INDICES OF FUNCTION SPACES AND THEIR RELATIONSHIP TO INTERPOLATION

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A special case of the theorem of Marcinkiewicz states that if T is a linear operator which satisfies the weak-type conditions (p, p) and (q, q), then T maps L^r continuously into itself for any r with $p < r < q$. In a recent paper (**5**), as part of a more general theorem, Calderón has characterized the spaces X which can replace L^r in the conclusion of this theorem, independent of the operator T. The conditions which X must satisfy are phrased in terms of an operator $S(\sigma)$ which acts on the rearrangements of the functions in X.

One of Calderón's results implies that if X is a function space in the sense of Luxemburg (**9**), then X must be a rearrangement-invariant space. In this paper, starting with the assumption that X is rearrangement invariant, we reduce the conditions which X must satisfy to conditions on a pair of numbers (α, β) called the indices of X. The result is that X may replace L^r in the theorem of Marcinkiewicz if and only if $p < \alpha^{-1}$ and $\beta^{-1} < q$.

A rearrangement-invariant space is given completely by a function norm ρ and a measure space Ω. In case ρ is the L^r norm, it is immediate that $\alpha = \beta = r^{-1}$. In general, though, α and β depend both on ρ and on Ω. This may be illustrated by calculating α and β when ρ is an Orlicz norm. To avoid unduly lengthening this paper we shall report on this elsewhere; see (**4**).

1. Function spaces. Let $(\Omega, \mathscr{T}, \mu)$ be a totally σ-finite measure space which satisfies one of the following restrictions:

(1) Ω is non-atomic with infinite measure;

(2) Ω is non-atomic with finite measure;

(3) Ω is purely atomic with atoms having equal measure 1.

Let $\mathscr{M}(\Omega)$ and $\mathscr{P}(\Omega)$ denote the class of measurable and non-negative measurable functions on Ω, respectively. According to Luxemburg (**9**, p. 3) a *function norm* ρ: $\mathscr{P}(\Omega) \rightarrow [0, \infty]$ is a mapping which satisfies the following conditions for all f, g, $\{f_n\}$ in $\mathscr{P}(\Omega)$, for all $E \in \mathscr{T}$ with $\mu(E) < \infty$ and characteristic function χ_E, and for all constants $a \geqq 0$:

(4) $\rho(f) = 0 \Leftrightarrow f = 0$ a.e., $\quad f \leqq g$ a.e. $\Rightarrow \rho(f) \leqq \rho(g)$,
$\rho(f + g) \leqq \rho(f) + \rho(g)$, $\rho(af) = a\rho(f)$;

(5) $\rho(\chi_E) < \infty$;

(6) there exists $A_E < \infty$ such that $\int_E f\, d\mu \leqq A_E \rho(f)$;

(7) $f_n \uparrow f$ a.e. $\Rightarrow \rho(f_n) \uparrow \rho(f)$ (Fatou property).

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The space $L^\rho(\Omega)$ consists of all $f \in \mathscr{M}(\Omega)$ such that $\rho(|f|) < \infty$, with norm $||f|| = \rho(|f|)$. $L^\rho(\Omega)$ is a Banach space when functions which differ at most on a null set are identified.

Two functions f, $g \in \mathscr{M}(\Omega)$ are said to be *equimeasurable* if, for all $y > 0$,

$$\mu\{x: |f(x)| > y\} = \mu\{x: |g(x)| > y\}.$$

In this case we write $f \sim g$.

We say that L^ρ is *rearrangement-invariant* if $f \sim g$ and $f \in L^\rho$ implies $g \in L^\rho$, and that ρ is a *rearrangement-invariant norm* if $f \sim g$ implies $\rho(|f|) = \rho(|g|)$. By an equivalent renorming we may assume that a rearrangement-invariant space has such a norm; see (**10**).

The non-increasing rearrangement of $f \in \mathscr{M}(\Omega)$ onto $\mathbf{R}^+ = [0, \infty)$ is the non-increasing, left-continuous function $f^* \in \mathscr{P}(\mathbf{R}^+)$ for which, if m denotes Lebesgue measure,

$$m\{t \in \mathbf{R}^+: f^*(t) > y\} = \mu\{x \in \Omega: |f(x)| > y\}, \quad \text{all } y > 0.$$

For the existence of f^* and more details, see (**5**).

