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# Global existence for the heat equation with nonlinear dynamical boundary conditions

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This paper deals with local and global existence for the solutions of the heat equation in bounded domains with nonlinear boundary damping and source terms. The typical problem studied is

$$\begin{aligned} u_t - \Delta u &= 0 && \text{in } (0,\infty)\times\Omega,\\ u &= 0 && \text{on } [0,\infty)\times\Gamma_0,\\ \frac{\partial u}{\partial \nu} &= -|u_t|^{m-2}u_t + |u|^{p-2}u && \text{on } [0,\infty)\times\Gamma_1,\\ u(0,x) &= u_0(x) && \text{on } \Omega, \end{aligned}$$

where $\Omega\subset\mathbb{R}^n$ ($n\geqslant 1$) is a regular and bounded domain, $\partial\Omega = \Gamma_0\cup\Gamma_1$, $m>1$, $2\leqslant p<r$, where $r=2(n-1)/(n-2)$ when $n\geqslant 3$, $r=\infty$ when $n=1,2$ and $u_0\in H^1(\Omega)$, $u_0=0$ on $\Gamma_0$. We prove local existence of the solutions in $H^1(\Omega)$ when $m>r/(r+1-p)$ or $n=1,2$ and global existence when $p\leqslant m$ or the initial datum is inside the potential well associated to the stationary problem.

## 1. Introduction

We consider the problem

$$\left.\begin{aligned} u_t - \Delta u &= 0 && \text{in } (0,\infty)\times\Omega,\\ u &= 0 && \text{on } [0,\infty)\times\Gamma_0,\\ \frac{\partial u}{\partial \nu} &= -Q(t,x,u_t) + f(x,u) && \text{on } [0,\infty)\times\Gamma_1,\\ u(0,x) &= u_0(x) && \text{in } \Omega, \end{aligned}\right\} \tag{1.1}$$

where $u=u(t,x)$, $t\geqslant 0$, $x\in\Omega$, $\Delta$ denotes the Laplacian operator, with respect to the $x$ variable, $\Omega$ is a bounded open subset of $\mathbb{R}^n$ ($n\geqslant 1$) of class $C^1$ (see [4]), $\partial\Omega=\Gamma_0\cup\Gamma_1$, $\Gamma_0\cap\Gamma_1=\emptyset$, $\Gamma_0$ and $\Gamma_1$ are measurable over $\partial\Omega$, endowed with the $(n-1)$-dimensional surface measure $\sigma$. These properties of $\Omega$, $\Gamma_0$ and $\Gamma_1$ will be assumed, without further comments, throughout the paper.

The initial datum $u_0$ belongs to $H^1(\Omega)$ and $u_0=0$ on $\Gamma_0$. Moreover, $Q$ represents a nonlinear boundary damping term, i.e. $Q(t,x,v)v\geqslant 0$, and $f$ represents a nonlinear source term, i.e. $f(x,u)u\geqslant 0$.

Local and global existence for solutions of problems like (1.1) has been widely studied when $Q\equiv 0$ (parabolic problems with nonlinear boundary conditions) or

$Q \equiv u_t$ (parabolic problems with *dynamical* boundary conditions). We respectively refer to [2, 7, 16–18] and to [8–10, 13–15]. We also would like to emphasize that problem (1.1), when $Q = u_t$, naturally models various physical problems involving diffusion (see [9, 10] and the references therein; see also Appendix C).

The quoted papers contain, roughly, three different kinds of results: local existence (with various regularity assumptions on $u_0$ and $f$); global existence when $f$ is sublinear in $u$ for $|u|$ large or when $\sigma(\Gamma_0) > 0$ and $u_0$ is small in a suitable sense, via the so-called *potential well* theory; and, finally, blow-up when $u_0$ is large enough and $f$ is superlinear in $u$ for $|u|$ large.

In this paper we study the case $Q(t, x, u_t) \simeq |u_t|^{m-2}u_t$ when $|u_t| \geqslant 1$, $m > 1$ and $f(x, u) \simeq |u|^{p-2}u$, $p > 2$, when $|u| \geqslant 1$. The interest in considering nonlinear terms in $u_t$ is mainly mathematical. However, a physical model involving $Q(t, x, u_t) = u_t + |u_t|^{m-2}u_t$ is given in Appendix C. For the sake of simplicity, we shall consider in the sequel the model problem

$$\left.\begin{aligned} u_t - \Delta u &= 0 && \text{in } (0,\infty) \times \Omega,\\ u &= 0 && \text{on } [0,\infty) \times \Gamma_0,\\ \frac{\partial u}{\partial \nu} &= -|u_t|^{m-2}u_t + |u|^{p-2}u && \text{on } [0,\infty) \times \Gamma_1,\\ u(0,x) &= u_0(x) && \text{in } \Omega. \end{aligned}\right\} \tag{1.2}$$

When the term $|u_t|^{m-2}u_t$ is not present in (1.2) and $\sigma(\Gamma_0) > 0$, this type of problem was considered in [18]. In particular, global existence was proven when the datum $u_0$ belongs to a suitable stable set $W$ (see (1.13)) using potential well arguments.

When $m = 2$, problem (1.2) can be considered as a particular case in the general theory developed in [9] and [10], where semigroup theory arguments were applied. In particular, local existence and uniqueness of a maximal solution

$$u \in C([0, T_{\max}); H^1(\Omega)) \cap C^{0,\delta}_{\text{loc}}([0, T_{\max}); H^{1,s}(\Omega))$$

(where $\delta = \frac{1}{2}(1 - s)$, $s \in [0, 1)$ and $C^{0,\delta}_{\text{loc}}$ denotes locally $\delta$-Hölder continuous functions) were proven when $u_0 \in H^1(\Omega)$, $\Gamma_0 = \emptyset$ and $2 \leqslant p < 2(n-1)/(n-2)$ or $n = 1, 2$. Moreover, a blow-up result was given when $2 < p < 2(n-1)/(n-2)$ or $n = 1, 2$ and $u_0$ is large enough.

Some related problems concerning wave equations with nonlinear damping and source terms have been considered in [12, 24, 27]. In particular, [12] deals with the Cauchy–Dirichlet problem

$$\left.\begin{aligned} u_{tt} - \Delta u + |u_t|^{m-2}u_t &= |u|^{p-2}u && \text{in } (0,\infty) \times \Omega,\\ u &= 0 && \text{on } [0,\infty) \times \partial\Omega,\\ u(0,x) = u_0(x), \quad u_t(0,x) &= u_1(x) && \text{in } \Omega, \end{aligned}\right\} \tag{1.3}$$

when $m \geqslant 2$ and $2 < p \leqslant 1 + \frac{1}{2}2^*$ (here, $2^*$ denotes the critical exponent of the embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$), while [24] deals with the Cauchy problem associated to (1.3) when $\Omega$ is replaced by the entire space $\mathbb{R}^n$. In [24], the more general case $m > 1$, $2 \leqslant p < 2^*$, $m > 2^*/(2^* + 1 - p)$ was considered for the first time. In [27] the author studied a modified version of (1.1), where the heat operator $\partial/\partial t - \Delta$ is replaced by the d'Alambertian operator $\Box = \partial^2/\partial t^2 - \Delta$ and $2 \leqslant p < 2(n-1)/(n-2)$ or $n = 1, 2$.

The first aim of the present paper is to show that, when $2 \leqslant p \leqslant m$ and $p < 2(n-1)/(n-2)$ or $n = 1, 2$, the solutions of (1.2) are global in time, for arbitrary (even large) initial datum, so that the blow-up phenomenon shown when $Q \equiv 0$ (see [7, 16]), and when $Q \equiv u_t$ (see [10, 15], dealing with a small perturbation of (1.2)) cannot occur.

This type of result was proved by the author in [27] in connection with wave-type equations. In general, methods employed to study hyperbolic problems cannot be employed to study parabolic problems, and vice versa. Nevertheless, the arguments of [27] can be conveniently adapted to problem (1.2). However, there are several important differences in the proofs, which make the adaptation non-trivial and that will be outlined at the end of this section. Moreover, the extension of these results to (1.2) highlights the fact that the superlinear boundary damping term is the main factor that determines the lifespan of the solutions, regardless of the parabolic or hyperbolic structure of the differential operator acting on $\Omega$. A natural conjecture arising from this result is that the wave operator can be replaced by the Laplacian operator too.

In order to state our results, we set

$$r = \begin{cases} \dfrac{2(n-1)}{n-2} & \text{if } n \geqslant 3, \\ \infty & \text{if } n = 1, 2, \end{cases} \tag{1.4}$$

as the critical value of the trace-Sobolev theorem (see [1, theorem 7.58]), $H^1(\Omega) \to L^p(\partial\Omega)$. Moreover, we introduce the notations

$$\|\cdot\|_q = \|\cdot\|_{L^q(\Omega)}, \quad \|\cdot\|_{q,\Gamma_1} = \|\cdot\|_{L^q(\Gamma_1)}, \quad 1 \leqslant q \leqslant \infty,$$

and

$$H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega) : u|_{\Gamma_0} = 0\}, \qquad \|u\|^2_{H^1_{\Gamma_0}(\Omega)} = \|u\|_2^2 + \|\nabla u\|_2^2.$$

For the sake of simplicity, we state our main results only in the case $n \geqslant 3$.

The first step in our study is the following local existence result.

THEOREM 1.1. *Let* $m > 1$,

$$2 \leqslant p < r \quad \text{and} \quad m > \frac{r}{r+1-p}. \tag{1.5}$$

*Then, given* $u_0 \in H^1_{\Gamma_0}(\Omega)$*, there is* $T > 0$ *and a weak solution* $u$[1] *of problem (1.2) on* $(0,T) \times \Omega$ *such that*

$$u \in C([0,T]; H^1_{\Gamma_0}(\Omega)), \tag{1.6}$$

$$u_t \in L^2((0,T) \times \Omega) \cap L^m((0,T) \times \Gamma_1) \tag{1.7}$$

*and the energy identity*

$$\tfrac{1}{2}\|\nabla u\|_2^2\big|_s^t + \int_s^t (\|u_t\|_2^2 + \|u_t\|^m_{m,\Gamma_1}) = \int_s^t \int_{\Gamma_1} |u|^{p-2}uu_t \tag{1.8}$$

[1]The precise definition of weak solution will be given in § 3.

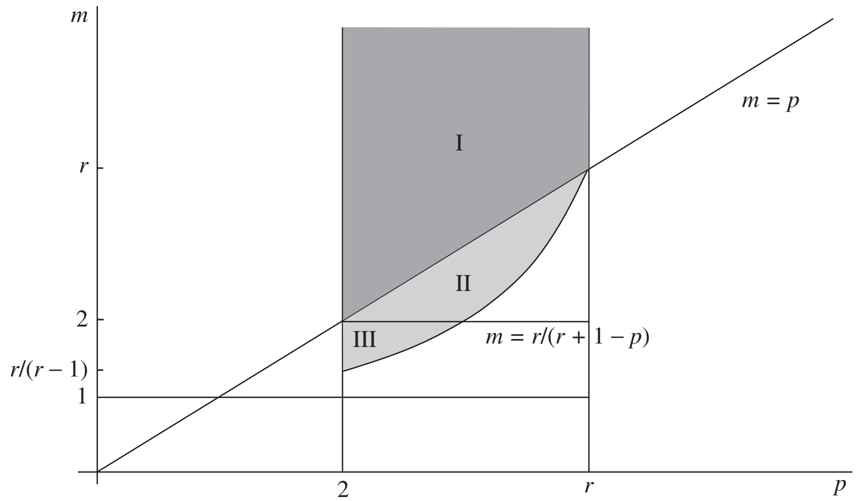


Figure 1. This picture describes the three regions in the $(p, m)$-plane related to the result we obtained for problem (1.2). In region I, global existence for arbitrary initial data is proved (theorem 1.2). In region II, local existence for arbitrary initial data (theorem 1.1) and global existence for initial data in the potential well (theorem 1.3) when $\sigma(\Gamma_0) > 0$ is obtained. On segment III, blow-up for a large datum was proven in [10].

*holds for* $0 \leqslant s \leqslant t \leqslant T$. *Moreover,*

$$T = T(\|u_0\|^2_{H^1_{\Gamma_0}(\Omega)}, m, p, \Omega, \Gamma_1)$$

*is decreasing in the first variable.*

The assumption $m > r/(r+1-p)$ in theorem 1.1 cuts a part of the full subcritical range $p < r$ and $m > 1$ (see figure 1). This type of condition also appears in [12, 24, 27], as recalled above, and depends on the presence of the nonlinear damping term $|u_t|^{m-2}u_t$ together with the low regularity required for $u_0$. When $m = 2$ and $\Gamma_0 = \emptyset$, this restriction is not necessary, as shown in [10]. On the other hand, problem (1.2) possesses in this case a *very particular* structure, which allows us to apply semigroup arguments, since it can be written as an abstract Cauchy problem of the form

$$z'(t) + A(z(t))z(t) = F(z(t)), \qquad z(0) = z_0$$

by setting $z = (u, u_{|\Gamma_1})$. An approach of this type seems very problematic when $m \neq 2$. Let us explain briefly where the assumption $m > r/(r+1-p)$ comes from. In order to handle with $|u_t|^{m-2}u_t$, we use the classical monotonicity method (see [20, 21]), in which the energy identity (1.8) plays an essential role. Since $u_0 \in H^1(\Omega)$, the best regularity of the solution $u$ that one can expect (at lest when $m \neq 2$) is that given by (1.6). Furthermore, *a priori* boundary estimates on $u_t$ only give $u_t \in L^m((0,T)\times\Gamma_1)$. Then, in view of the trace-Sobolev theorem $H^1(\Omega) \to L^p(\partial\Omega)$ and of Hölder's inequality, the integral in the right-hand side of (1.8) makes sense only if $m \geqslant r/(r+1-p)$.

