need to ensure that:

(i) for all $r\in D$,

$$(x,\omega)\to(\tilde{H}_t(x,\omega,r) := H_t(x,\omega)N_t(x,\omega,r))_{t\in[0,T]} \quad \text{is } \hat{\mathcal{F}}\otimes\mathcal{P}\text{-measurable},$$

where $\mathcal{P}$ is the σ-algebra generated by continuous $(\mathcal{F}_t^{x})_{t\in[0,T]}$-adapted processes,

(ii) the following integrability condition holds for all $r\in D$:

$$\int_{\mathcal{C}}\int_0^T\int_{\hat{\Omega}} \tilde{H}_t(x,\omega,r)^2\,\mathrm{d}\gamma(\omega)\,\mathrm{d}t\,\mathrm{d}P_r(x) < \infty.$$

The first hypothesis is a direct consequence of the regularity and measurability hypotheses of the lemma. We now demonstrate that the second hypothesis is valid. Indeed, for any $t\in[0,T]$,

$$\begin{aligned} \int_{\mathcal{C}}\int_{\hat{\Omega}} \tilde{H}_t(x,\omega,r)^2\,\mathrm{d}\gamma(\omega)\,\mathrm{d}P_r(x) &= \int_{\mathcal{C}} \mathcal{E}_\gamma[H_t(x,r)^2N_t(x,r)^2]\,\mathrm{d}P_r(x) \\ &\le \bigg\{\int_{\mathcal{C}} \mathcal{E}_\gamma[H_t^4(x)]\,\mathrm{d}P_r(x)\bigg\}^{1/2} \mathcal{E}_\gamma\bigg[\int_{\mathcal{C}} N_t^4(x,r)\,\mathrm{d}P_r(x)\bigg]^{1/2} \\ &\le C_T\mathcal{E}_\gamma\bigg[\int_{\mathcal{C}} \langle N\rangle_t^2(x)\,\mathrm{d}P_r(x)\bigg]^{1/2} \\ &\le C_T\bigg\{\int_{\mathcal{C}}\int_0^t \mathcal{E}_\gamma[H_s^4(x)]\,\mathrm{d}s\,\mathrm{d}P_r(x)\bigg\}^{1/2} \\ &< +\infty, \end{aligned}$$

using Cauchy–Schwarz and Fubini for the first and third inequalities and the Burkhölder–Davis–Gundy inequality for the second inequality. Hence, the theorem applies so that

$$\int_{\mathcal{C}\times D}\mathcal{E}_\gamma\bigg[\int_0^T H_t(x)N_t(x,r)\,\mathrm{d}W_t(x,r)\bigg]\,\mathrm{d}\rho_\mu(x,r)\,\mathrm{d}P(x,r) = \int_{\mathcal{C}\times D}\tilde{N}_T(x,r)\,\mathrm{d}\mu(x,r).$$

Note that [5, Equation (10)] in conjunction with Cauchy–Schwarz yields

$$\int_{\mathcal{C}\times D} \tilde{N}_T(x,r)\,\mathrm{d}\mu(x,r) \le 2\bigg(\int_{\mathcal{C}\times D} \langle \tilde{N}\rangle_T(x,r)\,\mathrm{d}\mu(x,r)\bigg)^{1/2} \times \bigg(I(\mu \mid P) + \log\bigg\{\int_{\mathcal{C}\times D} \exp\bigg\{\frac{\tilde{N}_T^2(x,r)}{4\langle \tilde{N}\rangle_T(x,r)}\bigg\}\,\mathrm{d}P(x,r)\bigg\}\bigg)^{1/2}.$$

As $\tilde{N}(\cdot,r)$ is a P_r-local martingale for every $r \in D$, the DDS theorem ensures that $\tilde{N}_T(\cdot,r)^2/4\langle \tilde{N}(\cdot,r)\rangle_T$ has the same law as $B^2_{\langle \tilde{N}\rangle_T}/4\langle \tilde{N}\rangle_T$, where B is some P_r-Brownian motion, so that there exists a constant $C > 0$ satisfying

$$\log\bigg\{\int_{\mathcal{C}\times D} \exp\bigg\{\frac{\tilde{N}_T^2(x,r)}{4\langle \tilde{N}\rangle_T(x,r)}\bigg\}\,\mathrm{d}P(x,r)\bigg\} \le C.$$

We can therefore conclude that there exist two constants: $\tilde{C} > 0$ independent of time and $C_T > 0$ increasing with T such that

$$\begin{aligned}
&\int_{\mathcal{C}\times D} \tilde{N}_t(x,r)\,\mathrm{d}\mu(x,r)\\
&\quad\le \tilde{C}\bigg(\int_{\mathcal{C}\times D}\int_0^T \mathcal{E}_\gamma[H_t(x)N_t(x,r)]^2\,\mathrm{d}t\,\mathrm{d}\mu(x,r)\bigg)^{1/2}(I(\mu\mid P)+1)^{1/2}\\
&\quad\le 2\tilde{C}\sup_{(x,t)\in\mathcal{C}\times[0,T]}\{\mathcal{E}_\gamma[\langle N\rangle_t(x)H_t^2(x)]\}^{1/2}\\
&\qquad\times\bigg(\int_0^T \mathcal{E}_\gamma\bigg[\int_{\mathcal{C}\times D}\frac{N_t^2(x,r)}{4\langle N\rangle_t(x)}\,\mathrm{d}\mu(x,r)\bigg]\,\mathrm{d}t\bigg)^{1/2}(I(\mu\mid P)+1)^{1/2}\\
&\quad\le C_T\sup_{\mathcal{C}\times[0,T]}\{\mathcal{E}_\gamma[H_s^2(x)H_t^2(x)]\}^{1/2}(I(\mu\mid P)+1),
\end{aligned}$$

using Cauchy–Schwarz and Fubini for the first inequality, and DDS in the second, and [5, Equation (10)]. An application of Isserlis' theorem then yields the result. □