One way of generating rearrangement-invariant norms for $\mathscr{M}(\Omega)$ is the following: let ρ be a rearrangement-invariant norm for $\mathscr{M}(\mathbf{R}^+)$ and define

$$\rho_\Omega(f) = \rho(f^*) \quad \text{for all } f \in \mathscr{P}(\Omega). \tag{8}$$

In (**1**) this was used as a definition. In (**10**) it is shown that for Ω satisfying (1), (2) or (3), all rearrangement-invariant norms arise in this way. We shall write $L^\rho(\Omega)$ for the space determined in $\mathscr{M}(\Omega)$ by ρ_Ω.

If Ω satisfies (2) with $\mu(\Omega) = a$, then $\operatorname{supp} f^* \subset [0, a]$, hence we sometimes regard f^* as being defined only on $[0, a]$ and will write $\Omega^* = [0, a]$. If Ω satisfies (3), then f^* is a step function constant on $(n - 1, n]$, for $n \in \mathbf{Z}^+ = \{1, 2, 3, \ldots\}$ thus we shall sometimes regard f^* as the sequence $\{f^*(n)\}$, and write $\Omega^* = \mathbf{Z}^+$. There will never be any confusion about using the notation f^* both for the function on $\mathbf{R}^+$ and for its restriction to Ω^*. If Ω satisfies (1), we define $\Omega^* = \mathbf{R}^+$.

The associate space of a function space $L^\rho(\Omega)$ plays an important rôle in the following discussion. Given a function norm ρ, the *associate norm* ρ' is defined on $\mathscr{P}(\Omega)$ by

$$\rho'(g) = \sup\left\{ \int_\Omega fg\, d\mu : \rho(f) \leqq 1\right\}. \tag{9}$$

The space $L^{\rho'}$ is called the *associate space* of L^ρ. If ρ_Ω is a rearrangement-invariant norm on $\mathscr{M}(\Omega)$ defined as in (8), then $(\rho_\Omega)' = (\rho')_\Omega$; see (**1**).

Furthermore, we have

$$(\rho_\Omega)'(g) = \rho'(g^*) = \sup\left\{ \int_{\Omega^*} f^*g^* : f \in \mathscr{M}(\Omega), \rho(f^*) \leqq 1\right\}. \tag{10}$$

Having defined ρ', we can define $\rho'' = (\rho')'$. A result due independently to Lorentz (unpublished) and Luxemburg (**9**, p. 9) states that $\rho'' = \rho$ for norms having the Fatou property (7).

We shall often use the notation $\langle f, g\rangle = \int_\Omega fg\,d\mu$, depending on context to indicate which Ω is meant.

2. Operators satisfying weak-type conditions. The notion of an operator of *weak type* (p, q) was introduced by Marcinkiewicz (see, e.g., **12**, p. 111, Chapter 12, § 4), and modified by Stein and Weiss (**11**). Calderón showed that the Stein and Weiss definition was equivalent to the operator being a continuous mapping between a pair of Lorentz spaces, except in one extreme case. In our situation, only the original Lorentz spaces, Λ_p and M_p, introduced in (**8**) are involved. These are rearrangement-invariant spaces defined by the norms

$$\lambda_p(f) = \gamma \int_0^\infty t^{\gamma-1} f^*(t)\,dt, \qquad \gamma = p^{-1}, \quad 1 \leqq p < \infty, \tag{11}$$

$$\mu_p(f) = \sup_{t>0} t^{\gamma-1} \int_0^t f^*(s)\,ds, \qquad \gamma = p^{-1}, \quad 1 \leqq p < \infty, \tag{12}$$

respectively. The space Λ_∞ is by definition the closure in L^∞ of the space of bounded functions with support in a set of finite measure, and M_∞ is defined to be L^∞. Λ_p is equivalent to the space $L_{p,1}$ and M_p is equivalent to $L_{p,\infty}$, as defined in (**5**, Theorem 6).

