POLICY INTERACTION, LEARNING, AND THE FISCAL THEORY OF PRICES

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We investigate both the rational explosive inflation paths studied by McCallum (2001), and the classification of fiscal and monetary policies proposed by Leeper (1991), for stability under least squares (LS) learning of the rational expectations equilibria (REE). Our first result is that the explosive fiscalist REE is not locally stable under LS learning. In contrast, in Leeper's setting, there are policy regimes for which the fiscalist solution, in which fiscal variables affect the price level, can be a locally stable outcome under LS learning. However, for other policy regimes the monetarist solution is, instead, the locally stable REE.

Keywords: Inflation, Expectations, Fiscal and Monetary Policy, Explosive Price Paths

1. INTRODUCTION

Interactions between fiscal and monetary policy in the determination of the price level have been the object of a great deal of new research in recent years. One relatively new strand of research, the fiscal theory of the price level (FTPL), asserts that fiscal policy can have an important influence on the price level in models in which one might expect prices to depend only on monetary variables. An extreme specific case of FTPL asserts that, in certain specific circumstances, fiscal variables can fully determine the price level independently of monetary variables.[1]

Clearly, this extreme result is the polar opposite of the monetarist contention that the price level and the inflation rate depend primarily on monetary variables. It is thus not surprising that the fiscal theory approach has aroused a great deal of debate and controversy. These debates consider various aspects of the theory. One

point of debate concerns the extreme specific case, in which the price level follows an explosive path. McCallum (2001) has argued that this fiscalist equilibrium is an implausible "bubble equilibrium."

The influence of fiscal variables on the price level is, however, not limited to extreme cases in which the system is nonstationary. In a local analysis around a unique steady state (Leeper 1991) made an important distinction between "active" and "passive" policies (the precise definitions will be given below). In a standard model, he showed that two combinations, either (i) active monetary and passive fiscal policy or (ii) active fiscal and passive monetary policy yield determinacy, that is, a unique stationary rational expectations equilibrium (REE). In case (i), the usual monetarist view that inflation depends only on monetary policy is confirmed. However, case (ii) is fiscalist in the sense that fiscal policy, in addition to monetary policy, has an effect on the inflation rate. Leeper (1991) also showed that the steady state is indeterminate, with multiple stationary solutions, when both policies are passive, whereas the economy is explosive when both policies are active.[2]

Another point of controversy concerns the nature of the intertemporal budget constraint of the government. Buiter (2002) argues that FTPL, which does not impose this constraint as an identity, can lead to inconsistencies or anomalies, and therefore should be rejected as a "starting point for further research in monetary economics." Buiter recommends restricting attention to Ricardian "fiscal-financial-monetary programmes" in which the government's intertemporal budget constraint "holds for all admissible sequences" of relevant variables. Woodford [e.g., see Woodford (2001) and Woodford (2003b, pp. 315–316)] argues against this position and for the validity of non-Ricardian fiscal policies in which the government need only satisfy its intertemporal budget constraint in equilibrium. Particular examples are the extreme case described earlier and the active fiscal/passive monetary regime of Leeper (1991), in which fiscal policy is non-Ricardian because taxes respond only weakly to debt levels.

The significance of FTPL continues to be debated. Two recent papers by Bassetto (2002) and Niepelt (2004) readdress the significance of FTPL by strengthening the notion of rationality used in the earlier literature. Bassetto (2002) casts the analysis in terms of a market game that generates REE as sequential equilibria of the game. He finds that versions of FTPL can emerge as a sequential equilibrium, which takes account of out-of-equilibrium behavior in a Bayesian fully rationalist way. In contrast, Niepelt (2004) strengthens the notion of REE in the literature on FTPL by requiring that the equilibrium also fully incorporates past optimal choices. Niepelt shows that debt valuation according to FTPL is inconsistent with his refined notion of REE.[3]

Strengthening the notion of rationality is not the only possible way to assess the implications of FTPL. Such approaches require even more information and coordination than the usual notion of REE on the part of the economic agents. In this paper, we take a very different approach by modeling out-of-equilibrium behavior as being boundedly rational in a specific sense. We employ the learning approach to macroeconomics, which has been developed in recent years[4] to provide, among

other things, a criterion to select "reasonable" outcomes when multiple REE exist. The approach is also useful in cases with unique REE as a way to assess the plausibility of an equilibrium. We reexamine some central results of FTPL from a learning viewpoint. Generally speaking, this view asserts that the REE of interest are those that are locally stable outcomes of a natural real-time learning process in which agents might temporarily deviate from rational expectations (RE), respond to these mistakes, and eventually arrive at correct forecast functions.

The learning processes that we consider are to be interpreted as a description of how private agents would plausibly update their forecast rules over time as new data become available. Thus, we are describing how learning by private agents might proceed, in a decentralized market economy, if the agents use statistical methods for estimation and forecasting. In general, stability is a joint property of the equilibrium under consideration and the learning rule followed by agents. However, the learning rules we consider employ standard econometric procedures, that is, least-squares (LS) regressions in stochastic models and sample means in nonstochastic models.[5] Indeed, these procedures are routinely used by economists when making forecasts and when empirically modelling RE. Furthermore, we give agents an appropriate functional form that corresponds to the REE of interest. We therefore regard local stability under LS learning as a minimal requirement for the economic plausibility of an equilibrium.

In many standard models, the REE of interest are indeed locally stable under these learning rules. This holds, for example, for the usual solutions of the Cagan model of inflation, the Samuelson overlapping generations model of money and the real business cycle model, and it holds for the New Keynesian model when appropriate interest-rate policies are followed. If an equilibrium is unstable under LS learning, then we regard it as one that would not be attainable. In such circumstances, depending on the outcome, a policy maker would do well to change its policy to ensure convergence to the equilibrium. Of course, in the unstable case, faced with increasing forecast errors, private agents (and policy makers) may alter their perceived functional form and updating rules, but there is no reason to expect this to redirect them to the equilibrium being considered.[6]

The version of the model under learning that we consider specifies household consumption and money demands as functions of the nominal interest rate and expected inflation over the coming period. These are obtained from the Euler equations that characterize the household optimal decisions under subjective expectations. In the deterministic cases, we employ the resulting nonlinear model for our analysis. However, in stochastic cases, the analysis is based on the linearized model around a nonstochastic steady state, which is standard practice.

Agents' expectations of inflation are based on a forecast function that is revised over time in accordance with adaptive LS learning. Given these expectations, the short-run state of the economy at each time is given by the temporary equilibrium. Our household behavior is boundedly rational in two respects. First, in contrast to "rational expectations" the parameters of the forecast function are not known *a priori* to be at their equilibrium values, although when learning is stable they will

converge to these values asymptotically. Second, the decision rules of the agents do not explicitly consider expectations of variables at more distant horizons.

Using this framework, we investigate both the rational explosive inflation paths studied by McCallum (2001), and the classification of fiscal and monetary policies proposed by Leeper (1991), for stability of the REE under LS learning.[7] We find that the fiscalist REE in the model of McCallum (2001) is not locally stable under LS learning, whereas the monetarist equilibrium is stable under LS learning when fiscal policy is instead "Ricardian." In contrast, in the setting of Leeper (1991), various cases arise. One plausible region of policy parameters arises when the nominal interest rate set by the policy maker responds strongly to inflation and when taxes are raised significantly in response to increases in real debt (active monetary and passive fiscal policy). In this case, the monetarist REE is the unique stationary REE and is stable under LS learning. In the monetarist REE, inflation is purely a monetary phenomenon and real balances are unaffected by fiscal variables. In this policy region, the orthodox view is confirmed.

If, instead, fiscal policy is active and monetary policy is passive, the fiscalist solution, in which fiscal variables affect the price level, is stable under LS learning, in line with FTPL. This is one of our most striking results because real balances are affected by the rule for setting lump-sum taxes. Hence, Ricardian equivalence fails in a setting where the traditional view would expect it to be satisfied. In this regime, taxes respond only weakly to debt and the interest rate responds only weakly to inflation. Nevertheless, the REE in question is both stochastically stationary and attainable by boundedly rational agents who follow LS learning rules.

In both of the cases just described, the stable REE is the unique stationary solution. For other combinations of monetary and fiscal policy, the results are intriguing: for some parameter values, all REE are unstable under LS learning, whereas for other parameter values there is incipient convergence to an explosive path. Taken together, our results clearly indicate that policy formulation should incorporate the implications of the stability properties, under LS learning, of the different REE.

