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Characterizations of right rejective chains

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Abstract. In this paper, we give characterizations of the category of finitely generated projective modules having a right rejective chain. By focusing on the characterizations, we give sufficient conditions for right rejective chains to be total right rejective chains. Moreover, we prove that Nakayama algebras with heredity ideals, locally hereditary algebras and algebras of global dimension at most two satisfy the sufficient conditions. As an application, we show that these algebras are right-strongly quasi-hereditary algebras.

1 Introduction

The notion of right rejective subcategories was introduced by Iyama and plays a crucial role in the proof of the finiteness theorem for representation dimensions of artin algebras [13]. Right rejective subcategories are a special class of coreflective subcategories which appear in the classical theory of localizations of abelian categories. It is known that right rejective subcategories provide effective tools to study representation-finite orders [12, 14, 19]. A subcategory $\mathcal{C}'$ of an additive category $\mathcal{C}$ is called a *right rejective subcategory* if for each $X \in \mathcal{C}$, there exists a monic right $\mathcal{C}'$-approximation of X. We call a chain of cosemisimple right rejective (respectively, coreflective) subcategories a *right rejective* (respectively, *coreflective*) *chain*.

Quasi-hereditary algebras were introduced by Cline, Parshall, and Scott to explore highest weight categories arising from the study of semisimple complex Lie algebras and algebraic groups [5, 20]. As a distinguished class of quasi-hereditary algebras, Ringel [18] introduced *right-strongly quasi-hereditary algebras* which frequently appear in the representation theory of algebras [7, 8, 10, 16, 21]. It is known that special right rejective chains called *total right rejective chains*, are characterized by right-strongly quasi-hereditary algebras.

Proposition 1.1 ([22, Theorem 3.22]) *Let A be an artin algebra and $(e_1, e_2, \dots, e_n)$ a complete set of primitive orthogonal idempotents of A. Let $\varepsilon_i := e_i + \cdots + e_n$. Then the following statements are equivalent.*

(1) *The following chain is a total right rejective chain.*

$$\mathsf{proj}\, A \supset \mathsf{add}\, \varepsilon_2 A \supset \cdots \supset \mathsf{add}\, \varepsilon_n A \supset 0.$$

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(2) *$A > A\varepsilon_2 A > \cdots > A\varepsilon_n A > 0$ is a heredity chain (namely A is a quasi-hereditary algebra), and the following chain is a coreflective chain.*

$$\operatorname{proj} A \supset \operatorname{add} \varepsilon_2 A \supset \cdots \supset \operatorname{add} \varepsilon_n A \supset 0.$$

(3) *$A > A\varepsilon_2 A > \cdots > A\varepsilon_n A > 0$ is a right-strongly heredity chain (namely A is a right-strongly quasi-hereditary algebra).*

Ágoston et al. [1] introduced neat algebras as a generalization of quasi-hereditary algebras. As an analog of Proposition 1.1, we give characterizations for proj A to admit a right rejective chain using neat algebras. For an idempotent e of A, let $S(e) := eA/eJ$, where J is the Jacobson radical of A.

Theorem 1.2 (Theorem 3.5) *Let A be an artin algebra and $(e_1, e_2, \ldots, e_n)$ a complete set of primitive orthogonal idempotents of A. Let $\varepsilon_i := e_i + \cdots + e_n$. Then the following statements are equivalent.*

(1) *The following chain is a right rejective chain.*

$$\operatorname{proj} A \supset \operatorname{add} \varepsilon_2 A \supset \cdots \supset \operatorname{add} \varepsilon_n A \supset 0. \tag{1.1}$$

(2) *$A > A\varepsilon_2 A > \cdots > A\varepsilon_n A > 0$ is a neat chain (namely A is a neat algebra), and the following chain is a coreflective chain.*

$$\operatorname{proj} A \supset \operatorname{add} \varepsilon_2 A \supset \cdots \supset \operatorname{add} \varepsilon_n A \supset 0.$$

(3) *For each $i \in [1, n]$, $\operatorname{pd}_{\varepsilon_i A \varepsilon_i} S(e_i)\varepsilon_i \leq 1$ holds.*

By focusing on the condition (3) in Theorem 1.2, we give sufficient conditions (see Theorem 3.9) for the right rejective chain (1.1) to be a total right rejective chain. Moreover, we prove that the following classes of algebras satisfy the sufficient conditions.

- Nakayama algebras with heredity ideals (Proposition 3.13).
- Locally hereditary algebras (Proposition 3.16).
- Algebras of global dimension at most two (Proposition 3.17).

Therefore, we have the following result.

Theorem 1.3 (Theorem 3.12) *Nakayama algebras with heredity ideals, locally hereditary algebras and algebras of global dimension at most two are right-strongly quasi-hereditary.*

1.1 Notation

Throughout this paper, A is a basic artin algebra. Let $J = J(A)$ be the Jacobson radical of A. By a module, we mean a finitely generated right A-module. Let $\operatorname{pd}_A M$ denote the projective dimension of an A-module M. We write mod A for the category of finitely generated right A-modules and proj A for the full subcategory of mod A consisting of projective A-modules. For $M \in \operatorname{mod} A$, let add M denote the full subcategory of mod A whose objects are direct summands of finite direct sums of M.

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2 Right rejective chains

In this section, we recall the definition of right rejective chains and collect related results (see [13, 15] for details). By a subcategory, we always mean a full subcategory which is closed under isomorphisms. For a subcategory $\mathcal{C}'$ of an additive category $\mathcal{C}$, we call a morphism $f : Y \to X$ in $\mathcal{C}$ a right $\mathcal{C}'$-approximation of X if $Y \in \mathcal{C}'$ and $\mathcal{C}(-, f) : \mathcal{C}(-, Y) \to \mathcal{C}(-, X)$ is an epimorphism on $\mathcal{C}'$. Dually, we define a left $\mathcal{C}'$-approximation.

Definition 2.1 Let $\mathcal{C}$ be an additive category and $\mathcal{C}'$ a subcategory of $\mathcal{C}$.

(1) We call $\mathcal{C}'$ a *coreflective subcategory* of $\mathcal{C}$ if for any $X \in \mathcal{C}$, there exists a right $\mathcal{C}'$-approximation $f \in \mathcal{C}(Y, X)$ of X such that $\mathcal{C}(-, f) : \mathcal{C}(-, Y) \to \mathcal{C}(-, X)$ is an isomorphism on $\mathcal{C}'$.
(2) We call $\mathcal{C}'$ a *right rejective subcategory* of $\mathcal{C}$ if for any $X \in \mathcal{C}$, there exists a right $\mathcal{C}'$-approximation of X such that it is a monomorphism in $\mathcal{C}$.

Dually, *reflective subcategories* and *left rejective subcategories* are defined.