On the space of measurable functions $\mathscr{M}(\Omega)$ we introduce the topology of convergence in measure on sets of finite measure. A continuous mapping of $\Lambda_p(\Omega)$ into $\mathscr{M}(\Omega)$ is said to be *quasilinear* if there is a constant A such that, for all f, $g \in \Lambda_p(\Omega)$, and $\lambda \in C$,

$$|T(f+g)| \leqq A(|Tf| + |Tg|) \text{ a.e.} \quad \text{and} \quad |T(\lambda f)| = |\lambda|\,|Tf| \text{ a.e.} \tag{13}$$

The mapping T is said to be of *weak type* (p_1, p_2) if T maps Λ_{p_1} continuously into $\mathscr{M}(\Omega)$ and there is a constant c such that, for all $f \in \Lambda_{p_1}$ and almost all $t > 0$,

$$(Tf)^*(t) \leqq c t^{-1/p_2} \lambda_{p_1}(f). \tag{14}$$

In case $p_2 > 1$, this is equivalent to requiring that, for some constant c_1, possibly different from c,

$$\mu_{p_2}(Tf) \leqq c_1 \lambda_{p_1}(f). \tag{15}$$

We shall be concerned entirely with the case $p_1 = p_2$.

The space $\Lambda_p + \Lambda_q$ is defined to be the function space consisting of all functions of the form $f + g$, with $f \in \Lambda_p$, $g \in \Lambda_q$, and the norm

$$\|f+g\| = \inf\{\lambda_p(f_1) + \lambda_q(g_1) : f_1 \in \Lambda_p,\ g_1 \in \Lambda_q, \text{ and } f_1 + g_1 = f + g\}. \tag{16}$$

If $1 \leqq p, q < \infty$, it can be shown that $f \in \Lambda_p + \Lambda_q$ if and only if

$$\int_0^\infty \min(t^{1/p}, t^{1/q}) f^*(t) t^{-1}\,dt < \infty. \tag{17}$$

We define the class of operators $W(p, q; \Omega)$ to consist of those linear operators T mapping $\Lambda_p + \Lambda_q$ continuously into $\mathscr{M}(\Omega)$ which are of weak types (p, p) and (q, q) simultaneously.

If X and Y are Banach spaces, $[X, Y]$ will denote the space of bounded linear operators from X into Y, and $[X]$ will denote $[X, X]$. With this notation, our problem is to characterize those function norms ρ for which

$$[L^\rho(\Omega)] \supset W(p, q; \Omega).$$

We first reduce the problem to the rearrangement-invariant case.

LEMMA 1. *If* $1 \leqq p < q \leqq \infty$, *and if* ρ *is a function norm on* $\mathscr{M}(\Omega)$, *such that* $[L^\rho(\Omega)] \supset W(p, q; \Omega)$, *then* $L^\rho(\Omega)$ *is rearrangement-invariant.*

Proof. The class $W(p, q; \Omega)$ contains all operators in $[L^1(\Omega), L^\infty(\Omega)]$, so that $[L^\rho(\Omega)] \supset W(p, q; \Omega) \supset [L^1(\Omega), L^\infty(\Omega)]$.

However, according to (**5**, Theorem 3(i)), this implies that L^ρ must satisfy the following condition:

$$f \in L^\rho, \text{ and for all } t > 0, \int_0^t g^*(s)\, ds \leqq \int_0^t f^*(s)\, ds \Rightarrow g \in L^\rho. \tag{18}$$

Thus, in particular, $f \in L^\rho$ and $g \sim f$ implies $g \in L^\rho$ so that L^ρ is rearrangement-invariant.

By earlier remarks and Lemma 1, we may assume without loss of generality that $\rho = \rho_\Omega$ is a rearrangement-invariant norm and even that it is of the form (8).

To state our next lemma, we need the following special operators which act on functions in $\mathscr{M}(\mathbf{R}^+)$. Let $\gamma = p^{-1}$, then

$$P_p f(t) = t^{-\gamma} \int_0^t s^{\gamma-1} f(s)\, ds, \tag{19}$$

$$Q_p f(t) = t^{-\gamma} \int_t^\infty s^{\gamma-1} f(s) ds. \tag{20}$$

The domains of the operators consist of all $f \in \mathscr{M}(\mathbf{R}^+)$ for which the respective integrals are finite a.e. By restriction, P_p and Q_p are defined for $f \in \mathscr{M}(\Omega^*)$. In case $\Omega^* = [0, a]$, formulas (19) and (20) are still valid for $0 \leqq t \leqq a$. In case $\Omega^* = \mathbf{Z}^+$, the operators take the form

$$P_p f(n) = n^{-\gamma} \sum_{k=1}^{n} c_{pk} f(k), \tag{21}$$

$$Q_p f(n) = n^{-\gamma} \sum_{k=n+1}^{\infty} c_{pk} f(k), \tag{22}$$

where $c_{pk} = \int_{k-1}^k s^{\gamma-1}\, ds$.