2. THE MODEL

We consider a stochastic optimizing model that is close to Leeper (1991) and McCallum (2001). For the basic model, notation and specification of monetary and fiscal policy rules we follow Leeper, but we use McCallum's more general class of utility functions and also his timing in which utility depends on beginning of period money balances.[8]

Households are assumed to maximize

$$\max E_t \left\{ \sum_{s=t}^{\infty} \beta^{s-t} \left[(1-\sigma_1)^{-1} c_s^{1-\sigma_1} + \mathcal{A}(1-\sigma_2)^{-1} \left(m_{s-1}\pi_s^{-1} \right)^{1-\sigma_2} \right] \right\}.$$

Here c_s denotes consumption in period s and $m_s = M_s/P_s$, where M_s is the money supply and P_s is the price level at s. Note that real money balances enter utility as $m_{s-1}\pi_s^{-1} = (M_{s-1}/P_{s-1})(P_{s-1}/P_s) = M_{s-1}/P_s$. The household's budget constraint is

$$c_s + m_s + b_s + \tau_s = y + m_{s-1}\pi_s^{-1} + R_{s-1}\pi_s^{-1}b_{s-1}, \tag{1}$$

where $b_s = B_s/P_s$, $\pi_s = P_s/P_{s-1}$ is the gross inflation rate and τ_s is real lump-sum taxes. Note that B_s is the end of period s nominal stock of bonds. R_{s-1} is the gross nominal interest rate on bonds, set at time $s-1$ but paid in the beginning of period s. The household has a constant endowment y of consumer goods each period. Throughout the paper, we assume $0 < \sigma_2 < 1$.

We assume that there is a constant flow of government purchases $g \geq 0$. As shown in Appendix A.1, household optimality and market clearing conditions imply the Fisher equation

$$R_t^{-1} = \beta E_t \pi_{t+1}^{-1}, \tag{2}$$

and the equation for money market equilibrium, in period t,

$$\mathcal{A}\beta m_t^{-\sigma_2} E_t \pi_{t+1}^{\sigma_2 - 1} = (y - g)^{-\sigma_1}\left(1 - \beta E_t \pi_{t+1}^{-1}\right). \tag{3}$$

Assuming a lower bound on real bonds, the transversality condition implies

$$\lim_{t \to \infty} \beta^t m_{t+1} = 0 \text{ and } \lim_{t \to \infty} \beta^t b_{t+1} = 0. \tag{4}$$

Equations (2) and (3) are usually derived under RE but, as argued in Appendix A.1, they also can be assumed to hold in a temporary equilibrium with given subjective expectations. As noted in the Introduction, these equations as decision rules of the agents do not explicitly consider expectations of variables at more distant horizons. However, we verify that the household transversality condition and the intertemporal budget constraint of the government are satisfied along the learnable temporary equilibrium paths that converge to an REE.

The specification of the model is completed by giving the government budget constraint and policy rules. The government budget constraint is $B_t + M_t + \tau_t P_t = g P_t + M_{t-1} + R_{t-1}B_{t-1}$, which in real terms can be written

$$b_t + m_t + \tau_t = g + m_{t-1}\pi_t^{-1} + R_{t-1}\pi_t^{-1}b_{t-1}. \tag{5}$$

For fiscal policy, we use Leeper's tax rule

$$\tau_t = \gamma_0 + \gamma b_{t-1} + \psi_t. \tag{6}$$

Monetary policy is given either by Leeper's interest rate rule

$$R_t = \alpha_0 + \alpha \pi_t + \theta_t, \tag{7}$$

or by a simple fixed money supply rule

$$M_t = M + \theta_t, \tag{8}$$

as in Sims (1999) or McCallum (2001). Here ψ_t and θ_t are exogenous random shocks, which for simplicity are *iid* with mean zero. We remark that the government nominal bond supply is jointly determined with the equilibrium price level by the government budget constraint once the tax and monetary policy rules have been specified.

In the terminology of Leeper (1991), fiscal policy is "active" if $|\beta^{-1} - \gamma| > 1$ and "passive" if $|\beta^{-1} - \gamma| < 1$, whereas under (7) monetary policy is active if $|\alpha\beta| > 1$ and passive if $|\alpha\beta| < 1$. For the empirically realistic case $0 \leq \gamma \leq \beta^{-1}$ active (passive, respectively) fiscal policy means that additional tax revenue from a small increase in the steady state level of debt is insufficient (sufficient, respectively) to cover the increased interest payments. Under the interest rate rule (7), with $\alpha > 0$, active monetary policy means that a small increase in the steady-state inflation rate leads to an increase in the real interest rate. As noted by Sims (1999), it is also natural to refer to monetary policy as active if the policy rule (8) is followed in place of (7).

We want to consider the RE solutions under different policy regimes and then to analyze their stability under LS learning. Leeper emphasized the cases of AM/PF (active monetary/passive fiscal policy) and AF/PM (active fiscal/passive monetary policy) in which, as discussed later, there is a unique stationary solution. We will be particularly interested in these cases but also will consider explosive regimes of the model and regimes with indeterminacy, that is, with multiple stationary solutions.

3. BUBBLES AND THE FISCAL THEORY OF PRICES

We begin our analysis with consideration of a prominent case of the fiscal theory of prices in which, under a constant money supply, the price level path is determined by fiscal policy. For example, see Sims (1999) or McCallum (2001). In this section, we use a nonstochastic version of the model in which $\psi_t \equiv 0$ and $\theta_t \equiv 0$. Monetary policy is given by (8) and fiscal policy is given by (6) with $\gamma = 0$. Thus, policy reduces to

$$\tau_t = \tau \text{ and } M_t = M,$$

which is a special case in which both monetary and fiscal policy are active.

With a nonstochastic model it is natural to assume point expectations, so that (3) becomes

$$m_t = (\mathcal{A}\beta)^{1/\sigma_2}(y - g)^{\sigma_1/\sigma_2}\big[\big(1 - \beta/\pi_{t+1}^e\big)\big(\pi_{t+1}^e\big)^{1-\sigma_2}\big]^{-1/\sigma_2}.$$

With constant nominal money stock, we can write

$$P_t = \hat{D}\big(\pi_{t+1}^e\big)^{(1-\sigma_2)/\sigma_2}\big[1 - \beta\big(\pi_{t+1}^e\big)^{-1}\big]^{1/\sigma_2}, \tag{9}$$

where $\hat{D} \equiv M(\mathcal{A}\beta)^{-1/\sigma_2}(y - g)^{-\sigma_1/\sigma_2}$.

Consider first the perfect foresight solutions. We take as given the initial money stock and the initial value of bonds redeemed $R_0 B_0$. With a constant money supply and taxes, we have $B_1 + \tau P_1 = g P_1 + R_0 B_0$ in period 1. Under perfect foresight, we have $R_t^{-1} = \beta \pi_{t+1}^{-1}$ for $t \geq 1$ and so (5) reduces to

$$b_t = g - \tau + \beta^{-1} b_{t-1}$$

for $t \geq 2$. This equation is explosive and will violate the transversality conditions unless $b_1 = B_1 / P_1 = (\tau - g)/(\beta^{-1} - 1)$. With $R_0 B_0$ given as an initial condition this equation uniquely determines, under perfect foresight, the price level P_1 through $(\tau - g)/(1 - \beta) = R_0 B_0 / P_1$. Under perfect foresight, the price equation (9) becomes

$$P_t = \mathcal{F}(P_{t+1}/P_t) \equiv \hat{D}(P_{t+1}/P_t)^{(1-\sigma_2)/\sigma_2}[1 - \beta(P_{t+1}/P_t)^{-1}]^{1/\sigma_2} \qquad \textbf{(10)}$$

for $t \geq 1$. We remark that the right-hand side of (10) is strictly increasing in P_{t+1}/P_t and therefore the equation can be solved uniquely for P_t given P_{t+1} (and vice versa). However, there is no explicit closed-form solution.

The equation (10) has a steady state at $\hat{P} = \hat{D}(1 - \beta)^{1/\sigma_2}$, but is explosive and will diverge unless initial conditions happen to be such that $P_1 = \hat{P}$. However, for $0 < \sigma_2 < 1$ and initial $P_1 > \hat{P}$ we obtain an explosive price path $P_t \to \infty$ that is consistent with the transversality conditions and the equilibrium equations. In this "fiscalist" equilibrium, the price level $P_1 = R_0 B_0 (1 - \beta)/(\tau - g)$ is entirely determined by fiscal variables and P_t follows an explosive "bubble" price path despite a constant money stock.

McCallum (2001) argues that this solution is less plausible than an alternative "bubble-free" monetarist solution $P_t = \hat{P}$ and $b_t = 0$ for all $t = 1, 2, 3, \dots$, in which (with our timing) the level of real taxes τ_t adjusts to satisfy $\tau_1 = g + R_0 B_0 / P_1$ and $\tau_t = g$ for $t = 2, 3, \dots$. One way to interpret McCallum's view, as he acknowledges, is as an argument that fiscal policy must be Ricardian, that is, must satisfy the government intertemporal budget constraint for all feasible sequences, not just for equilibrium sequences. [For a related argument see Buiter (2002).] However, the status of the fiscalist solution in this model remains controversial.

3.1. Fiscalist Case Under Learning

We now take a different tack, which nonetheless comes to the same conclusion as McCallum (2001), that is, that the fiscalist solution is not plausible in the case under scrutiny. We suppose that the government can indeed commit to $\tau_t = \tau$ for all $t = 1, 2, 3, \dots$, so that the only equilibrium perfect foresight price path is the explosive fiscalist solution given earlier. However, we drop the perfect foresight assumption and ask if the price path is learnable under a natural adaptive learning rule that is closely related to LS.