Right rejective subcategories are coreflective subcategories, but the converse does not hold in general.

To define a right rejective chain, we need the notion of cosemisimple subcategories. Let $\mathcal{J}_{\mathcal{C}}$ be the Jacobson radical of $\mathcal{C}$. For a subcategory $\mathcal{C}'$ of $\mathcal{C}$, let $[\mathcal{C}']$ denote the ideal of $\mathcal{C}$ consisting of morphisms which factor through some object of $\mathcal{C}'$, and let $\mathcal{C}/[\mathcal{C}']$ denote the factor category.

Definition 2.2 Let $\mathcal{C}$ be an additve category. A subcategory $\mathcal{C}'$ of $\mathcal{C}$ is called a *cosemisimple subcategory* of $\mathcal{C}$ if $\mathcal{J}_{\mathcal{C}/[\mathcal{C}']} = 0$ holds.

We recall characterizations of coreflective subcategories, right rejective subcategories and cosemisimple right rejective subcategories of $\operatorname{proj} A$. We write gldim A for the global dimension of A.

Proposition 2.3 ([15, Theorem 3.2]) *Let A be an artin algebra and ε an idempotent of A. Then the following statements hold.*

(1) $\operatorname{add}\varepsilon A$ *is a coreflective subcategory of* $\operatorname{proj} A$ *if and only if* $A\varepsilon \in \operatorname{proj} \varepsilon A\varepsilon$.
(2) $\operatorname{add}\varepsilon A$ *is a right rejective subcategory of* $\operatorname{proj} A$ *if and only if* $A\varepsilon A \in \operatorname{add}\varepsilon A$ *as a right A-module. In this case,* $\operatorname{gldim} \varepsilon A\varepsilon \le \operatorname{gldim} A$ *holds.*
(3) $\operatorname{add}\varepsilon A$ *is a cosemisimple right rejective subcategory of* $\operatorname{proj} A$ *if and only if* $eJ \in \operatorname{add}\varepsilon A$ *as a right A-module, where $e := 1 - \varepsilon$. In this case, the inclusion $eJ \to eA$ is a right* $\operatorname{add}\varepsilon A$*-approximation of eA.*

Now, we introduce the notion of coreflective chains, right rejective chains and total right rejective chains.

Definition 2.4 Let $\mathcal{C}$ be an additve category. Let

$$\mathcal{C} = \mathcal{C}_1 \supset \mathcal{C}_2 \supset \cdots \supset \mathcal{C}_n \supset \mathcal{C}_{n+1} = 0 \tag{2.1}$$

be a chain of subcategories of $\mathcal{C}$.

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(1) (2.1) is called a *coreflective chain* (of length n) if $\mathcal{C}_{i+1}$ is a cosemisimple coreflective subcategory of $\mathcal{C}_i$ for each $1 \le i \le n$.
(2) (2.1) is called a *right rejective chain* (of length n) if $\mathcal{C}_{i+1}$ is a cosemisimple right rejective subcategory of $\mathcal{C}_i$ for each $1 \le i \le n$.
(3) (2.1) is called a *total right rejective chain* (of length n) if the following conditions hold for each $1 \le i \le n$:
   (a) $\mathcal{C}_{i+1}$ is a right rejective subcategory of $\mathcal{C}$ and
   (b) $\mathcal{C}_{i+1}$ is a cosemisimple subcategory of $\mathcal{C}_i$.

One can easily check that total right rejective chains are right rejective chains, and right rejective chains are coreflective chains. However, the converses do not always hold. We refer to Example 3.6 which shows that right rejective chains are not necessarily total right rejective chains.

If $\mathsf{proj}\,A$ has a right rejective chain, then we give an upper bound of the global dimension of A.

Proposition 2.5 ([15, Theorem 3.3]) *Let A be an artin algebra. If* $\mathsf{proj}A$ *admits a right rejective chain of length n, then the global dimension of A is at most $2n-2$.*

Remark 2.6 The length of a right rejective chain of $\mathsf{proj}\,A$ is bounded by the number of indecomposable direct summands of A, or equivalently the number of isomorphism classes of simple A-modules. Hence, if A has m simple modules and $\mathsf{proj}\,A$ admits a right rejective chain, then the global dimension of A is at most $2m-2$ by Proposition 2.5.

Total right rejective chains are closely related to right-strongly quasi-hereditary algebras which are a special class of quasi-hereditary algebras. We recall the definitions of quasi-hereditary algebras and right-strongly quasi-hereditary algebras (see [5, 9] and [18], for details). We call an idempotent ideal $A\varepsilon A$ a *heredity ideal* of A if $A\varepsilon A \in \mathsf{proj}\,A$ and $\varepsilon A\varepsilon$ is semisimple.

Definition 2.7 Let A be an artin algebra.

(1) ([5, 9]) We call A a *quasi-hereditary algebra* if there exists a chain of idempotent ideals

$$A = A\varepsilon_1 A > A\varepsilon_2 A > \cdots > A\varepsilon_n A > A\varepsilon_{n+1} A = 0 \tag{2.2}$$

such that $A\varepsilon_i A/A\varepsilon_{i+1}A$ is a heredity ideal of $A/A\varepsilon_{i+1}A$ for each $1 \le i \le n$. In this case, (2.2) is called a *heredity chain*.
(2) ([18]) We call A a *right-strongly quasi-hereditary algebra* if there exists a chain of idempotent ideals

$$A = A\varepsilon_1 A > A\varepsilon_2 A > \cdots > A\varepsilon_n A > A\varepsilon_{n+1} A = 0$$

such that $A\varepsilon_i A \in \mathsf{proj}\,A$ and $\varepsilon_i A\varepsilon_i/\varepsilon_i A\varepsilon_{i+1}A\varepsilon_i$ is semisimple for each $1 \le i \le n$.

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Since $-\otimes_A A/A\varepsilon_{i+1}A$ preserves projective modules, right-strongly quasi-hereditary algebras are quasi-hereditary algebras. We give a characterization of right-strongly quasi-hereditary algebras by total right rejective chains.

Proposition 2.8 ([22, Theorem 3.22]) *Let A be an artin algebra. Then A is a right-strongly quasi-hereditary algebra if and only if* $\mathsf{proj}\,A$ *admits a total right rejective chain.*

3 Main results

In this section, we give characterizations of right rejective chains and constructions of total right rejective chains. Moreover, using the constructions, we provide total right rejective chains for several classes of algebras.

3.1 Characterizations of right rejective chains

In this subsection, using neat algebras, we give characterizations of $\mathsf{proj}\,A$ admitting a right rejective chain.