The next lemma shows that we can restrict our study entirely to the operators P_p and Q_q. In it, if $f \in L^\rho(\Omega)$, we regard f^* as being a function in

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$L^\rho(\Omega^*)$, and $P_p f^*$, $Q_q f^*$ are to be interpreted as in the remarks following equation (20).

LEMMA 2. *Let* ρ *be a rearrangement-invariant norm on* $\mathscr{M}(\mathbf{R}^+)$, *let* $1 \leqq p < q \leqq \infty$, *and let* Ω *satisfy* (1), (2) *or* (3). *Then* $[L^\rho(\Omega)] \supset W(p, q; \Omega)$ *if and only if there are constants* A, B *such that for all* $f \in L^\rho(\Omega)$,

$$\rho_{\Omega^*}(P_p f^*) \leqq A \rho_{\Omega^*}(f^*) \tag{23}$$

and

$$\rho_{\Omega^*}(Q_q f^*) \leqq B \rho_{\Omega^*}(f^*). \tag{24}$$

Proof. By (**5**, Theorem 8), interpreted in our notation, if $T \in W(p, q; \Omega)$, there is a constant $c = c(p, q; T)$ such that, for all $f \in \Lambda_p + \Lambda_q$,

$$(Tf)^* \leqq c(p^{-1}P_p + q^{-1}Q_q)f^*. \tag{25}$$

If (23) and (24) hold for all $f \in L^\rho(\Omega)$, then $L^\rho(\Omega) \subset \Lambda_p + \Lambda_q$ follows by using (10). Thus (23), (24), and (25) together imply that if $f \in L^\rho(\Omega)$, then

$$\rho_\Omega(Tf) = \rho_{\Omega^*}((Tf)^*) \leqq C\rho_{\Omega^*}(f^*) = C\rho_\Omega(f), \tag{26}$$

so that $T \in [L^\rho(\Omega)]$, with $||T|| \leqq C = c(p^{-1}A + q^{-1}B)$.

Conversely, assume that $[L^\rho(\Omega)] \supset W(p, q; \Omega)$. We observe that

$$P_p \in W(p, q; \Omega^*) \quad \text{and} \quad Q_q \in W(p, q, \Omega^*),$$

directly from the definitions involved. Certainly then, if Ω is one of the spaces $[0, a]$, $\mathbf{R}^+$ or $\mathbf{Z}^+$ so that $\Omega = \Omega^*$, then

$$P_p, Q_q \in W(p, q; \Omega^*) \subset [L^\rho(\Omega^*)],$$

which proves (23) and (24).

If $\Omega \neq \Omega^*$, we use the fact that there is an almost one-to-one measure-preserving transformation τ: $\mathscr{T} \to \mathscr{T}^*$, where $\mathscr{T}$ and $\mathscr{T}^*$ are the rings of measurable subsets of Ω and Ω^*, respectively. (See Halmos (**6**, pp. 173–174) for the non-atomic case; the atomic case is trivial.) This isomorphism is used to construct operators $\tilde{P}_p, \tilde{Q}_q \in W(p, q; \Omega)$ such that $\tilde{P}_p, \tilde{Q}_q \in [L^\rho(\Omega)]$ if and only if $P_p, Q_q \in [L^\rho(\Omega^*)]$.

For example, let $\Omega^* = \mathbf{R}^+$, and define

$$S_t = \tau^{-1}([0, t]), \quad \Delta_t = \tau^{-1}\{t\}, \qquad 0 \leqq t < \infty. \tag{27}$$

Then define

$$g(x) = \begin{cases} t^{\gamma-1}, & \text{if } x \in \Delta_t, \\ 0, & \text{if } x \notin \Delta_t \text{ for any } t. \end{cases} \tag{28}$$

Then $g \in \mathscr{M}(\Omega)$ and $g^*(t) = t^{\gamma-1}$. Now define

$$\tilde{P}_p f(x) = \begin{cases} t^{-\gamma}\int_{S_t} fg\, d\mu, & \text{if } x \in \Delta_t, \\ 0, & \text{if } x \notin \Delta_t \text{ for any } t. \end{cases} \tag{29}$$

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One readily shows that if $f \in L^\rho(\Omega)$, then there is an $\tilde{f} \in L^\rho(\Omega)$ with $\tilde{f} \sim f$ and such that $(\tilde{P}_p\tilde{f})^* = P_p f^*$. Then, we have $\tilde{P}_p \in W(p, q; \Omega) \subset [L^\rho(\Omega)]$, thus $P_p \in [L^\rho(\Omega^*)]$ and (16) holds.