Returning to the Proof of Theorem 3.2, it is easy to check that $B_1,\dots,B_4$ are of the form of the terms handled in Lemma 3.2, satisfying, in particular, the adaptability conditions (we recall that the law of $G_t^\nu(x)$ depends on the trajectory of x up to time t). To conclude, we emphasize that the quantities $\sup_{x\in\mathcal{C},\,t\in[0,T]}\mathcal{E}_\gamma[(G_t(x)-G'_t(x))^2]$ and $\sup_{x\in\mathcal{C},\,t\in[0,T]}(m_\mu(t,x)-m_\nu(t,x))^2$, are bounded by $(d_T^V(\mu,\nu)+\varepsilon)^2$ (see (3.7) for the term involving the means). □

3.2. Upper bound and tightness

We are now in a position to demonstrate a partial LDP relying on the exponential tightness of the family $(Q^N(\hat{\mu}_N \in \cdot))_N$, and an upper-bound inequality for closed subsets. To prove the first point, we rely on the exponential tightness of $P^{\otimes N}$ and the short-time hypothesis (2.1), and follow the approach proposed by Ben Arous and Guionnet in [1], [10]. The second point is a consequence of an upper bound for compact sets obtained similarly as in [5, Theorem 7], and both the exponential tightness of $(Q^N(\hat{\mu}_N \in \cdot))_N$ and the suitability of H extending this bound to every closed set.

Theorem 3.3. *Under the condition* $2\sigma^2\|b\|_\infty^2 T/\lambda^2 < 1$, *we have the following:*

(i) *the sequence* $(\hat{\mu}_N)_N$ *is exponentially tight under* Q^N,

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(ii) *for any closed subset F of $\mathcal{M}_1^+(\mathcal{C} \times D)$,*

$$\limsup_{N\to\infty} \frac{1}{N} \log Q^N(\hat{\mu}_N \in F) \le -\inf_F H.$$

Proof. (i) We use the exponential tightness of the sequence $(\hat{\mu}_N)_N$ under $P^{\otimes N}$ provided by Sanov's theorem. Let K_M be a compact of $\mathcal{M}_1^+(\mathcal{C} \times D)$ such that

$$\frac{1}{N} \log P^{\otimes N}(\hat{\mu}_N \notin K_M) \le -M,$$

and we remark that the Hölder inequality yields, for any conjugate exponents (p, q) with $(p+1)p^2\sigma^2\|b\|_\infty^2 T/\lambda^2 < 1$,

$$\begin{aligned} &Q^N(\hat{\mu}_N \notin K_M) \\ &\quad\le \bigg(\int_{(\mathcal{C}\times D)^N} \exp\{pN\bar{\Gamma}(\hat{\mu}_N)\}\,\mathrm{d}P^{\otimes N}(\boldsymbol{x}, \boldsymbol{r})\bigg)^{1/p} P^{\otimes N}(\hat{\mu}_N \notin K_M)^{1/q} \\ &\quad\le \bigg(\int_{(\mathcal{C}\times D)^N} \prod_{i=1}^N \mathcal{E}_\gamma(\exp\{pX^{\hat{\mu}_N}(x^i, r_i)\})\,\mathrm{d}P^{\otimes N}(\boldsymbol{x}, \boldsymbol{r})\bigg)^{1/p} P^{\otimes N}(\hat{\mu}_N \notin K_M)^{1/q}, \end{aligned}$$

using Jensen's inequality for the second inequality. Let $(\tilde{X}^{\hat{\mu}_N,i})_{1\le i\le N}$ be independent copies of $X^{\hat{\mu}_N}$ under the measure γ. Then, by independence, Hölder's inequality, and the martingale property, we have

$$\begin{aligned} &\int_{(\mathcal{C}\times D)^N} \prod_{i=1}^N \mathcal{E}_\gamma(\exp\{pX^{\hat{\mu}_N}(x^i, r_i)\})\,\mathrm{d}P^{\otimes N}(\boldsymbol{x}, \boldsymbol{r}) \\ &\quad= \mathcal{E}_\gamma\bigg[\int_{(\mathcal{C}\times D)^N} \exp\bigg\{p\sum_{i=1}^N \tilde{X}^{\hat{\mu}_N,i}(x^i, r_i)\bigg\}\,\mathrm{d}P^{\otimes N}(\boldsymbol{x}, \boldsymbol{r})\bigg] \\ &\quad\le \bigg(\int_{(\mathcal{C}\times D)^N} \prod_{i=1}^N \mathcal{E}_\gamma\bigg[\exp\bigg\{\frac{p^2(p+1)}{2}\int_0^T (G_t^{\hat{\mu}_N}(x^i) + m_{\hat{\mu}_N}(t, x^i))^2\,\mathrm{d}t\bigg\}\bigg] \\ &\qquad\qquad \times\,\mathrm{d}P^{\otimes N}(\boldsymbol{x}, \boldsymbol{r})\bigg)^{(p-1)/p}. \end{aligned} \tag{3.10}$$

We can now proceed as in the proof of Proposition 3.2(ii) to find that there exists a constant c_T such that

$$\int_{(\mathcal{C}\times D)^N} \prod_{i=1}^N \mathcal{E}_\gamma(\exp\{pX^{\hat{\mu}_N}(x^i, r_i)\})\,\mathrm{d}P^{\otimes N}(\boldsymbol{x}, \boldsymbol{r}) \le \exp\{(p-1)c_T N\}.$$

As a consequence,

$$\limsup_{N\to+\infty} \frac{1}{N} \log Q^N(\hat{\mu}_N \notin K_M) \le (p-1)c_T - \frac{M}{q}.$$

(ii) From (i) and since H is a good rate function, it is sufficient to prove the upper bound for compact sets. We follow the method of [5, Theorem 7], relying on the following lemma. □