For an idempotent e of A, $S(e)$ stands for a semisimple A-module eA/eJ. Let $P(e)$ denote a projective cover of $S(e)$. In the following, we fix a complete set $(e_1, e_2, \ldots, e_n)$ of primitive orthogonal idempotents of A. Then, we have a complete set $\{S(e_i) \mid 1 \le i \le n\}$ of representatives of isomorphism classes of simple A-modules. Let $\varepsilon_i := e_i + \cdots + e_n$ for $1 \le i \le n$ and $\varepsilon_{n+1} := 0$. We recall the definition of neat algebras (see [1] for details). A typical example of a neat algebra is a quasi-hereditary algebra.

Definition 3.1 [1] Let A be an artin algebra.

(1) An idempotent e of A is called a *neat idempotent* if $\mathrm{Ext}^i_A(S(e), S(e)) = 0$ holds for each $i \ge 1$. We call $(e_1, e_2, \ldots, e_n)$ a *neat sequence* if e_i is a neat idempotent of $\varepsilon_i A \varepsilon_i$ for each $1 \le i \le n$.

(2) An algebra A is called a *neat algebra* if there exists a neat sequence.

We give an upper bound of the global dimension of a neat algebra.

Proposition 3.2 ([1, Proposition 2]) *If A is a neat algebra with a neat sequence $(e_1, e_2, \ldots, e_n)$, then the global dimension of A is at most $2^n - 2$.*

We reformulate neat algebras in terms of a chain of strong idempotent ideals. Recall that an idempotent ideal $A\varepsilon A$ is called a *strong idempotent ideal* (or *stratifying ideal*) of A if $\mathrm{Ext}^j_{A/A\varepsilon A}(X, Y) \cong \mathrm{Ext}^j_A(X, Y)$ holds for each $X, Y \in \mathsf{mod}\,(A/A\varepsilon A)$ and $j \ge 1$ (see [3] and [6] for details).

Proposition 3.3 *Let A be an artin algebra. Then A is a neat algebra with a neat sequence $(e_1, e_2, \ldots, e_n)$ if and only if a chain of idempotent ideals*

$$A > A\varepsilon_2 A > \cdots > A\varepsilon_n A > A\varepsilon_{n+1} A = 0 \tag{3.1}$$

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satisfies that $\varepsilon_i A\varepsilon_{i+1}A\varepsilon_i$ *is a strong idempotent ideal of* $\varepsilon_i A\varepsilon_i$ *and* $\varepsilon_i A\varepsilon_i/\varepsilon_i A\varepsilon_{i+1}A\varepsilon_i$ *is semisimple for each* $1 \le i \le n$. *In this case, we call* (3.1) *a neat chain.*

Proof Let e be an idempotent of A and $\varepsilon := 1-e$. It suffices to show that e is a neat idempotent of A if and only if $A\varepsilon A$ is a strong idempotent ideal and $A/A\varepsilon A$ is semisimple. Indeed, one can apply this claim to each subalgebra $\varepsilon_i A\varepsilon_i$. By [1, Proposition 1.1], e is a neat idempotent of A if and only if it satisfies that (a) the multiplication map: $A\varepsilon \otimes_{\varepsilon A\varepsilon} \varepsilon A \to A\varepsilon A$ is an isomorphism, (b) $\mathrm{Tor}_j^{\varepsilon A\varepsilon}(A\varepsilon, \varepsilon A) = 0$ for each $j \ge 1$ and (c) $eA\varepsilon Ae = eJe$. Moreover, by [6, Remark 2.1.2(a)], the conditions (a) and (b) are equivalent to the condition that $A\varepsilon A$ is a strong idempotent ideal of A. Thus we show that the condition (c) holds if and only if $A/A\varepsilon A$ is semisimple. The "only if" part holds since $A/A\varepsilon A \cong eAe/eA\varepsilon Ae = eAe/J(eAe)$. We show the "if" part. By $\varepsilon = 1-e$, we obtain that $eA\varepsilon Ae \subseteq eJe$. Since $A/A\varepsilon A$ is semisimple, so is $eAe/eA\varepsilon Ae$. Thus $0 = J(eAe/eA\varepsilon Ae) = J(eAe)/eA\varepsilon Ae$, and hence the assertion holds. ■

By Proposition 3.3, we reprove the following result.

Corollary 3.4 ([1, Corollary in §0]) *Any quasi-hereditary algebra is a neat algebra.*

Proof We assume that A is a quasi-hereditary algebra. Then there exists a heredity chain

$$A > A\varepsilon_2 A > \cdots > A\varepsilon_n A > A\varepsilon_{n+1}A = 0.$$

We show that this chain is a neat chain. By definition, $\varepsilon_i A\varepsilon_i/\varepsilon_i A\varepsilon_{i+1}A\varepsilon_i$ is semisimple. Thus it suffices from Proposition 3.3 to prove that $\varepsilon_i A\varepsilon_{i+1}A\varepsilon_i$ is a strong idempotent ideal of $\varepsilon_i A\varepsilon_i$ for each $1 \le i \le n$. By $A\varepsilon_i A/A\varepsilon_{i+1}A \in \mathsf{proj}\,(A/A\varepsilon_{i+1}A)$, it follows from [3, Proposition 5.2] that $A\varepsilon_i A/A\varepsilon_{i+1}A$ is a strong idempotent ideal of $A/A\varepsilon_{i+1}A$. Since

$$\frac{A/A\varepsilon_{i+1}A}{A\varepsilon_i A/A\varepsilon_{i+1}A} \cong A/A\varepsilon_i A,$$

we obtain that $\mathrm{Ext}^j_{A/A\varepsilon_i A}(X,Y) \cong \mathrm{Ext}^j_{A/A\varepsilon_{i+1}A}(X,Y)$ for each $X, Y \in \mathsf{mod}\,(A/A\varepsilon_i A)$ and $j \ge 1$. Thus, $A\varepsilon_i A$ is inductively a strong idempotent ideal of A for each $2 \le i \le n$. Since $A\varepsilon_{i+1}A$ is a strong idempotent ideal of A, it follows from [3, Theorem 2.1] that $\varepsilon_i A\varepsilon_{i+1}A\varepsilon_i$ is also a strong idempotent ideal of $\varepsilon_i A\varepsilon_i$. Hence the assertion holds. ■

We give characterizations for $\mathsf{proj}\,A$ to admit a right rejective chain.

