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Analyses of velocity, acceleration, statics, and workspace of a 2(3-SPR) serial-parallel manipulator

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SUMMARY
The kinematics, statics, and workspace of a 2(3-SPR) serial-parallel manipulator (S-PM) are studied systematically in this paper. First, a 2(3-SPR) S-PM including an upper 3-SPR parallel manipulator (PM) and a lower 3-SPR PM is constructed, and the inverse/forward displacements, velocity, acceleration, and statics of the lower and upper 3-SPR PMs are studied, respectively. Second, the kinematics and statics of the lower and upper 3-SPR PMs are combined and the displacement, velocity, acceleration, and statics of a 2(3-SPR) S-PM are analyzed systematically. Third, a workspace of the 2(3-SPR) S-PM is constructed and analyzed. Finally, the analytic solved results are given and verified by the simulation mechanism.

KEYWORDS: Parallel manipulators; Serial-parallel manipulator; Kinematics; Statics; Workspace.

1. Introduction

Robot manipulators can be serial, parallel, or hybrid. The serial manipulators (SMs) have some merits such as larger workspace, more flexibility, and simple solution of forward kinematics.[1,2] The parallel manipulators (PMs) have some merits such as higher stiffness, greater load-to-weight ratio, good stability, and simple solution of inverse kinematics.[3,4] In general, either SMs or PMs have been limited in their applications, and the advantages of SMs and PMs are mutual beneficial for designing robots.[1–8] In order to make up the shortcomings of SMs and PMs, some PMs have been connected serially to form various serial-parallel manipulators (S-PMs).[5–17] The purpose is to bring some advantages of PMs into play, and meanwhile to increase workspace and flexibility of the end moving platform. Thanks to these advantages, S-PMs are appropriate for multi-tasking machining, such as milling, drilling, deburring, and grinding, and provide more flexibility in NC machining[18,19] and robot arms and legs. In this aspect, Romdhane[6] designed a hybrid serial-parallel Stewart-like mechanism and analyzed its displacement kinematics. Waldron *et al.*,[7] Shahinpoor,[8] and Tanev[9] analyzed the inverse/forward displacement kinematics of some hybrid serial-parallel robot manipulators. Using dual vectors and matrices, Bandyopadhyay and Ghosal[10] studied analytical determination of principal twists in serial, parallel, and hybrid manipulators. Based on two kinds of 3-UPU PMs, Zheng *et al.*[11] analyzed displacement kinematics of a hybrid S-PM. Lu and Leinonen[12] studied its displacement kinematics of a multi-3-PSR S-PM. Lu and Hu[13] proposed 2(3-SPR) S-PM, and solved its active forces by CAD variation geometry. Cha *et al.*[14] solved kinematic redundancy resolution S-PM by local optimization including joint constraints. Others designed or studied different S-PMs.[15–18] Kyung *et al.*[19] analyzed the joint reaction force and driving force of the actuator of a S-PM. Kindermann and Cruse[20] proposed a numerical approach to the kinematics of serial, parallel, and hybrid chain manipulators. However, up to now, there are no efforts made toward the study on velocity, acceleration, and statics of S-PMs.

This paper focuses on analyses of kinematics, statics, and workspace of a 2(3-SPR) S-PM. Based on analyses of velocity, acceleration, statics, and workspace of a 3-SPR PM, the velocity, acceleration, statics, and workspace of 2(3-SPR) S-PM are analyzed and verified by a simulation mechanism of 2(3-SPR) S-PM. Since the 2(3-SPR) S-PM possesses the merits of both the SMs and the PMs, it has some potential applications for the robot arms, the robot legs, the S-PM machine tools, the sensor, the surgical manipulator, the tunnel borer, the barbette of war ship, and the satellite surveillance platform.

2. The 2(3-SPR) S-PM and Its DOF

A 2(3-SPR) S-PM is consisted of a lower 3-SPR PM and an upper 3-SPR PM (see Fig. 1). Two 3-SPR PMs are connected serially, so that the workspace and the flexibility are enlarged obviously. The lower 3-SPR PM is composed of a middle moving platform m, a fixed base B, and 3-SPR (spherical joint–active prismatic joint–revolute joint) legs r_i ($i = 1, 2, 3$) with the linear actuator. The upper 3-SPR PM is composed of an upper moving platform m_1, a moving base c, and 3-SPR active legs r_{i1} with the linear actuator. Here, m is a regular triangle with three vertices b_1, b_2, and b_3, three sides $l_i = l$, and a center point o; B is a regular triangle with three vertices B_1, B_2, and B_3, three sides $L_i = L$, and a central point O; m_1 is a regular triangle with three vertices b_{11}, b_{21}, and b_{31}, three sides $l_{i1} = l_1$, and a central point o_1; c is a regular triangle with three vertices B_{11}, B_{21}, and B_{31}, three sides $L_{i1} = L_1$, and a central point o. Each of the SPR legs r_i ($i = 1$, 2, 3) of the lower 3-SPR PM connects m to B by a revolute joint R on m at b_i, a leg with an active prismatic joint P, and

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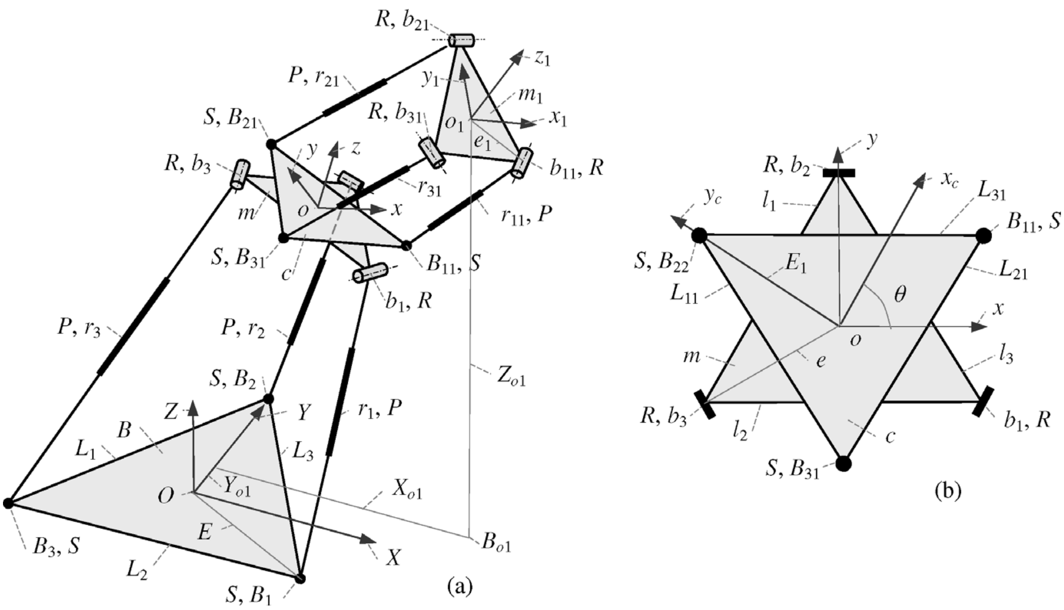

Fig. 1. The 2(3-SPR) S-PM and its composite platform.

a spherical joint S on B at B_i. Each of the SPR legs r_{i1} ($i = 1$, 2, 3) of the upper 3-SPR PM connects m_1 to c by a revolute joint R on m_1 at point b_{i1}, a leg with an active prismatic joint P, and a spherical joint S on c at B_{i1}. Let $\perp$ be a perpendicular geometric constraint and $\|$ be a parallel geometric constraint. Let $\{m_1\}$ be a coordinate o_1–$x_1y_1z_1$ attached on m_1 at o_1, $\{c\}$ be a coordinate o–$x_cy_cz_c$ attached on c at o, $\{m\}$ be a coordinate o–xyz attached on m at o, and $\{B\}$ be a coordinate O–XYZ attached on B at O. Let e_1 be the distance from o_1 to b_{i1}, e be the distance from o to b_i, E_1 be the distance from o to B_{i1}, and E be the distance from O to B_i. In structure, c and m are coplanar and form a hexagon plane with common central point o (see Fig. 1b). In addition, the structure constraints $r_{i1} \perp l_{i1}$ and $r_i \perp l_i$ ($i = 1$, 2, 3) are satisfied.

In the 2(3-SPR) S-PM, the number of links are $g_0 = 15$ for one platform m_1, one composite platform c/m, one base B, six cylinders, and six piston-rods; the number of joints is $g = 18$ for six prismatic joints, six revolute joint, and six spherical joints; the local degree of freedom (DOF) is $M_0 = 0$. Based on a revised Kutzbach–Grübler equation,[1,2] the DOF M of 2(3-SPR) S-PM is calculated as follows

$$M = 6(g_0 - g - 1) + \sum_{i=1}^{g} m_i - M_0 = 6 \times (15 - 18 - 1) + (12 \times 1 + 6 \times 3) = 6 \qquad (1)$$

3. Kinematics and Statics of the Lower 3-SPR PM

3.1. Inverse/forward displacement

A lower 3-SPR PM is shown in Fig. 2a. Its force situation is shown in Fig. 2b.