Lemma 3.3. *For any real number $q > 1$, if $2\sigma^2\|b\|_\infty^2 T/\lambda^2 < 1$ then there exist a strictly positive real number δ_q and a function $C_q : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{\delta\to 0} C_q(\delta) = 0$, and, for any $\delta < \delta_q$,*

$$\int_{\hat{\mu}_N \in K\cap B(\nu,\delta)} \mathcal{E}_\gamma\bigg[\prod_{i=1}^{N}(\exp\{q(\tilde{X}^{\hat{\mu}_N,i}(x^i,r_i) - \tilde{X}^{\nu,i}(x^i,r_i))\}\exp\{\tilde{X}^{\nu,i}(x^i,r_i)\})\bigg]\,\mathrm{d}P^{\otimes N}(\boldsymbol{x},\boldsymbol{r}) \le \exp\{C_q(\delta)N\}.$$

Proof. Let

$$B_N := \int_{\hat{\mu}_N \in K\cap B(\nu,\delta)} \mathcal{E}_\gamma\bigg[\prod_{i=1}^{N}(\exp\{q(\tilde{X}^{\hat{\mu}_N,i}(x^i,r_i) - \tilde{X}^{\nu,i}(x^i,r_i))\}\exp\{\tilde{X}^{\nu,i}(x^i,r_i)\})\bigg]\,\mathrm{d}P^{\otimes N}(\boldsymbol{x},\boldsymbol{r}).$$

We again split this quantity relying on the Hölder inequality with conjugate exponents (ρ,η), i.e.

$$B_N \le \Bigg\{\overbrace{\int_{(\mathcal{C}\times D)^N}\prod_{i=1}^{N}\mathcal{E}_\gamma[\exp\{\rho X^\nu(x^i,r_i)\}]\,\mathrm{d}P^{\otimes N}(\boldsymbol{x},\boldsymbol{r})}^{B_1^N}\Bigg\}^{1/\rho} \times \underbrace{\Bigg\{\int_{\hat{\mu}_N\in B(\nu,\delta)}\mathcal{E}_\gamma\bigg[\prod_{i=1}^{N}\exp\{q\eta(\tilde{X}^{\hat{\mu}_N,i}(x^i,r_i) - \tilde{X}^{\nu,i}(x^i,r_i))\}\bigg]\,\mathrm{d}P^{\otimes N}(\boldsymbol{x},\boldsymbol{r})\Bigg\}}_{B_2^N}^{1/\eta}.$$

On the one hand, we can proceed exactly as in (3.10) to obtain the existence of a constant c_T uniform in ρ and N such that

$$B_1^N \le \exp\{N(\rho-1)c_T\},$$

so that we have to choose the proper relation between $\rho - 1$ and δ. On the other hand, the second term can be handled exactly as in [5, Lemma 5]. □

4. Existence and characterization of the limit

4.1. Uniqueness of the minimum

This section is devoted to proving the existence and uniqueness of the minima of H in order to obtain exponential convergence of the empirical measure. We then proceed as in [5, Lemma 6] to obtain a convenient characterization of the minima of H.

Lemma 4.1. *Let μ be a probability measure on $\mathcal{C}\times D$ which minimizes H. Then*

$$\mu \simeq P, \qquad \mu = Q_\mu,$$

where $\mu \to Q_\mu$ introduced in (3.6) is well defined from $\mathcal{M}_1^+(\mathcal{C}\times D) \to \mathcal{M}_1^+(\mathcal{C}\times D)$.

This leads to the next result.

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Theorem 4.1. *The map $\mu \to Q_\mu$ admits a unique fixed point.*

Proof. As in [5, Lemma 3] or [1, Lemma 5.15], we can show that

$$\frac{\mathrm{d}Q_\mu}{\mathrm{d}P}(x,r) = \exp\bigg\{\int_0^T O_\mu(t,x,r)\,\mathrm{d}W_t(x,r) - \frac{1}{2}\int_0^T O_\mu^2(t,x,r)\,\mathrm{d}t\bigg\},$$

where

$$O_\mu(t,x,r) = \mathcal{E}_\gamma[\Lambda_t(G^\mu(x))G_t^\mu(x)L_t^\mu(x,r)] + m_\mu(t,x).$$

Moreover, as in [5, Theorem 6], we introduce $Q_{\mu,r} \in \mathcal{M}_1^+(\mathcal{C})$ such as

$$\mathrm{d}Q_\mu(x,r) = \mathrm{d}Q_{\mu,r}(x) \times \mathrm{d}\pi(r) \quad \text{for every } (x,r) \in \mathcal{C} \times D.$$

An application of Girsanov's theorem naturally leads us to introduce the following SDE with a putative solution that has a law equal to $Q_{\mu,r}$:

$$\mathrm{d}x_t^\mu(r) = f(r,t,x_t^\mu(r))\,\mathrm{d}t + \lambda O_\mu(t,x^\mu(r),r)\,\mathrm{d}t + \lambda\,\mathrm{d}W_t, \qquad x_0^\mu(r) = \bar{x}_0(r), \tag{4.1}$$

where W is a $\mathbb{P}$-Brownian motion and $\bar{x}_0(r) \in \mathbb{R}$ is the realization of the continuous version for the family of initial laws $(\mu_0(r))_{r\in D}$ evaluated at r; see [5, Equation (3)]. We show, in Lemma 4.2 below, that for any $(r,\mu) \in D \times \mathcal{M}_1^+(\mathcal{C}\times D)$, there exists a unique strong solution $(x_t^\mu(r))_{t\in[0,T]}$ to (4.1). Let $\nu \in \mathcal{M}_1^+(\mathcal{C}\times D)$, and define similarly $x_t^\nu(r)$ with same initial condition and Brownian

path.