Theorem 3.5 *Let A be an artin algebra. Then the following statements are equivalent.*

(1) *The following chain is a right rejective chain.*

$$\mathsf{proj}\,A \supset \mathsf{add}\,\varepsilon_2 A \supset \cdots \supset \mathsf{add}\,\varepsilon_n A \supset 0.$$

(2) *A is a neat algebra with a neat sequence* $(e_1, e_2, \dots, e_n)$ *and the following chain is a coreflective chain.*

$$\mathsf{proj}\,A \supset \mathsf{add}\,\varepsilon_2 A \supset \cdots \supset \mathsf{add}\,\varepsilon_n A \supset 0. \tag{3.2}$$

(3) *For each* $1 \le i \le n$, $\mathrm{pd}_{\varepsilon_i A\varepsilon_i} S(e_i)\varepsilon_i \le 1$ *holds.*

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Proof By an equivalence $\mathrm{Hom}_A(\varepsilon_i A, -) : \mathsf{add}\,\varepsilon_i A \xrightarrow{\sim} \mathsf{proj}\,\varepsilon_i A\varepsilon_i$, we obtain that $\mathsf{add}\,\varepsilon_{i+1}A$ is a cosemisimple right rejective (respectively, coreflective) subcategory of $\mathsf{add}\,\varepsilon_i A$ if and only if $\mathsf{add}\,\varepsilon_{i+1}A\varepsilon_i$ is a cosemisimple right rejective (respectively, coreflective) subcategory of $\mathsf{proj}\,\varepsilon_i A\varepsilon_i$.

(1)⇔(3): We consider an exact sequence in $\mathsf{mod}\,\varepsilon_i A\varepsilon_i$

$$0 \to e_i J\varepsilon_i \to e_i A\varepsilon_i \to S(e_i)\varepsilon_i \to 0.$$

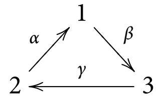
If $\mathsf{add}\,\varepsilon_{i+1}A$ is a cosemisimple right rejective subcategory of $\mathsf{add}\,\varepsilon_i A$, then $e_i J\varepsilon_i \in \mathsf{add}\,\varepsilon_{i+1}A\varepsilon_i$ by Proposition 2.3(3). Thus $\mathrm{pd}_{\varepsilon_i A\varepsilon_i} S(e_i)\varepsilon_i \leq 1$. Conversely, we assume that $\mathrm{pd}_{\varepsilon_i A\varepsilon_i} S(e_i)\varepsilon_i \leq 1$. Then $e_i J\varepsilon_i \in \mathsf{proj}\,\varepsilon_i A\varepsilon_i$. Since $e_i A\varepsilon_i$ is not a direct summand of $e_i J\varepsilon_i$, we have $e_i J\varepsilon_i \in \mathsf{add}\,\varepsilon_{i+1}A\varepsilon_i$. Hence $\mathsf{add}\,\varepsilon_{i+1}A$ is a cosemisimple right rejective subcategory of $\mathsf{add}\,\varepsilon_i A$ by Proposition 2.3(3). Thus the assertion holds.

(1)⇒ (2): Since right rejective chains are coreflective chains, it suffices to show that the chain $A > A\varepsilon_2 A > \cdots > A\varepsilon_n A > 0$ is a neat chain by Proposition 3.3. Due to our assumption, $\mathsf{add}\,\varepsilon_{i+1}A$ is a right rejective subcategory of $\mathsf{add}\,\varepsilon_i A$. By Proposition 2.3(2), we obtain that $\varepsilon_i A\varepsilon_{i+1}A\varepsilon_i \in \mathsf{proj}\,\varepsilon_i A\varepsilon_i$ for each $1 \leq i \leq n$. Thus $\varepsilon_i A\varepsilon_{i+1}A\varepsilon_i$ is a strong idempotent ideal of $\varepsilon_i A\varepsilon_i$ by [3, Proposition 5.2]. Let $\mathcal{C} := \mathsf{add}\,\varepsilon_i A/[\mathsf{add}\,\varepsilon_{i+1}A]$. Then $\mathcal{J}_{\mathcal{C}} = 0$ holds by cosemisimplicity. Since $J(\varepsilon_i A\varepsilon_i/\varepsilon_i A\varepsilon_{i+1}A\varepsilon_i) = J(\mathrm{End}_{\mathcal{C}}(\varepsilon_i A)) = \mathcal{J}_{\mathcal{C}}(\varepsilon_i A, \varepsilon_i A) = 0$, we have the assertion.

(2)⇒ (1): Since (3.2) is a coreflective chain, $\mathsf{add}\,\varepsilon_{i+1}A$ is a cosemisimple subcategory of $\mathsf{add}\,\varepsilon_i A$. Hence, it is enough to show that $\mathsf{add}\,\varepsilon_{i+1}A$ is a right rejective subcategory of $\mathsf{add}\,\varepsilon_i A$ for each $1 \leq i \leq n-1$. By Proposition 2.3(2), we prove that $\varepsilon_i A\varepsilon_{i+1}A\varepsilon_i \in \mathsf{proj}\,\varepsilon_i A\varepsilon_i$. Since A is a neat algebra with a neat sequence $(e_1, e_2, \ldots, e_n)$, it follows from Proposition 3.3 that $\varepsilon_i A\varepsilon_{i+1}A\varepsilon_i$ is a strong idempotent ideal of $\varepsilon_i A\varepsilon_i$. Due to our assumption, $\mathsf{add}\,\varepsilon_{i+1}A$ is a coreflective subcategory of $\mathsf{add}\,\varepsilon_i A$. Thus, $\varepsilon_i A\varepsilon_{i+1} \in \mathsf{proj}\,\varepsilon_{i+1}A\varepsilon_{i+1}$ by Proposition 2.3(1). Since $\varepsilon_i A\varepsilon_{i+1}A\varepsilon_i$ is a strong idempotent ideal of $\varepsilon_i A\varepsilon_i$ and $\varepsilon_i A\varepsilon_{i+1} \in \mathsf{proj}\,\varepsilon_{i+1}A\varepsilon_{i+1}$, it follows from [3, Corollary 3.8(b), Proposition 5.2] that $\varepsilon_i A\varepsilon_{i+1}A\varepsilon_i \in \mathsf{proj}\,\varepsilon_i A\varepsilon_i$. Thus the proof is complete. ∎

We give an example of a right rejective chain using Theorem 3.5.

Example 3.6 Let A be the algebra defined by the quiver



with relations $\alpha\beta\gamma\alpha$ and $\gamma\alpha\beta$. Then we can check that $\mathrm{pd}_A S(e_1) = 1$, $\mathrm{pd}_{\varepsilon_2 A\varepsilon_2} S(e_2)\varepsilon_2 = 1$ and $\mathrm{pd}_{\varepsilon_3 A\varepsilon_3} S(e_3)\varepsilon_3 = 0$. Thus $\mathsf{proj}\,A \supset \mathsf{add}\,(e_2 + e_3)A \supset \mathsf{add}\,e_3 A \supset 0$ is a right rejective chain by Theorem 3.5. However, this is not a total right rejective chain since $Ae_3 A \cong P(e_3)^{\oplus 2} \oplus P(e_3)/P(e_3)J^2 \notin \mathsf{proj}\,A$.