The position vectors $\boldsymbol{B}_i$ of B_i on B in $\{B\}$, the position vectors ${}^m\boldsymbol{b}_i$ of b_i on m in $\{m\}$, the position vectors $\boldsymbol{b}_i$ of b_i

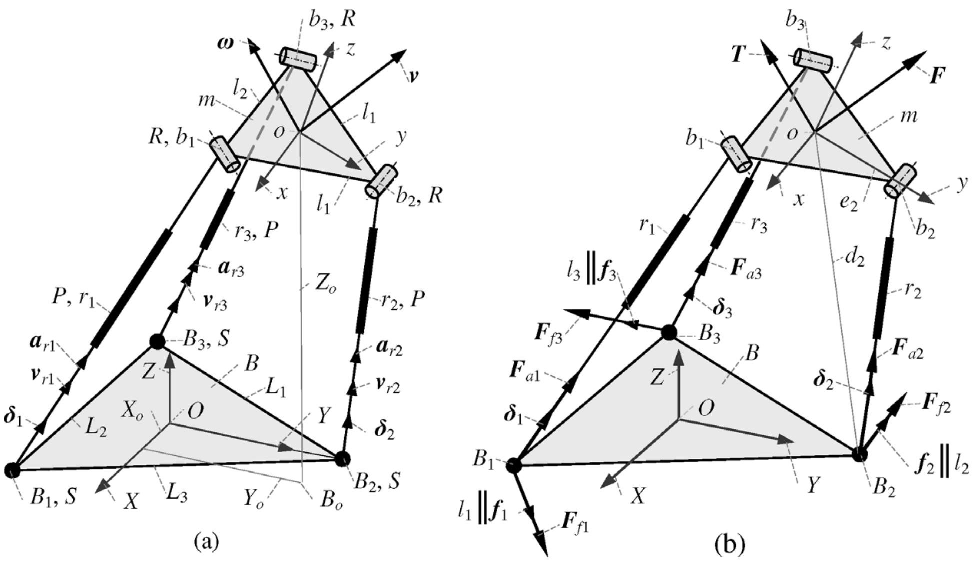

Fig. 2. The lower 3-SPR PM and it force situation.

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on m in $\{B\}$, and the position vector $\boldsymbol{o}$ of o on m in $\{B\}$ can be expressed as follows:

$$\boldsymbol{B}_i = \begin{bmatrix} X_{Bi} \\ Y_{Bi} \\ Z_{Bi} \end{bmatrix}, \quad {}^m\boldsymbol{b}_i = \begin{bmatrix} x_{bi} \\ y_{bi} \\ z_{bi} \end{bmatrix}, \quad \boldsymbol{b}_i = \begin{bmatrix} X_{ai} \\ Y_{ai} \\ Z_{ai} \end{bmatrix},$$
$$\boldsymbol{o} = \begin{bmatrix} X_o \\ Y_o \\ Z_o \end{bmatrix}, \quad {}^B_m\mathbf{R} = \begin{bmatrix} x_l & y_l & z_l \\ x_m & y_m & z_m \\ x_n & y_n & z_n \end{bmatrix}, \quad \boldsymbol{b}_i = {}^B_m\mathbf{R}\,{}^m\boldsymbol{b}_i + \boldsymbol{o}, \tag{2}$$

where $(X_o\ Y_o\ Z_o)$ are the components of $\boldsymbol{o}$ in $\{B\}$; ${}^B_m\mathbf{R}$ is a rotational transformation matrix from $\{m\}$ to $\{B\}$; $(x_l, x_m, x_n, y_l, y_m, y_n, z_l, z_m, z_n)$ are nine orientation parameters of m, their constrained equations can be obtained from refs. [1–3].

${}^m\boldsymbol{b}_i$, $\boldsymbol{b}_i$, and $\boldsymbol{B}_i$ $(i = 1, 2, 3)$ can be derived from Eq. (2) as follows:

$$ {}^m\boldsymbol{b}_1 = \frac{e}{2}\begin{bmatrix} q \\ -1 \\ 0 \end{bmatrix}, \quad {}^m\boldsymbol{b}_2 = \begin{bmatrix} 0 \\ e \\ 0 \end{bmatrix}, \quad {}^m\boldsymbol{b}_3 = \frac{e}{2}\begin{bmatrix} -q \\ -1 \\ 0 \end{bmatrix},$$
$$\boldsymbol{B}_1 = \frac{E}{2}\begin{bmatrix} q \\ -1 \\ 0 \end{bmatrix}, \quad \boldsymbol{B}_2 = \begin{bmatrix} 0 \\ E \\ 0 \end{bmatrix}, \quad \boldsymbol{B}_3 = \frac{E}{2}\begin{bmatrix} -q \\ -1 \\ 0 \end{bmatrix}, \tag{3}$$
$$\boldsymbol{b}_1 = \frac{1}{2}\begin{bmatrix} qex_l - ey_l + 2X_o \\ qex_m - ey_m + 2Y_o \\ qex_n - ey_n + 2Z_o \end{bmatrix}, \quad \boldsymbol{b}_2 = \begin{bmatrix} ey_l + X_o \\ ey_m + Y_o \\ ey_n + Z_o \end{bmatrix},$$
$$\boldsymbol{b}_3 = \frac{1}{2}\begin{bmatrix} -qex_l - ey_l + 2X_o \\ -qex_m - ey_m + 2Y_o \\ -qex_n - ey_n + 2Z_o \end{bmatrix}, \quad q = \sqrt{3}.$$

Let α, β, and λ be three Euler angles of m in $\{B\}$. Corresponding to XYX rotational orders of (α, β, λ), a rotational transformation matrix ${}^B_m\mathbf{R}$ from $\{m\}$ to $\{B\}$ can be expressed as follows[3,4]:

$$ {}^B_m\mathbf{R} = \begin{bmatrix} x_l & y_l & z_l \\ x_m & y_m & z_m \\ x_n & y_n & z_n \end{bmatrix} = \begin{bmatrix} c_\beta & s_\lambda s_\beta & c_\lambda s_\beta \\ s_\alpha s_\beta & c_\alpha c_\lambda - s_\alpha c_\beta s_\lambda & -c_\alpha s_\lambda - s_\alpha c_\beta c_\lambda \\ -c_\alpha s_\beta & s_\alpha c_\lambda + c_\alpha c_\beta s_\lambda & -s_\alpha s_\lambda + c_\alpha c_\beta c_\lambda \end{bmatrix}. \tag{4}$$

where φ is one of the $\theta, \alpha, \beta, \lambda, {}^c\alpha_1, {}^c\beta_1, {}^c\lambda_1$; $s_\varphi = \sin\varphi$, $c_\varphi = \cos\varphi$.

Obviously, $(x_l, x_m, x_n, y_l, y_m, y_n, z_l, z_m, z_n)$ can be expressed by (α, β, λ) from Eq. (4).

Based on the three structure constraints $r_i \perp l_i$ $(i = 1, 2, 3)$ and the orthogonal equations of $(x_l, x_m, x_n, y_l, y_m, y_n, z_l, z_m, z_n)$, three constraint equations can be derived as

$$X_o x_l + Y_o x_m + Z_o x_n = E x_m, \quad x_m = y_l,$$
$$X_o y_l + Y_o y_m + Z_o y_n = E(x_l - y_m)/2 \tag{5a}$$

From Eqs. (4) and (5a), $(x_l, x_m, x_n, y_l, y_m, y_n, z_l, z_m, z_n)$ can be expressed by (α, β) as follows:

$$x_l = c_\beta, \quad x_m = y_l = s_\alpha s_\beta, \quad x_n = -z_l = -c_\alpha s_\beta,$$
$$y_m = c_\alpha^2 - s_\alpha^2 c_\beta, \quad y_n = -z_m = s_\alpha c_\alpha (1 + c_\beta), \tag{5b}$$
$$z_n = -s_\alpha^2 + c_\alpha^2 c_\beta.$$

X_o and Y_o can be derived from Eqs. (4) and (5a) as follows:

$$\alpha = \lambda, \quad X_o = \frac{E x_m (3y_m - x_l) + 2Z_o z_l}{2z_n} = \frac{E\left(3c_\alpha^2 - 3s_\alpha^2 c_\beta - c_\beta\right)s_\alpha s_\beta + 2Z_o s_\beta c_\alpha}{2\left(c_\alpha^2 c_\beta - s_\alpha^2\right)},$$
$$Y_o = \frac{E x_l (x_l - y_m) - 2E y_l x_m + 2Z_o z_m}{2z_n} = \frac{E\left(c_\beta - c_\alpha^2 + s_\alpha^2 c_\beta\right)c_\beta - 2E s_\alpha^2 s_\beta^2 - 2Z_o(1 + c_\beta)c_\alpha s_\alpha}{2\left(c_\alpha^2 c_\beta - s_\alpha^2\right)}. \tag{5c}$$

Thus, X_o and Y_o can be expressed by (α, β, Z_o).