Note that

$$\begin{aligned}
W_t(x^\mu(r),r) &= \int_0^t O_\mu(s,x^\mu(r),r)\,\mathrm{d}s + W_t,\\
L_t^\mu(x^\mu(r),r) &= \int_0^t G_s^\mu(x^\mu(r))\hat{O}_\mu(s,x^\mu(r),r)\,\mathrm{d}s + \int_0^t G_s^\mu(x^\mu(r))\,\mathrm{d}W_s,
\end{aligned}$$

where for any $x \in \mathcal{C}$, $\hat{O}_\mu(s,x,r) := (O_\mu(s,x,r) - m_\mu(s,r))$. Then

$$\begin{aligned}
&(x_t^\mu(r) - x_t^\nu(r))\\
&\quad= \int_0^t (f(r,s,x_s^\mu(r)) + \lambda m_\mu(s,x^\mu(r)) - f(r,s,x_s^\nu(r)) - \lambda m_\nu(s,x^\nu(r)))\,\mathrm{d}s\\
&\qquad+ \lambda\int_0^t \{\hat{O}_\mu(s,x^\mu(r),r) - \hat{O}_\nu(s,x^\mu(r),r)\}\,\mathrm{d}s\\
&\qquad+ \lambda\int_0^t\int_0^s (\tilde{K}^s_{\nu,x^\mu(r)}(s,v)\hat{O}_\nu(v,x^\mu(r),r)\\
&\qquad\qquad\qquad - \tilde{K}^s_{\nu,x^\nu(r)}(s,v)\hat{O}_\nu(v,x^\nu(r),r))\,\mathrm{d}v\,\mathrm{d}s\\
&\qquad+ \lambda\int_0^t \bigg\{\mathcal{E}_\gamma\bigg[\Lambda_s(G^\nu(x^\mu(r)))G_s^\nu(x^\mu(r))\bigg(\int_0^s G_v^\nu(x^\mu(r))\,\mathrm{d}W_v\bigg)\bigg]\\
&\qquad\qquad - \mathcal{E}_\gamma\bigg[\Lambda_s(G^\nu(x^\nu(r)))G_s^\nu(x^\nu(r))\bigg(\int_0^s G_v^\nu(x^\nu(r))\,\mathrm{d}W_v\bigg)\bigg]\bigg\}\,\mathrm{d}s.
\end{aligned} \tag{4.2}$$

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Let $\xi \in \mathcal{M}_1^+((\mathcal{C} \times D)^2)$ with marginals μ and ν. We have

$$\begin{aligned}
&\lambda(m_\mu(t, x^\mu(r)) - m_\nu(t, x^\nu(r))) \\
&\quad = \bar{J} \int_{(\mathcal{C}\times D)^2} (b(x_t^\mu(r), y_t) - b(x_t^\nu(r), y_t)) \\
&\qquad\qquad + (b(x_t^\nu(r), y_t) - b(x_t^\nu(r), z_t))\, \mathrm{d}\xi((y, r'), (z, \tilde{r})) \\
&\quad \leq K_b \bar{J}(|x_t^\mu(r) - x_t^\nu(r)| + \int_{(\mathcal{C}\times D)^2} \|y - z\|_{\infty,t}\, \mathrm{d}\xi((y, r'), (z, \tilde{r}))) \\
&\quad \leq C(|x_t^\mu(r) - x_t^\nu(r)| + d_t^V(\mu, \nu)),
\end{aligned} \tag{4.3}$$

where we take the infimum on ξ.

Furthermore, let $(\tilde{G}, \tilde{G}')$ be a γ-bi-dimensional centered Gaussian process with covariance

$$\frac{\sigma^2}{\lambda^2} \int_{(\mathcal{C}\times D)^2} \begin{pmatrix} b(x_s^\mu(r), y_s) b(x_t^\mu(r), y_t) & b(x_s^\mu(r), y_s) b(x_t^\nu(r), y_t) \\ b(x_s^\nu(r), y_s) b(x_t^\mu(r), y_t) & b(x_s^\nu(r), y_s) b(x_t^\nu(r), y_t) \end{pmatrix} \mathrm{d}\nu(y, r). \tag{4.4}$$

Then

$$\begin{aligned}
&|\tilde{K}^t_{\nu, x^\mu(r)}(t, s) - \tilde{K}^t_{\nu, x^\nu(r)}(t, s)| \\
&\quad = |\mathcal{E}_\gamma[(\Lambda_t(\tilde{G}) - \Lambda_t(\tilde{G}'))\tilde{G}_t\tilde{G}_s + \Lambda_t(\tilde{G}')(\tilde{G}_t - \tilde{G}_t')\tilde{G}_s + \Lambda_t(\tilde{G}')\tilde{G}_t'(\tilde{G}_s - \tilde{G}_s')]|.
\end{aligned}$$

Moreover, using Cauchy–Schwarz and (3.1),

$$\begin{aligned}
\mathcal{E}_\gamma[\Lambda_t(\tilde{G})(\tilde{G}_t - \tilde{G}_t')\tilde{G}_s] &\leq C_T \mathcal{E}_\gamma[(\tilde{G}_t - \tilde{G}_t')^2]^{1/2} \\
&\leq C_T \bigg(\int_{\mathcal{C}\times D} (b(x_t^\mu(r), y_t) - b(x_t^\nu(r), y_t))^2\, \mathrm{d}\nu(y, r')\bigg)^{1/2} \\
&\leq C_T |x_t^\mu(r) - x_t^\nu(r)|
\end{aligned} \tag{4.5}$$

and, using Cauchy–Schwarz and (3.2),

$$\mathcal{E}_\gamma[(\Lambda_t(\tilde{G}) - \Lambda_t(\tilde{G}'))\tilde{G}_t\tilde{G}_s] \leq C_T |x_t^\mu(r) - x_t^\nu(r)|,$$

so that

$$|\tilde{K}^t_{\nu, x^\mu(r)}(t, s) - \tilde{K}^t_{\nu, x^\nu(r)}(t, s)| \leq C_T |x_t^\mu(r) - x_t^\nu(r)|. \tag{4.6}$$