In the rest of this subsection, we give constructions of total right rejective chains using Theorem 3.5. We need the following two lemmas.

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Lemma 3.7 *Let e and ε be idempotents of A such that* $\mathsf{add}\, eA \subset \mathsf{add}\, \varepsilon A$. *If $\varphi : Y \to X$ is a right* $\mathsf{add}\, eA\varepsilon$*-approximation of $X \in$* $\mathsf{proj}\, \varepsilon A\varepsilon$, *then $\varphi \otimes_{\varepsilon A\varepsilon} \varepsilon A : Y \otimes_{\varepsilon A\varepsilon} \varepsilon A \to X \otimes_{\varepsilon A\varepsilon} \varepsilon A$ is a right* $\mathsf{add}\, eA$*-approximation of $X \otimes_{\varepsilon A\varepsilon} \varepsilon A$.*

Proof Let $f \in \mathrm{Hom}_A(eA, X \otimes_{\varepsilon A\varepsilon} \varepsilon A)$. Then $f \otimes_A A\varepsilon : eA \otimes_A A\varepsilon \to X \otimes_{\varepsilon A\varepsilon} \varepsilon A \otimes_A A\varepsilon$. Since φ is a right $\mathsf{add}\, eA\varepsilon$-approximation, there exists $\varphi' : eA \otimes_A A\varepsilon \to Y$ such that $\alpha(f \otimes_A A\varepsilon) = \varphi\varphi'$, where $\alpha : X \otimes_{\varepsilon A\varepsilon} \varepsilon A \otimes_A A\varepsilon \to X$ is an isomorphism. Let $\beta : eA \otimes_A A\varepsilon \otimes_{\varepsilon A\varepsilon} \varepsilon A \to eA$ be an isomorphism via $ea \otimes b\varepsilon \otimes \varepsilon c \mapsto eab\varepsilon c$. Then $(\alpha \otimes_{\varepsilon A\varepsilon} \varepsilon A)(f \otimes_A A\varepsilon \otimes_{\varepsilon A\varepsilon} \varepsilon A) = f\beta$ holds. Since $(\varphi \otimes_{\varepsilon A\varepsilon} \varepsilon A)(\varphi' \otimes_{\varepsilon A\varepsilon} \varepsilon A)\beta^{-1} = (\varphi\varphi' \otimes_{\varepsilon A\varepsilon} \varepsilon A)\beta^{-1} = (\alpha \otimes_{\varepsilon A\varepsilon} \varepsilon A)(f \otimes_A A\varepsilon \otimes_{\varepsilon A\varepsilon} \varepsilon A)\beta^{-1} = f$, we have the assertion. ∎

Lemma 3.8 *Let ε be an idempotent of A such that* $\mathsf{add}\, \varepsilon A$ *is a cosemisimple subcategory of* $\mathsf{proj}\, A$. *If $\varphi : P \to Q$ is a monomorphism in* $\mathsf{add}\, \varepsilon A$, *then* $\mathrm{Ker}\, \varphi \in \mathsf{mod}\, (A/A\varepsilon A)$.

Proof Let $\varphi : P \to Q$ be a monomorphism in $\mathsf{add}\, \varepsilon A$. We show that $\mathrm{Hom}_A(\varepsilon A, \mathrm{Ker}\, \varphi) = 0$. Let $\psi \in \mathrm{Hom}_A(\varepsilon A, \mathrm{Ker}\, \varphi)$. Then, we have a composition map

$$\varepsilon A \xrightarrow{\psi} \mathrm{Ker}\, \varphi \xrightarrow{i} P \xrightarrow{\varphi} Q$$

such that $\varphi i \psi = 0$ holds. Since $\varphi : P \to Q$ is a monomorphism in $\mathsf{add}\, \varepsilon A$, we have $i\psi = 0$. Hence $\psi = 0$ holds. ∎

Now, we give constructions of total right rejective chains.

Theorem 3.9 *Let A be an artin algebra and*

$$\mathsf{proj}\, A \supset \mathsf{add}\, \varepsilon_2 A \supset \cdots \supset \mathsf{add}\, \varepsilon_n A \supset 0 \tag{3.3}$$

a chain of subcategories. Then the chain (3.3) is a total right rejective chain if one of the following two conditions is satisfied.

(1) *For each $1 \le i \le n$, $e_i J \varepsilon_i \in$* $\mathsf{add}\, \varepsilon_n A\varepsilon_i$ *and $A\varepsilon_n A \in$* $\mathsf{proj}\, A$ *hold.*
(2) *Assume that there exists $1 \le j \le n-1$ such that*

$$\mathrm{pd}_{\varepsilon_i A\varepsilon_i} S(e_i)\varepsilon_i = \begin{cases} 1 & 1 \le i \le j \\ 0 & j+1 \le i \le n. \end{cases}$$

If $\varphi : P \to Q$ is a monomorphism in $\mathsf{add}\, \varepsilon_{i+1} A\varepsilon_i$, *then* $\mathrm{Ker}\, \varphi \in \mathsf{proj}\, \varepsilon_i A\varepsilon_i$ *for all $1 \le i \le n-1$.*

Proof First, we assume (1). Then $e_i J \varepsilon_i \in \mathsf{add}\, \varepsilon_n A\varepsilon_i$ implies that $\mathrm{pd}_{\varepsilon_i A\varepsilon_i} S(e_i)\varepsilon_i \le 1$ holds. Thus, by Theorem 3.5, (3.3) is a right rejective chain. Since $\varphi_j : e_j J\varepsilon_j \to e_j A\varepsilon_j$ is a right $\mathsf{add}\, \varepsilon_{j+1} A\varepsilon_j$-approximation by Proposition 2.3(3), it follows from Lemma 3.7 that a composition map of $\varphi_j \otimes_{\varepsilon_j A\varepsilon_j} \varepsilon_j A : e_j J\varepsilon_j \otimes_{\varepsilon_j A\varepsilon_j} \varepsilon_j A \to e_j A\varepsilon_j \otimes_{\varepsilon_j A\varepsilon_j} \varepsilon_j A$ and an isomorphism $e_j A\varepsilon_j \otimes_{\varepsilon_j A\varepsilon_j} \varepsilon_j A \to e_j A$ is a right $\mathsf{add}\, \varepsilon_{j+1} A$-approximation of $e_j A$. By $e_j J\varepsilon_j \in \mathsf{add}\, \varepsilon_n A\varepsilon_j$, we obtain that $e_j J\varepsilon_j \otimes_{\varepsilon_j A\varepsilon_j} \varepsilon_j A \in \mathsf{add}\, \varepsilon_n A$. Take a minimal right $\mathsf{add}\, \varepsilon_{j+1} A$-approximation $\varphi'_j : P' \to e_j A$ of $e_j A$. Then $P' \in \mathsf{add}\, \varepsilon_n A$. On the other hand, we have that $\varepsilon_n J\varepsilon_n = 0$ since $\varepsilon_n J\varepsilon_n \in \mathsf{add}\, \varepsilon_n A\varepsilon_n$ and $\varepsilon_n A\varepsilon_n$ is an indecomposable $\varepsilon_n A\varepsilon_n$-module. Thus, we obtain that $A\varepsilon_n A$ is a heredity ideal of A since

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$A\varepsilon_n A \in \mathsf{proj}\, A$. By [4, Lemma 1.7], φ'_j is a monomorphism, and hence the assertion holds.