The length r_i $(i = 1, 2, 3)$ and the unit vectors $\boldsymbol{\delta}_i$ of active legs, and the vectors $\boldsymbol{e}_i$ of lines e_i, the unit vectors $\boldsymbol{f}_i$ of constrained forces $\boldsymbol{F}_{fi}$, and arm vector $\boldsymbol{d}_i$ from point o to $\boldsymbol{F}_{fi}$ have been derived in ref. [4] as follows:

$$r_1^2 = E^2 + e^2 + X_o^2 + Y_o^2 + Z_o^2 + EY_o - qEX_o - 2eEx_l + 2qeEy_l \tag{6a}$$
$$r_2^2 = E^2 + e^2 + X_o^2 + Y_o^2 + Z_o^2 - 3eEy_m + eEx_l - 2EY_o \tag{6b}$$
$$r_3^2 = E^2 + e^2 + X_o^2 + Y_o^2 + Z_o^2 + EY_o + qEX_o - 2eEx_l - 2qeEy_l. \tag{6c}$$

$$\boldsymbol{\delta}_1 = \frac{1}{2r_1}\begin{bmatrix} qex_l - ey_l + 2X_o - qE \\ qex_m - ey_m + 2Y_o + E \\ qex_n - ey_n + 2Z_o \end{bmatrix},$$
$$\boldsymbol{\delta}_2 = \frac{1}{r_2}\begin{bmatrix} ey_l + X_o \\ ey_m + Y_o - E \\ ey_n + Z_o \end{bmatrix},$$
$$\boldsymbol{\delta}_3 = \frac{1}{2r_3}\begin{bmatrix} -qex_l - ey_l + 2X_o + qE \\ -qex_m - ey_m + 2Y_o + E \\ -qex_n - ey_n + 2Z_o \end{bmatrix}, \tag{7}$$
$$\boldsymbol{e}_1 = \frac{e}{2}\begin{bmatrix} qx_l - y_l \\ qx_m - y_m \\ qx_n - y_n \end{bmatrix}, \quad \boldsymbol{e}_2 = e\begin{bmatrix} y_l \\ y_m \\ y_n \end{bmatrix},$$
$$\boldsymbol{e}_3 = -\frac{e}{2}\begin{bmatrix} qx_l + y_l \\ qx_m + y_m \\ qx_n + y_n \end{bmatrix}.$$

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$$\boldsymbol{f}_1=\frac{1}{2}\begin{bmatrix}x_l+qy_l\\x_m+qy_m\\x_n+qy_n\end{bmatrix},\quad \boldsymbol{f}_2=-\begin{bmatrix}x_l\\x_m\\x_n\end{bmatrix},$$
$$\boldsymbol{f}_3=\frac{1}{2}\begin{bmatrix}x_l-qy_l\\x_m-qy_m\\x_n-qy_n\end{bmatrix},\quad \boldsymbol{d}_1=\frac{1}{2}\begin{bmatrix}-2X_o+qE\\-2Y_o-E\\-2Z_o\end{bmatrix},\tag{8}$$
$$\boldsymbol{d}_2=\begin{bmatrix}-X_o\\-Y_o+E\\-Z_o\end{bmatrix},\quad \boldsymbol{d}_3=\frac{1}{2}\begin{bmatrix}-2X_o-qE\\-2Y_o-E\\-2Z_o\end{bmatrix}.$$

From Eqs. (3) to (8), (r_i, $\boldsymbol{\delta}_i$, $\boldsymbol{e}_i$, $\boldsymbol{f}_i$, and $\boldsymbol{d}_i$) can be expressed by (α, β, Z_o).

3.2. Forward kinematics of the lower 3-SPR PM

From Eq. (5b) and the orthogonal equations of ($x_l, x_m, x_n, y_l, y_m, y_n, z_l, z_m, z_n$), leads to

$$y_n=y_l z_l/(1-x_l),\quad y_m=\left(1-y_l^2-x_l\right)/(1-x_l).\tag{9a}$$

Two equations are derived from Eqs. (6a) and (6c) as follows:

$$\begin{gathered}r_1^2-r_3^2=2qE(2ey_l-X_o),\\ \left(r_1^2+r_3^2-2r_2^2\right)/(6E)=Y_oex_l+ey_m,\end{gathered}\tag{9b}$$
$$r_1^2+r_3^2=2\left(E^2+e^2+X_o^2+Y_o^2+Z_o^2-2eEx_l+EY_o\right).\tag{9c}$$

Equations (9a) and (9b) lead to

$$\begin{aligned}X_o&=2ey_l-\frac{r_1^2-r_3^2}{2qE},\\ Y_o&=\frac{r_1^2+r_3^2-2r_2^2}{6E}+e\left(\frac{y_l^2}{1-x_l}+x_l-1\right).\end{aligned}\tag{9d}$$

Equations (5a) and (5b) lead to

$$z_lZ_o=X_ox_l+Y_oy_l-Ey_l\Rightarrow Z_o^2=(X_ox_l+Y_oy_l-Ey_l)^2/\left(1-x_l^2-y_l^2\right).\tag{9e}$$

Equations (5a), (9a), and (9e) lead to

$$3Ey_l^2-2X_oy_l=2Y_o(1-x_l)+E(x_l-1)^2\tag{9f}$$

Equations (9d) and (9f) lead to

$$Y_o=\frac{1}{2-2x_l}\left[3Ey_l^2-4ey_l^2+\frac{y_l\left(r_1^2-r_3^2\right)}{qE}-E(x_l-1)^2\right].\tag{9g}$$

From Eqs. (9d) and (9g), an equation is derived as shown below

$$\frac{1}{2-x_l}\left[3Ey_l^2-4ey_l^2+\frac{y_l\left(r_1^2-r_3^2\right)}{qE}-E(x_l-1)^2\right]-\frac{r_1^2+r_3^2-2r_2^2}{6E}-\frac{ey_l^2}{1-x_l}+e(1-x_l)=0.\tag{9h}$$

From Eqs. (9c) and (9e), a constraint equation is derived as shown below

$$\left(\frac{r_1^2}{2}+\frac{r_3^2}{2}-E^2-e^2-X_o^2-Y_o^2+2eEx_l-EY_o\right)\times\left(1-x_l^2-y_l^2\right)=(X_ox_l+Y_oy_l-Ey_l)^2.\tag{9i}$$

From Eqs. (9g) and (9i), an equation for solving y_l is derived as shown below

$$\begin{aligned}&E^2\left(9y_l^4+x_l^4-3\right)-2y_l^2\left(3E^2x_l^2-12eEx_l+3E^2+6e^2\right.\\&\left.+r_3^2+r_1^2\right)+8y_l\left(r_1^2-r_3^2\right)(e/E-x_l)/q+2\left(E^2+2e^2\right.\\&\left.-r_1^2-r_3^2\right)x_l^2-4e^2+8eEx_l\left(1-x_l^2\right)+2r_1^2+2r_3^2\\&-\left(r_1^2-r_3^2\right)^2/(3E^2)=0.\end{aligned}\tag{9j}$$

When given r_i, the analytic results of (X_o, Y_o, Z_o), and ($\boldsymbol{\delta}_i$, $\boldsymbol{e}_i$, $\boldsymbol{f}_i$, and $\boldsymbol{d}_i$) can be solved by using Matlab software as follows: (1) Solve y_l and x_l from Eqs. (9h) and (9j); (2) Solve (X_o, Y_o, Z_o) by substituting y_l and x_l into Eqs. (9d) and (9e); (3) Determine the reasonable solutions of the multisolutions of (X_o, Y_o, Z_o) by simulation mechanism of 3-SPR PM; (4) Solve y_n and y_m from Eq. (9a); (5) Solve $\boldsymbol{\delta}_i$, $\boldsymbol{e}_i$, $\boldsymbol{f}_i$, and $\boldsymbol{d}_i$ from Eqs. (7) and (8).

3.3. General inverse/forward velocities and accelerations for lower 3-SPR PM

Let $\boldsymbol{V}$ be a general forward velocity of platform m at o in $\{B\}$. Let $\boldsymbol{v}$ and $\boldsymbol{\omega}$ be the linear velocity and the angular velocity of m at o in $\{B\}$, respectively. Let $\boldsymbol{A}$ be a general forward acceleration of the platform m at o in $\{B\}$. Let $\boldsymbol{a}$ and $\boldsymbol{\varepsilon}$ be the linear acceleration and the angular acceleration of m at o in $\{B\}$, respectively. They can be expressed as follows:

$$\begin{gathered}\boldsymbol{V}=\begin{bmatrix}\boldsymbol{v}\\\boldsymbol{\omega}\end{bmatrix},\quad \boldsymbol{v}=\begin{bmatrix}v_x\\v_y\\v_z\end{bmatrix},\quad \boldsymbol{\omega}=\begin{bmatrix}\omega_x\\\omega_y\\\omega_z\end{bmatrix},\\ \boldsymbol{A}=\begin{bmatrix}\boldsymbol{a}\\\boldsymbol{\varepsilon}\end{bmatrix},\quad \boldsymbol{a}=\begin{bmatrix}a_x\\a_y\\a_z\end{bmatrix},\quad \boldsymbol{\varepsilon}=\begin{bmatrix}\varepsilon_x\\\varepsilon_y\\\varepsilon_z\end{bmatrix}.\end{gathered}\tag{10}$$

Suppose there are two vectors $\boldsymbol{\eta}$ and ς, and a skew-symmetric matrix $S(\boldsymbol{\eta})$. They must satisfy following equations[1,2]

$$\begin{gathered}\boldsymbol{\eta}=\begin{bmatrix}\eta_x\\\eta_y\\\eta_z\end{bmatrix},\quad \varsigma=\begin{bmatrix}\varsigma_x\\\varsigma_y\\\varsigma_z\end{bmatrix},\quad S(\boldsymbol{\eta})=\begin{bmatrix}0&-\eta_z&\eta_y\\\eta_z&0&-\eta_x\\-\eta_y&\eta_x&0\end{bmatrix},\\ \boldsymbol{\eta}\times\varsigma=S(\boldsymbol{\eta})\varsigma,\\ S(\boldsymbol{\eta})^T=-S(\boldsymbol{\eta}),\\ S(\boldsymbol{\eta})^2=S(\boldsymbol{\eta})S(\boldsymbol{\eta}).\end{gathered}\tag{11}$$

Let $\boldsymbol{\eta}$ be one of the vectors $\boldsymbol{e}_i$, $\boldsymbol{\delta}_i$, $\boldsymbol{f}_i$, $\boldsymbol{d}_i$, ${}^c\boldsymbol{o}_1$, ${}^c\boldsymbol{\delta}_{i1}$, ${}^c\boldsymbol{e}_{i1}$, ${}^c\boldsymbol{f}_{i1}$, ${}^c\boldsymbol{d}_{i1}$, $\boldsymbol{\omega}$, and $\boldsymbol{\varepsilon}$.