To explain why the inequality $m \geqslant r/(r+1-p)$ is assumed in its stronger form, we have to give some detail from the proof. A standard contraction argument is not available and a compactness argument (Schauder fixed-point theorem) is used instead. This immediately explains why the case $p = r$ cannot be considered, since, in this case, the trace mapping $H^1(\Omega) \to L^p(\partial\Omega)$ is not compact. Moreover, since the estimate of the nonlinear damping term plays an essential role, we cannot apply a compactness arguments when $m = r/(r+1-p)$. A further consequence of the method applied is that we cannot assert uniqueness of the solutions of (1.2), as in [24] and [27]. We cannot use a contraction argument, even with some further (non-trivial) restrictions on the parameters $p$ and $m$, as in [12], since here we cannot estimate the boundary damping term using the $L^2$ norm.

Our global existence result is as follows.

Theorem 1.2. *Let*

$$2 \leqslant p \leqslant m \quad \text{and} \quad p < r. \tag{1.9}$$

*Then any weak solution given by theorem 1.1 can be extended to the whole of* $(0,\infty) \times \Omega$.

Theorem 1.2 is the first result, to the author's knowledge, which shows that a superlinear boundary damping term forces global existence for the solutions of the heat equations with source terms, with arbitrary initial data.

Since, in [12], the condition $p \leqslant m$ was proved to be optimal to ensure the global existence of the solutions of (1.3), we conjecture here that the same phenomenon happens for (1.2), i.e. that when $p > m$ solutions blow-up in finite time. The arguments of [10] show that this conjecture is correct when $m = 2$[2].

The second aim of the paper is to extend the potential well theory result of [18] to (1.2). The stable set we consider is the same introduced in [18, 19]. To recall it, we set

$$J(u) = \tfrac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_{p,\Gamma_1}^p \tag{1.10}$$

and

$$K(u) = \|\nabla u\|_2^2 - \|u\|_{p,\Gamma_1}^p \tag{1.11}$$

defined on $H^1_{\Gamma_0}(\Omega)$.

When $\sigma(\Gamma_0) > 0$, the Poincaré inequality holds (see [28]), so that $\|\nabla u\|_2$ is an equivalent norm on $H^1_{\Gamma_0}(\Omega)$ and the number

$$d = \inf_{u \in H^1_{\Gamma_0}(\Omega),\, u|_{\Gamma_1} \not\equiv 0} \sup_{\lambda > 0} J(\lambda u) \tag{1.12}$$

is positive (see [19] or lemma 3.3 below, where a different characterization of $d$ is given). The stable set $W$ of the initial data, for which global existence is proven in [18], is defined by

$$W = \{u_0 \in H^1_{\Gamma_0}(\Omega) : K(u_0) \geqslant 0 \text{ and } J(u_0) < d\}. \tag{1.13}$$

We can now state the following result.

[2]This blow-up result is stated in [10] for a vectorial version of (1.2) when $\Gamma_0 = \emptyset$, but the arguments there can be applied without essential changes to the scalar case and when $\Gamma_0 \neq \emptyset$.

THEOREM 1.3. *Suppose that* $m > 1$,

$$2 \leqslant p < r, \qquad m > \frac{r}{r+1-p}$$

*and* $\sigma(\Gamma_0) > 0$. *If* $u_0 \in W$, *then there is a global solution* $u$ *of (1.2) on* $(0,\infty)\times\Omega$ *such that* $u(t) \in W$ *for all* $t \geqslant 0$,

$$u \in C([0,\infty); H^1_{\Gamma_0}(\Omega)), \tag{1.14}$$

$$u_t \in L^m((0,\infty)\times\Gamma_1) \cap L^2((0,\infty)\times\Omega) \tag{1.15}$$

*and the energy identity*

$$\tfrac{1}{2}\|\nabla u\|_2^2\big|_s^t + \int_s^t \|u_t\|_{m,\Gamma_1}^m + \|u_t\|_2^2 = \int_s^t \int_{\Gamma_1} |u|^{p-2}uu_t \tag{1.16}$$

*holds for* $0 \leqslant s \leqslant t < \infty$.

REMARK 1.4. It is easy to see that the set $W$ is bounded in $H^1_{\Gamma_0}(\Omega)$, so that the solution $u$ also belongs to $L^\infty(0,\infty; H^1_{\Gamma_0}(\Omega))$.

As a byproduct of the proof of theorem 1.1, we also obtain the following global existence–uniqueness result for the problem,

$$\left.\begin{aligned} u_t - \Delta u &= 0 && \text{in } (0,T)\times\Omega,\\ u &= 0 && \text{on } [0,T)\times\Gamma_0,\\ \frac{\partial u}{\partial\nu} &= -|u_t|^{m-2}u_t + g(t,x) && \text{on } [0,T)\times\Gamma_1,\\ u(0,x) &= u_0(x) && \text{in } \Omega, \end{aligned}\right\} \tag{1.17}$$

where $m > 1$, $T > 0$ is arbitrary and $g$ is a given forcing term acting on $\Gamma_1$.

THEOREM 1.5. *Suppose that* $g \in L^{m'}((0,T)\times\Gamma_1)$, *where* $1/m + 1/m' = 1$. *If* $u_0 \in H^1_{\Gamma_0}(\Omega)$, *then there is a unique weak solution* $u$[3] *of (1.17) on* $(0,T)\times\Omega$. *Moreover,*

$$u \in C([0,T]; H^1_{\Gamma_0}(\Omega)), \tag{1.18}$$

$$u_t \in L^m((0,T)\times\Gamma_1) \cap L^2((0,T)\times\Omega) \tag{1.19}$$

*and the energy identity*

$$\tfrac{1}{2}\|\nabla u\|_2^2\big|_s^t + \int_s^t (\|u_t\|_2^2 + \|u_t\|_{m,\Gamma_1}^m) = \int_s^t \int_{\Gamma_1} gu_t \tag{1.20}$$

*holds for* $0 \leqslant s \leqslant t \leqslant T$.

Theorems 1.1–1.3 are extended in § 4 to problem (1.1), under suitable assumptions on the nonlinearities $f$ and $Q$. For the sake of simplicity, we first present the proofs for the model problem (1.2) and then we give in § 4 the necessary generalizations needed to handle with (1.1). This section is naturally addressed to a more

[3]The precise definition of weak solution will be given in § 2.

specialized audience and then a higher lever of mathematical expertise of the reader is supposed. In particular, the proofs are only sketched.

In order to explain the main difficulties that arise in the proofs of theorems 1.1–1.5, we now make some comparison with the arguments used in [27]. Theorem 1.5, which is the parabolic version of [27, theorem 4], is proved via the Faedo–Galerkin procedure, so that finite-dimensional approximations of (1.17) are considered. The first essential difference, with respect to [27], emerges here, since the boundary damping appearing in (1.17) is now a time derivative of the same (highest) order of $u_t$ acting on $\Omega$. Moreover, different *a priori* estimates are used in the proof of theorem 1.5. Theorem 1.1 is proven by using theorem 1.5 together with a fixed-point argument. The main estimates of [27] are here conveniently modified to handle (1.2). It is worth mentioning that the energy identity (1.20) used in the proof of theorem 1.1 differs from the usual energy identity used for the heat equation (see, for example, [11]), since it is obtained by formally multiplying the equation by $u_t$ as for wave-type equations while in parabolic equations one usually multiplies by $u$. This explains why the transposition of techniques motivated by hyperbolic problems gives original outcomes when it can be applied to parabolic problems. The proofs of theorems 1.2 and 1.3 do not essentially differ from the analogous ones given in [27], but they are explicitly given because of their simplicity and for the sake of completeness. Finally, we would like to mention that the techniques used in the paper also allow us to consider modified versions of problem (1.1), where the Laplacian operator $\Delta$ is replaced by more general second-order uniformly elliptic operators in divergence form, such as $Au = \operatorname{div}(A_0(x)\nabla u)$, where $A_0(x) = (a_{ij}(x))_{i,j=1,\dots,n}$, $a_{ij} \in L^\infty(\Omega)$, $i, j = 1, \dots, n$, where $v^t A(x) v \geqslant \alpha_0 |v|^2$ for all $v \in \mathbb{R}^n$ and some positive constant $\alpha_0$. Theorems 1.1–1.3 could also be extended to the vectorial case of systems of the form (1.1), using the same arguments. These generalizations are left to the interested reader.

### 1.1. Notation and preliminaries

We collect some notation and preliminaries that we will use during the paper,

| | |
|---|---|
| $q'$ | Hölder conjugate of $q > 1$, i.e. $1/q + 1/q' = 1$, |
| $(\cdot,\cdot)$ | scalar product in $L^2(\Omega)$, |
| $C_c^\infty(\mathcal{O})$ | space of compactly supported $C^\infty$ function on $\mathcal{O}$, |
| $C([a,b];X)$ | space of all norm continuous functions from $[a,b]$ to $X$, |
| $C_c((a,b);X)$ | space of compactly supported strongly continuous functions from $(a,b)$ to $X$, |
| $C_w([a,b];X)$ | space of weakly continuous functions from $[a,b]$ to $X$, |
| $X'$ | the dual space of $X$, |
| $\langle\cdot,\cdot\rangle_X$ | the duality product between $X$ and $X'$, |
| $H^{-1}(\Omega)$ | the dual space of $H_0^1(\Omega)$, |
| $\mathcal{L}(X,X')$ | the space of linear bounded operators from $X$ to $X'$, |

where $X$ is a Banach space and $\mathcal{O}$ any open subset of $\mathbb{R}^k$. Moreover, we call *the trace theorem* the existence of the continuous trace mapping $H^1_{\Gamma_0}(\Omega) \to L^2(\Gamma_1)$. We also call *the trace-Sobolev theorem* the existence of the continuous mapping (see [1])

$$H^1_{\Gamma_0}(\Omega) \to L^p(\Gamma_1) \quad \text{for } 2 \leqslant p < r.$$

## 2. Global existence and uniqueness for a boundary forced heat equation

This section is devoted to the proof of theorem 1.5. We first give the following definition.

DEFINITION 2.1. Let $u_0 \in H^1_{\Gamma_0}(\Omega)$. A weak solution of (1.17) is a function $u$ such that the following hold.

(a) $u \in L^\infty(0,T; H^1_{\Gamma_0}(\Omega))$, $u_t \in L^2((0,T)\times\Omega)$.

(b) The spatial trace of $u$ on $(0,T)\times\partial\Omega$ (which exists by the trace theorem) has a distributional time derivative on $(0,T)\times\partial\Omega$ belonging to $L^m((0,T)\times\partial\Omega)$.

(c) For all

$$\phi \in X := \{u \in H^1_{\Gamma_0}(\Omega) : u|_{\Gamma_1} \in L^m(\Gamma_1)\}, \tag{2.1}$$

and for almost all $t \in [0,T]$, we have

$$\int_\Omega u_t(t)\phi + \nabla u(t)\nabla\phi + \int_{\Gamma_1} |u_t(t)|^{m-2}u_t(t)\phi = \int_{\Gamma_1} g(t)\phi. \tag{2.2}$$

(d) $u(0) = u_0$.

Note that, in (d), $u(0)$ makes sense since, by (a),

$$u \in W^{1,2}(0,T; L^2(\Omega)) \hookrightarrow C([0,T]; L^2(\Omega)).$$

In the proof of theorem 1.5, we shall use the following result.

LEMMA 2.2. *Let*

$$\zeta \in L^{m'}((0,T)\times\Gamma_1) \tag{2.3}$$

*and suppose that $u$ is a weak solution of*

$$\left.\begin{aligned} u_t - \Delta u &= 0 \quad \text{in } (0,T)\times\Omega, \\ u &= 0 \quad \text{on } [0,T)\times\Gamma_0, \\ \frac{\partial u}{\partial\nu} &= \zeta \quad \text{on } (0,T)\times\Gamma_1, \end{aligned}\right\} \tag{2.4}$$

*i.e. a function*

$$u \in L^\infty(0,T; H^1_{\Gamma_0}(\Omega)) \tag{2.5}$$

*such that*

$$u_t \in L^2((0,T)\times\Omega) \cap L^m((0,T)\times\Gamma_1) \tag{2.6}$$

*and*

$$\int_\Omega u_t(\tau)\phi + \nabla u(\tau)\nabla\phi = \int_{\Gamma_1} \zeta(\tau)\phi \tag{2.7}$$

*for all* $\phi \in X$ *and almost all* $\tau \in [0,T]$. *Then*

$$u \in C([0,T]; H^1_{\Gamma_0}(\Omega)) \tag{2.8}$$

*and the energy identity*

$$\tfrac{1}{2}\|\nabla u\|_2^2\big|_s^t + \int_s^t \|u_t\|_2^2 = \int_s^t \int_{\Gamma_1} \zeta u_t \tag{2.9}$$

*holds for* $0 \leqslant s \leqslant t \leqslant T$.

Lemma 2.2 is an extension of [25, theorems 3.1 and 3.2], which cannot be directly applied to (2.4). Nevertheless, the technique of [25] continues to work in the present situation. Since the proof of this result is rather technical, we give it in Appendix A.