In a temporary equilibrium (9), agents make a forecast of the relevant variable π_{t+1}, for which they are assumed to employ a suitable parametric model. In each

period, agents estimate the key parameter, using the available data, and employ the estimated model to make their forecast π_{t+1}^e. The temporary equilibrium then generates a new data point, and in the following period agents update their parameter estimate and make forecasts using the reestimated model. The issue of interest is whether the estimated parameter converges asymptotically so that the economy converges to the perfect foresight path over time.

We first note that, using (10), $P_t \to \infty$ implies that $\pi_{t+1} \to \infty$ along the perfect foresight path.[9] Because there is no closed-form solution of (10) for P_{t+1} in terms of P_t we look for a suitable approximation for large P_t that could be used for forecasting within the setting of temporary equilibrium dynamics in the nonlinear model. Letting

$$\mathcal{G}(P_{t+1}/P_t) = \hat{D}(P_{t+1}/P_t)^{(1-\sigma_2)/\sigma_2},$$

it can be seen that

$$\lim_{\pi \to \infty} \frac{\mathcal{F}(\pi)}{\mathcal{G}(\pi)} = 1.$$

Thus, for P_{t+1}/P_t large, $\mathcal{G}(P_{t+1}/P_t)$ provides a good approximation to $\mathcal{F}(P_{t+1}/P_t)$. Solving $P_t = \mathcal{G}(P_{t+1}/P_t)$ gives

$$P_{t+1} = \bar{D} P_t^{1/(1-\sigma_2)}, \tag{11}$$

where $\bar{D} = \hat{D}^{-\sigma_2/(1-\sigma_2)}$. For all large P_t, along a fiscalist path, forecasts based on (11) would be close to the perfect foresight value of P_{t+1}.

We therefore endow private agents with the Perceived Law of Motion (PLM)

$$P_{t+1} = D P_t^{\bar{\phi}}, \quad \text{where } \bar{\phi} = 1/(1 - \sigma_2).$$

We assume that agents know the growth rate parameter $\bar{\phi} > 1$, but do not know D (equal to $\bar{D}$ under RE) and must estimate D from the data. Under this PLM the ratio $Q_t = P_t/P_{t-1}^{\bar{\phi}}$ provides the relevant data for estimating the unknown constant D. Allowing the estimation of both D and ϕ would require a stochastic framework because otherwise there is insufficient information to identify two parameters.[10] If the fiscalist solution is locally unstable under learning under the assumption that $\bar{\phi}$ is known, then this clearly would cast doubt on the plausibility of the fiscal theory in this policy setting.

Our learning rule specifies that the estimate D_t is updated as follows:

$$D_t = D_{t-1} + \kappa(Q_t - D_{t-1}) \quad \text{for } 0 < \kappa < 1. \tag{12}$$

This rule adjusts the estimate in the direction of the most recent value of the ratio Q_t. The rule (12) is a standard constant gain learning rule for estimating an unknown constant. The "gain" κ is usually taken to be a small positive value.[11]

Under this learning rule, forecasts of prices are given by $P_t^e = D_{t-1} P_{t-1}^{\bar{\phi}}$ and $P_{t+1}^e = D_{t-1}(P_t^e)^{\bar{\phi}}$, so that expected inflation is given by[12] π_{t+1}^e

$= P^e_{t+1}/P^e_t = D^{\bar{\phi}}_{t-1} P^{\bar{\phi}(\bar{\phi}-1)}_{t-1}$. Substituting this forecast into (9) and solving for Q_t yields

$$Q_t = \hat{D} D^{\bar{\phi}/(\bar{\phi}-1)}_{t-1} \left(1 - \beta D^{-\bar{\phi}}_{t-1} P^{\bar{\phi}(1-\bar{\phi})}_{t-1}\right)^{1/\sigma_2}. \tag{13}$$

Thus, the system under learning consists of (12), (13), and the definitional equation

$$P_t = Q_t P^{\bar{\phi}}_{t-1}, \tag{14}$$

and the fiscalist solution is stable under the learning rule (12) if $D_t, Q_t \to \bar{D} = \hat{D}^{-\sigma_2/(1-\sigma_2)}$ as $t \to \infty$.

PROPOSITION 1. *Under constant taxes and fixed money supply, the explosive fiscalist price path is locally unstable under the learning rule* (12) *for all* $0 < \kappa < 1$.

Proof. Suppose to the contrary that $D_t, Q_t \to \bar{D}$ as $t \to \infty$. Then $P_t \to \infty$ under the system (12), (13), and (14), so that $(1 - \beta D^{-\bar{\phi}}_{t-1} P^{\bar{\phi}(1-\bar{\phi})}_{t-1})^{1/\sigma_2} \to 1$. For large P_t, D_t approximately follows

$$D_t = D_{t-1} + \kappa \left(\hat{D} D^{\bar{\phi}/(\bar{\phi}-1)}_{t-1} - D_{t-1}\right). \tag{15}$$

This equation has the fixed point $\bar{D}$, but it is easily seen that $\bar{D}$ is unstable under (15) since $T'(\bar{D}) > 1$, where $T(D) = \hat{D} D^{\bar{\phi}/(\bar{\phi}-1)}$. ∎

This instability result indicates a lack of robustness of the perfect foresight price path, to small deviations, under a simple LS learning rule of a type that is known to yield stability in other contexts, and contrasts with cases here in which analogous learning rules converge.

We close this section with two remarks.

First, the contribution of Proposition 1 is not that it shows the possibility of an REE being unstable under LS learning. This possibility is well known. In some cases, instability under learning is due to a PLM specified in terms of a variable that explodes. For example, Bullard (1994) and Schönhofer (1996) give an example of instability in an overlapping generations model with constant money growth. Their instability result depends on a specification of LS updating in which the regressor follows an explosive path, and the solution becomes stable when the updating rule is instead specified in terms of the inflation rate.[13] In the current context, we are careful to avoid this problem by formulating learning in terms of the variable Q_t, which is asymptotically stationary along the explosive perfect foresight path. However, examples of instability under LS learning also arise when the variables of interest are nonexplosive in the REE. These include the high-inflation steady state in the hyperinflation model and the New Keynesian model with an inappropriate interest-rate rule. In the spirit of the instability results in these other settings, the contribution of Proposition 1 is to raise concerns about the plausibility of FTPL for the specification of policy under study.

Second, the instability result relies on using natural but simple rules for decision making and learning. These decision rules are discussed in Appendix A.1. In

particular, the household demand for real balances depends only on the interest rate and the expected rate of inflation over the coming period. More elaborate decision (and learning) rules can be imagined in which households choose their money demands based on a forecast of the whole future price path.[14] However, our decision rule is natural because it ensures that the household attempts each period to meet the first-order condition for maximizing utility given by the usual Euler equation.

3.2. Monetarist Solution Under Learning

We now consider LS learning stability of the monetarist solution. We set this case up by assuming that the government pays off the debt immediately, never resorting to bond finance thereafter. (We continue to assume that money supply is constant.) Clearly, this is an extreme form of Ricardian policies.[15] In consequence, there are no bonds in the economy and the only equation of interest is (9). Under the perfect foresight monetarist solution, there is no seigniorage because $\pi_t = 1$ and $\tau_t = g$ for all $t > 1$. Under learning, lump-sum taxes adjust each period to offset seigniorage.

The solution of interest is the steady state

$$P_t = \bar{P} \equiv \hat{D}(1 - \beta)^{1/\sigma_2} \quad \text{with } \pi_t = 1.$$

It is easily seen that this is the unique steady state solution for π_t. We refer to this as the monetarist solution and we now investigate its stability under (constant-gain) LS learning. A natural learning rule treats the equilibrium inflation rate as an unknown constant ζ to be estimated.[16] As in Section 3.1, we assume that agents use a constant-gain learning rule for estimating an unknown constant

$$\zeta_t = \zeta_{t-1} + \kappa(\pi_{t-1} - \zeta_{t-1}), \tag{16}$$

with expected inflation given by $\pi_{t+1}^e = \zeta_t$. Again, κ is a small positive constant.

Writing the nonlinear model (9) as $P_t = \mathcal{F}(\pi_{t+1}^e)$, we have $\pi_t = \mathcal{F}(\zeta_t)/\mathcal{F}(\zeta_{t-1})$ so that our system becomes

$$\zeta_t = \zeta_{t-1} + \kappa \left[\frac{\mathcal{F}(\zeta_{t-1})}{\mathcal{F}(\zeta_{t-2})} - \zeta_{t-1} \right]. \tag{17}$$

The monetarist solution is locally stable under (17) if $\zeta_t \to 1$ from nearby starting points as $t \to \infty$. The relevant stability condition is easily obtained by looking at the linearized second-order difference equation. This yields

$$\tilde{\zeta}_t = (1 - \kappa + \kappa L)\tilde{\zeta}_{t-1} - \kappa L \tilde{\zeta}_{t-2},$$

where $\tilde{\zeta}_t = \zeta_t - 1$ and $L = \sigma_2^{-1}[\beta(1 - \beta)^{-1} + 1 - \sigma_2] > 0$. Using standard results, the linearized system is stable if $|\kappa L| < 1$ and $|1 - \kappa + \kappa L| < 1 + \kappa L$. Since $L > 0$ these inequalities hold for $\kappa > 0$ sufficiently small. We conclude as follows:

PROPOSITION 2. *Under constant money supply and Ricardian fiscal policy, the monetarist solution is locally stable under the learning rule* (16) *for all κ > 0 sufficiently small.*

We conjecture that the monetarist solution is in fact globally stable under our simple learning rule. We have not investigated this because it is not central to the message of this paper. There are a number of examples of global stability results in the learning literature. In particular, for the standard overlapping generations model of money, Wenzelburger (2002) has recently developed a relatively complicated learning rule under which the monetary steady state is globally stable.