The general inverse/forward velocities and the active forces F_{ai} $(i = 1, 2, 3)$, the constrained forces F_{fi}, the general inverse acceleration $\boldsymbol{a}_r$, and the general forward acceleration $\boldsymbol{A}$ of the lower 3-SPR PM have been derived from ref. [4] as follows:

$$\boldsymbol{v}_r = \mathbf{J}V, \quad \boldsymbol{V} = \mathbf{J}^{-1}\boldsymbol{v}_r, \quad \boldsymbol{v}_r = \begin{bmatrix} v_{r1} \\ v_{r2} \\ v_{r3} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{12}$$

$$\mathbf{J} = \begin{bmatrix} \boldsymbol{\delta}_1^T & (\boldsymbol{e}_1 \times \boldsymbol{\delta}_1)^T \\ \boldsymbol{\delta}_2^T & (\boldsymbol{e}_2 \times \boldsymbol{\delta}_2)^T \\ \boldsymbol{\delta}_3^T & (\boldsymbol{e}_3 \times \boldsymbol{\delta}_3)^T \\ \boldsymbol{f}_1^T & (\boldsymbol{d}_1 \times \boldsymbol{f}_1)^T \\ \boldsymbol{f}_2^T & (\boldsymbol{d}_2 \times \boldsymbol{f}_2)^T \\ \boldsymbol{f}_3^T & (\boldsymbol{d}_3 \times \boldsymbol{f}_3)^T \end{bmatrix}_{6\times 6}, \quad \begin{bmatrix} F_{a1} \\ F_{a2} \\ F_{a3} \\ F_{f1} \\ F_{f2} \\ F_{f3} \end{bmatrix} = -(\mathbf{J}^T)^{-1} \begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{T} \end{bmatrix}.$$

$$\boldsymbol{a}_r = \mathbf{J}\boldsymbol{A} + \boldsymbol{V}^{\mathrm{T}}\mathbf{H}\boldsymbol{V}, \; \boldsymbol{A} = \mathbf{J}^{-1}(\boldsymbol{a}_r - \boldsymbol{V}^{\mathrm{T}}\mathbf{H}\boldsymbol{V}),$$
$$\boldsymbol{a}_r = [a_{r1} \quad a_{r2} \quad a_{r3} \quad 0 \quad 0 \quad 0]^T,$$
$$w\mathbf{H} = [\mathbf{h}_1 \quad \mathbf{h}_2 \quad \mathbf{h}_3 \quad \mathbf{h}_{f1} \quad \mathbf{h}_{f2} \quad \mathbf{h}_{f3}]^{\mathrm{T}}, \tag{13}$$
$$\mathbf{h}_i = \frac{1}{r_i}\begin{bmatrix} -S(\boldsymbol{\delta}_i)^2 & S(\boldsymbol{\delta}_i)^2 S(\boldsymbol{e}_i) \\ -S(\boldsymbol{e}_i)S(\boldsymbol{\delta}_i)^2 & r_i S(\boldsymbol{e}_i)S(\boldsymbol{\delta}_i) + S(\boldsymbol{e}_i)S(\boldsymbol{\delta}_i)^2 S(\boldsymbol{e}_i) \end{bmatrix}_{6\times 6},$$
$$\mathbf{h}_{fi} = \begin{bmatrix} \mathbf{0}_{3\times 3} & -S(f_i) \\ S(f_i) & -S(f_i)S(d_i) \end{bmatrix}_{6\times 6},$$

where, $\mathbf{J}$ is a 6×6 Jacobian matrix; $\mathbf{H}$ is $6 \times 6 \times 6$ Hessian matrix of the upper 3-SPR PM. Each of items in $\mathbf{J}$ and the sub-matrices of $\mathbf{H}$ can be solved from Eqs. (5) to (8). $(\boldsymbol{F}, \boldsymbol{T})$ is a workload wrench applied on m at o in $\{B\}$. $\boldsymbol{F}$ is a concentrated force, and $\boldsymbol{T}$ is a concentrated torque.

4. Kinematics and Statics of the Upper 3-SPR PM

4.1. *Inverse/forward displacement*

Let ${}^c\boldsymbol{B}_{i1}$ be the position vector of point B_{i1} on the composite platform m/c in $\{c\}$. Let ${}^{m1}\boldsymbol{b}_{i1}$ and ${}^c\boldsymbol{b}_{i1}$ be the position vectors of point b_{i1} on the end platform m_1 in $\{m_1\}$ and $\{c\}$, respectively. Let ${}^c\mathbf{o}_1$ and $({}^c\alpha_1, {}^c\lambda_1, {}^cX_{o1}, {}^cY_{o1}, {}^cZ_{o1})$ be the position vector of m_1 at point o_1 and its pose components in $\{c\}$. Let $({}^cx_{l1}, {}^cx_{m1}, {}^cx_{n1}, {}^cy_{l1}, {}^cy_{m1}, {}^cy_{n1}, {}^cz_{l1}, {}^cz_{m1}, {}^cz_{n1})$ be orientation parameters of m_1 in $\{c\}$.

Similarly, ${}^c\boldsymbol{B}_{i1}, {}^{m1}\boldsymbol{b}_{i1}$ and ${}^c\boldsymbol{b}_{i1}$, and ${}^c\boldsymbol{o}_1$ can be solved from Eq. (3) by replacing ${}^m\boldsymbol{b}_i, \boldsymbol{b}_i, \boldsymbol{B}_i, \boldsymbol{o}, X_o, Y_o, Z_o, x_l, x_m, x_n, y_l, y_m, y_n, z_l, z_m, z_n$ with ${}^{m1}\boldsymbol{b}_{i1}, {}^c\boldsymbol{b}_{i1}, {}^c\boldsymbol{B}_{i1}, {}^c\boldsymbol{o}_1, {}^cX_{o1}, {}^cY_{o1}, {}^cZ_{o1}, {}^cx_{l1}, {}^cx_{m1}, {}^cx_{n1}, {}^cy_{l1}, {}^cy_{m1}, {}^cy_{n1}, {}^cz_{l1}, {}^cz_{m1}, {}^cz_{n1}$, respectively.

Similarly, a rotational transformation matrix ${}^c{}_{m1}\mathbf{R}$ from $\{m_1\}$ to $\{c\}$ can be derived from Eq. (4) by replacing (α, β, λ) with $({}^c\alpha_1, {}^c\beta_1, {}^c\lambda_1)$, respectively.

Similarly, ${}^cX_{o1}$ and ${}^cY_{o1}$ can be derived from Eqs. (4) to (5c) by replacing (α, β, Z_o) with $({}^c\alpha_1, {}^c\beta_1\, {}^cZ_{o1})$, respectively.

Similarly, in the inverse displacement analysis, the extensions r_{i1} of active legs can be solved from Eq. (6) by replacing e, E, and $(r_i, \alpha, \beta, Z_o)$ with e_1, E_1, and $(r_{i1}, {}^c\alpha_1, {}^c\beta_1{}^cZ_{o1})$, respectively.

Similarly, the unit vector ${}^c\boldsymbol{\delta}_{i1}$ of active leg ${}^cr_{i1}$, the vector ${}^c\boldsymbol{e}_{i_1}$ of line ${}^ce_{i1}$ from point o_1 to b_{i1} in $\{c\}$, the unit vectors ${}^c\boldsymbol{f}_{i1}$ of three constrained forces ${}^c\boldsymbol{F}_{fi1}$, and the vector ${}^c\boldsymbol{d}_{i1}$ of the arm from point o_1 to ${}^c\boldsymbol{F}_{fi1}$ in $\{c\}$ can be solved from Eqs. (7) and (8) by replacing $(e, E, x_l, x_m, x_n, y_l, y_m, y_n, z_l, z_m, z_n, X_o, Y_o, Z_o)$ with $(e_1, E_1, {}^cx_{l1}, {}^cx_{m1}, {}^cx_{n1}, {}^cy_{l1}, {}^cy_{m1}, {}^cy_{n1}, {}^cz_{l1}, {}^cz_{m1}, {}^cz_{n1}, {}^cX_{o1}, {}^cY_{o1}, {}^cZ_{o1})$, respectively. Thus, $r_{i1}, {}^c\mathbf{o}_1, {}^c\boldsymbol{\delta}_{i1}, {}^c\boldsymbol{e}_{i1}, {}^c\boldsymbol{f}_{i1}$, and ${}^c\boldsymbol{d}_{i1}$ can be expressed by $({}^c\alpha_1, {}^c\beta_1, {}^cZ_{o1})$.

Similarly, in the forward displacement analysis, $({}^cX_{o1}, {}^cY_{o1}, {}^cZ_{o1})$ in $\{c\}$ can be solved from Eqs. (9a) to (9i) by replacing $(r_i, e, E, x_l, x_m, x_n, y_l, y_m, y_n, z_l, z_m, z_n, X_o, Y_o, Z_o)$ with $(r_{i1}, e_1, E_1, {}^cx_{l1}, {}^cx_{m1}, {}^cx_{n1}, {}^cy_{l1}, {}^cy_{m1}, {}^cy_{n1}, {}^cz_{l1}, {}^cz_{m1}, {}^cz_{n1}, {}^cX_{o1}, {}^cY_{o1}, {}^cZ_{o1})$, respectively.