*Proof of theorem 1.5.* We apply the Faedo–Galerkin procedure. Let $(w_k)_k$ be a sequence of linearly independent vectors in $X$ whose finite linear combinations are dense in $X$. By (eventually) using the Grahm–Schmidt orthogonalization process, we can take $(w_k)_k$ to be orthonormal in $L^2(\Omega)$. Since $X$ is dense in $H^1_{\Gamma_0}(\Omega)$ (see, for example, [27, Appendix A]), there are $u_{0k} = \sum_{j=1}^k y^j_{0k} w_j$, $k \in \mathbb{N}$, for some real numbers $y^j_{0k}$, $j = 1,\dots,k$, $k \in \mathbb{N}$, such that

$$u_{0k} \to u_0 \quad \text{in } H^1_{\Gamma_0}(\Omega). \tag{2.10}$$

For any fixed $k \in \mathbb{N}$, we look for approximate solutions of (1.17), that is, for solutions $u^k(t) = \sum_{j=1}^k y^j_k(t) w_j$, of the finite-dimensional problem

$$\left.\begin{aligned} (u^k_t, w_j) + (\nabla u^k, \nabla w_j) + \int_{\Gamma_1} |u^k_t|^{m-2} u^k_t w_j = \int_{\Gamma_1} g w_j, \quad j = 1,\dots,k,\\ u^k(0) = u_{0k}. \end{aligned}\right\} \tag{2.11}$$

In order to recognize that (2.11) has a local solution, we set

$$\begin{aligned} y_{0k} &= (y^1_{0k},\dots,y^k_{0k})^{\mathrm{T}},\\ y_k &= (y^1_k,\dots,y^k_k)^{\mathrm{T}},\\ A_k &= ((\nabla w_i, \nabla w_j))_{i,j=1,\dots,k},\\ B_k(x) &= (w_1(x),\dots,w_k(x))^{\mathrm{T}},\\ G_k(y) &= y + \int_{\Gamma_1} |B_k(x)\cdot y|^{m-2} B_k(x)\cdot y B_k(x)\,\mathrm{d}x, \quad y \in \mathbb{R}^k, \end{aligned}$$

and

$$H_k(t) = \int_{\Gamma_1} g(t,x) B_k(x)\,\mathrm{d}x. \tag{2.12}$$

Problem (2.11) can be rewritten in the vectorial form

$$\left.\begin{aligned} G_k(y'_k(t)) + A_k y_k(t) = H_k(t),\\ y_k(0) = y_{0k}. \end{aligned}\right\} \tag{2.13}$$

Since $w_j \in L^m(\Gamma_1)$ for $j = 1, \dots, k$, we have $|B_k| \in L^m(\Gamma_1)$ and $G_k \in C(\mathbb{R}^k)$. Moreover, $G_k = \nabla \mathcal{G}_k$, where

$$\mathcal{G}_k(y) = \tfrac{1}{2}|y|^2 + \frac{1}{m}\int_{\Gamma_1} |B_k(x)\cdot y|^m \, dx.$$

The first addendum in the previous formula is strictly convex, while the second one is convex since

$$\begin{aligned}\frac{1}{m}\int_{\Gamma_1} &|B_k(x)\cdot[\alpha y_1 + (1-\alpha y_2)]|^m \, dx \\ &= \frac{1}{m}\int_{\Gamma_1} |\alpha B_k(x)\cdot y_1 + (1-\alpha)B_k(x)\cdot y_2]|^m \, dx \\ &\leqslant \frac{1}{m}\int_{\Gamma_1} \alpha|B_k(x)\cdot y_1|^m + (1-\alpha)|B_k(x)\cdot y_2|^m \, dx\end{aligned}$$

for any $\alpha \in (0,1)$, where the convexity of the function $|y|^m$ was used. Then $\mathcal{G}_k$ is strictly convex in $\mathbb{R}^k$. Moreover, for any $y \neq 0$,

$$\lim_{\lambda\to+\infty} \frac{\mathcal{G}_k(\lambda y)}{\lambda} = \infty.$$

Then, using [22, theorems 2.5 and 2.6], we recognize that $G_k$ is an homeomorphism from $\mathbb{R}^k$ onto itself, with inverse $G_k^{-1}$. Then (2.13) can be written in the form

$$\left.\begin{aligned} y_k'(t) &= G_k^{-1}(H_k(t) - A_k y_k(t)), \\ y_k(0) &= y_{0k}. \end{aligned}\right\} \tag{2.14}$$

Now we note that, since $G_k(y)y \geqslant |y|^2$ for all $y \in \mathbb{R}^k$, by the Schwartz inequality, it follows that $|y| \leqslant |G_k(y)|$. Then $|G_k^{-1}(y)| \leqslant |y|$ for all $y \in \mathbb{R}^k$, so that

$$|G_k^{-1}(H_k(t) - A_k y_k)| \leqslant |H_k(t)| + \|A_k\||y_k|. \tag{2.15}$$

Since $|B_k| \in L^m(\Gamma_1)$ and $g \in L^{m'}((0,T)\times\Gamma_1)$, it follows that $H_k \in L^1(0,T)$. We can then apply Carathéodory's theorem (see [5, theorem 1.1]) to conclude that (2.14), and then (2.13) and (2.11), have a local solution on $(0, t_k)$ for some $t_k > 0$.

Multiplying (2.11) by $(y_k^j)'$ and summing for $j = 1, \dots, k$, we obtain the energy identity (here and in the sequel, explicit dependence on $t$ will be omitted, when clear)

$$\frac{d}{dt}(\tfrac{1}{2}\|\nabla u^k\|_2^2) + \|u_t^k\|_2^2 + \|u_t^k\|_{m,\Gamma_1}^m = \int_{\Gamma_1} g u_t^k. \tag{2.16}$$

Integrating over $(0,t)$, $0 < t < t_k$, and using Young's inequality, we obtain

$$\begin{aligned}\tfrac{1}{2}\|\nabla u^k\|_2^2 + \int_0^t (\|u_t^k\|_2^2 + \|u_t^k\|_{m,\Gamma_1}^m) \\ \leqslant \tfrac{1}{2}\|\nabla u_{0k}\|_2^2 + \frac{1}{m'}\|g\|_{L^{m'}((0,T)\times\Gamma_1)}^{m'} + \frac{1}{m}\int_0^t \|u_t^k\|_{m,\Gamma_1}^m.\end{aligned}$$

Then, using (2.10), there exists

$$K_1 = K_1(\|\nabla u_0\|_2, \|g\|_{L^{m'}((0,T)\times\Gamma_1)}, m) > 0$$

such that

$$\left.\begin{aligned} \|\nabla u^k\|_{L^\infty(0,t_k;L^2(\Omega))} &\leqslant K_1, \\ \|u_t^k\|_{L^2((0,t_k)\times\Omega)} &\leqslant K_1, \\ \|u_t^k\|_{L^m((0,t_k)\times\Gamma_1)} &\leqslant K_1, \\ \||u_t^k|^{m-2}u_t^k\|_{L^{m'}((0,t_k)\times\Gamma_1)} &\leqslant K_1 \end{aligned}\right\} \tag{2.17}$$

for $k \in \mathbb{N}$. By (2.10), (2.17) and Hölder's inequality in time, it follows that

$$\|u^k\|_2^2 \leqslant \|u_{0k}\|_2 + \int_0^t \|u_t^k\|_2 \leqslant \|u_{0k}\|_2 + T^{1/2}\biggl(\int_0^{t_k} \|u_t^k\|_2^2\biggr)^{1/2} \leqslant K_2 \tag{2.18}$$

for some

$$K_2 = K_1(\|u_0\|_{H^1_{\Gamma_0}(\Omega)}, \|g\|_{L^{m'}((0,T)\times\Gamma_1)}, T, m) > 0.$$

Since $(w_k)_k$ is orthonormal in $L^2(\Omega)$, we have $|y_k(t)| = \|u^k(t)\|_2$, so (2.18) yields that $|y_k(t)| \leqslant K_2^{1/2}$. Then, by (2.15),

$$|G_k^{-1}(H_k(t) - A_k y_k)| \leqslant |H_k(t)| + \|A_k\| K_2^{1/2} \in L^1(0,T).$$

We can then apply [5, theorem 1.3] to conclude that $t_k = T$ for $k = 1, \dots, n$. Next, by (2.17) and (2.18), it follows (see [20, 21]) that, up to a subsequence,

$$\left.\begin{aligned} u_k &\to u && \text{weakly}^* \text{ in } L^\infty(0,T;H^1_{\Gamma_0}(\Omega)), \\ u_t^k &\to u_t && \text{weakly in } L^2((0,T)\times\Omega), \\ u_t^k &\to \psi && \text{weakly in } L^m((0,T)\times\Gamma_1), \\ |u_t^k|^{m-2}u_t^k &\to \chi && \text{weakly in } L^{m'}((0,T)\times\Gamma_1). \end{aligned}\right\} \tag{2.19}$$

A consequence of the convergence (2.19) is that $u^k \to u$ weakly in $H^1(0,T;L^2(\Omega))$ and then, up to a subsequence, $u^k \to u$ strongly in $C([0,T];L^2(\Omega))$, so that $u(0) = u_0$. It follows in a standard way (see, for example, [27, p. 272]) that $\psi$ is the distribution time derivative of $u$ on $(0,T)\times\partial\Omega$, i.e. $\psi = u_t$.

Next, multiplying (2.11) by $\phi \in C_c^\infty(0,T)$, integrating on $(0,T)$, passing to the limit as $k \to \infty$ (using (2.19)) and finally using the density of the finite linear combinations of $(w_k)_k$ in $X$, we obtain

$$\int_0^T \biggl[(u_t, w)\phi + (\nabla u, \nabla w)\phi + \int_{\Gamma_1} \chi w\phi - \int_{\Gamma_1} g w\phi\biggr] = 0 \tag{2.20}$$

for all $w \in X$, $\phi \in C_c^\infty(0,T)$, and then

$$(u_t, w) + (\nabla u, \nabla w) + \int_{\Gamma_1} \chi w = \int_{\Gamma_1} g w$$

almost everywhere in $(0,T)$. Then, to prove that $u$ is a weak solution of (1.17), we have only to show that

$$\chi = |u_t|^{m-2}u_t \quad \text{a.e. on } (0,T)\times\Gamma_1. \tag{2.21}$$

By lemma 2.2, we obtain (1.18) and the energy identity (2.9), with $\zeta = g - \chi$, which, for $s = 0$ and $t = T$, reads as

$$\tfrac{1}{2}\|\nabla u\|_2^2|_0^T + \int_0^T \|u_t\|_2^2 + \int_0^T \int_{\Gamma_1} \chi u_t = \int_0^T \int_{\Gamma_1} g u_t. \tag{2.22}$$

The classical monotonicity method (see [20, 21]) then allows us to prove (2.21), together with the uniqueness of the solution. We briefly outline here the proof of these facts, for the reader's convenience. Integrating (2.16) over $(0, T)$, extracting a subsequence such that $\nabla u^k(T) \to \nabla u(T)$ weakly in $L^2(\Omega)$, passing to the limit as $k \to \infty$ and using (2.19) and the weakly lower semi-continuity of norms in Hilbert spaces, we get

$$\tfrac{1}{2}\|\nabla u(T)\|_2^2 + \int_0^T \|u_t\|_2^2 + \varliminf_k \int_0^T \|u_t^k\|_{m,\Gamma_1}^m \leqslant \tfrac{1}{2}\|\nabla u_0\|_2^2 + \int_0^T \int_{\Gamma_1} g u_t. \tag{2.23}$$

By combining (2.22) with (2.23), we have

$$\varliminf_k \int_0^T \|u_t^k\|_{m,\Gamma_1}^m \leqslant \int_0^T \int_{\Gamma_1} \chi u_t.$$

The last formula, together with system (2.19) and the monotonicity of the map $y \to |y|^{m-2}y$, gives

$$0 \leqslant \varliminf_k \int_0^T \int_{\Gamma_1} (|u_t^k|^{m-2}u_t^k - |\phi|^{m-2}\phi)(u_t^k - \phi) \leqslant \int_0^T \int_{\Gamma_1} (\chi - |\phi|^{m-2}\phi)(u_t - \phi)$$

for all $\phi \in L^m((0,T) \times \Gamma_1)$. Setting $\phi = u_t - \lambda\phi_1$, $\lambda > 0$, letting $\lambda \to 0^+$ and dividing by $\lambda$, we get

$$\int_0^T \int_{\Gamma_1} (\chi - |u_t|^{m-2}u_t)\phi_1 \geqslant 0 \quad \text{for all } \phi_1 \in L^m((0,T) \times \Gamma_1),$$

and then (2.21) follows. The uniqueness of the solutions follows by applying equation (2.9) to the difference $w = \tilde{u} - \hat{u}$ of any two solutions (which solves (2.4) with $\zeta = -|\tilde{u}_t|^{m-2}\tilde{u}_t + |\hat{u}_t|^{m-2}\hat{u}_t$ and $w(0) = 0$) and using the monotonicity of $|y|^{m-2}y$. □

## 3. Proof of main results

In this section we prove theorems 1.1–1.3. We first give the precise definition of a weak solution of (1.2).

Definition 3.1. A weak solution of (1.2) is a function $u$ such that (a)–(d) of definition 2.1 hold, with (2.2) replaced by

$$\int_\Omega u_t(t)\phi + \nabla u(t)\nabla\phi + \int_{\Gamma_1} |u_t(t)|^{m-2}u_t(t)\phi = \int_{\Gamma_1} |u(t)|^{p-2}u(t)\phi. \tag{3.1}$$

To prove theorem 1.1, we use the following compactness result. The proof can be found in [27].

LEMMA 3.2. *Let $m > 1$ and $1 < p_0 < r$. Set*

$$Z_T := \{u \in L^\infty(0,T; H^1_{\Gamma_0}(\Omega)) : u_t \in L^m((0,T) \times \Gamma_1)\}$$

*endowed with the natural norm*

$$\|u\|^2_{Z_T} = \|u\|^2_{L^\infty(0,T;H^1_{\Gamma_0}(\Omega))} + \|u_t\|^2_{L^m((0,T)\times\Gamma_1)}.$$

*Then the operator $Z_T \to C([0,T]; L^{p_0}(\Gamma_1))$ is compact.*

We can now give the following proof.