Next, we assume (2). We show that (3.3) is a total right rejective chain by induction on n. If $n = 1$, then this is clear. Let $n \geq 2$. If $j = 1$, then $\varepsilon_2 A \varepsilon_2$ satisfies (1). Thus, we obtain a total right rejective chain of $\mathsf{proj}\, \varepsilon_2 A \varepsilon_2$. If $j > 1$, then we also have a total right rejective chain of $\mathsf{proj}\, \varepsilon_2 A \varepsilon_2$ by induction hypothesis. Since $\mathrm{pd}\, S(e_1) = 1$, it follows from Proposition 2.3(3) that $\mathsf{add}\, \varepsilon_2 A$ is a cosemisimple right rejective subcategory of $\mathsf{proj}\, A$. Since an equivalence $\mathsf{add}\, \varepsilon_i A \varepsilon_2 \simeq \mathsf{add}\, \varepsilon_i A$ holds for each $2 \leq i \leq n$, the chain of subcategories

$$\mathsf{proj}\, A \supset \mathsf{add}\, \varepsilon_2 A \supset \cdots \supset \mathsf{add}\, \varepsilon_n A \supset 0.$$

satisfies that $\mathsf{add}\, \varepsilon_2 A$ is a cosemisimple right rejective subcategory of $\mathsf{proj}\, A$ and $\mathsf{add}\, \varepsilon_2 A \supset \cdots \supset \mathsf{add}\, \varepsilon_n A \supset 0$ is a total right rejective chain. Thus, each minimal right $\mathsf{add}\, \varepsilon_i A$-approximation $\varphi : P \to Q$ of $Q \in \mathsf{add}\, \varepsilon_2 A$ is a monomorphism in $\mathsf{proj}\, \varepsilon_2 A \varepsilon_2$. Since $\mathsf{add}\, \varepsilon_2 A$ is a cosemisimple subcategory of $\mathsf{proj}\, A$, there exists $l \geq 0$ such that $\mathrm{Ker}\, \varphi \cong S(e_1)^{\oplus l}$ by Lemma 3.8. Suppose to the contrary that $l > 0$. Due to our assumption, we have that $\mathrm{Ker}\, \varphi \in \mathsf{proj}\, A$. Thus $S(e_1) \in \mathsf{proj}\, A$, a contradiction. Hence, the proof is complete. ■

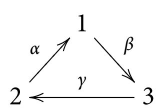
We give a naive sufficient condition for $\mathsf{proj}\, A$ to satisfy the condition (1) in Theorem 3.9.

Corollary 3.10 *Keep the notation in* (3.3). *If* $\mathrm{pd}_{\varepsilon_i A \varepsilon_i} S(e_i)\varepsilon_i = 0$ *holds for each* $1 \leq i \leq n$*, then* A *satisfies the condition* (1) *in Theorem* 3.9. *In particular, the chain* (3.3) *is a total right rejective chain.*

Proof Since $e_i J \varepsilon_i = 0 \in \mathsf{add}\, \varepsilon_n A \varepsilon_i$, it is enough to prove $A\varepsilon_n A \in \mathsf{proj}\, A$. By $\mathrm{rad}_A(\varepsilon_i A, e_i A) = e_i J \varepsilon_i = 0$, we obtain that $\mathrm{Hom}_A(P(e_n), e_i J) = 0$ holds for each $1 \leq i \leq n$. Hence $A\varepsilon_n A = (e_1 + \cdots + e_n) A \varepsilon_n A = \varepsilon_n A \in \mathsf{add}\, \varepsilon_n A$. ■

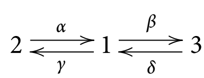
The following examples show that the conditions (1) and (2) in Theorem 3.9 are independent.

Example 3.11 (1) Let A be the algebra defined by the quiver



with relations $\alpha\beta$ and $\gamma\alpha$. Then we can easily check that the complete set (e_1, e_2, e_3) of primitive orthogonal idempotents satisfies the condition (1). However, it dose not satisfy the condition (2). Indeed, $\varphi : P(e_2) \to P(e_3)$ is a monomorphism in $\mathsf{add}\, (e_2 + e_3)A$ and $\mathrm{Ker}\, \varphi \notin \mathsf{proj}\, A$.

(2) Let A be the algebra defined by the quiver



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with relations $\alpha\beta, \alpha\gamma, \delta\beta$, and $\delta\gamma$. Then we can check that the complete set (e_1, e_2, e_3) of primitive orthogonal idempotents satisfies the condition (2). However, it dose not satisfy the condition (1) since $e_1 J \cong P(e_2) \oplus P(e_3) \notin \mathsf{add}\, \varepsilon_3 A$.

3.2 Application

In this subsection, we construct total right rejective chains for three classes of algebras: Nakayama algebras with heredity ideals, locally hereditary algebras and algebras of global dimension at most two. Hence, these algebras are right-strongly quasi-hereditary algebras by Proposition 2.8.

The following theorem is a main result of this subsection.

Theorem 3.12 *The following classes of algebras are right-strongly quasi-hereditary algebras.*

(1) *Nakayama algebras with heredity ideals.*
(2) *Locally hereditary algebras.*
(3) *Algebras of global dimension at most two.*

It is known that algebras in Theorem 3.12 are quasi-hereditary by [4, Proposition 2.3], [4, Proposition 1.6] and [9, Theorem 2]. Hence, Theorem 3.12 is a refinement of their results. Moreover, Theorem 3.12(3) is proven in [22, Theorem 4.1].

In the rest of this subsection, we give a proof of Theorem 3.12. First, we construct right rejective chains and total right rejective chains for Nakayama algebras. We say that an algebra A is a *Nakayama algebra* if every indecomposable projective module and every indecomposable injective module are uniserial.