In forward kinematics analysis, when given r_{i1} $(i = 1, 2, 3)$, $({}^c\alpha_1, {}^c\beta_1, {}^cZ_{o1})$ can be solved by relevant implicit equations and Matlab.[4] Then, ${}^c\boldsymbol{o}_1$, ${}^c\boldsymbol{\delta}_{i1}$, ${}^c\boldsymbol{e}_{i1}$, ${}^c\boldsymbol{f}_{i1}$, and ${}^c\boldsymbol{d}_{i1}$ can be solved.

4.2. *General inverse/forward velocities and accelerations and statics*

Let ${}^c\boldsymbol{V}_1$ be a general forward velocity of m_1 at o_1 in $\{c\}$; ${}^c\boldsymbol{v}_1$ and ${}^c\boldsymbol{\omega}_1$ be the linear velocity and the angular velocity of m_1 at o_1 in $\{c\}$, respectively. Let ${}^c\boldsymbol{A}_1$ be a general forward acceleration of platform m_1 in $\{c\}$; ${}^c\boldsymbol{a}_1$ and ${}^c\boldsymbol{\varepsilon}_1$ be the linear acceleration and the angular acceleration of m_1 at o_1 in $\{c\}$, respectively. They can be expressed as follows:

$$ {}^c\boldsymbol{V}_1 = \begin{bmatrix} {}^c\boldsymbol{v}_1 \\ {}^c\boldsymbol{\omega}_1 \end{bmatrix}, \quad {}^c\boldsymbol{v}_1 = \begin{bmatrix} {}^cv_{x1} \\ {}^cv_{y1} \\ {}^cv_{z1} \end{bmatrix}, \quad {}^c\boldsymbol{\omega}_1 = \begin{bmatrix} {}^c\omega_{x1} \\ {}^c\omega_{y1} \\ {}^c\omega_{z1} \end{bmatrix},$$
$$ {}^c\boldsymbol{A}_1 = \begin{bmatrix} {}^c\boldsymbol{a}_1 \\ {}^c\boldsymbol{\varepsilon}_1 \end{bmatrix}, \quad {}^c\boldsymbol{a}_1 = \begin{bmatrix} {}^ca_{x1} \\ {}^ca_{y1} \\ {}^ca_{z1} \end{bmatrix}, \quad {}^c\boldsymbol{\varepsilon}_1 = \begin{bmatrix} {}^c\varepsilon_{x1} \\ {}^c\varepsilon_{y1} \\ {}^c\varepsilon_{z1} \end{bmatrix}. \tag{14}$$

The general inverse velocity $\boldsymbol{v}_{r1}$, general forward velocity ${}^c\boldsymbol{V}_1$, and the active forces ${}^c\boldsymbol{F}_{ai1}$ $(i = 1, 2, 3)$, and the constrained ${}^c\boldsymbol{F}_{fi1}$ in $\{c\}$ of the upper manipulator can be derived from Eq. (12) as follows:

$$\boldsymbol{v}_{r1} = \mathbf{J}_1{}^c\boldsymbol{V}_1, \quad {}^c\boldsymbol{V}_1 = \mathbf{J}_1^{-1}\boldsymbol{v}_{r1},$$

$$\boldsymbol{v}_{r1} = \begin{bmatrix} v_{r11} \\ v_{r21} \\ v_{r31} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{J}_1 = \begin{bmatrix} {}^c\boldsymbol{\delta}_{11}^T & ({}^c\boldsymbol{e}_{11} \times {}^c\boldsymbol{\delta}_{11})^T \\ {}^c\boldsymbol{\delta}_{21}^T & ({}^c\boldsymbol{e}_{21} \times {}^c\boldsymbol{\delta}_{21})^T \\ {}^c\boldsymbol{\delta}_{31}^T & ({}^c\boldsymbol{e}_{13} \times {}^c\boldsymbol{\delta}_{31})^T \\ {}^c\boldsymbol{f}_{11}^T & ({}^c\boldsymbol{d}_{11} \times {}^c\boldsymbol{f}_{11})^T \\ {}^c\boldsymbol{f}_{21}^T & ({}^c\boldsymbol{d}_{21} \times {}^c\boldsymbol{f}_{21})^T \\ {}^c\boldsymbol{f}_{31}^T & ({}^c\boldsymbol{d}_{31} \times {}^c\boldsymbol{f}_{31})^T \end{bmatrix}_{6\times 6},$$

$$\begin{bmatrix} F_{a11} \\ F_{a21} \\ F_{a31} \\ F_{f11} \\ F_{f21} \\ F_{f31} \end{bmatrix} = -(\mathbf{J}_1^T)^{-1} \begin{bmatrix} {}^c\boldsymbol{F}_1 \\ {}^c\boldsymbol{T}_1 \end{bmatrix}, \tag{15}$$

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where, $\mathbf{J}_1$ is a 6 × 6 Jacobian matrix for the upper 3-SPR PM; $({}^c\boldsymbol{F}_1, {}^c\boldsymbol{T}_1)$ is a workload wrench applied on m_1 at o_1 in $\{c\}$, ${}^c\boldsymbol{F}_1$ is a concentrated force, and ${}^c\boldsymbol{T}_1$ is a concentrated torque.

The inverse acceleration $\boldsymbol{a}_{r1}$ along active legs r_{i1} (i = 1, 2, 3) and forward acceleration of platform m_1 at o_1 in $\{c\}$ can be derived from Eqs. (11) and (13) as follows:

$$\begin{aligned}
&\boldsymbol{a}_{r1} = \mathbf{J}_1{}^c\boldsymbol{A}_1 + {}^c\boldsymbol{V}_1^{\mathrm{T}}\mathbf{H}_1{}^c\boldsymbol{V}_1, \quad {}^c\boldsymbol{A}_1 = \mathbf{J}_1^{-1}(\boldsymbol{a}_{r1} - {}^c\boldsymbol{V}_1^{\mathrm{T}}\mathbf{H}_1{}^c\boldsymbol{V}_1),\\
&\boldsymbol{a}_{r1} = \begin{bmatrix} a_{r11} & a_{r21} & a_{r31} & 0 & 0 & 0\end{bmatrix}^{\mathrm{T}},\\
&\mathbf{H}_1 = \begin{bmatrix} \mathbf{h}_{11} & \mathbf{h}_{21} & \mathbf{h}_{31} & \mathbf{h}_{f11} & \mathbf{h}_{f21} & \mathbf{h}_{f31}\end{bmatrix}^{\mathrm{T}},\\
&\mathbf{h}_{i1} = \frac{1}{r_{i1}}\begin{bmatrix} -S({}^c\delta_{i1})^2 & S({}^c\delta_{i1})^2 S({}^c e_{i1})\\ -S({}^c e_{i1})S({}^c\delta_{i1})^2 & \mathbf{h}\end{bmatrix}_{6\times 6},\\
&\mathbf{h}_{fi1} = \begin{bmatrix} \mathbf{0}_{3\times3} & -S({}^c f_{i1})\\ S({}^c f_{i1}) & -S({}^c f_{i1})S({}^c d_{i1})\end{bmatrix}_{6\times 6},\\
&\quad \mathbf{h} = r_{i1}S({}^c e_{i1})S({}^c\delta_{i1}) + S({}^c e_{i1})S({}^c\delta_{i1})^2 S({}^c e_{i1}),
\end{aligned} \tag{16}$$

where, $\mathbf{H}_1$ is a 6 × 6 × 6 Hessian matrix of the upper 3-SPR PM. Each of items in the sub-matrices of $\mathbf{H}_1$ can be solved from Eqs. (11) and (16).

5. Kinematics and Statics of the 2(3-SPR) S-PM

Let $\boldsymbol{o}_1$ be a position vector of point o_1 on platform m_1 in $\{B\}$; $\boldsymbol{V}_1$ be a general forward velocity of m_1 at o_1 in $\{B\}$; $\boldsymbol{v}_1$ and $\boldsymbol{\omega}_1$ be a linear velocity and an angular velocity of m_1 at o_1 in $\{B\}$. Let $\boldsymbol{A}_1$ be a general forward acceleration of m_1 at o_1 in $\{B\}$; $\boldsymbol{a}_1$ and $\boldsymbol{\varepsilon}_1$ be a linear acceleration and an angular acceleration of m_1 at o_1 in $\{B\}$. They can be expressed as follows:

$$\begin{aligned}
&\boldsymbol{o}_1 = \begin{bmatrix} X_{o1}\\ Y_{o1}\\ Z_{o1}\end{bmatrix}, \quad \boldsymbol{V}_1 = \begin{bmatrix} \boldsymbol{v}_1\\ \boldsymbol{\omega}_1\end{bmatrix}, \quad \boldsymbol{v}_1 = \begin{bmatrix} v_{x1}\\ v_{y1}\\ v_{z1}\end{bmatrix},\\
&\boldsymbol{\omega}_1 = \begin{bmatrix} \omega_{x1}\\ \omega_{y1}\\ \omega_{z1}\end{bmatrix}, \quad \boldsymbol{A}_1 = \begin{bmatrix} \boldsymbol{a}_1\\ \boldsymbol{\varepsilon}_1\end{bmatrix}, \quad \boldsymbol{a}_1 = \begin{bmatrix} a_{x1}\\ a_{y1}\\ a_{z1}\end{bmatrix},\\
&\boldsymbol{\varepsilon}_1 = \begin{bmatrix} \varepsilon_{x1}\\ \varepsilon_{y1}\\ \varepsilon_{z1}\end{bmatrix}.
\end{aligned} \tag{17}$$

A composite rotational matrix ${}^B_m\mathbf{R}$ from $\{m\}$ to $\{B\}$ can be derived as follows[1]:

$$\begin{aligned}
{}^B_c\mathbf{R} = {}^B_m\mathbf{R}{}^m_c\mathbf{R} &= \begin{bmatrix} c_\beta & s_\alpha s_\beta & c_\alpha s_\beta\\ s_\alpha s_\beta & c_\alpha^2 - s_\alpha^2 c_\beta & -c_\alpha s_\alpha - s_\alpha c_\alpha c_\beta\\ -c_\alpha s_\beta & s_\alpha c_\alpha + s_\alpha c_\alpha c_\beta & -s_\alpha^2 + c_\alpha^2 c_\beta\end{bmatrix}\\
&\times \begin{bmatrix} c_\theta & -s_\theta & 0\\ s_\theta & c_\theta & 0\\ 0 & 0 & 1\end{bmatrix},\\
{}^c_B\mathbf{R} &= {}^B_c\mathbf{R}^{-1} = {}^B_c\mathbf{R}^T,
\end{aligned} \tag{18}$$

where θ is an angle between x and x_c (see Fig. 1b).