*Proof of theorem 1.1.* We set, for any $T > 0$, the Banach space

$$Y_T = \{u \in C([0,T]; H^1_{\Gamma_0}(\Omega)),\ u_t \in L^m((0,T) \times \Gamma_1)\},$$

endowed with the norm $\|u\|_{Y_T} = \|u\|_{Z_T}$, and the closed convex set

$$X_T = X_T(u_0) = \{u \in Y_T : u(0) = u_0\}.$$

Take $u \in X_T$. Assumption (1.5) yields that $(p-1)m' < r$ and then, since $\Gamma_1$ is bounded, using Hölder's inequality, we have $|u|^{p-2}u \in L^{m'}((0,T) \times \Gamma_1)$. Then, by theorem 1.5, there is a unique solution $v$ of the problem

$$\left.\begin{aligned} v_t - \Delta v &= 0 && \text{in } (0,T) \times \Omega, \\ v &= 0 && \text{on } [0,T) \times \Gamma_0, \\ \frac{\partial v}{\partial \nu} &= -|v_t|^{m-2}v_t + |u|^{p-2}u && \text{on } [0,T) \times \Gamma_1, \\ v(0,x) &= u_0(x) && \text{in } \Omega. \end{aligned}\right\} \tag{3.2}$$

We denote by $v = \Phi(u)$ the solution $v$ of (3.2) that corresponds to $u$, so that $\Phi : X_T \to X_T$ by (1.18) and (1.19). We apply Schauder's fixed-point theorem (see [6, corollary 3.6.2]) to $\Phi : B_R \to B_R$, where $B_R = \{u \in X_T : \|u\|_{Y_T} \leqslant R\}$, provided that $R$ is suitably large and $T$ is suitably small. Note that $B_R$ is closed and convex, and it is non-empty for $R \geqslant R_0 := \|u_0\|_{H^1_{\Gamma_0}(\Omega)}$.

STEP 1. We show that $\Phi$ maps $B_R$ into itself for $R$ sufficiently large and $T$ small enough. Let $u \in B_R$. Writing the energy identity (1.20) for $v$, with $g = |u|^{p-2}u$ and $s = 0$ and using Hölder's inequality, we obtain, for $0 \leqslant t \leqslant T$,

$$\begin{aligned} \tfrac{1}{2}\|\nabla v(t)\|_2^2 + \int_0^t \|v_t\|_2^2 + \int_0^t \|v_t\|^m_{m,\Gamma_1} \\ &\leqslant \tfrac{1}{2}\|\nabla u_0\|_2^2 + \int_0^T \int_{\Gamma_1} |u|^{p-1}|v_t| \\ &\leqslant \tfrac{1}{2}\|\nabla u_0\|_2^2 + \int_0^T \left(\int_{\Gamma_1} |u|^{(p-1)m'}\right)^{1/m'} \|v_t\|_{m,\Gamma_1}. \end{aligned} \tag{3.3}$$

Since $(p-1)m' < r$ by (1.5), applying Hölder's inequality in the space variable again, and then the trace-Sobolev embedding and Hölder's inequality in the time

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variable, we obtain, for $0 \leqslant t \leqslant T$ (here and in the rest of the proof $k_i$ will denote positive constants depending only on $p$, $m$, $\Gamma_1$ and $\Omega$),

$$\begin{aligned}\tfrac{1}{2}\|\nabla v(t)\|_2^2 + \int_0^t \|v_t\|_2^2 + \int_0^t \|v_t\|_{m,\Gamma_1}^m \\ &\leqslant \tfrac{1}{2}\|\nabla u_0\|_2^2 + k_1 \int_0^T \|u\|_{r,\Gamma_1}^{p-1}\|v_t\|_{m,\Gamma_1} \\ &\leqslant \tfrac{1}{2}\|\nabla u_0\|_2^2 + k_2 R^{p-1} T^{1/m'} \|v_t\|_{L^m((0,T)\times\Gamma_1)}. \qquad (3.4)\end{aligned}$$

Writing (3.4) with $t = T$, disregarding the first and the second addendum in the left-hand side and using Young's inequality, we get

$$\begin{aligned}\|v_t\|_{L^m((0,T)\times\Gamma_1)}^m &\leqslant \tfrac{1}{2}\|\nabla u_0\|_2^2 + k_2 R^{p-1} T^{1/m'} \|v_t\|_{L^m((0,T)\times\Gamma_1)} \\ &\leqslant \tfrac{1}{2}\|\nabla u_0\|_2^2 + \frac{1}{m}\|v_t\|_{L^m((0,T)\times\Gamma_1)}^m + k_3 T R^{(p-1)m'}.\end{aligned}$$

Using the inequality

$$(A+B)^\tau \leqslant \max\{1, 2^{\tau-1}\}(A^\tau + B^\tau) \quad \text{for } A, B \geqslant 0,\ \tau > 0, \qquad (3.5)$$

we have

$$\|v_t\|_{L^m((0,T)\times\Gamma_1)} \leqslant k_4 (R_0^{2/(m-1)} + R^{(p-1)/(m-1)} T^{1/m}). \qquad (3.6)$$

Using (3.6) in (3.4) we obtain, for $0 \leqslant t \leqslant T$,

$$\tfrac{1}{2}\|\nabla v(t)\|_2^2 + \int_0^t \|v_t\|_2^2 \leqslant \tfrac{1}{2}R_0^2 + k_5 R^{p-1} T^{1/m'} (R_0^{2/(m-1)} + R^{(p-1)/(m-1)} T^{1/m}) \qquad (3.7)$$

and then, for $t = T$,

$$\|v_t\|_{L^2((0,T)\times\Omega)}^2 \leqslant \tfrac{1}{2}R_0^2 + k_5 R^{p-1} T^{1/m'} (R_0^{2/(m-1)} + R^{(p-1)/(m-1)} T^{1/m}). \qquad (3.8)$$

Using (3.5) with $\tau = 2$ and Hölder's inequality in time, we have

$$\begin{aligned}\|v(t)\|_2^2 \leqslant \biggl(\|u_0\|_2 + \int_0^t \|v_t\|_2\biggr)^2 &\leqslant 2\|u_0\|_2^2 + 2T\|v_t\|_{L^2((0,T)\times\Omega)}^2 \\ &\leqslant 2R_0^2 + 2T\|v_t\|_{L^2((0,T)\times\Omega)}^2,\end{aligned}$$

and consequently, using (3.8),

$$\|v(t)\|_2^2 \leqslant (2+T)R_0^2 + 2k_5 R^{p-1} T^{1+1/m'} (R_0^{2/(m-1)} + R^{(p-1)/(m-1)} T^{1/m}). \qquad (3.9)$$

Combining (3.7) and (3.9), we have

$$\begin{aligned}&\|v\|_{L^\infty(0,T;H^1_{\Gamma_0}(\Omega))}^2 \\ &\quad\leqslant (3+T)R_0^2 + 2k_5 R^{p-1} T^{1/m'} (1+T)(R_0^{2/(m-1)} + R^{(p-1)/(m-1)} T^{1/m}). \qquad (3.10)\end{aligned}$$

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Thanks to (3.6) and (3.10), in order to prove that $v \in B_R$, it is enough to show that

$$k_4(R_0^{2/(m-1)} + R^{(p-1)/(m-1)}T^{1/m}) \leqslant \tfrac{1}{2}R \tag{3.11}$$

and

$$(3+T)R_0^2 + 2k_5R^{p-1}T^{1/m'}(1+T)(R_0^{2/(m-1)} + R^{(p-1)/(m-1)}T^{1/m}) \leqslant \tfrac{1}{4}R^2. \tag{3.12}$$

We first fix

$$R := \max\{1, 4k_4R_0^{2/(m-1)}, 6R_0\}, \tag{3.13}$$

so that

$$k_4R_0^{2/(m-1)} \leqslant \tfrac{1}{4}R \quad \text{and} \quad 3R_0^2 \leqslant \tfrac{1}{8}R^2. \tag{3.14}$$

We then take $T \leqslant 1$ small enough so that

$$\left.\begin{aligned} k_4R^{(p-1)/(m-1)}T^{1/m} &\leqslant \tfrac{1}{2}, \\ TR_0^2 &\leqslant \tfrac{1}{16}, \\ 4K_5R^{p-1}T^{1/m'}R_0^{2/(m-1)} &\leqslant \tfrac{1}{32}, \\ 4k_5R^{p-1}T^{1/m'}R^{(p-1)/(m-1)}T^{1/m} &\leqslant \tfrac{1}{32}. \end{aligned}\right\} \tag{3.15}$$

Solving (3.15) with respect to $T$ and using equation (3.13), one immediately see that $T = T(R_0^2, m, p, \Omega, \Gamma_1)$ is decreasing in $R_0$. Moreover, since by (3.13) one has $R \geqslant 1$, formulae (3.14) and (3.15) immediately yield (3.11) and (3.12), proving that with such a choice of $R$ and $T$ one has $v \in B_R$.

Step 2. We now prove that, with this choice of $R$ and $T$, the map $\Phi$ is continuous on $B_R$ and that $\Phi(B_R)$ is relatively compact in $Y_T$. Let $u, \bar{u} \in B_R$, and denote $v = \Phi(u)$, $\bar{v} = \Phi(\bar{u})$, $w = v - \bar{v}$. Clearly, $w$ is a solution of the problem

$$\left.\begin{aligned} w_t - \Delta w &= 0 && \text{in } [0,T) \times \Omega, \\ w &= 0 && \text{on } (0,T) \times \Gamma_0, \\ \frac{\partial w}{\partial \nu} &= -|v_t|^{m-2}v_t + |\bar{v}_t|^{m-2}\bar{v}_t \\ &\qquad + |u|^{p-2}u - |\bar{u}|^{p-2}\bar{u} && \text{on } [0,T) \times \Gamma_1, \\ w(0,x) &= 0 && \text{in } \Omega. \end{aligned}\right\} \tag{3.16}$$

Since $v_t, \bar{v}_t \in L^m((0,T) \times \Gamma_1)$, we also know that $|v_t|^{m-2}v_t$ and $|\bar{v}_t|^{m-2}\bar{v}_t$ belong to $L^{m'}((0,T) \times \Gamma_1)$. Moreover, by (1.5), the functions $|u|^{p-2}u$ and $|\bar{u}|^{p-2}\bar{u}$ belong to $L^{m'}((0,T) \times \Gamma_1)$. Then we can apply lemma 2.2, so that the energy identity (2.9) becomes

$$\begin{aligned} \tfrac{1}{2}\|\nabla w\|_2^2 + \int_0^t \|w_t\|_2^2 + \int_0^t \int_{\Gamma_1} [|v_t|^{m-2}v_t - |\bar{v}_t|^{m-2}\bar{v}_t]w_t \\ = \int_0^t \int_{\Gamma_1} [|u|^{p-2}u - |\bar{u}|^{p-2}\bar{u}]w_t \end{aligned} \tag{3.17}$$

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for $0 \leqslant t \leqslant T$. Estimating from above the right-hand side and from below the third term in the left-hand side of (3.17) exactly as in [27, starting from formula (72) onward], we obtain the estimates

$$\|v_t - \bar{v}_t\|^2_{L^2((0,T)\times\Omega)} \leqslant K\|u - \bar{u}\|_{L^\infty(0,T;L^{r_0}(\Gamma_1))}, \tag{3.18}$$

$$\|\nabla v - \nabla\bar{v}\|^2_{L^\infty(0,T;L^2(\Omega))} \leqslant K\|u - \bar{u}\|_{L^\infty(0,T;L^{r_0}(\Gamma_1))} \tag{3.19}$$

and

$$\|v_t - \bar{v}_t\|^m_{L^m((0,T)\times\Gamma_1)} \leqslant K\|u - \bar{u}\|_{L^\infty(0,T;L^{r_0}(\Gamma_1))}, \tag{3.20}$$

when $m \geqslant 2$, while

$$\||v_t|^{m-2}v_t - |\bar{v}_t|^{m-2}\bar{v}_t\|^{m'}_{L^{m'}((0,T)\times\Gamma_1)} \leqslant K\|u - \bar{u}\|_{L^\infty(0,T;L^{r_0}(\Gamma_1))}, \tag{3.21}$$

when $1 < m < 2$, where $K = K(p, m, \Gamma_1, \Omega, T, R) > 0$. From $v(0) = \bar{v}(0) = u_0$, we get that, for $0 \leqslant t \leqslant T$,

$$\|v(t) - \bar{v}(t)\|_2 \leqslant \int_0^T \|v_t - \bar{v}_t\|_2 \leqslant T^{1/2}\|v_t - \bar{v}_t\|_{L^2((0,T)\times\Omega)}.$$

Hence, by (3.18), we get

$$\|v(t) - \bar{v}(t)\|^2_{L^\infty(0,T;L^2(\Omega))} \leqslant KT\|u - \bar{u}\|_{L^\infty(0,T;L^{r_0}(\Gamma_1))}. \tag{3.22}$$

Estimates (3.18)–(3.22) show that $\Phi$ is continuous on $B_R$. To prove that $\Phi(B_R)$ is relatively compact in $Y_T$, let $(u^k)_k$ be a sequence in $B_R$. By lemma 3.2, the sequence $u^k$ is relatively compact and then, up to a subsequence, it is a Cauchy sequence, in $C([0,T]; L^{r_0}(\Gamma_1))$. Hence, applying (3.18)–(3.20) and (3.22) with $u = u^i$, $\bar{u} = u^j$, $i, j \in \mathbb{N}$, it follows that, when $m \geqslant 2$, $v^k = \Phi(u^k)$ is a Cauchy sequence in $Y_T$.

When $1 < m < 2$, by (3.21), $|v^k_t|^{m-2}v^k_t$ is a Cauchy sequence in the space $L^{m'}((0,T)\times\Gamma_1)$. Then, by [4, théorème IV.9], there is $\chi_0 \in L^{m'}((0,T)\times\Gamma_1)$ such that (up to a subsequence) $|v^k_t|^{m-1} \leqslant \chi_0$ on $(0,T)\times\Gamma_1$ and $v^k_t$ is a.e. convergent in $(0,T)\times\Gamma_1$. Then, by the Lebesgue dominated convergence theorem, $v^k_t$ is convergent in $L^m((0,T)\times\Gamma_1)$. Then, using (3.18), (3.19) and (3.22), $v^k$ is also a Cauchy sequence in $Y_T$ when $1 < m < 2$. Since $Y_T$ is complete, this concludes the proof. □

Now we can give the following proof.

