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HALL SUBGROUPS AND 2-COCYCLE REGULARITY

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Abstract

Let H be a subgroup of a finite group G and let α be a complex-valued 2-cocycle of G. Conditions are found to ensure there exists a nontrivial element of H that is α-regular in G. However, a new result is established allowing a prime by prime analysis of the Sylow subgroups of $C_G(x)$ to determine the α-regularity of a given $x \in G$. In particular, this result implies that every α_H-regular element of a normal Hall subgroup H is α-regular in G.

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1. Introduction

Throughout this paper, G will denote a finite group.

DEFINITION 1.1. A 2-*cocycle* of G over $\mathbb{C}$ is a function $\alpha : G \times G \to \mathbb{C}^*$ such that $\alpha(x, 1) = 1$ and $\alpha(x, y)\alpha(xy, z) = \alpha(x, yz)\alpha(y, z)$ for all $x, y, z \in G$.

The set of all such 2-cocycles of G forms a group $Z^2(G, \mathbb{C}^*)$ under multiplication. Let $\delta : G \to \mathbb{C}^*$ be any function with $\delta(1) = 1$. Then $t(\delta)(x, y) = \delta(x)\delta(y)/\delta(xy)$ for all $x, y \in G$ is a 2-cocycle of G, which is called a *coboundary*. Two 2-cocycles α and β are *cohomologous* if there exists a coboundary $t(\delta)$ such that $\beta = t(\delta)\alpha$. This defines an equivalence relation on $Z^2(G, \mathbb{C}^*)$ and the *cohomology classes* $[\alpha]$ form a finite abelian group, called the *Schur multiplier* $M(G)$.

DEFINITION 1.2. Let α be a 2-cocycle of G. Then $x \in G$ is α-*regular* if $\alpha(x, g) = \alpha(g, x)$ for all $g \in C_G(x)$.

Obviously, if $x \in G$ is α-regular, then it is α^k-regular for any integer k; also setting $y = 1$ and $z = x$ in Definition 1.1 yields $\alpha(1, x) = 1$ for all $x \in G$ and hence 1 is α-regular. Let $\beta \in [\alpha]$. Then x is α-regular if and only if it is β-regular and any conjugate of x is also α-regular (see [5, Lemma 2.6.1]), so that one may refer to the α-regular conjugacy classes of G. Using this notation and $o(\ )$ for the order of a group element, we quote [3, Lemma 1.2(b)] for future reference.

Lemma 1.3. *Suppose $o(x)$ and $o([\alpha])$ are relatively prime. Then x is α-regular.*

Let H be a subgroup of G. Given a 2-cocycle α of G, one can define the 2-cocycle α_H of H by $\alpha_H(x,y) = \alpha(x,y)$ for all $x, y \in H$. The mapping from $Z^2(G,\mathbb{C}^*) \to Z^2(H,\mathbb{C}^*)$ defined by $\alpha \mapsto \alpha_H$ maps coboundaries of G to those of H and consequently induces the *restriction* homomorphism $\mathrm{Res}_{G,H} : M(G) \to M(H)$ defined by $[\alpha] \mapsto [\alpha_H]$. Clearly, an element $h \in H$ that is α-regular in G is α_H-regular, but the converse is in general false. The twin aims of this paper are to find conditions under which first there exists a nontrivial element of H that is α-regular in G and second that every α_H-regular element of H is α-regular in G.

There are some circumstances in which it is possible to produce a nontrivial element $x \in G$ that is α-regular for all $[\alpha] \in M(G)$. For example, this is true if $C_G(x) = \langle x\rangle$, since the Schur multiplier of a cyclic group is trivial (see [4, Proposition 2.1.1]). However, in general, α-regularity very much depends upon the choice of $[\alpha]$ as the next example demonstrates, using the *inflation* homomorphism. Let N be a normal subgroup of G. Then the mapping from $Z^2(G/N,\mathbb{C}^*) \to Z^2(G,\mathbb{C}^*)$, $\beta \mapsto \alpha$, where $\alpha(x,y) = \beta(xN, yN)$ for all $x, y \in G$ maps coboundaries of G/N to those of G and hence induces $\mathrm{Inf} : M(G/N) \to M(G), [\beta] \mapsto [\alpha]$. Using this notation, it is clear that every element of N is α-regular.