3.3. Discussion

Propositions 1 and 2 cast doubt on the plausibility of FTPL for the special case of constant money and taxes. If the government follows non-Ricardian policies and the money supply is held fixed, the only REE is the explosive price bubble path, but this equilibrium is not stable under (constant-gain) LS learning. The economy under the specified learning rule may follow some explosive path for a period of time, but this path will not converge to the fiscalist solution.

However, there are other policy regimes in which FTPL has been proposed as the relevant solution. In particular, Leeper (1991) studied situations in which the inflation rate is affected by government tax and bond variables but with finite steady state inflation. We now turn to an analysis of LS learning under policy rules (6) and (7) based on a linearization around the steady state of the model with random shocks. We will be particularly interested in the policy regimes in which the interaction of monetary and fiscal policy rules leads to a unique stationary solution under RE, but we also will consider other policy regimes.

4. LINEARIZED MODEL WITH STOCHASTIC SHOCKS

We thus return to monetary policy following an interest rate rule, with the system specified by (2), (3), (5), and the policy rules given by (6) and (7). This is a stochastic, nonlinear system with exogenous shocks, lagged endogenous variables and expectations of future quantities that are themselves nonlinear functions of the endogenous economic variables.[17] Equilibria of such nonlinear stochastic systems do not in general have an explicit functional representation in terms of a finite number of parameters, which would provide a natural PLM that agents could estimate. However, in a neighborhood of the nonstochastic steady state, one can analyze the linearization of the model, provided that the random shocks are sufficiently small. This procedure is standard in stochastic RE macroeconomic models and is valid (that is, yields a close approximation) provided the stochastic disturbances have a sufficiently small support. For a justification see Appendix A.3 of Woodford (2003b). Analysis of the linearized model is also standard in the study of local convergence of LS learning. Although a formal justification for the linearization technique in general nonlinear settings has not

been developed, the RE-based arguments would appear to carry over and the technique has been verified numerically for many specific models in the literature. In addition, linearization techniques to obtain local stability results under LS learning have formally been shown to apply in forward-looking stochastic nonlinear models with small *iid* shocks when an REE is described using a finite number of parameters, see Chapters 11–12 of Evans and Honkapohja (2001).

In Appendix A.2, we show that the linearized system takes the form

$$\pi_t = (\alpha\beta)^{-1} E_t^* \pi_{t+1} - \alpha^{-1}\theta_t \tag{18}$$

$$0 = b_t + \varphi_1\pi_t + \varphi_2\pi_{t-1} - (\beta^{-1} - \gamma)b_{t-1} + \psi_t + \varphi_3\theta_t + \varphi_4\theta_{t-1}, \tag{19}$$

where $E_t^* \pi_{t+1}$ denotes inflation expectations formed at t. The notation $E_t^* \pi_{t+1}$ is used to emphasize that the reduced form (18)–(19) applies whether or not expectations are rational. The coefficients $\varphi_1, \ldots, \varphi_4$ are given in Appendix A.2.[18] From now on, we make the assumptions $\alpha \neq 0, \alpha\beta \neq 1, \gamma\beta \neq 1$ and $\beta^{-1} - \gamma \neq 1$.

Linear stochastic expectations models such as (18)–(19) may or may not have a unique stationary REE, depending on the parameters. When there is a unique stationary RE solution, the steady state of the model is said to be (locally) "determinate" (the terminology "regular" is also used). When there are multiple stationary RE solutions, the steady state (or model) is called "indeterminate" (or "irregular"), and when there are no stationary RE solutions the steady state of the model is called "explosive."[19]

In univariate one-step ahead forward-looking models with positive feedback, there is a simple connection between determinacy of a steady state and the stability of the unique stationary solution under LS learning.[20] However, this correspondence does not generalize, even for linear models, to multivariate setups and models with lagged endogenous variables. There are many examples in the literature of determinate models that are unstable under LS learning, indeterminate models in which some of the solutions are stable under LS learning and explosive solutions that are stable under LS learning. Determinacy of a model and stability of the various solutions under LS learning are, therefore, distinct properties that must be assessed separately.

Sections 4.1 and 4.2 consider, respectively, determinacy and LS learnability of the different REE. Section 4.3 combines these analyses into an overall assessment of the plausibility of the FTPL.

4.1. Determinacy of the Different Equilibria

For models such as (18)–(19) the REE can be represented as vector autoregressions (VARs) depending also on exogenous variables, and by stationarity we mean that the VAR is asymptotically covariance stationary in the usual time-series sense. In the current context, stationarity is equivalent to the solution being nonexplosive in conditional mean. We begin by considering the determinate cases, which motivates the monetarist and fiscalist solutions, and present the fully general treatment a bit later.
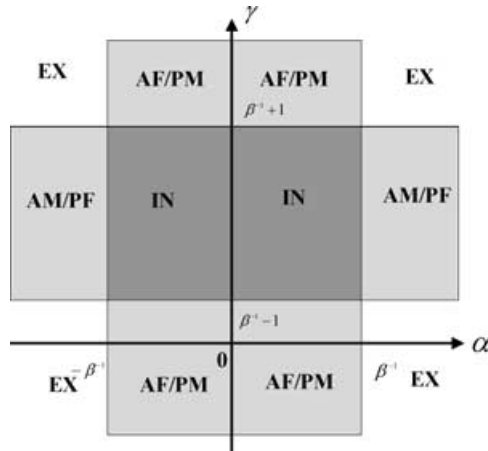
FIGURE 1. Determinate, indeterminate and explosive regions.

In Appendix A.3, it is shown that the determinate case arises when either $|\alpha\beta| > 1$ and $|\beta^{-1} - \gamma| < 1$, that is, active monetary policy and passive fiscal policy (AM/PF), or $|\alpha\beta| < 1$ and $|\beta^{-1} - \gamma| > 1$, that is, active fiscal policy and passive monetary policy (AF/PM). As shown in Appendix A.3, either condition $|\alpha\beta| > 1$ or $|\beta^{-1} - \gamma| > 1$ leads to a linear restriction of the form

$$\pi_t = K_1 b_t + K_2 \theta_t \tag{20}$$

when nonexplosiveness of the solution is imposed. This equation together with (19) defines the unique stationary solution in the determinate case.

In the AM/PF regime, we obtain $K_1 = 0$ and $K_2 = -\alpha^{-1}$, so that

$$\pi_t = -\alpha^{-1}\theta_t.$$

We will refer to this solution as the "monetarist solution," as π_t is independent of both b_{t-1} and ψ_t. In the AF/PM regime, we obtain

$$\pi_t = \frac{\alpha\beta\varphi_1 + \varphi_2}{\beta^{-1} - \gamma - \alpha\beta} b_t + K_2\theta_t. \tag{21}$$

From (21) and (19), it is apparent that inflation now depends on b_{t-1} and ψ_t as well as on monetary policy. We therefore refer to this REE as the "fiscalist solution."

In addition to the determinate cases, two other regimes are possible, depending on policy parameters. If $|\alpha\beta| < 1$ and $|\beta^{-1} - \gamma| < 1$, so that both policies are passive, the model is indeterminate and there are multiple stationary solutions, which include the monetarist, the fiscalist REE and, as shown here, a class of nonfundamental solutions. If $|\alpha\beta| > 1$ and $|\beta^{-1} - \gamma| > 1$, so that both policies are active, the model is explosive, and there are no stationary solutions.[21] The different regimes are shown in Figure 1, where IN and EX refer to indeterminate and explosive regions, respectively.

We now systematically examine the possible REE. Introducing vector notation $y_t = (\pi_t, b_t)'$, the linearized model (18)–(19) can be written

$$y_t = M E_t^* y_{t+1} + N y_{t-1} + P v_t + R v_{t-1}, \tag{22}$$

where

$$M = \begin{pmatrix} (\alpha\beta)^{-1} & 0 \\ -\varphi_1(\alpha\beta)^{-1} & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 \\ -\varphi_2 & \beta^{-1} - \gamma \end{pmatrix},$$

$$P = \begin{pmatrix} -\alpha^{-1} & 0 \\ \varphi_1\alpha^{-1} - \varphi_3 & -1 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 \\ -\varphi_4 & 0 \end{pmatrix}, \quad v_t = \begin{pmatrix} \theta_t \\ \psi_t \end{pmatrix}.$$

Using the method of undetermined coefficients, we consider equilibria of the form

$$y_t = A + B y_{t-1} + C v_t + D v_{t-1}. \tag{23}$$

The form (23) excludes exogenous sunspot variables by assumption (we will briefly consider such solutions later). Computing the expectation[22]

$$\begin{aligned} E_t^* y_{t+1} &= A + B(A + B y_{t-1} + C v_t + D v_{t-1}) + D v_t \\ &= (I + B)A + B^2 y_{t-1} + (BC + D)v_t + B D v_{t-1} \end{aligned} \tag{24}$$

and inserting into (22) we obtain

$$\begin{aligned} y_t &= M(I + B)A + (M B^2 + N)y_{t-1} \\ &\quad + [M(BC + D) + P]v_t + (M B D + R)v_{t-1}. \end{aligned} \tag{25}$$

The possible solutions to (A, B, C, D) can then be obtained by requiring that (23) and (25) describe the same random process.