Proposition 3.13 *Let A be a Nakayama algebra. Then the following statements hold.*

(1) *If A has a simple projective module or a heredity ideal, then* $\mathsf{proj}\, A$ *admits a total right rejective chain.*
(2) *If* $\operatorname{gldim} A < \infty$, *then* $\mathsf{proj}\, A$ *admits a right rejective chain.*

Proof Let A be a Nakayama algebra and ε an idempotent of A. Note that $\varepsilon A \varepsilon$ is also a Nakayama algebra since $\operatorname{Hom}_A(\varepsilon A, -)$ is an exact and dense functor. We may assume that A is connected and fix a complete set $(e_1, e_2, \ldots, e_n)$ of primitive orthogonal idempotents. By [2, Theorem 32.4], we can order the primitive idempotents such that there are projective covers

$$e_i A \to e_{i+1} J \to 0 \tag{3.4}$$

for each $1 \leq i \leq n-1$ and

$$e_n A \to e_1 J \to 0 \tag{3.5}$$

if $e_1 J \neq 0$.

(1) If A has a simple projective module $S(e)$, then $(1-e)A(1-e)$ is also a Nakayama algebra with a simple projective module by [2, Theorem 32.4]. It follows from Corollary 3.10 that $\mathsf{proj}\, A$ admits a total right rejective chain.

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We assume that there exists a heredity ideal of A. We prove that A satisfies the condition (1) in Theorem 3.9 by induction on n. If $n = 1$, then this is clear. Let $n \geq 2$. We assume that A has no simple projective modules. Then $e_i J \neq 0$ holds for each $1 \leq i \leq n$. By (3.4) and (3.5), we obtain an exact sequence $e_i A \to e_{i+1} J \to 0$ for each $1 \leq i \leq n$, where $e_{n+1} := e_1$. Since A has a heredity ideal, we may assume that $Ae_n A$ is a heredity ideal of A. Then we have a composition map $e_n A \to e_1 J \to e_1 A$. Since $Ae_n A$ is a heredity ideal of A, this composition map is a monomorphism by [4, Lemma 1.7]. Thus, we obtain that $e_n A \cong e_1 J \neq 0$, and hence $\operatorname{pd} S(e_1) = 1$. Let $\varepsilon_2 := 1 - e_1$. Then $\varepsilon_2 A e_n A \varepsilon_2 \in \operatorname{proj} \varepsilon_2 A \varepsilon_2$ and $e_n J(\varepsilon_2 A \varepsilon_2) e_n = e_n J(A) e_n = 0$ hold. Thus, $\varepsilon_2 A \varepsilon_2$ is a Nakayama algebra with a heredity ideal $\varepsilon_2 A e_n A \varepsilon_2$. By induction hypothesis, $\varepsilon_2 A \varepsilon_2$ satisfies the condition (1) in Theorem 3.9. Since $Ae_n A \in \operatorname{proj} A$ and $e_1 J \in \operatorname{add} e_n A$, we have the assertion.

(2) Let A be an algebra of finite global dimension. By (1), we assume that A has no simple projective modules. Take an indecomposable projective module $e_i A$ such that its Loewy length $LL(e_i A)$ is maximal. By (3.4) and (3.5), we have $LL(e_{i-1} A) \geq LL(e_i A) - 1$. If $LL(e_{i-1} A) = LL(e_i A) - 1$, then $\operatorname{pd} S(e_i) = 1$. Thus, we assume that $LL(e_{i-1} A) > LL(e_i A) - 1$. Then $LL(e_{i-1} A) = LL(e_i A)$ by maximality of $LL(e_i A)$. By $\operatorname{gldim} A < \infty$, A is non-self-injective. Therefore, we obtain a simple module with projective dimension exactly one by replacing $e_i A$ with $e_{i-1} A$ and repeating this argument. Let $S(e)$ be the simple module with projective dimension one. Since it follows from Proposition 2.3(2) that $\operatorname{gldim}(1-e)A(1-e) \leq \operatorname{gldim} A < \infty$, we inductively obtain that A satisfies the condition (3) in Theorem 3.5. Hence, $\operatorname{proj} A$ admits a right rejective chain. ■

We give applications of Proposition 3.13. In [11, Theorem], it is shown that if A is a Nakayama algebra with n simple modules and $\operatorname{gldim} A < \infty$, then $\operatorname{gldim} A \leq 2n - 2$. We give another proof by Proposition 3.13. Moreover, we give a refinement of [4, Proposition 2.3].

Corollary 3.14 *Let A be a Nakayama algebra. Then the following statements hold.*

(1) *The following statements are equivalent.*
   (a) $\operatorname{gldim} A < \infty$.
   (b) *A is a neat algebra.*
   (c) $\operatorname{proj} A$ *admits a right rejective chain.*
   In this case, if A has n simple modules, then $\operatorname{gldim} A \leq 2n - 2$.
(2) *The following statements are equivalent.*
   (a) *A has a heredity ideal.*
   (b) *A is a quasi-hereditary algebra.*
   (c) *A is a right-strongly quasi-hereditary algebra.*

Proof (1) (c)⇒ (b) and (b)⇒ (a) follow from Theorem 3.5 and Proposition 3.2, respectively. (a)⇒ (c) follows from Proposition 3.13(2). Moreover, if A has n simple modules, then $\operatorname{gldim} A \leq 2n - 2$ by Remark 2.6.

(2) (c)⇒ (b) and (b)⇒ (a) are clear. By Proposition 3.13(1), if A has a heredity ideal, then $\operatorname{proj} A$ has a total right rejective chain. Thus, (a)⇒ (c) holds by Proposition 2.8. ■

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In Corollary 3.14, (2)⇒ (1) clearly holds. However, the converse does not hold in general. Indeed, Example 3.6 satisfies the equivalent conditions in Corollary 3.14(1), but it does not satisfy the equivalence conditions in Corollary 3.14(2).

Next, we prove that locally hereditary algebras satisfy the conditions (1) and (2) in Theorem 3.9. An algebra A is called a *locally hereditary algebra* if each nonzero morphism between indecomposable projective A-modules is a monomorphism (see [17] for details). We need the following lemma.

Lemma 3.15 *Let A be a nonsemisimple locally hereditary algebra. Then the following statements hold.*

(1) *Let e be a primitive idempotent of A and $\varepsilon := 1 - e$. Assume that* $\mathsf{add}\,\varepsilon A$ *is a cosemisimple subcategory of* $\mathsf{proj}\,A$. *If $\varphi : P \to Q$ is a monomorphism in* $\mathsf{add}\,\varepsilon A$, *then* $\operatorname{Ker}\varphi \in \mathsf{proj}\,A$.
(2) $\operatorname{soc} A \in \mathsf{proj}\,A$ *holds. In particular, there exists a simple projective module.*
(3) *There exists a simple module such that its projective dimension is exactly one.*

Proof (1) Let $\varphi : P \to Q$ be a monomorphism in $\mathsf{add}\,\varepsilon A$. Since $\mathsf{add}\,\varepsilon A$ is a cosemisimple subcategory of $\mathsf{proj}\,A$, there exists $l \geq 0$ such that $\operatorname{Ker}\varphi \cong S(e)^{\oplus l}$ by Lemma 3.8. Thus $\rho^{\oplus l} : P(e)^{\oplus l} \to \operatorname{Ker}\varphi$ is a projective cover of $\operatorname{Ker}\varphi$, where $\rho : P(e) \to S(e)$ is a projective cover of $S(e)$. On the other hand, there exists an indecomposable direct summand P' of P such that $S(e)$ is a direct summand of $\operatorname{soc} P'$. Let $\iota : S(e) \to P'$ be the inclusion. Then a composition map $\iota\rho : P(e) \to P'$ is a nonzero morphism between indecomposable projective modules. Since $\iota\rho$ is a monomorphism, ρ is an isomorphism. Thus $\operatorname{Ker}\varphi \in \mathsf{proj}\,A$.