Example 1.4. Let $C_n^{(m)}$ denote the direct product of m copies of the cyclic group of order n. Let $G \cong C_{n_1} \times \cdots \times C_{n_k}$, where $n_{i+1} \mid n_i$ for $i = 1, \dots, k-1$ and $k \geq 2$. Then $M(G) \cong C_{n_2} \times C_{n_3}^{(2)} \times \cdots \times C_{n_k}^{(k-1)}$ (see [4, Corollary 2.2.12]). Also, the group of elements that are α-regular for all $[\alpha] \in M(G)$ is isomorphic to C_{n_1/n_2} (see [5, Theorem 11.8.19]). Let $R \cong C_2^{(2)}$, then $M(R) \cong C_2$ and so only the trivial element of $C_2^{(2)}$ is α-regular for $[\alpha]$ nontrivial. However, if $H \neq R$ is a subgroup of R, then every element of H is α_H-regular. Now let $S \cong C_2^{(3)}$, so that $M(S) \cong C_2^{(3)}$. Let x be a nontrivial element of S. Then $\mathrm{Inf} : M(S/\langle x\rangle) \to M(S)$ is an injective map (see [4, Theorem 2.3.10]) that produces a subgroup $\langle[\alpha]\rangle$ of order 2 of $M(S)$ in which 1 and x are the only α-regular elements. Thus, for any two different nontrivial elements $[\alpha], [\beta] \in M(S)$, the intersection of the set of α-regular elements and β-regular elements of S contains only the identity element.

2. Subgroups and regularity

Definition 2.1. Let α be a 2-cocycle of G. Then an *α-representation* of G of *dimension n* is a function $P : G \to GL(n,\mathbb{C})$ such that $P(x)P(y) = \alpha(x,y)P(xy)$ for all $x, y \in G$.

An α-representation P is also called a *projective* representation of G with 2-cocycle α, its trace function ξ is its *α-character* and $\xi(1)$, which is the dimension of P, is called the *degree* of ξ.

To avoid repetition, all α-representations of G in this section are defined over $\mathbb{C}$. Let $\mathrm{Proj}(G,\alpha)$ denote the set of all irreducible α-characters of G, the relationship between $\mathrm{Proj}(G,\alpha)$ and α-representations is much the same as that between $\mathrm{Irr}(G)$

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and (ordinary) representations of G (see [5, page 184] for details) so, for example, $\sum_{\xi\in\mathrm{Proj}(G,\alpha)}\xi(1)^2=|G|$ (see [6, Lemma 1.4.4]). Next, $x\in G$ is α-regular if and only if $\xi(x)\neq 0$ for some $\xi\in\mathrm{Proj}(G,\alpha)$ (see [6, Proposition 1.6.3]) and $|\mathrm{Proj}(G,\alpha)|$ is the number of α-regular conjugacy classes of G (see [6, Theorem 1.3.6]).

For $[\beta]\in M(G)$, there exists $\alpha\in[\beta]$ such that $o(\alpha)=o([\beta])$ and α is *class-preserving*, that is, the elements of $\mathrm{Proj}(G,\alpha)$ are class functions (see [6, Corollary 4.1.6]). Henceforward, it will be assumed, without loss of generality, that the initial choice of 2-cocycle α has these two properties. Under these assumptions, the 'standard' inner product $\langle\ ,\ \rangle$ may be defined on α_H-characters of subgroups H of G and the 'normal' orthogonality relations hold (see [6, Section 1.11.D]).

The main result in this section is the following simple observation.

LEMMA 2.2. *Let α be a 2-cocycle of G and let H be a subgroup of G. Let $\xi\in\mathrm{Proj}(G,\alpha)$ and $\gamma\in\mathrm{Proj}(H,\alpha_H)$. Suppose that either $\langle\xi_H,\gamma\rangle=0$ or $|H|\nmid\xi(1)\gamma(1)$. Then there exists a nontrivial $h\in H$ such that $\xi(h)\overline{\gamma(h)}\neq 0$ and, in particular, all such elements are α-regular in G.*

PROOF. The inner product of ξ_H and γ, which is a nonnegative integer, is defined by

$$\langle\xi_H,\gamma\rangle=\frac{1}{|H|}\Big(\xi(1)\gamma(1)+\sum_{h\in H-\{1\}}\xi(h)\overline{\gamma(h)}\Big).$$

Thus, under the two specified conditions, the summation on the right-hand side must be nonzero. □

Using Frobenius reciprocity, similar results can be obtained to those in Lemma 2.2 using induction instead of restriction and replacing $|H|$ by $|G|$.