Equivalently, we can use the terminology from the study of LS learning and E-stability, so that the guess (23) is the PLM and equation (25) is the Actual Law of Motion (ALM). The mapping from the PLM to the ALM is

$$A \longrightarrow T_A(A, B) = M(I + B)A \tag{26}$$

$$B \longrightarrow T_B(B) = M B^2 + N$$

$$C \longrightarrow T_C(B, C, D) = M B C + M D + P$$

$$D \longrightarrow T_D(B, D) = M B D + R$$

and the fixed points of this mapping correspond to REE of the form (23).

The second component of the mapping gives a quadratic matrix equation in B, which can have more than one solution. Given any solution for B, the first component gives the unique solution $A = 0$, provided that $I - M(I + B)$ is nonsingular. Similarly, for given B the third and fourth components of the mapping are linear equations for C and D. Imposing the zero elements of M and N in the matrix quadratic for B, three types of equilibria can easily be shown to emerge:[23]

PROPOSITION 3. *There are three types of REE, I. the monetarist solution, II. the fiscalist solution and III. nonfundamental solutions, taking the form* (23), *as listed below.*

   I. $B = N, C = P$ and $D = R$ with $A = 0$. This is the monetarist solution identified earlier in the determinate case.

   II. $B = \chi^{-1} \begin{pmatrix} -(\beta\gamma + \alpha\beta^2 - 1)\varphi_2 & -\beta^{-1}(\beta\gamma - 1)(\beta\gamma + \alpha\beta^2 - 1) \\ \beta\varphi_2(\alpha\beta\varphi_1 + \varphi_2) & (\beta\gamma - 1)(\alpha\beta\varphi_1 + \varphi_2) \end{pmatrix}$,   where $\chi = (\beta\gamma - 1)\varphi_1 - \beta\varphi_2$, $A = 0$ and C and D are also uniquely determined by the fixed point.[24] This is a way of representing the fiscalist solution identified above in the determinate case. Although this may appear to be a complicated representation, it can be verified that the eigenvalues of B are 0 and $\alpha\beta$. The zero eigenvalue corresponds to the static linear relationship (21) between π_t and b_t, which can be used to obtain alternative representations of the REE irrespective of stationarity (see Appendix A.3).

   III. $B = \begin{pmatrix} \alpha\beta & 0 \\ -(\varphi_1\alpha\beta + \varphi_2) & \beta^{-1} - \gamma \end{pmatrix}$, $A = 0$. For C and D the solution is not unique. For D there is a two-dimensional continuum and, given a value for D, the equation for C also yields a two-dimensional continuum. We call this class of solutions the non-fundamental solutions, because of the indeterminacy in the C and D coefficients.[25] We remark that this solution set can be expanded to allow for dependence on an exogenous sunspot variable.

The next proposition summarizes the determinacy properties of the different solutions shown in the preceding proposition.

PROPOSITION 4. *(i) Under AM/PF policy, the monetarist solution is stationary while the fiscalist solution and nonfundamental solutions are explosive. (ii) Under AF/PM policy, the fiscalist solution is stationary while the monetarist solution and nonfundamental solutions are explosive. (iii) Under PM/PF policy, all the REE are stationary. (iv) Under AM/AF policy, all solutions are explosive.*[26]

We now turn to an examination of whether these solutions are stable under learning.

4.2. Stability Under LS Learning

In stochastic setups of type (22), in a temporary equilibrium with possibly nonrational expectations, agents need to make forecasts of endogenous variables in the next period. Agents are assumed to employ a parametric class of stochastic processes for the endogenous variables, the PLM (23), to estimate its parameters using past data and to make their forecasts using the estimated model. More specifically, agents use recursive least squares (RLS) to estimate the parameters $\xi = (A, B, C, D)$ of the PLM (23) from past data. Thus, at time t, agents have estimates $\xi_t = (A_t, B_t, C_t, D_t)$ and these are used to make forecasts $E_t^* y_{t+1}$ in (24) with ξ replaced by ξ_t. These forecasts together with y_{t-1}, v_t and v_{t-1} determine y_t according to (22). In the following period $t + 1$ the estimates ξ_t are updated

using RLS to ξ_{t+1} and the additional data point. An REE is said to be locally stable under LS learning if ξ_t locally converges to an REE as $t \to \infty$.[27]

The phrase "locally stable under LS learning" can be made precise in several different senses. First, for initial parameter estimates in an appropriate compact set strictly containing the fixed point of interest, there is convergence with probability that can be made arbitrarily close to one if the adaption speed is sufficiently slow. Second, by adding a projection facility to the algorithm that constrains parameter estimates to a suitable compact set, one can obtain probability one convergence.[28] For further details, see Evans and Honkapohja (1998) and Chapters 6 and 10 of Evans and Honkapohja (2001).

In this context, E-stability is known to give the conditions for local convergence under LS learning for stationary (and in many cases for nonstationary) REE. In contrast, if E-stability of the REE does not hold, with some additional mild assumptions one can show that LS learning converges with probability zero even for initial estimates arbitrarily close to the RE values. To define E-stability, given PLM parameters ξ let $T(\xi)$ denote the corresponding values of the ALM given by the mapping (26). The three types of RE solutions, given in Proposition 3, correspond to fixed points of this map. The RE solution (or solution set) of interest is E-stable if it is a locally asymptotically stable fixed point of the differential equation

$$d\xi/du = T(\xi) - \xi \qquad (27)$$

operating under the notional time u.

We now present the results giving stability under LS learning of the different solutions. Appendix A.4 establishes the result by examining E-stability for the different RE solutions.

PROPOSITION 5. *Assume that the REE of interest is stationary.* (I) *The monetarist solution is locally stable under LS learning if*

$$(\alpha\beta)^{-1} < 1 \quad and \quad \frac{\beta^{-1} - \gamma}{\alpha\beta} < 1.$$

(II) *The fiscalist solution is locally stable under LS learning if*

$$\frac{\beta^{-1} - \gamma}{\alpha\beta} > 1 \quad and \quad \frac{\gamma + 1 - \beta^{-1}}{\alpha\beta} < 0.$$

and

(III) *The nonfundamental solutions are not locally stable under LS learning.*

We next examine how the conditions for stability under LS learning related to the earlier determinacy results and then turn to the implications for FTPL.
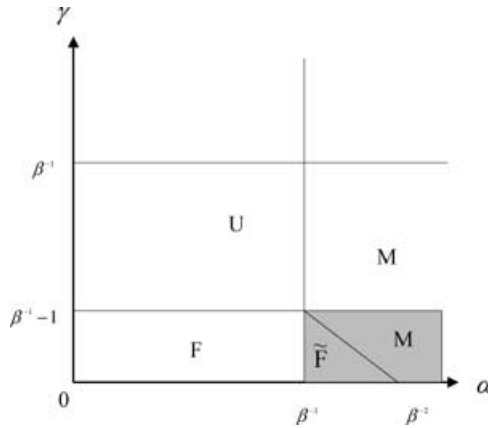
FIGURE 2. Regions of E-stable REE.

4.3. Economic Implications

Looking at the economic model, it is evident that the most natural policy rules entail the parameter restrictions $\alpha > 0$ and $\gamma \geq 0$. $\alpha > 0$ means that the nominal interest rate responds positively to current inflation and $\gamma > 0$ means that the lump-sum tax responds positively to beginning-of-period debt b_{t-1}. In the case $\gamma = 0$, taxes are set independently of the debt level. Realistic values of γ would also appear to be below β^{-1}, since $\gamma > \beta^{-1}$ implies that, at the non-stochastic steady state, any shock to debt levels would lead to a tax increase that would more than pay off the debt, including interest, within one period. We therefore focus on the region $\alpha > 0$ and $\beta^{-1} > \gamma \geq 0$ of the policy parameter space.

Figure 2 shows the results on learning and E-stability for the monetarist and fiscalist solutions in this part of the parameter space. In the figure, M indicates that the monetarist solution is locally stable under LS learning. F indicates that the fiscalist solution is locally stable and U indicates that neither solution is locally stable under LS learning. In the region marked $\tilde{F}$, stability of the fiscalist solution is fragile, as discussed later. In none of the areas are both solutions simultaneously locally stable under LS learning. In the shaded region, defined by $\alpha > \beta^{-1}$ and $0 \leq \gamma < \beta^{-1} - 1$, the solutions are not stationary.