The other cases are treated similarly.

3. Indices of rearrangement-invariant spaces. In this section we reduce the question of whether (23) and (24) hold for a given function norm ρ to a consideration of the indices of $L^\rho(\Omega)$.

We begin by introducing certain semigroups of operators acting on functions in the classes $\mathscr{M}(\Omega^*)$. If $\Omega^* = [0, a]$ or $\mathbf{R}^+$, we define

$$(E_s f)(t) = f(st) \quad \text{for } 0 < s < \infty, \quad t \in \Omega^*, \tag{30}$$

where, for $\Omega^* = [0, a]$ and $t > a$, we set $f(t) = 0$.

If $\Omega^* = \mathbf{Z}^+$, we define

$$(E_m f)(n) = f(mn), \qquad m, n \in \mathbf{Z}^+. \tag{31}$$

To keep the notation uniform, let $S(\Omega) = (0, \infty)$ if $\Omega^* = [0, a]$ or $\mathbf{R}^+$, and $S(\Omega) = \mathbf{Z}^+$ if $\Omega^* = \mathbf{Z}^+$.

If ρ is a rearrangement-invariant norm, then

$$h(s, L^\rho(\Omega)) = \sup\{\rho(E_s f^*) : f \in L^\rho(\Omega^*), \rho(f) \leqq 1\}, \quad \text{for } s \in S(\Omega). \tag{32}$$

In case Ω is non-atomic, it can be seen that $h(s; L^\rho(\Omega))$ is the norm of E_s as a member of $[L^\rho(\Omega^*)]$. However, in case $\Omega^* = \mathbf{Z}^+$, the norm of E_n in $[L^\rho(\mathbf{Z}^+)]$ is 1 for all m, since if $f(n) = 0$ for all $n \not\equiv 0 \pmod m$, then $(E_m f)^* = f^*$. By restricting consideration to non-increasing functions in (32), we shall have $h(m; L^\rho(\mathbf{Z}^+)) < 1$ for some norms ρ.

LEMMA 3. *Let* $h(s) = h(s; L^\rho(\Omega))$. *Then*

(a) h *is non-increasing;*

(b) *For* $s, t \in S(\Omega)$, $h(st) \leqq h(s)h(t)$;

(c) *If* $\theta(s) = -\log h(s)/\log s$, *and if* $S(\Omega) = (0, \infty)$ *then the following limits exist,*

$$\alpha = \lim_{s \to 0+} \theta(s) = \inf_{0<s<1} \theta(s), \tag{33}$$

$$\beta = \lim_{s \to \infty} \theta(s) = \sup_{s>1} \theta(s). \tag{34}$$

(d) *If* $S(\Omega) = \mathbf{Z}^+$, *and* θ *is as above, then*

$$\beta = \lim_{n \to \infty} \theta(n) = \sup_{n \in \mathbf{Z}^+} \theta(n). \tag{35}$$

Proof. (a) is obvious, since $f^*(s_1 t) \leqq f^*(s_2 t)$ for $s_1 > s_2$, for all $f^* \in L^\rho(\Omega^*)$.

(b) In case $\Omega^* = \mathbf{R}^+$ or $\mathbf{Z}^+$, we have $E_s E_t = E_{st}$ for all $s, t \in S(\Omega)$, thus $h(st) \leqq h(s)h(t)$ is clear.

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For $\Omega^* = [0, a]$, we have $E_s E_t = E_{st}$ unless $s > 1$ and $t < 1$, in which case we have $E_s E_t f = \chi_{[0, s^{-1}]} E_{st} f$. This is enough to show that (b) holds.

(c) This is a consequence of (b) which is easily derived from (**7**, p. 244) by writing $f(x) = \log h(e^x)$.