Some formulae for solving $\boldsymbol{o}_1$, $\boldsymbol{v}_1$, $\boldsymbol{\omega}_1$, $\boldsymbol{a}_1$, and $\boldsymbol{\varepsilon}_1$ can be derived from Eqs. (10), (11), and (18) as follows[1,2]:

$$\begin{aligned}
\boldsymbol{o}_1 &= \boldsymbol{o} + {}^B_c\mathbf{R}{}^c\boldsymbol{o}_1,\\
\boldsymbol{v}_1 &= \boldsymbol{v} + {}^B_c\mathbf{R}{}^c\boldsymbol{v}_1 + S(\boldsymbol{\omega}){}^B_c\mathbf{R}{}^c\boldsymbol{o}_1 = \boldsymbol{v} - S({}^B_c\mathbf{R}{}^c\boldsymbol{o}_1)\boldsymbol{\omega} + {}^B_c\mathbf{R}{}^c\boldsymbol{v}_1,\\
\boldsymbol{a}_1 &= \boldsymbol{a} + {}^B_c\mathbf{R}{}^c\boldsymbol{a}_1 + 2S(\boldsymbol{\omega}){}^B_c\mathbf{R}{}^c\boldsymbol{v}_1 + S(\varepsilon){}^B_c\mathbf{R}{}^c\boldsymbol{o}_1\\
&\quad + S(\boldsymbol{\omega})S(\boldsymbol{\omega}){}^B_c\mathbf{R}{}^c\boldsymbol{o}_1\\
&= \boldsymbol{a} - S({}^B_c\mathbf{R}{}^c\boldsymbol{o}_1)\varepsilon + {}^B_c\mathbf{R}{}^c\boldsymbol{a}_1 + 2S(\boldsymbol{\omega}){}^B_c\mathbf{R}{}^c\boldsymbol{v}_1\\
&\quad + S(\boldsymbol{\omega})S(\boldsymbol{\omega}){}^B_c\mathbf{R}{}^c\boldsymbol{o}_1,\\
\boldsymbol{\omega}_1 &= \boldsymbol{\omega} + {}^B_c\mathbf{R}{}^c\boldsymbol{\omega}_1,\ \varepsilon_1 = \varepsilon + {}^B_c\mathbf{R}{}^c\varepsilon_1 + S(\boldsymbol{\omega}){}^B_c\mathbf{R}{}^c\boldsymbol{\omega}_1.
\end{aligned} \tag{19}$$

A general forward velocity $\boldsymbol{V}_1$ of o_1 in $\{B\}$ is derived from Eqs. (11), (12), and (14)–(19) as follows:

$$\begin{aligned}
&\boldsymbol{V}_1 = \mathbf{J}_v\boldsymbol{V} + \mathbf{J}_R{}^c\boldsymbol{V}_1 = \mathbf{J}_v\mathbf{J}^{-1}\boldsymbol{v}_r + \mathbf{J}_R(\mathbf{J}_1^{-1})\boldsymbol{v}_{r1},\\
&\mathbf{J}_v = \begin{bmatrix} \mathbf{E}_{3\times3} & -S({}^B_c\mathbf{R}{}^c\boldsymbol{o}_1)\\ \mathbf{0}_{3\times3} & \mathbf{E}_{3\times3}\end{bmatrix}_{6\times6}, \ \mathbf{J}_R = \begin{bmatrix} {}^B_c\mathbf{R} & \mathbf{0}_{3\times3}\\ \mathbf{0}_{3\times3} & {}^B_c\mathbf{R}\end{bmatrix}_{6\times6}.
\end{aligned} \tag{20}$$

When given (r_i, r_{i1}, v_{ri}, v_{ri1}, $i = 1, 2, 3$), $\boldsymbol{V}_1$ can be solved from Eqs. (11), (12), (15), and (20).

A general forward acceleration $\boldsymbol{A}_1$ of o_1 in $\{B\}$ is derived from Eqs. (11)–(13), and (14)–(19) as follows:

$$\begin{aligned}
\boldsymbol{A}_1 &= \mathbf{J}_v\boldsymbol{A} + \mathbf{J}_R{}^c\boldsymbol{A}_1 + \begin{bmatrix} 2S(\boldsymbol{\omega}){}^B_c\mathbf{R} & \mathbf{0}_{3\times3}\\ \mathbf{0}_{3\times3} & S(\omega){}^B_c\mathbf{R}\end{bmatrix}{}^c\boldsymbol{V}_1\\
&\quad + \begin{bmatrix} S(\boldsymbol{\omega})S(\boldsymbol{\omega}){}^B_c\mathbf{R}{}^c\boldsymbol{o}_1\\ \mathbf{0}_{3\times1}\end{bmatrix},\\
\boldsymbol{A} &= \mathbf{J}^{-1}(\boldsymbol{a}_r - \boldsymbol{V}^{\mathrm{T}}\mathbf{H}\boldsymbol{V}),\ \boldsymbol{V} = \mathbf{J}^{-1}\boldsymbol{v}_r,\\
{}^c\boldsymbol{A}_1 &= \mathbf{J}_1^{-1}\left(a_{r1} - {}^c\boldsymbol{V}_1^{\mathrm{T}}\mathbf{H}_1{}^c\boldsymbol{V}_1\right),\ {}^c\boldsymbol{V}_1 = \mathbf{J}_1^{-1}\boldsymbol{v}_{r1}.
\end{aligned} \tag{21}$$

When given (r_i, r_{i1}, v_{ri}, v_{ri1}, a_{ri}, a_{ri1}, $i = 1, 2, 3$), ${}^c\mathbf{o}_1$ and $\boldsymbol{\omega}$ can be solved from relevant equations presented in Section 4.1 and Eq. (12). Then, $\boldsymbol{A}_1$ can be solved from Eqs. (11)–(16) and (21).

When given a workload wrench $(\boldsymbol{F}_1, \boldsymbol{T}_1)$ applied on m_1 at o_1 in $\{B\}$, a workload wrench $({}^c\boldsymbol{F}_1, {}^c\boldsymbol{T}_1)$ applied on m_1 at o_1 in $\{c\}$ and a workload wrench $(\boldsymbol{F}, \boldsymbol{T})$ applied on m at o in $\{B\}$ can be derived as follows:

$$\begin{aligned}
&\begin{bmatrix} {}^c\boldsymbol{F}_1\\ {}^c\boldsymbol{T}_1\end{bmatrix} = \begin{bmatrix} {}^c_B\mathbf{R} & \mathbf{0}_{3\times3}\\ \mathbf{0}_{3\times3} & {}^c_B\mathbf{R}\end{bmatrix}_{6\times6}\begin{bmatrix} F_1\\ T_1\end{bmatrix} = \mathbf{J}_R^{\mathrm{T}}\begin{bmatrix} \boldsymbol{F}_1\\ \boldsymbol{T}_1\end{bmatrix},\\
&\begin{bmatrix} \boldsymbol{F}\\ \boldsymbol{T}\end{bmatrix} = \begin{bmatrix} \boldsymbol{F}_1\\ S({}^B_c\mathbf{R}{}^c o_1)\boldsymbol{F}_1 + \boldsymbol{T}_1\end{bmatrix} = \mathbf{G}\begin{bmatrix} \boldsymbol{F}_1\\ \boldsymbol{T}_1\end{bmatrix},\\
&\mathbf{G} = \begin{bmatrix} \mathbf{E}_{3\times3} & \mathbf{0}_{3\times3}\\ S({}^B_c\mathbf{R}{}^c\boldsymbol{o}_1) & \mathbf{E}_{3\times3}\end{bmatrix}_{6\times6}.
\end{aligned} \tag{22}$$

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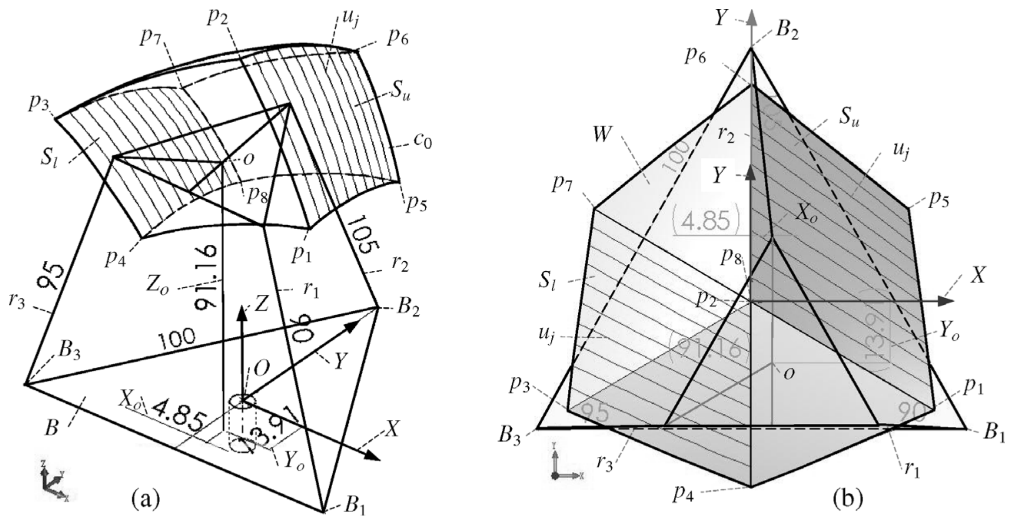

Fig. 3. A reachable workspace of the lower 3-SPR PM. (a) The isometric view and (b) the top view.