Within the parameter region described by Figure 2, the AM/PF regime lies within region M. In this regime, the monetarist equilibrium is the unique stationary solution and it is also locally stable under LS learning. In the AF/PM regime, which coincides with region F, the fiscalist REE is the unique stationary solution and is locally stable under LS learning.[29] In the indeterminacy region with policy combination PM/PF both solutions are stationary, but they fail to be locally stable under LS learning.[30]

The shaded explosive region with policy combination AM/AF is divided into two cases with either the fiscalist or the monetarist solution being stable under LS

learning. We emphasize that our results are local: they are valid in a neighborhood of the steady state and here suggest an incipient tendency for the economy under LS learning to follow the indicated explosive equilibrium. We now briefly discuss the nonlinear dynamics in the explosive region.

Consider the monetarist solution in the nonlinear model. In this solution, the inflation rate is stationary, while bonds b_t explode at rate $\beta^{-1} - \gamma$ (in the mean). Because the temporary equilibrium equation for π_t is independent of bonds, the monetarist solution is E-stable for univariate PLMs throughout the (shaded) explosive region in Figure 2. Moreover, the transversality condition for b_t is satisfied because $\beta(\beta^{-1} - \gamma) < 1$. However, this is only a weak form of E-stability. Our linearization suggests that with VAR-type PLMs (23), stability of the monetarist solution holds only in the shaded region M. We remark that there are two technical complications with this last argument. First, as b_t is explosive, the linearization coefficients change over time. This is unlikely to matter because the coefficients that change, φ_1 and φ_2, do not affect the E-stability conditions. Second, LS learning would include one explosive regressor. However, in related setups, this has not been found to be a difficulty, see Section 9.6 of Evans and Honkapohja (2001) and the references therein.

In the fiscalist solution, both b_t and π_t are explosive. π_t grows away from the steady state at rate $\alpha\beta$. Initially b_t also grows at this rate, as can be seen from the relation (21). However, asymptotically the growth rate of b_t becomes $\beta^{-1} - \gamma$, which can be seen from the nonlinear bond equation (5) by making use of approximations of money demand and $R_{t-1}\pi_t^{-1}$ for large π_t. The transversality condition for b_t is satisfied throughout the shaded explosive region, provided $\gamma > 0$.[31] Using again approximations based on large π_t we obtain

$$\pi_t = -\alpha^{-1}\alpha_0 + (\alpha\beta)^{-1}\pi_{t+1}^e - \alpha^{-1}\theta_t,$$

$$b_t + (\gamma - \beta^{-1})b_{t-1} \approx g - \gamma_0 - \psi_t + \varkappa\theta_t.$$

This is a linear system in which the b_t equation has become effectively decoupled from π_t. The coefficients of the endogenous variables are the same as in (18)–(19) with $\varphi_1 = \varphi_2 = \varphi_4 = 0$. Applying the analysis of Section 4.2, it can be seen that DT_A has an eigenvalue larger than one, indicating a failure of E-stability of the fiscalist solution. These results suggest that in the shaded explosive region $\tilde{F}$, the fiscalist solution is E-stable in a region near the steady state but at some point loses stability along an explosive path.

In summary, we have found that when there is a unique stationary REE (the determinate case), this REE is stable under LS learning. Depending on the values of policy parameters α and γ the solution is either monetarist or fiscalist. When there are multiple stationary REE, there is no learnable REE. These results provide support for FTPL in the specific case of active fiscal and passive monetary policy (AF/PM). In commenting on the results of our paper, Woodford (2003a) considers the AF/PM regime the "primary case with which the literature on FTPL has been concerned." In the AM/PF case the REE chosen by LS learning is the monetarist

solution. This is one of the points noted by McCallum (2003a), (2003b) in his discussion of our results. Woodford (2003a) contends that the monetarist solution in this region is consistent with the predictions of FTPL. Intriguingly, in the explosive region $\tilde{\text{F}}$, there is some support for FTPL in the sense that the incipient E-stability property of the fiscalist solution corresponds to the fact that the fiscalist solution has a less explosive bond growth rate than does the monetarist solution.[32] Even more intriguingly, in this region the fiscalist solution asymptotically loses E-stability and the asymptotic rate of growth of b_t becomes identical to that in the monetarist solution.

We close this section with a brief discussion of the connection of the results of this section and those of Section 3. In Section 3, monetary policy is given by a fixed money rule, whereas in the framework of this section an interest rate rule is used instead. Strictly speaking, these rules are non-nested. However, they can be compared (under perfect foresight) as follows. Assuming high values of π_t and using (10), we obtain $P_t = \hat{D} P_{t+1}^{1-\sigma_2}$ and hence $\pi_t = (\pi_{t+1})^{1-\sigma_2}$. Also using the Fisher equation $\pi_{t+1} = \beta R_t$ yields $R_t = \beta^{-1} \pi_t^{1/(1-\sigma_2)}$. This can be compared to the interest rate rule (7) and it is immediate that $dR_t/d\pi_t \to \infty$ as $\pi_t \to \infty$. Thus, the fixed money supply is like a very active (large α) interest rate policy. Because in Section 3 we also have $\gamma = 0$, this in effect places us at the lower boundary of the shaded M region, where the fiscalist solution is unstable and the monetarist solution is stable under LS learning. Note that although the transversality condition is not satisfied at the boundary $\gamma = 0$, it holds for arbitrarily small $\gamma > 0$.

5. CONCLUSIONS

We have considered local stability under LS learning of the RE solutions in a simple stochastic optimizing monetary model in which the interaction between monetary and fiscal policy is central. Our first finding was that in the case of constant money supply and constant taxes, the equilibrium explosive price paths dictated by FTPL are not locally stable under LS learning. In contrast, if fiscal policy is Ricardian, then the monetarist equilibrium is locally stable under LS learning. These results appear to cast doubt on the plausibility of FTPL in this policy setting.

We then examined an alternative setting in which interest rates are set as a linear function of inflation and taxes are set as a linear function of real debt. The usual monetarist solution is locally stable under LS learning in the active monetary/passive fiscal policy regime in which it is the unique stationary solution. However, the fiscalist solution, in which inflation depends on debt and on tax shocks, is stable under LS learning for a plausible subregion of the active fiscal/passive monetary regime, in which the fiscalist solution is the unique stationary solution. This provides some support for FTPL.

There also are regions of plausible policy parameter values in which the economy is indeterminate, with multiple stationary solutions. However, in this parameter domain none of the REE are locally stable under LS learning.

Overall, our results provide significant, although limited, support for the fiscalist solution. Whether the fiscalist solution emerges under learning depends on the joint fiscal and monetary policy regime. Careful consideration of the interaction of these policies is therefore required to understand the qualitative characteristics of inflation and debt dynamics. Naturally, local stability under natural learning rules is only a minimal criterion in policy design. Examination of sizes of the domains of attraction would be important in practical applications.

NOTES

1. For a long list of references on the fiscal theory of prices, see Woodford (2001), Cochrane (1999), Cochrane (2005), and Buiter (2002).

2. It should be pointed out that the FTPL does not predict that the "fiscalist" solution will always be followed, regardless of fiscal and monetary policies. For example, in the active monetary/passive fiscal policy regime, there is general agreement that the appropriate solution is the monetarist solution.

3. Niepelt also suggests a different definition of FTPL in terms of valuation of flows.

4. See Evans and Honkapohja (2001) for a recent treatise. Surveys of the literature are provided, for example, in Evans and Honkapohja (1999), Marimon (1997), and Sargent (1983).

5. The sample mean can, of course, be regarded as least squares regression on a constant.

6. If an alternative convergent learning rule exists, the policy maker could in principle try to induce agents to coordinate on using a such rule. However, achieving this coordination is likely to be far more difficult than changing fiscal or monetary policies.

7. In independent work, McCallum (2003a) considers stability under LS learning of the fiscalist bubble solution using a simplified linear model. He does not analyze the case of interest rate rules.

8. The question of whether beginning- or end-of-period real balances is used leads to subtle differences in the model and can in some cases have major implications; compare Carlstrom and Fuerst (2001).

9. If instead we had $P_t \to \infty$ and $P_{t+1}/P_t \to \hat{\pi}$ where $0 < \hat{\pi} < \infty$, the right-hand side of (10) would tend to a finite value. This is a contradiction. (If $\hat{\pi} = 0$, there would be deflation, that is, $P_{t+1} < P_t$ for sufficiently large t, which would violate the assumption $P_t \to \infty$.)

10. It can be shown that when both D and ϕ are unknown the equilibrium fails to be E-stable, which indicates that instability would arise in a stochastic setup of the model. Learning in terms of some other parameters could also be considered without changing the results as E-stability is invariant to smooth $1-1$ transformations of the parameters.

11. In stochastic models, a decreasing gain such as $\kappa_t = t^{-1}$ would normally be used. In this case the rule would be a recursive form for estimating an unknown constant by least squares.

12. We are here treating the information set at the time expectations are formed as including P_{t-1} but not P_t. (However, including current P_t in the information set would not make the price bubble paths stable.)

13. See Section 14.5 of Evans and Honkapohja (2001) for details.

14. For example, Woodford (2001) considers an analysis along these lines, drawing on the calculation equilibrium approach of Evans and Ramey (1998). Long-horizon decision rules with learning also have been considered by Preston (2005) in the context of a New Keynesian model of monetary policy.