(2) Let $\operatorname{soc} A = S(e_{i_1}) \oplus \cdots \oplus S(e_{i_t})$, where $S(e_{i_j})$ is a simple module. For each $1 \leq j \leq t$, there exists an indecomposable projective module P_j such that $S(e_{i_j})$ is a direct summand of $\operatorname{soc} P_j$. Since a composition map of a projective cover $p : P(e_{i_j}) \to S(e_{i_j})$ and the inclusion $S(e_{i_j}) \to P_j$ is a monomorphism, p is an isomorphism. Thus $\operatorname{soc} A \in \mathsf{proj}\,A$.

(3) We show that there exists an indecomposable projective module P such that its Loewy length $LL(P)$ is two. If $LL(A) = 2$, then this is clear. We assume that $LL(A) \geq 3$. Let Q be an indecomposable projective module with $LL(Q) =: l \geq 3$. Then $QJ^{l-2}/QJ^{l-1} \cong S(e_{i_1}) \oplus \cdots \oplus S(e_{i_s}) \neq 0$. Note that there exists $1 \leq j \leq s$ such that $S(e_{i_j}) \notin \mathsf{proj}\,A$. Thus, $LL(P(e_{i_j})) \geq 2$. Since a composition map of $P(e_{i_j}) \to QJ^{l-2}$ and $QJ^{l-2} \to Q$ is a monomorphism, so is $P(e_{i_j}) \to QJ^{l-2}$. Since $LL(P(e_{i_j})) \leq 2$, $P(e_{i_j})$ is a desired projective module. Thus, we obtain $\operatorname{pd} S(e_{i_j}) = 1$ since it follows from (2) that $\operatorname{soc} P(e_{i_j}) \in \mathsf{proj}\,A$. ∎

Now, we state the following proposition.

Proposition 3.16 *Any locally hereditary algebra satisfies the conditions* (1) *and* (2) *in Theorem* 3.9. *Namely, if A is a locally hereditary algebra, then* $\mathsf{proj}\,A$ *admits a total right rejective chain.*

Proof Let A be a locally hereditary algebra. If A is semisimple, then it clearly satisfies the conditions (1) and (2) in Theorem 3.9. Thus, we assume that A is not

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semisimple. First, we show that A satisfies the condition (1). By Lemma 3.15(2), there exists a simple module $S(e)$ such that $\operatorname{pd} S(e) = 0$. Let $\varepsilon := 1 - e$. Then $\varepsilon A \varepsilon$ is also a locally hereditary algebra by an equivalence of categories $\operatorname{add} \varepsilon A \simeq \operatorname{proj} \varepsilon A \varepsilon$. Due to Corollary 3.10, A inductively satisfies the condition (1) in Theorem 3.9.

Next, we show that A satisfies the condition (2). By Lemma 3.15(3), there exists a simple module $S(e)$ with $\operatorname{pd} S(e) = 1$. Let $\varepsilon := 1 - e$. Then $\operatorname{add} \varepsilon A$ is a cosemisimple subcategory of $\operatorname{proj} A$ by Proposition 2.3(3). For each monomorphism $\varphi : P \to Q$ in $\operatorname{add} \varepsilon A$, we obtain that $\operatorname{Ker} \varphi \in \operatorname{proj} A$ by Lemma 3.15(1). Since $\varepsilon A \varepsilon$ is also a locally hereditary algebra, A inductively satisfies the condition (2) in Theorem 3.9. ■

Finally, we show the following proposition. This proposition is proven in [15, Theorem 3.6] and [22, Theorem 4.1]. In this paper, we give a proof using Theorem 3.9.

Proposition 3.17 *If* $\operatorname{gldim} A \leq 2$, *then* $\operatorname{proj} A$ *admits a total right rejective chain.*

Proof If $\operatorname{gldim} A \leq 1$, then A is a locally hereditary algebra. Thus, the assertion follows from Proposition 3.16. We assume that $\operatorname{gldim} A = 2$. Let S be an A-module such that its length $\ell(S)$ is minimal among A-modules with the projective dimension one. We show that S is a simple module. Suppose to the contrary that S is not simple. Then there exists an exact sequence $0 \to S' \to S \to S/S' \to 0$ such that $S' \neq 0$ and $S/S' \neq 0$. Since $\operatorname{pd} S' \leq \max\{\operatorname{pd} S, \operatorname{pd} S/S' - 1\} = 1$, it follows from the assumption on S that $\operatorname{pd} S' = 0$. Thus, we have $\operatorname{pd} S/S' \leq \max\{\operatorname{pd} S, \operatorname{pd} S' + 1\} = 1$. By the assumption on S, we have $\operatorname{pd} S/S' = 0$, a contradiction. Thus S is a simple module. We put $S(e) := S$ and $\varepsilon := 1 - e$. Then $\operatorname{add} \varepsilon A$ is a cosemisimple right rejective subcategory of $\operatorname{proj} A$ by Proposition 2.3(3). Let $\varphi : P \to Q$ be a monomorphism in $\operatorname{add} \varepsilon A$. Then, we have an exact sequence in $\operatorname{mod} A$.

$$\operatorname{Ker} \varphi \to P \to Q \to \operatorname{Cok} \varphi \to 0.$$

By $\operatorname{gldim} A = 2$, we have that $\operatorname{Ker} \varphi \in \operatorname{proj} A$. Since $\operatorname{gldim} \varepsilon A \varepsilon \leq \operatorname{gldim} A = 2$ by Proposition 2.3(2), we inductively obtain that A satisfies the condition (2) in Theorem 3.9. Hence, we have the assertion. ■

Now we are ready to prove Theorem 3.12.

Proof of Theorem 3.12 Let A be an algebra in Theorem 3.12. By Proposition 2.8, it is enough to show that there exists a total right rejective chain of $\operatorname{proj} A$. Thus, the assertion follows from Propositions 3.13, 3.16 and 3.17. ■

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