Thus, the active/constrained forces (F_{a11}, F_{a21}, F_{a31}, F_{f11}, F_{f21}, F_{f31}) of the upper 3-SPR PM and the active/constrained forces (F_{a1}, F_{a2}, F_{a3}, F_{f1}, F_{f2}, F_{f3}) of the lower 3-SPR PM can be solved from Eqs. (12), (15), and (22) as follows:

$$\begin{bmatrix} F_{a11} \\ F_{a21} \\ F_{a31} \\ F_{f11} \\ F_{f21} \\ F_{f31} \end{bmatrix} = -(\mathbf{J}_1^{\mathrm{T}})^{-1}\mathbf{J}_R^{\mathrm{T}} \begin{bmatrix} \boldsymbol{F}_1 \\ \boldsymbol{T}_1 \end{bmatrix},$$
$$\begin{bmatrix} F_{a1} \\ F_{a2} \\ F_{a3} \\ F_{f1} \\ F_{f2} \\ F_{f3} \end{bmatrix} = -(\mathbf{J}^{\mathrm{T}})^{-1}\mathbf{G} \begin{bmatrix} \boldsymbol{F}_1 \\ \boldsymbol{T}_1 \end{bmatrix}. \tag{23}$$

6. Reachable Workspace of the 2(3-SPR) S-PM

A reachable workspace W_1 of 2(3-SPR) S-PM in $\{B\}$ is defined as all positions that can be reached by the central point o_1 of the platform m_1.

When given the maximum extension $r_{\max}$ and the minimum extension $r_{\min}$ of the active legs r_i, a reachable workspace W of the lower 3-SPR PM in $\{B\}$ has been constructed by means of its simulation mechanism or some relative analytic formulae.[21] W is a volume formed by three symmetric upper surfaces about axis Z and three symmetric lower surfaces about axis Z (see Fig. 3). Each of upper surface S_u and lower surface S_l is formed by a family of spatial curves u_j ($j = 0, 1, \ldots, k$) by means of the lofting technique in CAD.

Similarly, when given the maximum extension $r_{\max 1}$ and the minimum extension $r_{\min 1}$ of the active legs r_{i1}, a reachable workspace ${}^cW_1$ of upper 3-SPR PM in $\{c\}$ can be constructed by means of its simulation mechanism.[21] ${}^cW_1$ is a volume formed by three symmetric upper surfaces ${}^cS_{u1}$ about z and three symmetric lower surfaces ${}^cS_{l1}$ about z (see Fig. 3). Each of ${}^cS_{u1}$ and ${}^cS_{l1}$ is formed by a family of spatial curves u_{j1} ($j = 1, \ldots, n$) by means of the lofting technique in CAD.[21]

In fact, ${}^cW_1$ is a sub-workspace of the reachable workspace W_1 of the 2(3-SPR) S-PM. W_1 can be constructed from W and a family of ${}^cW_1$. Its construction processes are explained as follows:

Step 1: Construct W and ${}^cW_1$ (see Fig. 3) by means of the simulation mechanism of the lower 3-SPR PM and the upper 3-SPR PM.[21]

Step 2: Move and copy ${}^cW_1$ of the upper 3-SPR PM along a curve u_1 on one of the three upper surfaces S_u of W, and construct a family of the same ${}^cW_{k1} = {}^cW_1$ ($k = 1, 2, \ldots, n_1$) by the move and copy command.

Step 3: Repeat Step 2, except that curve u_1 is replaced by u_j ($j = 2, \ldots, n$). Thus, $n_1 \times n$ same sub-workspaces ${}^cW_{k1} = {}^cW_1$ can be constructed and arranged above surface S_u by the copy command (see Fig. 4).

Step 4: Construct one of the three upper surfaces S_{u1} of W_1 from the upper surfaces ${}^cS_{u1}$ of ${}^cW_{k1}$ ($k = 1, 2, \ldots, n_1 \times n$) by the surface loft command.

Step 5: Repeat Steps 2, 3, and 4, except that the three upper surfaces S_u are replaced by three lower surfaces S_l of W, and three upper surfaces ${}^cS_{u1}$ are replaced by three lower surfaces ${}^cS_{l1}$ of ${}^cW_{k1}$. Thus, one of the three lower surfaces S_{l1} of W_1 can be constructed.

Step 6: Construct W_1 from S_{u1} and S_{l1} by circumference pattern command about Z (see Fig. 5).

7. Analytic Solved Example

Set $L = 120$ cm, $L_1 = l = 80$ cm, $l_1 = 60$ cm; $\theta = 60°$; $\boldsymbol{F}_1 = -[20\ 30\ 60]^T$ kN; $\boldsymbol{T}_1 = [-30\ -30\ 100]^T$ kN·cm. The kinematics and statics of the upper and lower 3-SPR PMs and the 2(3-SPR) S-PM are solved by using relevant analytic formulae and Matlab (see Figs. 6 and 7).

In inverse kinematic analysis, when given the independent pose parameters (Z_o, α, β, ${}^cZ_{o1}$, ${}^c\alpha_1$, ${}^c\beta_1$) and their velocities

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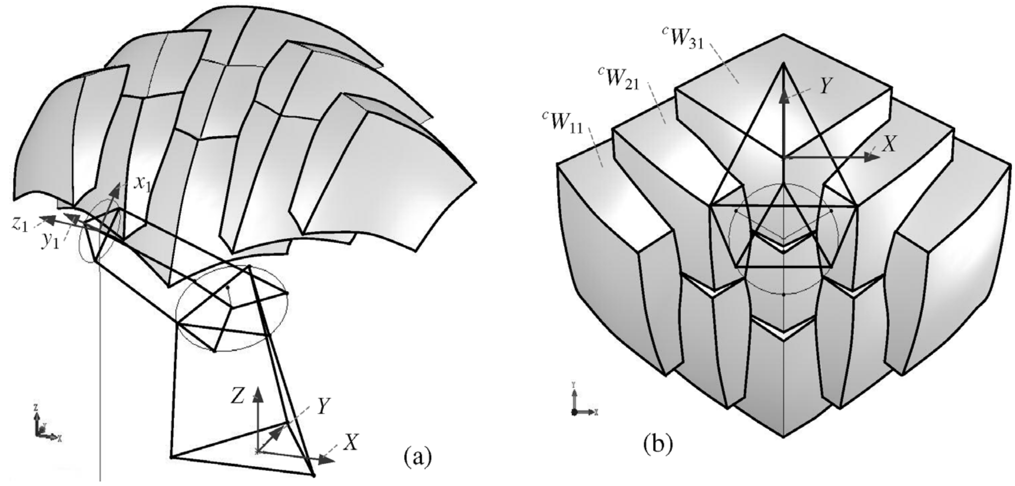

Fig. 4. Groups of reachable workspaces $^cW_{k1}$ of the upper 3-SPR PM vs. $\theta = 60°$. (a) The isometric view of groups of $^cW_1$ and (b) the top view of a family of $^cW_1$.

(see Fig. 6b, c), the extension, velocity, and acceleration of active legs r_i and r_{i1} $(i = 1, 2, 3)$ are solved (see Fig. 6a, i, j).

In forward kinematic analysis, when given the extension and velocity of active legs r_i and r_{i1} $(i = 1, 2, 3)$ (see Fig. 6a, i, j), the position, velocity, angular velocity, acceleration, and angular acceleration of platforms m in $\{B\}$ and m_1 in $\{c\}$ are solved (see Fig. 6c–h).

In forward kinematics, the position, the velocity, the angular velocity, the acceleration, and the angular acceleration of tope platform m_1 in $\{B\}$ are solved (see Fig. 7a–f).