*Proof of Theorem 1.2.* Since

$$T = T(\|u_0\|^2_{H^1_{\Gamma_0}(\Omega)}, m, p, \Omega, \Gamma_1)$$

is decreasing in the first variable, we can apply the standard continuation procedure of ordinary differential equations (see also [23]) to conclude that either the solution $u$ is global or there is $T_{\max} < \infty$ such that

$$\lim_{t\to T^-_{\max}} \|u(t)\|^2_{H^1_{\Gamma_0}(\Omega)} = \infty. \tag{3.23}$$

We prove that (3.23) cannot happen.

We note that $u \in W^{1,p}(0,T;L^p(\Gamma_1))$, since $p \leqslant m$. Moreover, the potential operator $u \to \|u\|_{p,\Gamma_1}^p$ is Fréchet differentiable in $L^p(\Gamma_1)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{p,\Gamma_1}^p = p\int_{\Gamma_1}|u(t)|^{p-2}u(t)u_t(t), \tag{3.24}$$

for almost all $t \in (0,T)$, and the energy identity (1.8) can be written in the equivalent form

$$\tfrac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_{p,\Gamma_1}^p|_s^t + \int_s^t (\|u_t\|_2^2 + \|u_t\|_{m,\Gamma_1}^m) = 0 \tag{3.25}$$

for $0 \leqslant s \leqslant t < T_{\max}$. We now introduce the auxiliary functional

$$\mathcal{H}(t) = \tfrac{1}{2}\|\nabla u\|_2^2 + \frac{1}{p}\|u\|_{p,\Gamma_1}^p. \tag{3.26}$$

By (3.24) and (3.25),

$$\mathcal{H}'(t) = -\|u_t\|_2^2 - \|u_t\|_{m,\Gamma_1}^m + 2\int_{\Gamma_1}|u|^{p-2}uu_t \leqslant -\|u_t\|_{m,\Gamma_1}^m + 2\int_{\Gamma_1}|u|^{p-2}uu_t. \tag{3.27}$$

Repeating verbatim the arguments of [27, proof of theorem 2, eqn (88) onward], we then prove the estimate

$$\mathcal{H}'(t) \leqslant C(1+\mathcal{H}(t)) \tag{3.28}$$

with a suitable $C(m,p,\Gamma_1) > 0$. By Gronwall's lemma, we have $\mathcal{H} \in L^\infty(0,T_{\max})$. Then, by (3.24),

$$\|u\|_{p,\Gamma_1}, \|\nabla u\|_2 \in L^\infty(0,T_{\max}). \tag{3.29}$$

Moreover, by (3.25) and (3.29),

$$\int_0^t \|u_t\|_2^2 \leqslant \tfrac{1}{2}\|\nabla u_0\|_2^2 + \frac{1}{p}\|u\|_{p,\Gamma_1}^p \leqslant \tfrac{1}{2}\|\nabla u_0\|_2^2 + \frac{1}{p}\|u\|_{L^\infty(0,T_{\max};L^p(\Gamma_1))}^p,$$

so that

$$\|u\|_2^2 \leqslant 2\|u_0\| + 2\int_0^t \|u_t\|_2^2 \leqslant 2\|u_0\|_2^2 + \|\nabla u_0\|_2^2 + \frac{2}{p}\|u\|_{L^\infty(0,T_{\max};L^p(\Gamma_1))}^p, \tag{3.30}$$

which, together with (3.29), contradicts (3.23), concluding the proof. □

We can now turn to the proof of Theorem 1.3. We first give an alternative and more explicit characterization of the set $W$ given in (1.13) and of the number $d$ defined in (1.12). Since $\sigma(\Gamma_0) > 0$, as recalled, Poincaré's inequality holds (see [28]) and we can take $\|\nabla u\|_2$ as an equivalent norm in $H^1_{\Gamma_0}(\Omega)$. Then, using the trace-Sobolev theorem, since $p < r$,

$$B_\infty := \sup_{u\in H^1_{\Gamma_0}(\Omega),u\not\equiv 0} \frac{\|u\|_{p,\Gamma_1}}{\|\nabla u\|_2} < \infty. \tag{3.31}$$

We set

$$\lambda_\infty = B_\infty^{-p/(p-2)} \quad \text{and} \quad E_\infty = \left(\frac{1}{2}-\frac{1}{p}\right)\lambda_\infty^2,$$

and

$$W_1 = \{u_0 \in H^1_{\Gamma_0}(\Omega) : \|\nabla u_0\|_2 < \lambda_\infty,\ J(u_0) < E_\infty\}.$$

We can then give the following result.

LEMMA 3.3. *Suppose $\sigma(\Gamma_0) > 0$ and let $d$ be the number and $W$ the set defined in (1.12) and (1.13), respectively. Then $E_\infty = d$ and $W = W_1$.*

The proof of lemma 3.3 is a straightforward modification of that of [27, lemma 3]. It is given, for the reader's convenience, in Appendix B.

REMARK 3.4. The number $d = E_\infty$ above is equal (see [26]) to the Mountain-Pass level associated to the elliptic problem

$$\begin{aligned} \Delta u &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_0, \\ \frac{\partial u}{\partial \nu} &= |u|^{p-2}u && \text{on } \Gamma_1, \end{aligned}$$

that is, to the number (see, for example, [3])

$$\inf_{\gamma\in\Lambda} \sup_{t\in[0,1]} J(\gamma(t)),$$

where

$$\Lambda = \{\gamma \in C([0,1]; H^1_{\Gamma_0}(\Omega)) : \gamma(0) = 0,\ J(\gamma(1)) < 0\}.$$

*Proof of theorem 1.3.* By (3.25), the function $t \mapsto J(u(t))$ is decreasing. Then

$$J(u(t)) \leqslant J(u_0) \quad \text{for } t < T_{\max}. \tag{3.32}$$

As in the proof of theorem 1.2, using theorem 1.1, it is enough to prove that (3.23) leads to a contradiction.

By lemma 3.3, we have

$$\|\nabla u_0\|_2 < \lambda_\infty \quad \text{and} \quad J(u_0) < d = E_\infty. \tag{3.33}$$

By (3.31),

$$J(u(t)) \geqslant \tfrac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}B_\infty^p \|\nabla u\|_2^p := h(\|\nabla u\|_2). \tag{3.34}$$

Starting from (3.33) and (3.34) and repeating verbatim the arguments of [27, proof of theorem 3, eqn (98) onward] (with $E(t) = J(u(t))$), we get

$$\|\nabla u(t)\|_2 < \lambda_\infty \quad \text{for all } t \in [0, T_{\max}). \tag{3.35}$$

Still using Poincaré's inequality, this contradicts (3.23), concluding the proof. □

REMARK 3.5. To prove theorem 1.3, we used, in an essential way, the statement of theorem 1.1, that is, local existence for solutions of (1.2). It is also possible to give a more direct proof of theorem 1.3, without using this argument. This alternative proof is simpler than the combination of the proofs of theorems 1.1 and 1.3, so should be considered preferable from an 'abstract' point of view. We refer the interested reader to the proof of theorem 4.10 in § 4, where, handling with more general equations, we follow this different approach. We gave this proof for the sake of brevity.

## 4. More general equations

This section is devoted to giving the extension of our results to problem (1.1), where $Q$ and $f$ satisfy some specific assumptions that generalize the specific behaviour of $|u_t|^{m-2}u_t$ and $|u|^{p-2}u$. As for the term $Q$, we suppose that there is a $\Theta > 0$ such that the following assumptions hold.

(Q1) $Q$ is a Carathéodory real function defined on $(0,\Theta) \times \Gamma_1 \times \mathbb{R}$ such that $Q(t,x,0) = 0$ for almost all $(t,x) \in (0,\Theta) \times \Gamma_1$, and there exists and exponent $m > 1$ and positive constants $c_1$, $c_2$, $c_3$ and $c_4$, possibly dependent on $\Theta$, such that

$$c_1|v|^{m-1} \leqslant |Q(t,x,v)| \leqslant c_2|v|^{m-1} \quad \text{when } |v| \geqslant 1$$

and

$$c_3|v|^{m-1} \leqslant |Q(t,x,v)| \leqslant c_4 \qquad \text{when } |v| \leqslant 1$$

for almost all $(t,x) \in (0,\Theta) \times \Gamma_1$ and all $v \in \mathbb{R}$.

(Q2) The function $Q(t,x,\cdot)$ is increasing for almost all $(t,x) \in (0,\Theta) \times \Gamma_1$.

In most (but not all) of our results, we also use a stronger version of (Q2), as follows.

(Q3) There is $c_5 > 0$ such that, when $m \geqslant 2$,

$$(Q(t,x,v) - Q(t,x,w))(v-w) \geqslant c_5|v-w|^m, \tag{4.1}$$

while, when $1 < m < 2$,

$$(Q(t,x,v) - Q(t,x,w))(v-w) \geqslant c_5\big||v|^{m-2}v - |w|^{m-2}w\big|^{m'} \tag{4.2}$$

for almost all $(t,x) \in (0,\Theta) \times \Gamma_1$ and all $v, w \in \mathbb{R}$.

Remark 4.1. Assumptions (Q1)–(Q3) are satisfied *for all* $\Theta > 0$ by any $Q = Q(v)$ such that

$$Q \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}), \qquad Q(0) = 0, \qquad \liminf_{v \to 0} \frac{|Q(v)|}{|v|^{m-1}} > 0, \tag{4.3}$$

$$\limsup_{|v| \to \infty} \frac{|Q(v)|}{|v|^{m-1}} < \infty, \qquad \liminf_{|v| \to \infty} \frac{|Q(v)|}{|v|^{m-1}} > 0 \tag{4.4}$$

and

$$Q'(v) \geqslant q_0|v|^{m-2} \quad \text{for almost all } v \in \mathbb{R} \tag{4.5}$$

for some positive constant $q_0$. This assertion can be easily checked using the elementary inequality

$$(|v_1|^{\ell-2}v_1 - |v_2|^{\ell-2}v_2)(v_1 - v_2) \geqslant \text{const.}|v_1 - v_2|^{\ell} \tag{4.6}$$

for all $\ell \geqslant 2$, $v_1, v_2 \in \mathbb{R}$, with $\ell = m$ when $m \geqslant 2$ and with $\ell = m'$ when $1 < m < 2$. Of course, the damping model term $|u_t|^{m-2}u_t$ considered in previous sections satisfies (4.3)–(4.5). Two explicit non-power-like examples satisfying these requirements are given by

$$Q_0(v) = \begin{cases} |v|^{m-2}v & \text{for } |v| \geqslant 1, \\ |v|^{\mu-2}v & \text{for } |v| \leqslant 1, \end{cases} \qquad 1 < \mu < m, \tag{4.7}$$

and by

$$Q_1(v) = |v|^{\mu-2}v + |v|^{m-2}v, \qquad 1 < \mu < m. \tag{4.8}$$

A time-dependent example satisfying (Q1)–(Q3) *for all* $\Theta > 0$ is given by

$$Q_2(t, v) = \beta(t)|v|^{m-2}v, \quad m > 1, \quad \beta \in C([0, \infty); \mathbb{R}^+). \tag{4.9}$$

Remark 4.2. Let us note that (Q1), (Q2) yield that there are positive constants $c_6$ and $c_7$ (possibly dependent on $\Theta$) such that

$$|Q(t, x, v)| \leqslant c_6(1 + |v|^{m-1}) \tag{4.10}$$

and

$$Q(t, x, v)v \geqslant c_7|v|^m \tag{4.11}$$

for almost all $(t, x) \in (0, \Theta) \times \Gamma_1$ and all $v \in \mathbb{R}$.

Concerning the term $f$, for the time being, we assume only that $f$ is a Carathéodory real function defined on $\Gamma_1 \times \mathbb{R}$. We shall give case by case the assumptions we need.

### 4.1. Forced heat equation

Our first result generalizes theorem 1.5 of § 2 to the problem

$$\left.\begin{aligned} u_t - \Delta u &= 0 && \text{in } (0, T) \times \Omega, \\ u &= 0 && \text{on } [0, T) \times \Gamma_0, \\ \frac{\partial u}{\partial \nu} &= -Q(t, x, u_t) + g(t, x) && \text{on } [0, T) \times \Gamma_1, \\ u(0, x) &= u_0(x) && \text{on } \Omega, \end{aligned}\right\} \tag{4.12}$$

where $g$ is a given forcing term acting on $\Gamma_1$ and $T > 0$ is fixed.

Theorem 4.3. *Suppose that (Q1), (Q2) hold with* $\Theta = T$ *and that*

$$g \in L^{m'}((0, T) \times \Gamma_1).$$

*Then, given any initial datum* $u_0 \in H^1_{\Gamma_0}(\Omega)$, *there is a unique weak solution* $u$ *of (4.12) on* $(0, T) \times \Omega$. *Moreover, (1.18) and (1.19) hold, and* $u$ *satisfies the energy identity*

$$\tfrac{1}{2}\|\nabla u\|_2^2\Big|_s^t + \int_s^t \|u_t\|_2^2 + \int_s^t \int_{\Gamma_1} Q(\cdot, \cdot, u_t)u_t = \int_s^t \int_{\Gamma_1} g u_t \tag{4.13}$$

*for* $0 \leqslant s \leqslant t \leqslant T$.

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REMARK 4.4. According to definition 2.1, a solution of (4.12) is a function

$$u \in L^\infty(0,T; H^1_{\Gamma_0}(\Omega))$$

such that $u_t \in L^2((0,T)\times\Omega) \cap L^m((0,T)\times\Gamma_1)$, (2.2) is verified with $|u_t|^{m-2}u_t$ replaced by $Q(\cdot,\cdot,u_t)$ and $u(0)=u_0$.