We now focus on controlling the second term of (4.2). Choose another $\xi \in \mathcal{M}_1^+((\mathcal{C} \times D)^2)$ with marginals μ and ν, and the couple (G, G') of centered γ-Gaussian process with covariance $K_\xi(\cdot, \cdot, x^\mu(r))$ given in (3.8). Replacing the couple $(G^\mu(x^\mu(r)), G^\nu(x^\mu(r)))$ by (G, G') in the term of interest, we obtain

$$\hat{O}_\mu(s, x^\mu(r), r) - \hat{O}_\nu(s, x^\mu(r), r) = \mathcal{E}_\gamma[\Lambda_s(G) G_s L_s - \Lambda_s(G') G_s' L_s'],$$

where

$$\begin{aligned}
L_t &:= \int_0^t G_s(\mathrm{d}W_s(x^\mu(r), r) - m_\mu(s, x^\mu(r))\, \mathrm{d}s), \\
L_t' &:= \int_0^t G_s'(\mathrm{d}W_s(x^\mu(r), r) - m_\nu(s, x^\mu(r))\, \mathrm{d}s).
\end{aligned}$$

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With the shorthand notation $\hat{O}(t)$ for $\hat{O}_\mu(t, x^\mu(r), r)$ and $\hat{O}'(t)$ for $\hat{O}_\nu(t, x^\mu(r), r)$, we can prove, as in [5, Theorem 9], the following:

$$
\begin{aligned}
&\hat{O}(t) - \hat{O}'(t) \\
&\quad \le \mathcal{E}_\gamma C_T \biggl([L_t^2]^{1/2} \biggl(\int_{(\mathcal{C}\times D)^2} \|y - z\|_{\infty,t}^2 \,\mathrm{d}\xi((y, r'), (z, \tilde{r})) \biggr)^{1/2} + \mathcal{E}_\gamma[(L_t - L_t')^2]^{1/2} \biggr) \\
&\quad \le C_T \biggl(1 + \mathcal{E}_\gamma \biggl[\biggl(\int_0^s G_v \,\mathrm{d}W_v \biggr)^2 \biggr] \biggr) \biggl(\int_{(\mathcal{C}\times D)^2} \|y - z\|_{\infty,s}^2 \,\mathrm{d}\xi((y, r'), (z, \tilde{r})) \biggr) \\
&\qquad + C_T \mathcal{E}_\gamma \biggl[\biggl(\int_0^s (G_v - G_v') \,\mathrm{d}W_v \biggr)^2 \biggr].
\end{aligned}
$$

As a consequence, (4.2) becomes

$$
\begin{aligned}
&\|x^\mu(r) - x^\nu(r)\|_{\infty,t}^2 \\
&\quad \le C_T \int_0^t \biggl\{ \|x^\mu(r) - x^\nu(r)\|_{\infty,s}^2 + \mathcal{E}_\gamma \biggl[\biggl(\int_0^s (G_v - G_v') \,\mathrm{d}W_v \biggr)^2 \biggr] \\
&\qquad + \biggl(1 + \mathcal{E}_\gamma \biggl[\biggl(\int_0^s G_v \,\mathrm{d}W_v \biggr)^2 \biggr] \biggr) \biggl(\int_{(\mathcal{C}\times D)^2} \|y - z\|_{\infty,s}^2 \,\mathrm{d}\xi((y, r'), (z, \tilde{r})) \biggr) \\
&\qquad + \biggl| \mathcal{E}_\gamma \biggl[\Lambda_s(G^\nu(x^\mu(r))) G_s^\nu(x^\mu(r)) \biggl(\int_0^s G_v^\nu(x^\mu(r)) \,\mathrm{d}W_v \biggr) \biggr] \\
&\qquad - \mathcal{E}_\gamma \biggl[\Lambda_s(G^\nu(x^\nu(r))) G_s^\nu(x^\nu(r)) \biggl(\int_0^s G_v^\nu(x^\nu(r)) \,\mathrm{d}W_v \biggr) \biggr] \biggr|^2 \biggr\} \,\mathrm{d}s.
\end{aligned}
$$

Relying on Gronwall's lemma, taking the expectation over both initial conditions and the Brownian path, making use of Fubini's theorem, Itô isometry, and eventually taking the infimum in ξ yields

$$
\begin{aligned}
&\mathbb{E}[\|x^\mu(r) - x^\nu(r)\|_{\infty,t}^2] \\
&\quad \le C_T \int_0^t \biggl\{ \biggl(\int_{(\mathcal{C}\times D)^2} \|y - z\|_{\infty,s}^2 \,\mathrm{d}\xi((y, r'), (z, \tilde{r})) \biggr) \\
&\qquad + \mathbb{E}\biggl[\biggl| \mathcal{E}_\gamma \biggl[\Lambda_s(G^\nu(x^\mu(r))) G_s^\nu(x^\mu(r)) \biggl(\int_0^s G_v^\nu(x^\mu(r)) \,\mathrm{d}W_v \biggr) \biggr] \\
&\qquad - \mathcal{E}_\gamma \biggl[\Lambda_s(G^\nu(x^\nu(r))) G_s^\nu(x^\nu(r)) \biggl(\int_0^s G_v^\nu(x^\nu(r)) \,\mathrm{d}W_v \biggr) \biggr] \biggr|^2 \biggr] \biggr\} \,\mathrm{d}s.
\end{aligned}
$$