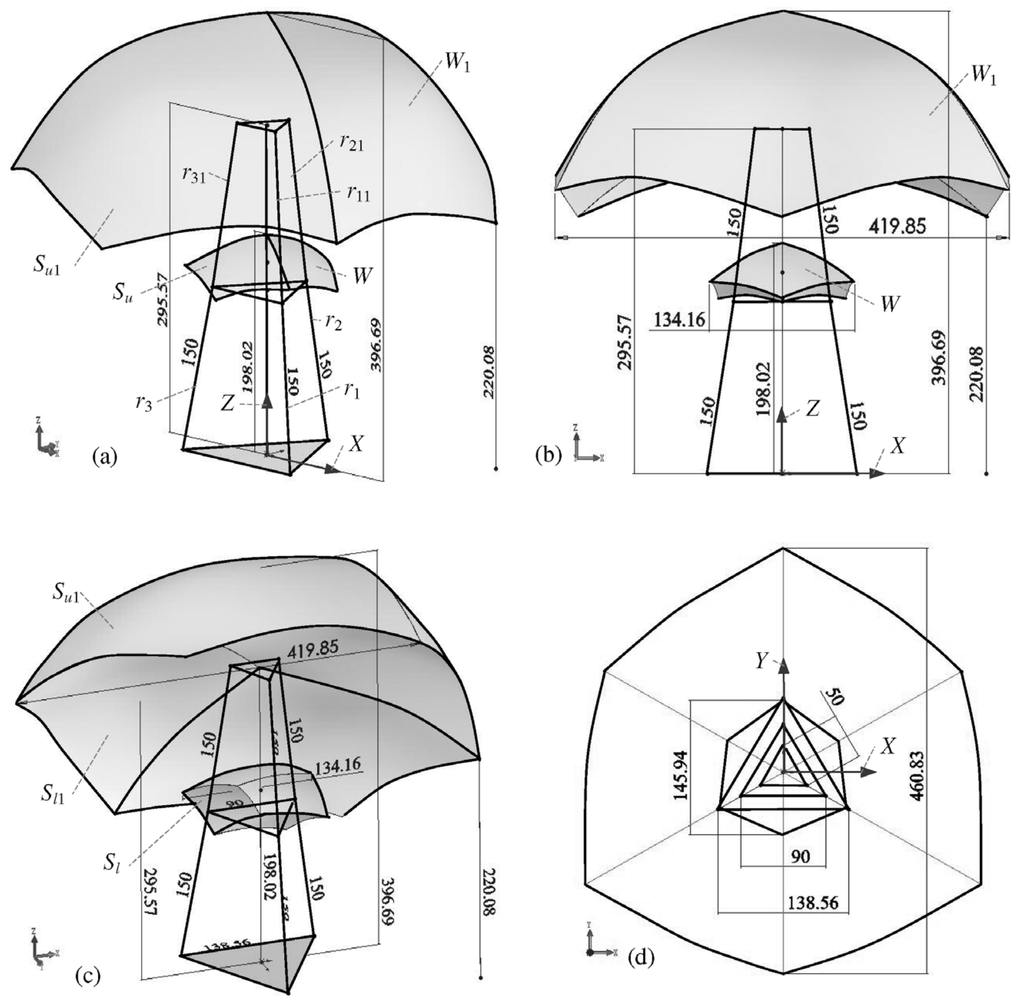

Fig. 5. A reachable workspace W_1 of 2(3-SPR) S-PM vs. $\theta = 0°$. (a) The isometric view, (b) the front view, (c) the bottom view, and (d) the top view.

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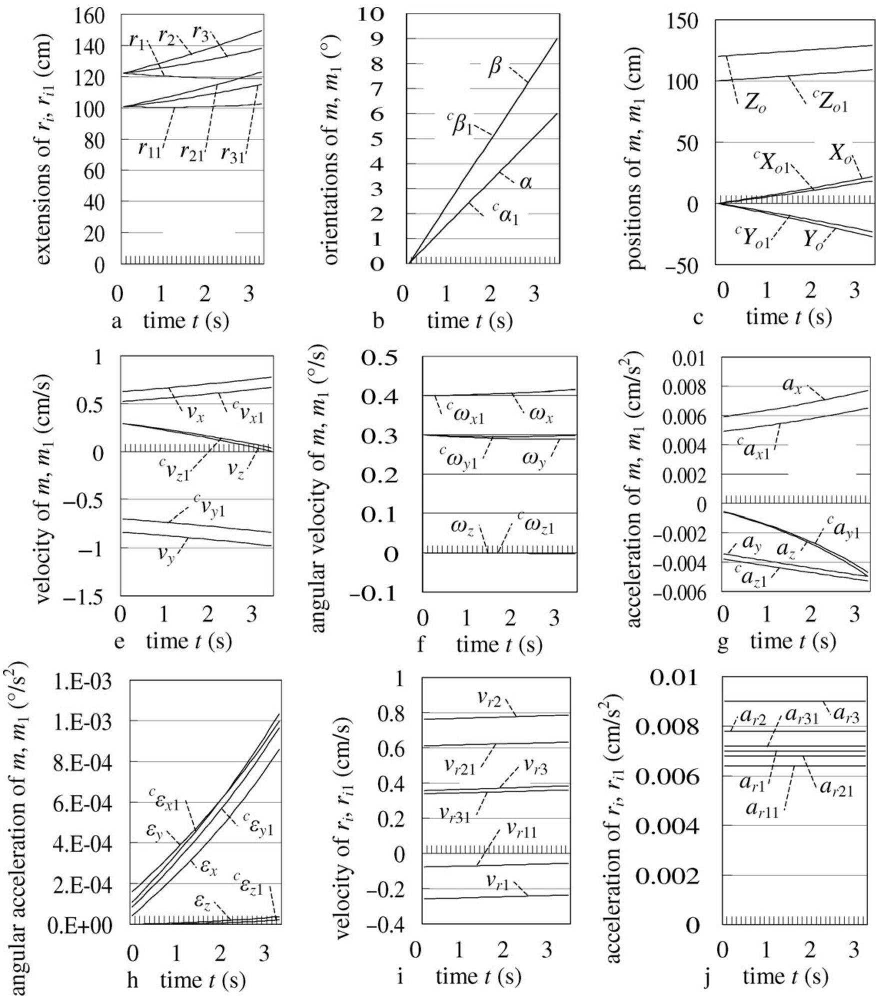

Fig. 6. Analytic solved results of the lower 3-SPR PM and the upper 3-SPR PM.

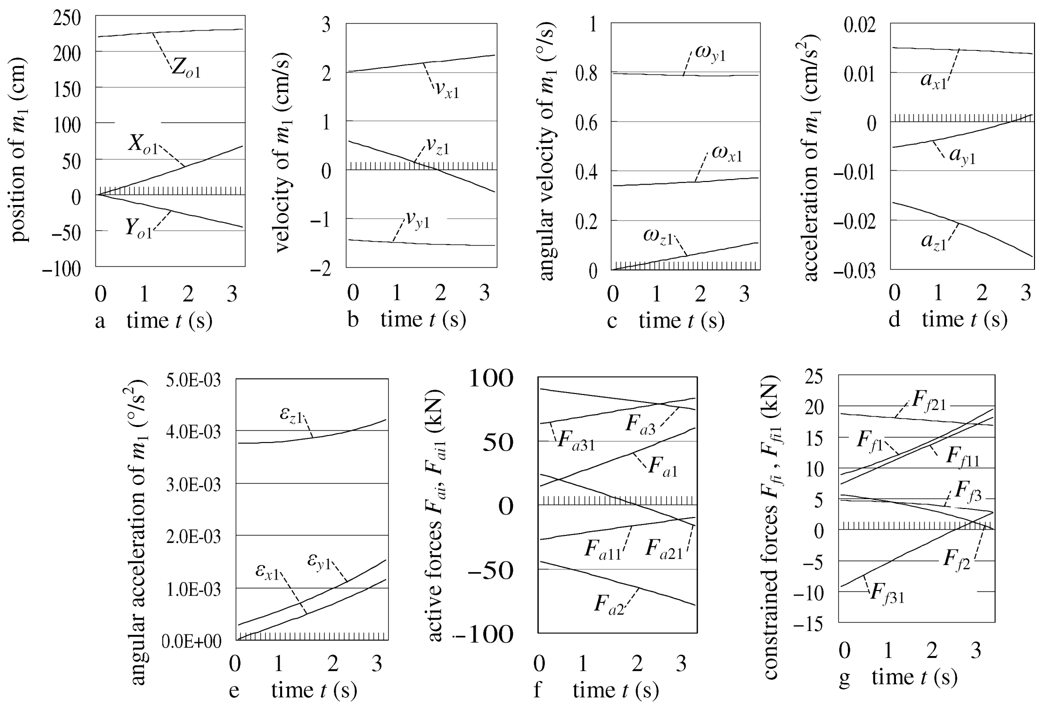

Fig. 7. Analytic solved results of 2(3-SPR) S-PM.

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The active forces along active legs r_i and r_{i1} are solved (see Fig. 7f). The constrained forces exerted onto active legs r_i and r_{i1} ($i = 1, 2, 3$) are solved (see Fig. 7g).

8. Conclusions

A 2(3-SPR) S-PM has 6-DOF and possesses merits of both SM and PM. It is composed of a 3-DOF upper 3-SPR PM and a 3-DOF lower 3-SPR PM. Each of 3-SPR PMs includes three spherical joint–active prismatic joint–revolute joint legs with linear actuator.

The workspace and the flexibility of the 2(3-SPR) S-PM is much larger than that of the upper 3-SPR PM or the lower 3-SPR PM.

The analytic formulae for solving the forward displacement, the inverse/forward velocity, the inverse/forward acceleration, and the statics of the 2(3-SPR) S-PM can be derived from the analytic formulae for solving the inverse displacement, inverse/forward velocity, and inverse/forward acceleration of the 3-SPR PM. The analytic results are verified by the simulation mechanism of 2(3-SPR) S-PM.

The 2(3-SPR) S-PM has some potential applications for the 6-DOF robot arms, the 6-DOF robot legs, the 6-DOF S-PM machine tools, the 6-DOF sensor, the 6-DOF surgical manipulator, the tunnel borer, the barbette of war ship, and the satellite surveillance platform.

This approach for solving the kinematics, statics, and workspace of 2(3-SPR) S-PM can be used to solve the kinematics, the statics, and the workspace of other kinds of S-PMs.

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