(d) The proof depends on both (a) and (b) and can be found in (**3**, Lemma 2).

Definition. Let ρ be a rearrangement-invariant norm, and Ω a measure space satisfying (1), (2) or (3). The number $\beta = \beta(\rho, \Omega)$ defined by (34) and (35) above is called the *lower index* of $L^\rho(\Omega)$. If Ω is non-atomic, the number $\alpha(\rho, \Omega)$ defined by (33) is called the *upper index* of $L^\rho(\Omega)$. For Ω atomic, we define $\alpha(\rho, \Omega) = 1 - \beta(\rho', \Omega)$.

The next lemma gives an alternative definition of α in case Ω is atomic. We introduce two new operators on sequences:

(36) $$F_m f(n) = f([(n-1)/m] + 1)$$

and

(37) $$G_m f(n) = F_m E_m f(n) = f(m[(n-1)/m] + m).$$

Here $[x]$ denotes the integer part of x.

LEMMA 4. *Let ρ be a rearrangement-invariant function norm, and let F_m be as in* (36). *Define*

(38) $$k(m, L^\rho(\mathbf{Z}^+)) = \sup\{\rho(F_m f^*) : f \in L^\rho(\mathbf{Z}^+), \rho(f) \leqq 1\}.$$

Then

(39) $$mh(m; L^{\rho'}(\mathbf{Z}^+)) = k(m; L^\rho(\mathbf{Z}^+))$$

and

(40) $$\alpha(\rho, \Omega) = \lim_{m \to \infty} \frac{\log k(m; L^\rho(\mathbf{Z}^+))}{\log m}.$$

Proof. Let $f \in L^\rho(\mathbf{Z}^+)$, $g \in L^{\rho'}(\mathbf{Z}^+)$. Then it is easy to see that

(41) $$\begin{aligned} \langle F_m f^*, G_m g^* \rangle &= m\langle f^*, E_m G_m g^* \rangle \\ &\leqq \rho(f^*) mh(m; L^{\rho'}(\mathbf{Z}^+))\rho'(G_m g^*). \end{aligned}$$

Now $G_m(L^{\rho'}(\mathbf{Z}^+))$, consists of all sequences which are constant in blocks of length m. Furthermore, $F_m f^*$ is such a sequence, thus by the levelling property of rearrangement-invariant norms (see **10**, p. 99), we have

(42) $$\sup_{g^* \neq 0} \frac{\langle F_m f^*, G_m g^* \rangle}{\rho'(G_m g^*)} = \rho(F_m f^*).$$

Thus, by (41) and (42) we have

(43) $$\rho(F_m f^*) \leqq \rho(f^*) mh(m; L^{\rho'}(\mathbf{Z}^+)) \quad \text{for all } f \in L^\rho(\mathbf{Z}^+),$$

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which shows that

$$k(m; L^\rho(\mathbf{Z}^+)) \leqq mh(m; L^{\rho'}(\mathbf{Z}^+)). \tag{44}$$

To prove the reverse inequality is even easier since

$$m\langle f^*, E_m g^*\rangle \leqq \langle F_m f^*, g^*\rangle, \quad \text{for } f \in L^\rho,\ g \in L^{\rho'}. \tag{45}$$

Finally, (40) is an immediate consequence of (39).

LEMMA 5. *Let α, β denote the indices of $L^\rho(\Omega)$ and α', β' the indices of $L^{\rho'}(\Omega)$. Then $\alpha = 1 - \beta'$, $\beta = 1 - \alpha'$ and $0 \leqq \beta \leqq \alpha \leqq 1$.*

Proof. For $\Omega^* = \mathbf{Z}^+$, $\alpha = 1 - \beta'$ and $\alpha' = 1 - \beta$ by definition.
If $\Omega^* = [0, a]$ and if $f \in L^\rho$, $g \in L^{\rho'}$, and $s < 1$, one has

$$\langle E_s f^*, g^*\rangle = \int_0^a f^*(st)g^*(t)\,dt \tag{46}$$

$$= s^{-1}\int_0^{as} f^*(t)g^*(t/s)\,dt = s^{-1}\langle f^*, E_{s^{-1}}g^*\rangle.$$