15. McCallum (2001) points out that the monetarist solution is an REE even if the government attempts to implement a non-Ricardian fiscal policy because fully rational agents would not have a positive demand for bonds in this REE.

16. Equivalently, in analogy with the previous section, agents have the PLM $P_t = \zeta P_{t-1}$ and forecast $P_t^e = \zeta P_{t-1}$, $P_{t+1}^e = \zeta^2 P_{t-1}$ so that $\pi_{t+1}^e = \zeta$.

17. One might be tempted to initially examine a nonstochastic version of the model, but as will become evident, for assessing the FTPL it is essential to look at the stochastic system. The fiscalist equilibrium is a bivariate VAR with feedback from both endogenous variables. To capture these

dynamics, LS learning must be conducted in a stochastic setting in order to avoid asymptotic perfect multicollinearity. For a discussion of this point, see Section 3c of Evans and Honkapohja (1998).

18. These reduced form equations are identical to the reduced form given by Leeper, but with coefficients that differ slightly because of differences in timing and the more general utility function used here. See Leeper (1991), p. 136.

19. Conditions for determinacy and indeterminacy (equivalently regularity and irregularity) are given, for example, in Blanchard and Kahn (1980), Chapter 3 of Farmer (1999) and Chapter 10, Appendix 2, of Evans and Honkapohja (2001).

20. This is sometimes thought of as a reversal of dynamics in going from RE, or perfect foresight, to learning.

21. In this case requiring nonexplosiveness would impose two linear restrictions of the form (20). Because these are independent conditions, they can hold for all t only with probability 0.

22. We make the frequently employed assumption that when agents compute forecasts, using the PLM, they observe current values of the exogenous variables, but only lagged values of the endogenous variables. It can be shown that the key results do not change under the alternative information assumption that agents also observe current endogenous variables.

23. A Mathematica program with the details is available on request.

24. Explicit formulas for C and D are available on request. This assumes $\chi \neq 0$.

25. Note that this value of B is equal to J^{-1} in Appendix A.3. If both eigenvalues of J have absolute values greater than one, there is indeterminacy and this entire class of solutions is stationary.

26. The stationarity properties of the different REE are direct consequences of the earlier discussion in the text and in Appendix A.3.

27. In the indeterminate case, "locally stable" means local convergence under RLS to the set of fixed points.

28. Various positive probability of convergence results without a projection facility also can be stated.

29. Note that active monetary policy requires $\alpha > \beta^{-1}$. This is a somewhat stronger condition than given by the usual formulation of the "Taylor principle."

30. Cases in which policy leads to unstable REE under learning have appeared in the literature, see in particular the treatment of interest rate pegging by Howitt (1992) and Evans and Honkapohja (2003).

31. Even if agents believe that b_t will continue to grow forever at rate $\alpha\beta$, their perceived transversality condition is satisfied provided $\alpha < \beta^{-2}$.

32. This accords with an interpretation of FTPL that predicts selection only of solutions with the least explosive growth rate for real public debt. This interpretation also would pick the monetarist solution in the shaded region M.

33. The transversality condition for the household is $\lim_{t \to \infty} \beta^t (m_t + b_t) = 0$, as consumption in this model is constant. As noted by McCallum (2001), provided bonds have a (possibly negative) lower bound, this is equivalent to (4).

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APPENDIX

A.1. HOUSEHOLD OPTIMALITY CONDITIONS AND TEMPORARY EQUILIBRIUM

Define the variables $W_{t+1} = m_t + b_t$ and $x_{t+1} = m_t$. Following Chow (1996), Section 2.3, introduce the Lagrange multipliers λ_t for the budget constraint and μ_t for the equation

$x_{t+1} = m_t$ and write the Lagrangian

$$L = E_0 \sum_{t=0}^{\infty} \{ \beta^t \big[(1-\sigma_1)^{-1} c_t^{1-\sigma_1} + \mathcal{A}(1-\sigma_2)^{-1} (x_t \pi_t^{-1})^{1-\sigma_2} \big]$$
$$ - \beta^{t+1} \lambda_{t+1} \big[W_{t+1} - y + c_t + \tau_t - x_t \pi_t^{-1} - R_{t-1} \pi_t^{-1} (W_t - x_t) \big]$$
$$ - \beta^{t+1} \mu_{t+1} (x_{t+1} - m_t) \}.$$

Here W_t, x_t are the state and c_t, m_t the control variables.

The first-order conditions are

$$c_t^{-\sigma_1} - \beta E_t \lambda_{t+1} = 0, \tag{A.1}$$

$$E_t \mu_{t+1} = 0, \tag{A.2}$$

$$\lambda_t = \beta \big(R_{t-1} \pi_t^{-1} \big) E_t \lambda_{t+1}, \tag{A.3}$$

$$\mu_t = \mathcal{A} \pi_t^{-1} \big(x_t \pi_t^{-1} \big)^{-\sigma_2} + \beta \big(\pi_t^{-1} - R_{t-1} \pi_t^{-1} \big) E_t \lambda_{t+1}. \tag{A.4}$$

In addition, the optimal choices satisfy the transversality conditions (4).[33]

These equations hold under RE, but they also hold for an optimizing agent who is solving a dynamic optimization problem under his subjective expectations. We now derive the consumption and money demand equations under subjective expectations. Substituting (A.3) into (A.4), one eliminates $E_t \lambda_{t+1}$. Advancing the resulting equation and using (A.1) and (A.2) leads to

$$\mathcal{A} m_t^{-\sigma_2} E_t^* \pi_{t+1}^{\sigma_2 - 1} + \big(R_t^{-1} - 1 \big) \beta^{-1} c_t^{-\sigma_1} = 0. \tag{A.5}$$

We use $E_t^*(.)$ to emphasize that (A.5) holds for subjective as well as RE.

To derive the Euler equation for consumption, combine (A.1) and (A.3) to obtain $\lambda_t = R_{t-1} \pi_t^{-1} c_t^{-\sigma_1}$ and $c_t^{-\sigma_1} = \beta R_t E_t^* (\pi_{t+1}^{-1} c_{t+1}^{-\sigma_1})$. Assuming that all agents have identical expectations, market clearing implies that $c_t = y - g$ for all agents. It is, therefore, natural to assume that agents forecast their future consumption as $c_{t+1} = y - g$. We arrive at the consumption schedule

$$c_t^{-\sigma_1} = (y - g)^{-\sigma_1} \beta R_t E_t^* \pi_{t+1}^{-1}.$$

The temporary equilibrium is obtained by imposing market clearing $c_t = y - g$, which immediately gives the Fisher equation

$$R_t^{-1} = \beta E_t^* \pi_{t+1}^{-1}, \tag{A.6}$$

which under RE is (2). The equation for money market equilibrium is

$$\mathcal{A} \beta m_t^{-\sigma_2} E_t^* \pi_{t+1}^{\sigma_2 - 1} = (y - g)^{-\sigma_1} \big(1 - \beta E_t^* \pi_{t+1}^{-1} \big), \tag{A.7}$$

which, together with money supply rule (8), determines the current price level, for given expectations, in a temporary equilibrium. Under RE, we arrive at (3). If instead monetary policy is conducted using the interest rate rule (7), the money supply becomes endogenous and, given expectations, is jointly determined with the interest rate and price level.

A.2 LINEARIZATION OF THE MODEL

Rearranging (34), we can write money market clearing as

$$m_t = (\mathcal{A}\beta)^{1/\sigma_2}(y - g)^{\sigma_1/\sigma_2}\big[(1 - \beta E_t^* \pi_{t+1}^{-1})\big(E_t^* \pi_{t+1}^{\sigma_2 - 1}\big)^{-1}\big]^{-1/\sigma_2}. \tag{A.8}$$

Its linearization at the nonstochastic steady state π is

$$\tilde{m}_t = \left[\left(\frac{-\hat{C}\beta}{\sigma_2}\right)(\pi - \beta)^{-(1+\sigma_2)/\sigma_2} + \left(\frac{\sigma_2 - 1}{\sigma_2}\right)\hat{C}(\pi - \beta)^{-1/\sigma_2}\right]E_t^* \tilde{\pi}_{t+1},$$

where $\hat{C} = (\mathcal{A}\beta)^{1/\sigma_2}(y - g)^{\sigma_1/\sigma_2}$, or $\tilde{m}_t \equiv \tilde{C}E_t^* \tilde{\pi}_{t+1}$. Here $\tilde{m}_t$ and $E_t^* \tilde{\pi}_{t+1}$ denote the deviations from the steady state.

Linearizing the Fisher relation (A.6) at the steady state π, R, we get $E_t^* \tilde{\pi}_{t+1} = \beta \tilde{R}_t$, where $\tilde{R}_t$ is the deviation from the nonstochastic steady state. This can be substituted into money demand to yield $\tilde{m}_t = \tilde{C}\beta \tilde{R}_t$. Then linearize the budget constraint, taking note that m_t is a function of R_t:

$$0 = \tilde{b}_t + \tilde{C}\beta \tilde{R}_t + \gamma \tilde{b}_{t-1} + \psi_t - \pi^{-1}\tilde{C}\beta \tilde{R}_{t-1} + \frac{m}{\pi^2}\tilde{\pi}_t$$

$$- R\pi^{-1}\tilde{b}_{t-1} - \pi^{-1}b\tilde{R}_{t-1} + Rb\pi^{-2}\tilde{\pi}_t,$$

where π, b, R are the steady state values. We also note that $\tilde{R}_t = \alpha \tilde{\pi}_t + \theta_t$ as a result of centering.