*Sketch of the proof of theorem 4.3.* Repeat the proof of theorem 1.5, using (4.10), (4.11) and (Q2), starting from the problem

$$\left.\begin{aligned}(u^k_t, w_j) + (\nabla u^k, \nabla w_j) + \int_{\Gamma_1} Q(\cdot,\cdot,u^k_t)w_j = \int_{\Gamma_1} g w_j, \quad j=1,\dots,k,\\ u^k(0) = u_{0k},\end{aligned}\right\} \tag{4.14}$$

instead of from (2.11), and correspondingly redefining

$$G_k(y) = y + \int_{\Gamma_1} Q(\cdot,\cdot,B_k(x)\cdot y)B_k(x)\,\mathrm{d}x, \quad y\in\mathbb{R}^k, \tag{4.15}$$

and

$$\mathcal{G}_k(y) = \tfrac{1}{2}|y|^2 + \int_{\Gamma_1} \tilde{Q}(\cdot,\cdot,B_k(x)\cdot y)\,\mathrm{d}x, \tag{4.16}$$

where

$$\tilde{Q}(t,x,y) := \int_0^y Q(t,x,s)\,\mathrm{d}s, \quad y\in\mathbb{R}. \tag{4.17}$$

□

### 4.2. Local existence

To generalize theorem 1.1, we assume that the following assumption holds.

(F1) $f(x,0)=0$ and there are exponents $1<q<2\leqslant p$ and $c_8>0$ such that, for almost all $x\in\Gamma_1$ and all $u_1,u_2\in\mathbb{R}$,

$$|f(x,u_1)-f(x,u_2)| \leqslant c_8[|u_1-u_2|(1+|u_1|^{p-2}+|u_2|^{p-2}) + |u_1-u_2|^{q-1}].$$

Assumption (F1) is clearly satisfied by

$$f_0(x,u) = \gamma(x)|u|^{q-2}u + \delta(x)|u|^{p-2}u, \quad 1<q<p, \tag{4.18}$$

provided that $\gamma,\delta\in L^\infty(\Gamma_1)$. This can be seen using the elementary inequality

$$||u_1|^{p-2}u_1 - |u_2|^{p-2}u_2| \leqslant \text{const.}|u_1-u_2|(|u_1|^{p-2}+|u_2|^{p-2}) \tag{4.19}$$

for $u_1,u_2\in\mathbb{R}$, $p\geqslant 2$.

Moreover, (F1) is satisfied by any $f=f(u)$, derivable for large $|u|$, such that

$$f\in C^{0,\alpha}(\mathbb{R}), \qquad f(0)=0, \qquad |f'| = O(|u^{p-2}) \quad \text{as } |u|\to\infty, \tag{4.20}$$

where $0<\alpha\leqslant 1$, $p\geqslant 2$ and $C^{0,\alpha}(\mathbb{R})$ denotes the space of $\alpha$-Hölder continuous real functions on $\mathbb{R}$. An explicit non-algebraic example of such a function is given by

$$f_1(u) = \begin{cases}|u|^{\nu_1-2}u, & |u|\geqslant 1,\\ |u|^{\nu_2-2}u, & |u|\leqslant 1,\end{cases} \qquad \nu_1,\nu_2>1, \tag{4.21}$$

with $\alpha=\min\{1,\nu_1,\nu_2\}$ and $p=\max\{2,\nu_1,\nu_2\}$.

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REMARK 4.5. We remark that an immediate consequence of (F1) is that

$$|f(x,u)| \leqslant c_9(1+|u|^{p-1}) \tag{4.22}$$

for almost all $x \in \Gamma_1$ and all $u \in \mathbb{R}$, for some constant $c_9 > 0$.

We can now state the following result.

THEOREM 4.6. *Suppose that (Q1), (Q3) and (F1) hold, together with (1.5) when* $n \geqslant 3$. *Then, given any initial datum* $u_0 \in H^1_{\Gamma_0}(\Omega)$, *there is* $T > 0$ *and a weak solution* $u$ *of (1.1) on* $(0,T)\times\Omega$. *Moreover, equations (1.18) and (1.19) hold, together with the energy identity*

$$\tfrac{1}{2}\|\nabla u\|_2^2\big|_s^t + \int_s^t \|u_t\|_2^2 + \int_s^t\int_{\Gamma_1} Q(\cdot,\cdot,u_t)u_t = \int_s^t\int_{\Gamma_1} f(\cdot,u)u_t \tag{4.23}$$

*for* $0 \leqslant s \leqslant t \leqslant T$. *Moreover,*

$$T = T(\|u_0\|_{H^1_{\Gamma_0}(\Omega)}, m, p, \Omega, \Gamma_1, \Theta, c_7, c_9)$$

*is decreasing in the first variable and increasing in* $c_7$.

*Sketch of the proof.* We repeat the proof of theorem 1.1, where $v = \Phi(u)$ is now the unique solution of

$$\left.\begin{aligned} v_t - \Delta v &= 0 && \text{in } (0,T)\times\Omega,\\ v &= 0 && \text{on } [0,T)\times\Gamma_0,\\ \frac{\partial v}{\partial\nu} &= -Q(t,x,v_t) + f(x,u) && \text{on } [0,T)\times\Gamma_1,\\ v(0,x) &= u_0(x) && \text{in } \Omega \end{aligned}\right\} \tag{4.24}$$

given by theorem 4.3. We proceed exactly as in the proof of theorem 1.1, with the estimate (3.3) in step 1 being replaced by

$$\begin{aligned} \tfrac{1}{2}\|\nabla v(t)\|_2^2 &+ \int_0^t \|u_t\|_2^2 + c_7\int_0^t \|v_t\|_{m,\Gamma_1}^m \\ &\leqslant \tfrac{1}{2}\|\nabla u_0\|_2^2 + c_9\int_0^T\int_{\Gamma_1}(1+|u|^{p-1})|v_t| \\ &\leqslant \tfrac{1}{2}\|\nabla u_0\|_2^2 + 2^{m'-1}c_9\int_0^T\biggl(\int_{\Gamma_1} 1+|u|^{(p-1)m'}\biggr)^{1/m'}\|v_t\|_{m,\Gamma_1} \end{aligned} \tag{4.25}$$

and the estimate (3.4) by

$$\begin{aligned} \tfrac{1}{2}\|\nabla v(t)\|_2^2 + \int_0^t \|u_t\|_2^2 + c_7\int_0^t \|v_t\|_{m,\Gamma_1}^m \\ \leqslant \tfrac{1}{2}\|\nabla u_0\|_2^2 + k_1(1+R^{p-1})T^{1/m'}\|v_t\|_{L^m((0,T)\times\Gamma_1)}, \end{aligned} \tag{4.26}$$

where

$$k_1 = k_1(p,m,\Gamma_1,\Omega,c_5,c_7,c_9) > 0.$$

Taking $R \geqslant 1$, we can estimate $1+R^{p-1} \leqslant 2R^{p-1}$ and then complete the first step as in the proof of theorem 1.1, where now $T = T(R_0^2, m, p, \Omega, \Gamma_1, \Theta, c_7, c_9) \leqslant \Theta$. In this procedure, in particular, equation (3.6) is still obtained, where $k_4$ also depends on $c_5$, $c_7$ and $c_9$. The form of (4.11) immediately yields that $T$ can be taken increasing in $c_7$.

Repeating step 2, the energy identity (3.17) becomes

$$\tfrac{1}{2}\|\nabla w\|_2^2 + \int_0^t \|u_t\|_2^2 + \int_0^t \int_{\Gamma_1} [Q(\cdot,\cdot,v_t) - Q(\cdot,\cdot,\bar{v}_t)]w_t = \int_0^t \int_{\Gamma_1} [f(\cdot,u) - f(\cdot,\bar{u})]w_t$$

for $0 \leqslant t \leqslant T$. Estimating from above the right-hand side and from below the third term in the left-hand side of the last formula, exactly as in [27, proof of theorem 6], we get

$$\|v_t - \bar{v}_t\|_{L^2((0,T)\times\Omega)}^2 \leqslant K(\|u-\bar{u}\|_{L^\infty(0,T;L^{r_0}(\Gamma_1))} + \|u-\bar{u}\|_{L^\infty(0,T;L^{r_0}(\Gamma_1))}^{q-1})$$

$$\|\nabla v - \nabla\bar{v}\|_{L^\infty(0,T;L^2(\Omega))}^2 \leqslant K(\|u-\bar{u}\|_{L^\infty(0,T;L^{r_0}(\Gamma_1))} + \|u-\bar{u}\|_{L^\infty(0,T;L^{r_0}(\Gamma_1))}^{q-1})$$

and

$$\|v_t - \bar{v}_t\|_{L^m((0,T)\times\Gamma_1)}^m \leqslant K(\|u-\bar{u}\|_{L^\infty(0,T;L^{r_0}(\Gamma_1))} + \|u-\bar{u}\|_{L^\infty(0,T;L^{r_0}(\Gamma_1))}^{q-1})$$

when $m \geqslant 2$, while

$$\begin{aligned}\||v_t|^{m-2}v_t - |\bar{v}_t|^{m-2}\bar{v}_t\|_{L^{m'}((0,T)\times\Gamma_1)}^{m'} \\ \leqslant K(\|u-\bar{u}\|_{L^\infty(0,T;L^{r_0}(\Gamma_1))} + \|u-\bar{u}\|_{L^\infty(0,T;L^{r_0}(\Gamma_1))}^{q-1})\end{aligned} \quad (4.27)$$

when $1 < m < 2$, where $K = K(p, m, \Gamma_1, \Omega, T, R, c_5, c_7, c_9) > 0$. Starting from these estimates, which generalize (3.18)–(3.21), the proof can be completed exactly as for theorem 1.1. □

### 4.3. Global existence for arbitrary initial data

Theorem 1.2 is generalized as follows.

THEOREM 4.7. *Suppose that (Q1) and (Q3) hold for all $\Theta > 0$ and that $f$ satisfies (F1). Furthermore, suppose that the following assumption holds.*

(F2) *There exist $c_{10} > 0$ and $\kappa \geqslant 0$ such that*

$$F(x,u) \geqslant c_{10}|u|^p - \kappa$$

*for almost all $x \in \Gamma_1$ and all $u \in \mathbb{R}$, where*

$$F(x,u) = \int_0^u f(x,\eta)\,\mathrm{d}\eta. \quad (4.28)$$

*Finally, suppose that*

$$2 \leqslant p \leqslant m \quad \text{and} \quad p < r.$$

*Then any solution given by theorem 4.6 can be extended to the whole of $(0,\infty)\times\Omega$.*

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REMARK 4.8. It is worth noting that $f_0$ defined in (4.18) satisfies (F1) and (F2), provided that $\gamma, \delta \in L^\infty(\Gamma_1)$ and $\inf_{\Gamma_1}\delta > 0$. The same assumption are verified by any $f = f(u)$, derivable for large $u$, such that

$$f \in C^{0,\alpha}_{\rm loc}(\mathbb{R}), \qquad f(0) = 0, \qquad f'(u) \sim \tilde{c}|u|^{p-2} \quad \text{as } |u| \to \infty, \tag{4.29}$$

with $\tilde{c} > 0$, as, for example, the function $f_1$ defined in (4.21).

*Sketch of the proof of theorem 4.7.* At first, we apply the standard continuation procedure of ordinary differential equations to conclude that either the solution $u$ is global or there is $T_{\max} < \infty$ such that (3.23) holds. Let $T_{\max}$ be the right endpoint of the maximal interval of existence. Suppose, by contradiction, that

$$T_{\max} < \infty \quad \text{and} \quad \varliminf_{t \to T_{\max}^-} \|u(t)\|_{H^1_{\Gamma_0}(\Omega)} < \infty.$$

Then there is a sequence $T_n \to T_{\max}^-$ with $\|u(T_n)\|_{H^1_{\Gamma_0}(\Omega)}$ bounded. For all $n \in \mathbb{N}$, the time-translated damping term $Q(t - T_n, \cdot, \cdot)$ satisfies (Q1) and (Q3), with $c_1$, $c_3$ (and then $c_7$) independent on $n$ (it is enough to take $c_i = c_i(T_{\max})$). Hence theorem 4.6 can be applied to the Cauchy problem with initial time $T_n$, and the length $T'_n$ of the maximal interval of existence $[T_n, T_n + T'_n)$ is independent on $n$. This leads to a contradiction, since, in this way, we can continue the solution to the right of $T_{\max}$.

To prove that (3.23) cannot occur, we generalize the arguments given in the proof of theorem 1.2. The energy identity can now be written as

$$\tfrac{1}{2}\|\nabla u\|_2^2 - \int_{\Gamma_1} F(\cdot, u)\Big|_s^t + \int_s^t \|u_t\|_2^2 + \int_s^t \int_{\Gamma_1} Q(\cdot, \cdot, u_t)u_t = 0 \tag{4.30}$$

for $0 \leqslant s \leqslant t < T_{\max}$. We now introduce the modified auxiliary functional

$$\mathcal{H}(t) = \tfrac{1}{2}\|\nabla u\|_2^2 + \int_{\Gamma_1} F(\cdot, u) + \kappa\sigma(\Gamma_1). \tag{4.31}$$

Clearly,

$$\begin{aligned}\mathcal{H}'(t) &= -\|u_t\|_2^2 - \int_{\Gamma_1} Q(\cdot, \cdot, u_t)u_t + 2\int_{\Gamma_1} f(\cdot, u)u_t \\ &\leqslant -\int_{\Gamma_1} Q(\cdot, \cdot, u_t)u_t + 2\int_{\Gamma_1} f(\cdot, u)u_t.\end{aligned}$$

Then, arguing as in the proof of [27, theorem 7], we prove that $\mathcal{H} \in L^\infty(0, T_{\max})$. The proof can be completed as for theorem 1.2. □

### 4.4. Global existence in the potential well

It would be possible to generalize theorem 1.3 to problem (1.1) using theorem 4.6 together with a straightforward generalization of the arguments used in the proof of theorem 1.3. This approach would have the disadvantage that the nonlinearities $f$ and $Q$ have to verify assumptions (Q1), (Q3) and (F1) (together with assumption (F3) given later). The proof of this fact is left to the interested reader, since we are giving in the sequel a more general result, using a more direct approach inspired by [18] (see also remark 3.5).

In order to generalize the potential well arguments, we need $Q$ to verify (Q1), (Q2) and $f$ to verify (4.22), together with the following assumption on its primitive $F$ (see (4.28)).