Thus, taking supremums over f and g with $\rho(f) \leqq 1$, $\rho(g) \leqq 1$, we have

$$h(s, L^\rho) = s^{-1}h(s^{-1}, L^{\rho'}), \tag{47}$$

which clearly shows that $\alpha = 1 - \beta'$ and dually $\beta = 1 - \alpha'$.
A similar calculation works in case $\Omega^* = \mathbf{R}^+$.
The fact that $\beta \geqq 0$ follows immediately from the fact that h is non-increasing. Then $\alpha \leqq 1$ follows from $\alpha = 1 - \beta'$.
To prove that $\beta \leqq \alpha$, we deal first with Ω non-atomic. Then Lemma 3(b) applies to show that $h(s)h(s^{-1}) \geqq h(1) = 1$. Thus, for $s < 1$, $\theta(s) \geqq \theta(s^{-1})$, which proves that $\alpha \geqq \beta$.
For the case $\Omega^* = \mathbf{Z}^+$, we use Lemma 4. Note that if F_m is defined by (36), then $E_m F_m f = f$ for all $f \in L^\rho(\mathbf{Z}^+)$. Thus, if we write $k(m) = k(m; L^\rho(\mathbf{Z}^+))$, as defined by (38), then

$$\rho(E_m F_m f^*) \leqq h(m)k(m)\rho(f^*), \quad \text{for } f \in L^\rho, \tag{48}$$

so that $h(m)k(m) \geqq 1$.
But then $\log k(m)/\log m \geqq \theta(m)$, proving that $\alpha \geqq \beta$ upon using (40).

Our main theorem may now be proved. It will be convenient in the proof to use the notation $P_p \in [\mathscr{D}^\rho]$, $Q_q \in [\mathscr{D}^\rho]$ to mean that there are constants A and B so that (23) and (24) hold, respectively. (The $\mathscr{D}$ refers to the fact that only non-increasing functions are considered in equations (23) and (24).)

THEOREM 1. *Let ρ be a rearrangement-invariant function norm and let Ω be a measure space satisfying* (1), (2) *or* (3). *Then $[L^\rho(\Omega)] \supset W(p, q; \Omega)$ if and only if*

$$\alpha(\rho, \Omega) < p^{-1} \quad \textit{and} \quad \beta(\rho, \Omega) > q^{-1}. \tag{49}$$

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Proof. By Lemma 2, we need only show that

$$\alpha p < 1 \Leftrightarrow P_p \in [\mathscr{D}^\rho] \tag{50}$$

and

$$\beta q > 1 \Leftrightarrow Q_q \in [\mathscr{D}^\rho]. \tag{51}$$

In fact, only one of the implications (50) or (51) need be proved, since if $f \in L^\rho$, $g \in L^{\rho'}$, then

$$\langle P_p f^*, g^* \rangle = \langle f^*, Q_{p'} g^* \rangle, \tag{52}$$

where $p' = p(p-1)^{-1}$. From this, it follows immediately that

$$P_p \in [\mathscr{D}^\rho] \Leftrightarrow Q_{p'} \in [\mathscr{D}^{\rho'}]. \tag{53}$$

However, by Lemma 4, we have

$$\alpha p < 1 \Leftrightarrow \beta' p' > 1. \tag{54}$$

Combining (53) with (54) proves our assertion.

In case Ω is non-atomic, we choose to prove (50). In fact,

$$P_p \in [\mathscr{D}^\rho] \Leftrightarrow P_p \in [L^\rho(\Omega^*)]$$

in this case. This is a consequence of $(P_p f)^* \leqq P_p f^*$ which is a special case of the following well-known inequality of Hardy, Littlewood, and Pólya (see, e.g., **1**, p. 601, formula (1)):

$$\int_\Omega |fg| \, d\mu \leqq \int_{\Omega^*} f^* g^* \, d\mu^*. \tag{55}$$

By (**2**, Theorem 1), $P_p \in [L^\rho(\mathbf{R}^+)]$ if and only if

$$\int_0^1 s^{1/p-1} h(s, L^\rho(\mathbf{R}^+)) \, ds < \infty, \tag{56}$$

which is the case if and only if $\alpha p < 1$.

For $\Omega^* = [0, a]$, the proof that $P_p \in [L^\rho(\Omega^*)]$ is not substantially different from that for $\Omega^* = \mathbf{R}^+$ just referred to.