In order to deal with the last term on the right-hand side, we again let $(\tilde{G}, \tilde{G}')$ be a bi-dimensional centered Gaussian process on the probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \gamma)$ with covariance given by (4.4). Also let $\mathbb{E}_\gamma[\cdot] := \mathbb{E}[\mathcal{E}_\gamma[\cdot]]$. Then, using the Cauchy–Schwarz inequality for the two inequalities,

$$
\begin{aligned}
&\mathbb{E}\biggl[\biggl| \mathcal{E}_\gamma \biggl[\Lambda_s(G^\nu(x^\mu(r))) G_s^\nu(x^\mu(r)) \biggl(\int_0^s G_v^\nu(x^\mu(r)) \,\mathrm{d}W_v \biggr) \biggr] \\
&\qquad - \mathcal{E}_\gamma \biggl[\Lambda_s(G^\nu(x^\nu(r))) G_s^\nu(x^\nu(r)) \biggl(\int_0^s G_v^\nu(x^\nu(r)) \,\mathrm{d}W_v \biggr) \biggr] \biggr|^2 \biggr] \\
&\quad = \mathbb{E}\biggl[\mathcal{E}_\gamma \biggl[\Lambda_s(\tilde{G}) \tilde{G}_s \biggl(\int_0^s \tilde{G}_v \,\mathrm{d}W_v \biggr) - \Lambda_s(\tilde{G}') \tilde{G}_s' \biggl(\int_0^s \tilde{G}_v' \,\mathrm{d}W_v \biggr) \biggr]^2 \biggr]
\end{aligned}
$$

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$$\leq \mathbb{E}_\gamma\bigg[\bigg\{\Lambda_s(\tilde{G})\tilde{G}_s\bigg(\int_0^s \tilde{G}_v\,\mathrm{d}W_v\bigg) - \Lambda_s(\tilde{G}')\tilde{G}'_s\bigg(\int_0^s \tilde{G}'_v\,\mathrm{d}W_v\bigg)\bigg\}^2\bigg]$$

$$\leq 3\mathbb{E}_\gamma\bigg[\bigg(\int_0^s \tilde{G}_v\,\mathrm{d}W_v\bigg)^4\bigg]^{1/2}(\mathbb{E}_\gamma[(\Lambda_t(\tilde{G}) - \Lambda_t(\tilde{G}'))^4\tilde{G}_t^4]^{1/2}$$
$$+ \mathbb{E}_\gamma[\Lambda_t(\tilde{G}')^4(\tilde{G}_t - \tilde{G}'_t)^4]^{1/2})$$
$$+ 3\mathbb{E}_\gamma[\Lambda_t(\tilde{G}')^4\tilde{G}'^4_t]^{1/2}\mathbb{E}_\gamma\bigg[\bigg(\int_0^s (\tilde{G}_v - \tilde{G}'_v)\,\mathrm{d}W_v\bigg)^4\bigg]^{1/2}.$$

Gaussian calculus and (4.5) yield

$$\mathcal{E}_\gamma[(\tilde{G}_t - \tilde{G}'_t)^4] = C\mathcal{E}_\gamma[(\tilde{G}_t - \tilde{G}'_t)^2]^2 \leq C_T|x_t^\mu(r) - x_t^\nu(r)|^2.$$

Then relying on (3.1), (3.2), and the Burkhölder–Davis–Gundy inequality, we obtain

$$\mathbb{E}[\|x^\mu(r) - x^\nu(r)\|^2_{\infty,t}]$$
$$\leq C_T\int_0^t\bigg\{\bigg(\int_{(\mathcal{C}\times D)^2}\|y - z\|^2_{\infty,s}\,\mathrm{d}\xi((y,r'),(z,\tilde{r}))\bigg) + \mathbb{E}[\|x^\mu(r) - x^\nu(r)\|^2_{\infty,s}]\bigg\}\,\mathrm{d}s.$$

Another use of Gronwall's lemma then yields, for any $\xi \in \mathcal{M}_1^+((\mathcal{C}\times D)^2)$ with marginals μ and ν,

$$\mathbb{E}[\|x^\mu(r) - x^\nu(r)\|^2_{\infty,t}] \leq C_T\int_0^t\bigg(\int_{(\mathcal{C}\times D)^2}\|y - z\|^2_{\infty,s}\,\mathrm{d}\xi((y,r'),(z,\tilde{r}))\bigg)\,\mathrm{d}s. \tag{4.7}$$

We now prove the regularity in space of the left-hand side in the above inequality. In fact, fix $r' \neq r \in D$, and consider $x^\mu(r')$ to be the strong solution of (4.1) with the same W but with initial condition given by $\bar{x}_0(r')$. Developing a similar analysis as above, we find that

$$\mathbb{E}[\|x^\mu(r) - x^\mu(r')\|^2_{\infty,t}] \leq C_T\{\mathbb{E}[(\bar{x}_0(r) - \bar{x}_0(r'))^2] + \|r - r'\|^2_{\mathbb{R}^d}\},$$

so that $\mathbb{E}[\|x^\mu(r) - x^\mu(r')\|^2_{\infty,t}] \to 0$ as $\|r' - r\|_{\mathbb{R}^d} \searrow 0$, by using the continuity of the initial condition; see [5, Equation (3)]. We then conclude, exactly as in [5, Theorem 9], that $r \to \mathbb{E}[d_t((x^\mu(r),r),(x^\nu(r),r)^2]$ is continuous, and that $\mu \to Q_\mu$ admits a unique fix point relying on (4.7) and a Picard iteration. □

Lemma 4.2. *For any $r \in D$ and $\mu \in \mathcal{M}_1^+(\mathcal{C}\times D)$, there exists a unique strong solution to the SDE*

$$\mathrm{d}x_t^\mu(r) = f(r,t,x_t^\mu(r))\,\mathrm{d}t + \lambda O_\mu(t,x^\mu(r),r)\,\mathrm{d}t + \lambda\,\mathrm{d}W_t, \qquad x_0^\mu(r) = \bar{x}_0(r),$$

where W is a $\mathbb{P}$-Brownian motion and $\bar{x}_0(r) \in \mathbb{R}$ is the realization of the continuous version for the family of initial laws $(\mu_0(r))_{r\in D}$.