Collecting everything together, we have the two Leeper-type equations

$$E_t^* \tilde{\pi}_{t+1} = \alpha\beta \tilde{\pi}_t + \beta\theta_t,$$

$$0 = \tilde{b}_t + \tilde{\pi}_t\left(\tilde{C}\beta\alpha + \frac{m}{\pi^2} + Rb\pi^{-2}\right) + \tilde{\pi}_{t-1}(-\pi^{-1}\tilde{C}\beta\alpha - \pi^{-1}b\alpha) \tag{A.9}$$

$$+ \tilde{b}_{t-1}(\gamma - \beta^{-1}) + \tilde{C}\beta\theta_t + \psi_t + \theta_{t-1}\left(-\pi^{-1}\tilde{C}\beta - \frac{b}{\pi}\right).$$

Equation (A.9) implicitly specifies the coefficients φ_1, φ_2, φ_3, φ_4 of equation (19). The nonstochastic steady state values are given by equations

$$\beta R = \pi, \ R = \alpha_0 + \alpha\pi,$$

$$b + m + \gamma_0 + \gamma b = g + m\pi^{-1} + R\pi^{-1}b,$$

$$m = \mathcal{A}^{1/\sigma_2}(y - g)^{\sigma_1/\sigma_2}\beta R(R - 1)^{-1/\sigma_2}.$$

A.3 DETERMINACY CONDITIONS

For either specification, the system under RE can be rewritten as

$$\begin{pmatrix} 1 & 0 \\ -\varphi_2 & \beta^{-1} - \gamma \end{pmatrix}\begin{pmatrix} \pi_t \\ b_t \end{pmatrix} = \begin{pmatrix} (\alpha\beta)^{-1} & 0 \\ \varphi_1 & 1 \end{pmatrix}\begin{pmatrix} \pi_{t+1} \\ b_{t+1} \end{pmatrix} + \begin{pmatrix} (\alpha\beta)^{-1} \\ 0 \end{pmatrix}\eta_{t+1}$$

$$+ \begin{pmatrix} 0 \\ \varphi_3 \end{pmatrix}\theta_{t+1} + \begin{pmatrix} -\alpha^{-1} \\ \varphi_4 \end{pmatrix}\theta_t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\psi_{t+1},$$

or

$$\begin{pmatrix} \pi_t \\ b_t \end{pmatrix} = J \begin{pmatrix} \pi_{t+1} \\ b_{t+1} \end{pmatrix} + F_1 \eta_{t+1} + F_2 \theta_{t+1} + F_3 \theta_t + F_4 \psi_{t+1},$$

where $\eta_{t+1} = \pi_{t+1} - E_t \pi_{t+1}$ and

$$J = \begin{pmatrix} (\alpha\beta)^{-1} & 0 \\ (\beta^{-1} - \gamma)^{-1}(\varphi_1 + \varphi_2(\alpha\beta)^{-1}) & (\beta^{-1} - \gamma)^{-1} \end{pmatrix}.$$

The eigenvalues of J are $(\alpha\beta)^{-1}$ and $(\beta^{-1} - \gamma)^{-1}$. The model has one predetermined and one free variable and thus determinacy requires that exactly one of these eigenvalues has absolute value less than one and the other greater than one. Imposing non-explosiveness gives a linear restriction between π_t, b_t and θ_t corresponding to the root with absolute value less than one. This restriction is obtained as follows. Diagonalize J as $J = Q \Lambda Q^{-1}$, where $\Lambda = \text{diag}((\alpha\beta)^{-1}, (\beta^{-1} - \gamma)^{-1})$. Explicit computation using Mathematica gives

$$Q^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{\alpha\beta\varphi_1 + \varphi_2}{\gamma - \beta^{-1} + \alpha\beta} & 1 \end{pmatrix}.$$

Letting $(x_t, z_t)' = Q^{-1}(\pi_t, b_t)'$, we have

$$\begin{pmatrix} x_t \\ z_t \end{pmatrix} = \Lambda \begin{pmatrix} x_{t+1} \\ z_{t+1} \end{pmatrix} + Q^{-1}(F_1 \eta_{t+1} + F_2 \theta_{t+1} + F_3 \theta_t + F_4 \psi_{t+1}).$$

Let $(C_1, C_2)' = -Q^{-1}F_3$. If $|(\alpha\beta)^{-1}| < 1$ then nonexplosiveness of the solution requires that $E_t x_{t+1} = \alpha\beta(x_t + C_1\theta_t) = 0$, as otherwise $|E_t x_{t+s}| \to \infty$ as $s \to \infty$. This gives the restriction $x_t + C_1\theta_t = 0$. Using $x_t = \pi_t$ we obtain the static linear relationship satisfied by the monetarist solution. If $|(\beta^{-1} - \gamma)^{-1}| < 1$, then analogously nonexplosiveness requires that $z_t + C_2\theta_t = 0$. Using z_t as the linear function of π_t and b_t specified earlier gives the static linear relationship satisfied by the fiscalist solution.

Finally, we remark that in Section 4.1 the fiscalist solution II can be shown to satisfy the fiscalist static relationship whether or not the model is determinate. Because the matrix B is singular, one row is proportional to the other row and it can be verified that the proportionality factor is $\frac{\alpha\beta\varphi_1 + \varphi_2}{\beta^{-1} - \gamma - \alpha\beta}$, which is the same as the coefficient in (21).

A.4 PROOF OF PROPOSITION 5

The proposition is established by considering the E-stability conditions. First we note that the B component in this differential equation is nonlinear, with local stability determined by its linearization at the fixed point of interest. The B, C and D components are matrix-valued and need to be vectorized. Moreover, it is seen that the B component of (27) is an independent subsystem, the A and D subsystems, respectively, depend on B, and the C subsystem depends on both B and D. The stability conditions for (27) can be given in terms

of the following matrices:

$$D_A T_A(\bar{A}, \bar{B}) = M(I + \bar{B}),$$

$$D_B T_B(\bar{B}) = \bar{B}' \otimes M + I \otimes M\bar{B},$$

$$D_C T_C(\bar{B}, \bar{C}, \bar{D}) = I \otimes M\bar{B},$$

$$D_D T_D(\bar{B}, \bar{D}) = I \otimes M\bar{B},$$

where $\otimes$ denotes the Kronecker product and $\bar{A}$, $\bar{B}$, and soon denote the values of A, B etc. at the different REE in Proposition 3.

The E-stability condition for REE of type I and II is that the real parts of all eigenvalues of all four matrices $D_i T_i$, $i = A, B, C, D$, are less than one. For the class of nonfundamental solutions III, the matrices $D_C T_C$ and $D_D T_D$ will have some eigenvalues equal to one, because of the continuum of solutions. A necessary condition for E-stability is that the other eigenvalues of the four matrices have real parts less than one.

The explicit E-stability conditions for the three types of REE can be analytically computed using Mathematica (routines available on request). We give details for the monetarist solution and summarize the E-stability conditions in the other cases.

I. The monetarist solution: Inserting M and the value of $\bar{B}$ in the monetarist solution, we get

$$D_A T_A = \begin{pmatrix} \dfrac{1}{\alpha\beta} & 0 \\ \dfrac{-\varphi_1}{\alpha\beta} & 0 \end{pmatrix}, \quad D_B T_B = \begin{pmatrix} 0 & 0 & \dfrac{-\varphi_2}{\alpha\beta} & 0 \\ 0 & 0 & \dfrac{\varphi_1\varphi_2}{\alpha\beta} & 0 \\ 0 & 0 & \dfrac{1-\beta\gamma}{\alpha\beta^2} & 0 \\ 0 & 0 & \dfrac{-(1-\beta\gamma)\varphi_1}{\alpha\beta^2} & 0 \end{pmatrix}$$

and $D_C T_C = D_D T_D = 0$. The eigenvalues of $D_A T_A$ are 0 and $(\alpha\beta)^{-1}$. The nonzero eigenvalue of $D_B T_B$ is $(\beta^{-1} - \gamma)/\alpha\beta$. This yields the E-stability conditions given.

II. The fiscalist solution: The nonzero eigenvalues of $D_i T_i$, $i = A, B, C, D$, are $1 + (\gamma + 1 - \beta^{-1})/\alpha\beta$, $1 + (\gamma - \beta^{-1})/\alpha\beta$, and $2 + (\gamma - \beta^{-1})/\alpha\beta$. This yields the E-stability conditions given. Although the matrix $\bar{B}$ depends on φ_1 and φ_2, the eigenvalues of $D_i T_i$, $i = A, B, C, D$, are in fact independent of φ_1 and φ_2, as can be verified using Mathematica.

III. The nonfundamental solutions are not E-stable, as $D_B T_B$ has an eigenvalue equal to 2.