In the remaining case, $\Omega^* = \mathbf{Z}^+$, we shall prove (51). We begin by introducing an operator T_q which is easier to handle than Q_q.

If $f \in L^\rho(\Omega)$, then

$$T_q f^*(n) = \sum_{m=1}^{\infty} c_{q,m+1} f^*(mn), \tag{57}$$

where $c_{q,k}$ is defined by $\int_{k-1}^k u^{\gamma-1} \, du$, $\gamma = q^{-1}$. Now, note that

$$\begin{aligned} Q_q f^*(n) &= n^{-\gamma} \sum_{k=n+1}^{\infty} f^*(k) \int_{k-1}^{k} u^{\gamma-1} \, du \\ &= n^{-\gamma} \sum_{m=1}^{\infty} \sum_{j=1}^{n} f^*(mn+j) \int_{mn+j-1}^{mn+j} u^{\gamma-1} \, du \leqq \sum_{m=1}^{\infty} f^*(mn) n^{-\gamma} \sum_{j=1}^{n} \int_{mn+j-1}^{mn+j} u^{\gamma-1} \, du \\ &= \sum_{m=1}^{\infty} \left(\int_m^{m+1} u^{\gamma-1} \right) f^*(mn) = T_q f^*(n). \end{aligned} \tag{58}$$

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Similarly, using the estimate $f^*(mn + j) \geqq f^*((m + 1)n)$, and the fact that $\{c_{qn}\}$ is a non-increasing sequence, we obtain

(59) $$Q_q f^*(n) \geqq T_q f^*(n) - c_{q,2} f^*(1).$$

Thus, $Q_q \in [\mathscr{D}^p]$ if and only if $T_q \in [\mathscr{D}^p]$.

The operator T_q is of the type discussed in (**3**), and it is shown there that $T_q \in [\mathscr{D}^p]$ if and only if $\beta > \sigma_0$, where σ_0 is the abscissa of convergence of the Dirichlet series

(60) $$\zeta(s, T_q) = \sum_{m=1}^{\infty} c_{q,m+1} m^{-s},$$

which in this case is $\sigma_0 = q^{-1}$. Thus, $T_q \in [\mathscr{D}^p]$ if and only if $\beta q > 1$, proving (51) and hence (50).

4. Indices for special spaces. In (**1**), we showed how to compute $h(s; L^p(\mathbf{R}^+))$ in case L^p is an Orlicz space L^Φ or a Lorentz space $\Lambda(\phi, p)$. From the expressions given there, the indices α and β can be computed using (33) and (34). The situation $\Omega^* = \mathbf{R}^+$ is somewhat simpler than either of the cases $\Omega^* = [0, a]$ or $\Omega^* = \mathbf{Z}^+$. This is apparently due to the fact that $(0, \infty)$ is a group under multiplication.

In (**4**), we compute the indices α and β for $L^\Phi([0, a])$ and $L^\Phi(\mathbf{Z}^+) = l^\Phi$.

References

1. D. W. Boyd, *The Hilbert transform on rearrangement-invariant spaces*, Can. J. Math. *19* (1967), 599–616.

2. ——— *The spectral radius of averaging operators*, Pacific J. Math. *24* (1968), 19–28.

3. ——— *Monotone semigroups of operators on cones*, Can. Math. Bull. *12* (1969), 299–310.

4. ——— *Indices and exponents for Orlicz spaces* (unpublished manuscript).

5. A. P. Calderón, *Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz*, Studia Math. *26* (1966), 273–299.

6. P. Halmos, *Measure theory* (Van Nostrand, New York, 1950).

7. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloq. Publ., Vol. 31, rev. ed. (Amer. Math. Soc., Providence, R.I., 1957).

8. G. G. Lorentz, *Some new functional spaces*, Ann. of Math. (2) *51* (1950), 37–55.

9. W. A. J. Luxemburg, *Banach function spaces*, Thesis, Delft Technical University, 1955.

10. ——— *Rearrangement-invariant Banach function spaces*, Queen's papers in Pure and Applied Mathematics *10* (1967), 83–144, Queen's University, Canada.

11. E. M. Stein and G. Weiss, *An extension of a theorem of Marcinkiewicz and some of its applications*, J. Math. Mech. *8* (1959), 263–284.

12. A. Zygmund, *Trigonometric series*, Vol. II (Cambridge, at the University Press, 1959).

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