Proof. The proof relies on Picard iterations. Let $x^0 \in \mathcal{C}$ with $x_0^0 = \bar{x}_0(r)$, and define recursively the sequence $(x_t^n, 0 \leq t \leq T)_{n\in\mathbb{N}^*}$ by

$$x_t^{n+1} = \bar{x}_0(r) + \int_0^t f(r,s,x_s^n)\,\mathrm{d}s + \lambda\int_0^t O_\mu(s,x^n,r)\,\mathrm{d}s + \lambda W_t \quad \text{for all } t \in [0,T].$$

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As a consequence, for any $t \in [0, T]$, we obtain, as in (4.2),

$$\begin{aligned}
x_t^{n+1} - x_t^n &= \int_0^t (f(r, s, x_s^n) - f(r, s, x_s^{n-1}))\,\mathrm{d}s \\
&\quad + \int_0^t \lambda(O_\mu(s, x^n, r) - O_\mu(s, x^{n-1}, r))\,\mathrm{d}s \\
&= \int_0^t (f(r, s, x_s^n) + \lambda m_\mu(s, x_s^n) - f(r, s, x_s^{n-1}) - \lambda m_\mu(s, x_s^{n-1}))\,\mathrm{d}s \\
&\quad + \lambda \int_0^t \int_0^s (\widetilde{K}^s_{\mu,x^n_\cdot}(s, v)\hat{O}_\mu(v, x^n_\cdot, r) - \widetilde{K}^s_{\mu,x^{n-1}_\cdot}(s, v)\hat{O}_\mu(v, x^{n-1}_\cdot, r))\,\mathrm{d}v\,\mathrm{d}s \\
&\quad + \lambda \int_0^t \bigg\{ \mathcal{E}_\gamma\bigg[\Lambda_s(G^\mu(x^n_\cdot))G^\mu_s(x^n_\cdot)\bigg(\int_0^s G^\mu_v(x^n_\cdot)\,\mathrm{d}W_v\bigg)\bigg] \\
&\qquad - \mathcal{E}_\gamma\bigg[\Lambda_s(G^\mu(x^{n-1}_\cdot))G^\mu_s(x^{n-1}_\cdot)\bigg(\int_0^s G^\mu_v(x^{n-1}_\cdot)\,\mathrm{d}W_v\bigg)\bigg]\bigg\}\,\mathrm{d}s.
\end{aligned}$$

Then, using (4.3) and (4.6) to deal with the two first terms on the right-hand side, and controlling the last term as in the proof of theorem 4.1, we find that, taking the expectation,

$$\mathbb{E}[\|x^{n+1} - x^n\|^2_{\infty,t}] \le C_T \int_0^t \mathbb{E}[\|x^n - x^{n-1}\|^2_{\infty,s}]\,\mathrm{d}s.$$

The conclusion follows using classical arguments. □

4.2. Convergence of the process and quenched results.

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1. Let $\delta > 0$, and $B(Q, \delta)$ be the open ball of radius δ centered in Q for the Wasserstein distance. We prove that $Q^N(\hat{\mu}_N \notin B(Q, \delta)) \to 0$ exponentially fast as $N \to \infty$. In fact, the upper bound of the LDP for the closed set $B(Q, \delta)^c$ yields

$$\limsup_{N\to\infty} \frac{1}{N} \log Q^N(\hat{\mu}_N \notin B(Q, \delta)) \le -\inf_{B(Q,\delta)^c} H < 0,$$

where the last inequality comes from the fact that H attains its unique minimum at Q. This implies that $Q^N(\hat{\mu}_N \notin B(Q, \delta)) \to 0$ at least exponentially fast, so that the result is proved. □

Proof of Theorem 2.5. Given a closed set $F \subset \mathcal{M}_1^+(\mathcal{C} \times D)$, we can show that the average upper bound of Theorem 3 is also valid for almost all realizations of the connectivity matrix J. The proof relies on the Chebychev inequality and the Borel–Cantelli lemma, and proceeds exactly as in [1, Theorem 2.7, Appendix C.]. To conclude on the quenched convergence, we will however need a stronger result ensuring not only that for a given closed set an upper bound is valid for almost all realizations of the environment variables, but that for almost all environments, the upper bound is valid for all closed sets.

To show that this quenched upper bound holds almost surely for all closed sets, we proceed as follows. Since $\mathcal{M}_1^+(\mathcal{C} \times D)$ is a Polish space, there exists a sequence of closed sets $(F_i)_{i\in\mathbb{N}}$ of $\mathcal{M}_1^+(\mathcal{C} \times D)$ such that for any closed set $F \subset \mathcal{M}_1^+(\mathcal{C} \times D)$, there exists a countable set $A_F \subset \mathbb{N}$ such that

$$F = \bigcap_{i\in A_F} F_i.$$

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We now consider $A_F^m = A_F \cap [[1, m]]$, the set of the first m indices in A_F for some $m \in \mathbb{N}^*$, and $F^{(m)} = \bigcap_{i \in A_F^m} F_i$. The sequence of closed sets $F^{(m)}$ is decreasing and converges towards F. Moreover, since the subsets of $[[1, m]]$ are countable for any $m \in \mathbb{N}$, we have for any $m \in \mathbb{N}$, the $\mathcal{P}$-almost sure upper bound

$$\limsup_{N\to\infty} \frac{1}{N} \log Q_{\boldsymbol{r}}^N(J)(\hat{\mu}_N \in F) \le -\inf_{F^{(m)}} H \quad \text{for all closed set } F \subset \mathcal{M}_1^+(\mathcal{C} \times D)\,\mathcal{P}\text{-a.s.}$$

Taking the limit superior of the right-hand side of the inequality as $m \to \infty$ and using the lower semi-continuity of the good rate function H yields the desired inequality, ensuring the $\mathcal{P}$-almost sure upper bound for any closed set.

Moreover, a classical result of large-deviations theory ensures that the sequence of empirical measures is $\mathcal{P}$-a.s. exponentially tight, as a consequence of [7, Exercice 4.1.10(c)] (relying on the results of [13, Lemma 2.6] and [18, Theorem P]).