(F3) There exists $c_{11} > 0$ such that

$$F(x,u) \leqslant \frac{c_{11}}{p}|u|^p$$

for almost all $x \in \Gamma_1$ and all $u \in \mathbb{R}$.

All examples of damping terms $Q$ given in remark 4.1 clearly satisfy (Q1), (Q2) for all $\Theta > 0$. Moreover, a further example is given by any $Q = Q(v)$ such that

$$Q \in C(\mathbb{R}), \qquad Q(0) = 0, \qquad \liminf_{v\to 0} \frac{|Q(v)|}{|v|^{m-1}} > 0, \tag{4.32}$$

$$\limsup_{|v|\to\infty} \frac{|Q(v)|}{|v|^{m-1}} < \infty, \qquad \liminf_{|v|\to\infty} \frac{|Q(v)|}{|v|^{m-1}} > 0 \tag{4.33}$$

and

$$Q \text{ is increasing.} \tag{4.34}$$

Examples of functions $Q$ satisfying (4.32)–(4.34) but not (4.3)–(4.5) are given by

$$Q_3(v) = |v + \sin v|^{m-2}(v + \sin v), \qquad m > 1$$

and

$$Q_4(v) = \begin{cases} |v|^{\mu-2}v & \text{for } |v| \leqslant 1, \\ \operatorname{sgn} v & \text{for } 1 \leqslant |v| \leqslant 1, \\ (|v|-1)^{\mu-2}(|v|-1)\operatorname{sgn} v & \text{for } |v| \geqslant 2, \end{cases} \qquad 1 < \mu < m.$$

Examples of nonlinearities $f$ satisfying (F3) and (4.22) are given by $f_0$ defined in (4.18), provided that $\gamma, \delta \in L^\infty(\Gamma_1)$ and $\gamma \leqslant 0$ almost everywhere on $\Gamma_1$. In particular, when $\gamma = -1$ and $\delta = 1$, we can consider $f_2(u) = |u|^{p-2} - |u|^{q-2}u$, $1 < q < p$. A further example is given by any $f = f(u)$, $f \in C(\mathbb{R})$, $|f(u)| = O(|u|^{p-1})$ as $|u| \to \infty$ and $f(u)u \leqslant \text{const.}|u|^p$ for small $u$. In particular, $f_3(u) = \sin(\mathrm{e}^{|u|})|u|^{p-2}u$, $p \geqslant 2$, satisfies (F3) and (4.22), but not (F1), as can be easily seen. Examples $Q_3$, $Q_4$ and $f_3$ motivate the different approaches we are using to handle with (1.1).

To extend theorem 1.3, we first need to suitably modify the definition of the stable set $W$ given in (1.13). We shall use the characterization of $W$ given in § 3. With this aim, we note that, by (3.31) and (F3), when $\sigma(\Gamma_0) > 0$,

$$K_\infty := \sup_{u \in H^1_{\Gamma_0}(\Omega), u \neq 0} \frac{\int_{\Gamma_1} F(\cdot, u)}{\|\nabla u\|_2^p} \leqslant \frac{c_{11}}{p} B_\infty^p. \tag{4.35}$$

We set

$$\lambda_\infty = (pK_\infty)^{-1/(p-2)}, \qquad E_\infty = \left(\frac{1}{2} - \frac{1}{p}\right)\lambda_\infty^2$$

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if $K_1 > 0$ and $\lambda_\infty = E_\infty = +\infty$ if $K_1 \leqslant 0$. We define

$$W = \{u_0 \in H^1_{\Gamma_0}(\Omega) : \|\nabla u_0\|_2 < \lambda_\infty \text{ and } J(u_0) < E_\infty\}.$$

Remark 4.9. Clearly, when $f(x,u) = |u|^{p-2}u$, the above definitions of $\lambda_\infty$, $E_\infty$ and $W$ are in agreement with those given in § 3. Moreover, when $f(x,u) = \tilde\sigma(x)|u|^{p-2}u$, $\tilde\sigma \in L^\infty(\Gamma_1)$, $\tilde\sigma \geqslant 0$, it can be shown as in [26] that $E_\infty$ is the Mountain-Pass level associated to the elliptic problem

$$\begin{aligned} \Delta u = 0 &\quad \text{in } \Omega,\\ u = 0 &\quad \text{on } \Gamma_0,\\ \frac{\partial u}{\partial \nu} = \tilde\sigma |u|^{p-2}u &\quad \text{on } \Gamma_1. \end{aligned}$$

This would not be true if we had kept the definition of $E_\infty$ given in § 3, which is not optimal in this more general case. The last example is $f \equiv 0$, so that $\lambda_\infty = E_\infty = +\infty$ and $W = H^1_{\Gamma_0}(\Omega)$, as expected, since, in this case, there are no source terms in the equation.

We can now state the following result.

Theorem 4.10. *Let (Q1) and (Q2) hold for any $\Theta > 0$ and let (F3), (4.22) hold. Suppose that $m > 1$,*

$$2 \leqslant p < r \quad \text{and} \quad m > \frac{r}{r+1-p}$$

*and that $\sigma(\Gamma_0) > 0$. Then, given any initial datum $u_0 \in W$, there is a global weak solution of (1.1) on $(0,\infty)\times\Omega$. Moreover, $u(t) \in W$ for all $t \geqslant 0$,*

$$u \in C([0,\infty); H^1_{\Gamma_0}(\Omega)), \qquad u_t \in L^2((0,\infty)\times\Omega), \qquad u_t \in L^m_{\mathrm{loc}}([0,\infty)\times\Gamma_1)$$

*and the energy identity (4.23) holds. Finally, if $c_1$ and $c_3$ can be taken independent on $\Theta$, then $u_t \in L^m((0,\infty)\times\Gamma_1)$, and if $c_2$ and $c_4$ can also be taken independent on $\Theta$, then $Q(\cdot,\cdot,u_t) \in L^{m'}((0,\infty)\times\Gamma_1)$.*

Remark 4.11. Of course, the constants $c_i$, $i = 1,\dots,4$, are independent on $\Theta$ when $Q$ is time independent. Concerning the only time-dependent example we gave in this section, i.e. $Q_2$ (see (4.9)), clearly $c_1$ and $c_3$ are independent on $\Theta$ if and only if $\underline{\lim}_{t\to\infty}\beta(t) > 0$, while $c_2$ and $c_4$ are independent on $\Theta$ if and only if $\overline{\lim}_{t\to\infty}\beta(t) < \infty$.

*Sketch of the proof of theorem 4.10.* Let $X$ and $(w_k)_k$ and $u_{0k}$ be as in the proof of theorem 1.5. We apply the Faedo–Galerkin procedure, so, with fixed $k \in \mathbb{N}$, we consider the problem

$$\left.\begin{aligned} (u^k_t, w_j) + (\nabla u^k, \nabla w_j) + \int_{\Gamma_1} Q(\cdot,\cdot,u^k_t) w_j = \int_{\Gamma_1} f(\cdot,u^k) w_j,&\\ j = 1,\dots,k,&\\ u^k(0) = u_{0k}.& \end{aligned}\right\} \tag{4.36}$$

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To conclude that (4.36) has a local solution

$$u_k(t) = \sum_{j=1}^{k} y_k^j(t) w_j$$

for some $y_j^k \in W^{1,1}(0, t_k)$, $j = 1, \dots, k$, and $t_k > 0$, we use the arguments of theorem 1.5. In this process, equations (2.13), (2.14) and (2.15) are replaced by

$$\left.\begin{aligned} G_k(y_k'(t)) + A_k y_k(t) = \mathcal{F}_k(y_k(t)), \\ y_k(0) = y_{0k}, \end{aligned}\right\} \tag{4.37}$$

$$\left.\begin{aligned} &y_k'(t) = G_k^{-1}(\mathcal{F}(y_k(t)) - A_k y_k(t)), \\ &y_k(0) = y_{0k} \end{aligned}\right\} \tag{4.38}$$

and

$$|G_k^{-1}(\mathcal{F}_k(y_k) - A_k y_k)| \leqslant |\mathcal{F}_k(y_k)| + \|A_k\| |y_k|, \tag{4.39}$$

respectively, where $G_k$ is defined in (4.15) and

$$\mathcal{F}_k(y) = \int_{\Gamma_1} f(\cdot, B_k(x) \cdot y) B_k(x) \, \mathrm{d}x, \quad y \in \mathbb{R}^k,$$

is continuous in $\mathbb{R}^k$ (use (4.22) and the fact that $w_j \in L^p(\Gamma_1)$, $j = 1, \dots, k$).

The corresponding energy function and the energy identity associated to (4.36) are

$$E^k(t) = \tfrac{1}{2} \|\nabla u^k(t)\|_2^2 - \int_{\Gamma_1} F(\cdot, u^k(t)) \tag{4.40}$$

and

$$E^k(t) - E^k(s) = -\int_s^t \|u_t^k\|_2^2 - \int_s^t \int_{\Gamma_1} Q(\cdot, \cdot, u_t^k) u_t^k \leqslant 0, \tag{4.41}$$

respectively, for $0 \leqslant s \leqslant t < t_k$. We note that, by integrating (4.22), we easily see that

$$|F(x, u)| \leqslant c_{12}(1 + |u|^p) \tag{4.42}$$

for almost all $x \in \Gamma_1$, all $u \in \mathbb{R}$ and $c_{12} = c_{12}(p, c_9) > 0$.

Arguing exactly as in [27, proof of theorem 8], we then prove the following estimates,

$$\left.\begin{aligned} \|\nabla u^k\|_2 &\leqslant C_0, \\ \|u_t^k\|_{L^2((0,t_k)\times\Omega)} &\leqslant C_0, \\ \|u_t^k\|_{L^m((0,t_k)\times\Gamma_1)} &\leqslant C_0, \\ \|Q(\cdot, \cdot, u_t^k)\|_{L^{m'}((0,t_k)\times\Gamma_1)} &\leqslant C_0, \end{aligned}\right\} \tag{4.43}$$

for some $C_0 = C_0(u_0, m, p, \Gamma_1, c_6(T), c_7(T), c_{12}) > 0$ and for $k \geqslant \bar{k}$, with $\bar{k}$ sufficiently large (here, $c_6$ and $c_7$ are the constant appearing in (4.10) and (4.11) for $\Theta = T$).

Starting from estimates (4.43), and using lemma 3.2 with $p_0 = p$, a standard argument yields that $t_k = T$, $k \geqslant \bar{k}$ and that, up to a subsequence,

$$\begin{aligned} u_k \to u &\quad \text{weakly}^* \text{ in } L^\infty(0,T;H^1_{\Gamma_0}(\Omega)) \text{ and strongly in } C([0,T];L^p(\Gamma_1)),\\ u_t^k \to u_t &\quad \text{weakly in } L^2((0,T)\times\Omega) \text{ and in } L^m((0,T)\times\Gamma_1),\\ Q(\cdot,\cdot,u_t^k) \to \chi &\quad \text{weakly in } L^{m'}((0,T)\times\Gamma_1). \end{aligned}$$

Arguing as in the proof of theorem 1.5, we use lemma 2.2 to obtain the energy identity (which is possible since (1.5) and (4.22) yield $f(\cdot,u) \in L^{m'}((0,T)\times\Gamma_1)$) and (Q2) to prove that $\chi = Q(\cdot,\cdot,u_t)$. This yields the existence of a solution on $(0,T)\times\Omega$ for all $T > 0$. Since the same argument can be applied for the Cauchy problem with any initial time, the solution can be extended to the whole of $(0,\infty)\times\Omega$.

Finally, by the energy identity and (F3), we obtain, for all $t > 0$, that

$$\begin{aligned} \int_0^t \|u_t\|_2^2 + \int_0^t\int_{\Gamma_1} Q(\cdot,\cdot,u_t)u_t &\leqslant \tfrac{1}{2}\|\nabla u_0\|_2^2 + \int_{\Gamma_1} F(\cdot,u) - \int_{\Gamma_1} F(\cdot,u_0)\\ &\leqslant \tfrac{1}{2}\|\nabla u_0\|_2^2 - \int_{\Gamma_1} F(\cdot,u_0) + \frac{c_{11}}{p}\|u\|_{p,\Gamma_1}^p, \end{aligned}$$

and then, arguing as in the proof of theorem 1.3, we prove that $\|u\|_{p,\Gamma_1} \in L^\infty(0,\infty)$, so that we have $u_t \in L^2((0,\infty)\times\Omega)$ and $Q(\cdot,\cdot,u_t)u_t \in L^1((0,\infty)\times\Gamma_1)$. When $c_1$, $c_3$ (and then $c_7$) can be taken independent on $\Theta$, we then have $u_t \in L^m((0,\infty)\times\Gamma_1)$, and if $c_2$, $c_4$ (and then $c_6$) can also be taken independent on $\Theta$, we also obtain that $Q(\cdot,\cdot,u_t) \in L^{m'}((0,\infty)\times\Gamma_1)$, completing the proof. □

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## Appendix A. Proof of lemma 2.2

Denote

$$H = L^2(\Omega), \qquad V = H^1_{\Gamma_0}(\Omega), \qquad W = L^m(\Gamma_1), \qquad X = \{u \in V : u|_{\Gamma_1} \in W\}.$$

Since $V$ is dense in $H$, using [25, theorem 2.1] and (2.5), (2.6), we obtain that

$$u \in C_{\mathrm{w}}([0,T];V). \tag{A 1}$$

The key point is to show that (2.9) holds. With this aim and fixed $0 \leqslant s \leqslant t \leqslant T$, we set $\theta_0$ to be the characteristic function of the interval $[s,t]$. For small $\delta > 0$, let $\theta(\tau) = \theta_\delta(\tau)$ be 1 for $\tau \in [s+\delta, t-\delta]$, zero for $\tau \notin (s,t)$ and linear in the intervals $[s,s+\delta]$ and $[t-s,t]$. Next let $\eta_\varepsilon$ be a standard mollifying sequence, that is, $\eta = \eta_\varepsilon \in C^\infty(\mathbb{R})$, $\operatorname{supp}\eta_\varepsilon \subset (-\varepsilon,\varepsilon)$, $\int_{-\infty}^{+\infty}\eta_\varepsilon = 1$, $\eta_\varepsilon$ even and non-negative, and $\eta_\varepsilon = \varepsilon^{-1}\eta_1(\tau/\varepsilon)$. Let $*$ denote time convolution.