COROLLARY 2.3. *Let α be a 2-cocycle of G and let P be a Sylow p-subgroup of G.*

(a) *Suppose that G contains a nontrivial α-regular element. Then G contains a nontrivial α-regular element of prime power order.*
(b) *Suppose that P contains a nontrivial α_P-regular element. Then P contains a nontrivial α-regular element of G.*

PROOF. Let $c_\alpha(G)$ denote the greatest common divisor of the degrees of the elements of $\mathrm{Proj}(G,\alpha)$. Then $(c_\alpha(G))_p=\min\{\gamma(1):\gamma\in\mathrm{Proj}(P,\alpha_P)\}$ (see [6, Lemma 1.4.11]), where n_p denotes the pth part of n.

For item (a), $|\mathrm{Proj}(G,\alpha)|>1$ and so there exists a prime number q such that $(c_\alpha(G))_q^2<|Q|$, where Q is a Sylow q-subgroup of G. Let $\xi\in\mathrm{Proj}(G,\alpha)$ and let $\gamma\in\mathrm{Proj}(Q,\alpha_Q)$ with $(\xi(1))_q=\gamma(1)=(c_\alpha(G))_q$. Then Q contains a nontrivial α-regular element of G from Lemma 2.2.

For item (b), $|\mathrm{Proj}(P,\alpha_P)|>1$ and the proof is the same as for item (a). □

These results give little control over the nontrivial α-regular element of G produced, so in the next section, we will seek conditions under which a given element of G is α-regular.

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3. Hall subgroups and regularity

Let H be a subgroup of G and let α be a 2-cocycle of H. Then for $g \in G$, one can define the 2-cocycle α^g of $Z^2(gHg^{-1}, \mathbb{C}^*)$ by $\alpha^g(x, y) = \alpha(g^{-1}xg, g^{-1}yg)$ for all $x, y \in gHg^{-1}$. The mapping from $Z^2(H, \mathbb{C}^*) \to Z^2(gHg^{-1}, \mathbb{C}^*)$ defined by $\alpha \mapsto \alpha^g$ maps coboundaries of H to those of gHg^{-1} and therefore induces a homomorphism called *conjugation* by g, $\mathrm{Con}^g_H : M(H) \to M(gHg^{-1})$ defined by $[\alpha] \mapsto [\alpha^g]$. So, in particular, $h \in H$ is α-regular if and only if ghg^{-1} is α^g-regular in gHg^{-1}. Next, $[\alpha]$ is *G-stable* if for all $g \in G$,

$$\mathrm{Res}_{H,H(g)}([\alpha]) = \mathrm{Res}_{gHg^{-1},H(g)}(\mathrm{Con}^g_H([\alpha])),$$

where $H(g) = H \cap gHg^{-1}$. The G-stable elements of $M(H)$ form a subgroup $M(H)^G$ of $M(H)$. In the next result, another homomorphism is mentioned, this is *corestriction* from $M(H)$ into $M(G)$, but as it will not subsequently be used, the reader is referred to [4, page 10] for details.

Next, some notation and definitions. Let π denote a set of prime numbers and let n be a positive integer. Then n_π denotes the πth part of n and n is a *π-number* if $n_\pi = n$. An element $x \in G$ and a (sub)group H of G are a *π-element* and *π-(sub)group* if $o(x)$ and $|H|$ are respectively π-numbers. Also let x_π and $x_{\pi'}$ be the unique elements in $\langle x \rangle$ such that $x = x_\pi x_{\pi'}$ with $o(x_\pi)$ a π-number and $o(x_{\pi'})$ a π'-number, where π' is the complement to π in the set of all prime numbers. A *Sylow π-subgroup S* of G is a maximal π-subgroup of G; S is a *Hall π-subgroup* of G if, in addition, $|G : S|$ is relatively prime to $|S|$. The first result generalises to Hall subgroups a theorem on the connection between the Schur multiplier of G and those of its Sylow subgroups (see [4, Theorem 2.1.2]).