Both the almost sure upper bound and exponential tightness in turn imply the $\mathcal{P}$-almost sure convergence of the empirical measure. Indeed, for any $\varepsilon > 0$, we have

$$Q_{\boldsymbol{r}}^N(J)(\hat{\mu}_N \notin B(Q, \varepsilon)) = Q_{\boldsymbol{r}}^N(J)(d_T(\hat{\mu}_N, Q) \ge \varepsilon),$$

which is summable in N. We can thus conclude, using the Borel–Cantelli lemma, on the quenched convergence result. □

5. Perspectives and open problems

In this paper we have investigated the dynamics of randomly interacting diffusions with complex interactions depending on the state of both particles. From the mathematical viewpoint, we have extended existing estimates on large deviations initially developed for spin-glass systems [1], [10] to the present setting. The proof entailed a combination of Sanov's theorem and an extension of Varadhan's lemma to a functional that does not directly satisfy the canonical assumptions. The limit of the system is a complex non-Markovian process with dynamics that are relatively difficult to understand at this level of generality. However, the limits obtained are valid only in the presence of noise, since Girsanov's theorem is used to relate the dynamics of the coupled system to the uncoupled system. The limit of randomly connected systems in the absence of noise is a complex issue with numerous applications, and this has received little attention in the literature. One outstanding contribution that addresses a similar question is the work of Ben Arous *et al.* [3] for spherical spin glasses. In that work, the authors characterized the thermodynamic limit of this system and analyze its long-term behavior, providing a mathematical approach for ageing. This approach uses the rotational symmetry of the Hamiltonian allowing us, by a change of orthogonal basis, to rely on results on the eigenvalues of the coupling matrix. A similar approach seems unlikely to readily extend to the setting of the present work.

In the context of neuroscience, it may be useful to consider spatially extended systems with delays in the communication, and possibly non-Gaussian interactions. It is not difficult to combine our methods to those in [5] and the specific methods developed here to extend the present results to spatially dependent interactions with space-dependent delays. Moreover, we expect that the limit obtained is universal with respect to the distribution of the connectivity coefficient as soon as their tails have a sufficiently fast decay, as demonstrated for a discrete-time neuronal network in [14]. Eventually, the results hold in cases where the intrinsic dynamics is not Lipschitz-continuous as soon as sufficient nonexplosion estimates are obtained on the solutions of the uncoupled system, as was the case in [1] and [10]. However, we mention that in this case, the original fixed-point method developed in this paper to prove existence and uniqueness of the solutions to the mean-field equations are no more valid and adequate methods need to be employed such as the ones presented in [1] and [10].

References

[1] Ben Arous, G. and Guionnet, A. (1995). Large deviations for Langevin spin glass dynamics. *Prob. Theory Relat. Fields* **102,** 455–509.

[2] Ben Arous, G. and Guionnet, A. (1997). Symmetric Langevin spin glass dynamics. *Ann. Prob.* **25,** 1367–1422.

[3] Ben Arous, G., Dembo, A. and Guionnet, A. (2001). Aging of spherical spin glasses. *Prob. Theory Relat. Fields* **120,** 1–67.

[4] Cabana, T. and Touboul, J. (2013). Large deviations, dynamics and phase transitions in large stochastic disordered neural networks. *J. Statist. Phys.* **153,** 211–269.

[5] Cabana, T. and Touboul, J. (2018). Large deviations for randomly connected neural networks: I. Spatially extended systems. *Adv. Appl. Prob.* **50,** 944–982.

[6] Carrillo, J. A., Fornasier, M., Toscani, G. and Vecil, F. (2010). Particle, kinetic, and hydrodynamic models of swarming. In *Mathematical Modeling of Collective Behavior in Socio-Economic and Life Sciences*, Birkhäuser, Boston, MA, pp. 297–336.

[7] Dembo, A. and Zeitouni, O. (2010). *Large Deviations Techniques and Applications*, 2nd edn. Springer, Berlin.

[8] Ermentrout, G. B. and Terman, D. H. (2010). *Mathematical Foundations of Neuroscience*. Springer, New York.

[9] Gerstner, W. and Kistler, W. M. (2002). Mathematical formulations of Hebbian learning. *Biol. Cybernet.* **87,** 404–415.

[10] Guionnet, A. (1997). Averaged and quenched propagation of chaos for spin glass dynamics. *Prob. Theory Relat. Fields* **109,** 183–215.

[11] Kuramoto, Y. (1975). Self-entrainment of a population of coupled non-linear oscillators. In *International Symposium on Mathematical Problems in Theoretical Physics*, Springer, Berlin, pp. 420–422.

[12] Luçon, E. (2012). Oscillateurs couplés, désordre et synchronisation. Doctoral Thesis, Université Pierre et Marie Curie.

[13] Lynch, J. and Sethuraman, J. (1987). Large deviations for processes with independent increments. *Ann. Prob.* **15,** 610–627.

[14] Moynot, O. and Samuelides, M. (2002). Large deviations and mean-field theory for asymmetric random recurrent neural networks. *Prob. Theory Relat. Fields* **123,** 41–75.

[15] Neveu, J. (1968). Processus aléatoires gaussiens. Presses de l'Université de Montréal.

[16] Pareschi, L. and Toscani, G. (2014). *Interacting Multiagent Systems: Kinetic Equations and Monte Carlo Methods*. Oxford University Press.

[17] Protter, P. E. (2005). Stochastic differential equations. In *Stochastic Integration and Differential Equations*, Springer, Berlin, pp. 249–361.

[18] Pukhalskii, A. A. (1991). On functional principle of large deviations. In *New Trends in Probability and Statistics*, VSP International Science Publishers, Moscow, pp. 198–218.

[19] Strogatz, S. (2003). *Sync: How Order Emerges from Chaos in the Universe, Nature, and Daily Life*. Hyperion, New York.

[20] Strogatz, S. H. (2000). From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators. *Physica D* **143,** 1–20.

[21] Sznitman, A.-S. (1984). Équations de type de Boltzmann, spatialement homogènes. *Z. Wahrscheinlichkeitsth.* **66,** 559–592.