We approximate $u$, extended as zero outside $[0,T]$, with $v=\eta*(\theta u)\in C_{\mathrm{c}}^{\infty}(\mathbb{R};V)$. Then

$$0=\int_{-\infty}^{+\infty}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v\|_2^2=\int_{-\infty}^{+\infty}(\nabla v,\nabla v_t). \tag{A 2}$$

Using standard convolution properties and the Leibnitz rule, we see that

$$v_t=\eta*(\theta' u)+\eta*(\theta u_t)\quad \text{in } H,$$

so that $\eta*(\theta u_t)\in C_{\mathrm{c}}^{\infty}(\mathbb{R};V)$. Then, by (A 2),

$$0=\int_{-\infty}^{+\infty}(\eta*(\theta\nabla u),\eta*(\theta'\nabla u))+\int_{-\infty}^{+\infty}(\eta*(\theta\nabla u),\nabla(\eta*(\theta u_t))). \tag{A 3}$$

Using (2.6) we can take $\phi=\eta*\eta*(\theta u_t)$ in (2.7). Then, multiplying by $\theta$, integrating from $-\infty$ to $\infty$ and using standard properties of convolution, we can evaluate the second term in (A 3) in the following way:

$$\int_{-\infty}^{+\infty}(\eta*(\theta\nabla u),\nabla(\eta*(\theta u_t)))=\int_{-\infty}^{+\infty}\int_{\Gamma_1}\eta*(\theta\zeta)\eta*(\theta u_t)-\int_{-\infty}^{+\infty}\|\eta*(\theta u_t)\|_2^2. \tag{A 4}$$

Combining (A 3) and (A 4), and recalling that $\theta=\theta_\delta$, we obtain the first approximate energy identity,

$$\begin{aligned}0&=\int_{-\infty}^{+\infty}(\eta*(\theta_\delta\nabla u),\eta*(\theta_\delta'\nabla u))\\&\quad-\int_{-\infty}^{+\infty}\|\eta*(\theta_\delta u_t)\|_2^2+\int_{-\infty}^{+\infty}\int_{\Gamma_1}\eta*(\theta_\delta\zeta)\eta*(\theta_\delta u_t)\\&:=I_1+I_2+I_3.\end{aligned} \tag{A 5}$$

Now we examine each term in (A 5) separately as $\delta\to 0$ and $\varepsilon$ (i.e. $\eta$) is fixed. Since $\theta_\delta\to\theta_0$ a.e., we have

$$\|\eta*(\theta_\delta\zeta)\|_{m',\Gamma_1}\leqslant\|\zeta\|_{m',\Gamma_1},$$
$$\|\eta*(\theta_\delta u_t)\|_{m,\Gamma_1}\leqslant\|u_t\|_{m,\Gamma_1}$$

and

$$\|\eta*(\theta_\delta u_t)\|_2\leqslant\|u_t\|_2.$$

Using (2.3), (2.6) and Lebesgue's dominated converge theorem together with the last estimates, we have

$$I_2\to\int_{-\infty}^{+\infty}\|\eta*(\theta_0 u_t)\|_2^2 \tag{A 6}$$

and

$$I_3\to\int_{-\infty}^{+\infty}\int_{\Gamma_1}\eta*(\theta_0\zeta)\eta*(\theta_0 u_t). \tag{A 7}$$

Next we decompose the term $I_1$ as

$$I_1 = \int_{-\infty}^{+\infty} (\eta * (\theta_0 \nabla u), \eta * (\theta'_\delta \nabla u)) + \int_{-\infty}^{+\infty} (\eta * [(\theta_\delta - \theta_0)\nabla u], \eta * (\theta'_\delta \nabla u)) \quad \text{(A 8)}$$
$$:= I_4 + I_5.$$

Since $\theta_\delta \to \theta_0$ in $L^1(\mathbb{R})$, by (2.5), we have that $\eta * [(\theta_\delta - \theta_0)\nabla u] \to 0$ strongly in $L^\infty(0, T; H)$. Moreover, by (2.5),

$$\begin{aligned}\|\eta * (\theta'_\delta \nabla u)\|_{L^1(0,T;H)} &\leqslant \|\theta'_\delta\|_{L^1(\mathbb{R})} \|\eta\|_{L^\infty(\mathbb{R})} \|\nabla u\|_{L^\infty(0,T;H)} \\ &\leqslant 2\|\eta\|_{L^\infty(\mathbb{R})} \|\nabla u\|_{L^\infty(0,T;H)},\end{aligned}$$

so that

$$I_5 \to 0 \quad \text{as } \delta \to 0. \quad \text{(A 9)}$$

Next we note that, by the properties of convolution and the specific form of $\theta_\delta$,

$$\begin{aligned} I_4 &= \int_{-\infty}^{+\infty} \theta'_\delta (\eta * \eta * (\theta_0 \nabla u), \nabla u) \\ &= \frac{1}{\delta} \int_s^{s+\delta} (\eta * \eta * (\theta_0 \nabla u), \nabla u) - \frac{1}{\delta} \int_{t-\delta}^{t} (\eta * \eta * (\theta_0 \nabla u), \nabla u). \end{aligned}$$

By (A 1), the function $(\eta * \eta * (\theta_0 \nabla u), \nabla u)$ is continuous, so

$$I_4 \to (\eta * \eta * (\theta_0 \nabla u)(s), \nabla u(s)) - (\eta * \eta * (\theta_0 \nabla u)(t), \nabla u(t)) \quad \text{(A 10)}$$

as $\delta \to 0$.

Combining the convergences (A 7)–(A 10), recalling that $\eta = \eta_\varepsilon$ and letting $\rho_\varepsilon = \eta_\varepsilon * \eta_\varepsilon$, we obtain the second approximate energy identity

$$(\rho_\varepsilon * (\theta_0 \nabla u), \nabla u) = \int_{-\infty}^{+\infty} \int_{\Gamma_1} \eta_\varepsilon * (\theta_0 \zeta) \eta_\varepsilon * (\theta_0 u_t) - \int_{-\infty}^{+\infty} \|\eta_\varepsilon * (\theta_0 u_t)\|_2^2. \quad \text{(A 11)}$$

Now we consider the convergence of the two sides of (A 11) as $\varepsilon \to 0$. By standard arguments, using (2.6), $\rho_\varepsilon * (\theta_0 u_t) \to \theta_0 u_t$ strongly in $L^m((0, T) \times \Gamma_1)$ and in $L^2((0, T) \times \Omega)$, so that, using (2.3), the right-hand side of (A 11) tends to

$$\int_{-\infty}^{+\infty} \int_{\Gamma_1} \theta_0^2 \zeta u_t - \int_{-\infty}^{+\infty} \|\theta_0 u_t\|_2^2 = \int_s^t \int_{\Gamma_1} \zeta u_t - \int_{-\infty}^{+\infty} \|u_t\|_2^2.$$

For the left-hand side of (A 11), we note that $\operatorname{supp} \rho_\varepsilon \subset (-2\varepsilon, 2\varepsilon)$, $0 \leqslant \rho_\varepsilon = O(\varepsilon^{-1})$ and

$$\int_0^{+\infty} \rho_\varepsilon = \int_{-\infty}^0 \rho_\varepsilon = \frac{1}{2} \int_{-\infty}^{+\infty} \rho_\varepsilon = \tfrac{1}{2}.$$

Therefore, for sufficiently small $\varepsilon$,

$$(\rho_\varepsilon * (\theta_0 \nabla u)(t), \nabla u(t)) - \tfrac{1}{2}\|\nabla u(t)\|_2^2 = \int_0^{+\infty} \rho_\varepsilon(\tau)(\nabla u(t - \tau) - \nabla u(t), \nabla u(t))\, \mathrm{d}\tau.$$

Since, by (A 1), $\tau \mapsto (\nabla u(t-\tau) - \nabla u(t), \nabla u(t))$ is continuous and is zero for $\tau = 0$, we conclude that, as $\varepsilon \to 0$,

$$(\rho_\varepsilon * (\theta_0 \nabla u)(t), \nabla u(t)) \to \tfrac{1}{2}\|\nabla u(t)\|_2^2.$$

The same result, of course, continues to hold when $t$ is replaced by $s$. Then we can pass to the limit in (A 11) and conclude the proof of (2.9).

To show that (2.8) holds, we note that, by (2.9), it follows that $t \mapsto \|\nabla u(t)\|_2^2$ is continuous. Then, by (2.5) and (2.6), $t \mapsto \|u(t)\|_V^2$ is also continuous. Now fix $t$ in $[0, T]$ and let $t_k \to t$. Using (A 1), we have

$$\|u(t_k) - u(t)\|_V^2 = \|u(t_k)\|_V^2 + \|u(t)\|_V^2 - 2\langle u(t_k), u(t)\rangle_V \to 0$$

as $k \to \infty$, concluding the proof.

## Appendix B. Proof of lemma 3.3

An easy calculation shows that, for any $u \in H^1_{\Gamma_0}(\Omega)$ such that $u|_{\Gamma_1} \not\equiv 0$, we have

$$\sup_{\lambda > 0} J(\lambda u) = J(\lambda(u)u) = \left(\frac{1}{2} - \frac{1}{p}\right)\left(\frac{\|\nabla u\|_2}{\|u\|_{p,\Gamma_1}}\right)^{2p/(p-2)},$$

where

$$\lambda(u) = \frac{\|\nabla u\|_2^{2/(p-2)}}{\|u\|_{p,\Gamma_1}^{p/(p-2)}}.$$

Then, by (3.31), it is easy to see that

$$d = \left(\frac{1}{2} - \frac{1}{p}\right) B_\infty^{-2p/(p-2)},$$

so that $d = E_\infty$.

To show that $W = W_1$, we first prove that $W \subset W_1$. Let $u_0 \in W$. Hence $K(u_0) \geqslant 0$, and so $\|u_0\|_{p,\Gamma_1}^p \leqslant \|\nabla u_0\|_2^2$ by (1.10). Moreover, $J(u_0) < d = E_\infty$. Then

$$E_\infty > \left(\frac{1}{2} - \frac{1}{p}\right)\|\nabla u_0\|_2^2.$$

Now, if $\|\nabla u_0\|_2 \geqslant \lambda_\infty$, it follows that

$$E_\infty > \left(\frac{1}{2} - \frac{1}{p}\right)\lambda_\infty^2 = E_\infty,$$

a contradiction. In conclusion, $W \subset W_1$.

To prove that $W_1 \subset W$, let $u_0 \in W_1$ and suppose, by contradiction, that $K(u_0) < 0$. Then, using (3.31),

$$\|\nabla u_0\|_2^2 < \|u_0\|_{p,\Gamma_1}^p \leqslant B_\infty^p \|\nabla u_0\|_2^p,$$

and hence $\|\nabla u_0\|_2 > B_\infty^{-p/(p-2)} = \lambda_\infty$, a contradiction.

## Appendix C. A physical model

This section is devoted to giving a physical model for problem (1.1). Let $\Omega$ represent a solid body placed in a fluid denoted by $\Omega^c$. We suppose that a classical heat diffusion process occurs inside $\Omega$, so if $u = u(t,x)$ represents the temperature at point $x$ and time $t$, the process can be modelled by the classical heat equation

$$u_{tt} - \rho\Delta u = 0 \quad \text{in } (0,T)\times\Omega, \tag{C 1}$$

where the thermal conductivity $\rho > 0$ is taken to be 1 for simplicity. The surrounding fluid is supposed to be a perfect conductor of heat, so the temperature in $\Omega^c$ is spatially homogeneous and can be described by a number $v = v(t)$ for any $t \geqslant 0$. In particular, there is no diffusion in the fluid. Such an assumption is realistic if the fluid is well stirred. Moreover, we suppose that a reaction process occurs in the fluid such that the quantity of heat produced by the reaction is proportional to a superlinear power of the temperature, i.e. to $v^{p-1}$ with $p > 2$. Let $j = j(t,x)$ be the heat flux from $\Omega$ to $\Omega^c$. Then the rate of change of the temperature $v'(t)$ is given by

$$v'(t) = |v|^{p-2}(t)v(t) + \int_{\partial\Omega} j(t,x)\,\mathrm{d}S. \tag{C 2}$$

On the other hand, the heat flux $j(t,x)$ is given by the classical conductivity rule by

$$j(t,x) = -\frac{\partial u}{\partial \nu}, \tag{C 3}$$

since $\rho = 1$. Finally, the thermal contact of the fluid at $\partial\Omega$ yields the continuity condition

$$u(t,x) = v(t), \quad x \in \partial\Omega, \quad t \geqslant 0. \tag{C 4}$$

Combining (C 1)–(C 4), we obtain (1.1) with $\Gamma_0 = \emptyset$, $f = |u|^{p-2}u$ and $Q = u_t$. Now, since it is well known that solutions of this problem with sufficiently large initial datum blow-up in finite time, it is reasonable to try to introduce some control on the combustion process in order to slow down the reaction. This can be done by refrigerating the fluid $\Omega^c$. If the refrigerating system is controlled in such a way that the heat absorbed from the fluid is proportional to a power of the rate of change of the temperature, as $|v'(t)|^{m-2}v'(t)$, then the balance equation (C 2) has to be modified to

$$v'(t) = |v|^{p-2}(t)v(t) - |v'(t)|^{m-2}v'(t) + \int_{\partial\Omega} j(t,x)\,\mathrm{d}S. \tag{C 5}$$

Combining (C 1), (C 3), (C 4) and (C 5), we obtain (1.1) with $\Gamma_0 = \emptyset$, $f = |u|^{p-2}u$ and $Q = u_t + |u_t|^{m-2}u_t$. These nonlinear terms are included in the theory developed in § 4. In particular, theorem 1.2 shows that this type of refrigeration avoids explosions.

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