PROPOSITION 3.1. *Suppose H is a Hall π-subgroup of G. Then:*

(a) *corestriction from $M(H)$ into $M(G)$ maps $M(H)^G$ isomorphically onto the Hall π-subgroup of $M(G)$;*
(b) *restriction from $M(G)$ into $M(H)$ induces an injective homomorphism,* res*, from the Hall π-subgroup of $M(G)$ into $M(H)$;*
(c) *$M(H)^G$ is a direct factor of $M(H)$ and $M(H)^G$ is the image of* res.

The proof is the same as for the aforementioned theorem with a few very minor modifications, but it relies on the fact that $|H|$ and $|G : H|$ are relatively prime. Consequently, Proposition 3.1 does not hold in general for a Sylow π-subgroup of G. However, the next result is an immediate consequence of Proposition 3.1(a).

COROLLARY 3.2. *Suppose H_1 and H_2 are Hall π-subgroups of G. Then $M(H_1)^G$ and $M(H_2)^G$ are isomorphic.*

Despite this corollary, it is possible for two Hall π-subgroups to possess nonisomorphic Schur multipliers as the following example illustrates.

EXAMPLE 3.3. Using the nomenclature and results from [2], the Mathieu group M_{23} has trivial Schur multiplier and has two conjugacy classes of Hall π-subgroups for

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$\pi = \{2, 3, 5, 7\}$. Also, these Hall π-subgroups are either isomorphic to $L_3(4) : 2_2$ or $2^4 : A_7$ and the first of these groups has a cyclic Schur multiplier of order 4, whereas for the second, it is cyclic of order 6 using Magma [1].

Given the close relationship between the Schur multiplier of a Hall π-subgroup H of G and the Hall π-subgroup of $M(G)$, one might expect a corresponding relationship between the α_H-regular elements of H and the α-regular π-elements of G.

THEOREM 3.4. *Let α be a 2-cocycle of G. Let $x \in G$ and let π be the set of prime numbers that divide $o(x)$. For each $p_i \in \pi$, let P_i be a Sylow p_i-subgroup of $C = C_G(x)$ and suppose that $\alpha(g, x) = \alpha(x, g)$ for all $g \in P_i$. Then x is α-regular in G.*

PROOF. Using the assumption that $o(\alpha) = o([\alpha]), x$ is α-regular if and only if it is α_π-regular and $\alpha_{\pi'}$-regular. Now, x is $\alpha_{\pi'}$-regular from Lemma 1.3, so we may assume $\alpha = \alpha_\pi$. Now, $\alpha' : C \times \langle x \rangle \to \mathbb{C}^*$, defined by $\alpha'(g, x^i) = \alpha(g, x^i)/\alpha(x^i, g)$ for all $g \in C$ and all integers i, is a pairing (see [4, Lemma 2.3.8]). The kernel K of the linear character $\alpha'(g, x)$ for all $g \in C$ has order divisible by $|P|$ for all Sylow p-subgroups P of C, by supposition for $p \in \pi$ and by Lemma 1.3 otherwise. (Alternatively, $|K|$ is divisible by $|P_i|$ for all $p_i \in \pi$ by supposition and the group generated by the pairing α' is isomorphic to a subgroup of $C/K \otimes \langle x \rangle$. This tensor product is trivial since the first group is a π'-group whereas the second is a π-group.) □

Two applications of Theorem 3.4 are recorded in the following corollaries.

COROLLARY 3.5. *Let α be a 2-cocycle of G and let $x \in S$ be α_S-regular for S, a Sylow π-subgroup of G. For each prime number $p_i \in \pi$, let P_i be a Sylow p_i-subgroup of $C_S(x)$ and suppose that P_i is a Sylow p_i-subgroup of $C_G(x)$. Then x is α-regular in G.*

PROOF. The set of prime numbers that divide $o(x)$ is a subset of π and so x is α-regular in G from Theorem 3.4. □

COROLLARY 3.6. *Let α be a 2-cocycle of G and let S be a Sylow π-subgroup of G. If S is normal in G, then every α_S-regular element of S is α-regular in G.*

PROOF. Let $x \in S$ be α_S-regular. Then $C_S(x) = C_G(x) \cap S$ is a normal Sylow π-subgroup of $C_G(x)$ and Corollary 3.5 applies. □

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