Probability in the Engineering and Informational Sciences, **31**, 2017, 121–138.
doi:10.1017/S0269964816000486

PRICING VULNERABLE AMERICAN PUT OPTIONS UNDER JUMP–DIFFUSION PROCESSES

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This paper evaluates vulnerable American put options under jump–diffusion assumptions on the underlying asset and the assets of the counterparty. Sudden shocks on the asset prices are described as a compound Poisson process. Analytical pricing formulae of vulnerable European put options and vulnerable twice-exercisable European put options are derived. Employing the two-point Geske and Johnson method, we derive an approximate analytical pricing formula of vulnerable American put options under jump–diffusions. Numerical simulations are performed for investigating the impacts of jumps and default risk on option prices.

Keywords: credit risk, jump–diffusion processes, multi-exercisable options, vulnerable american options

1. INTRODUCTION

Over-the-counter markets have grown rapidly in recent years. Default risk in over-the-counter transactions has attracted special attentions since the global financial crisis in 2007, as evidenced by the collapse of Lehman Brothers. This paper focuses on the valuation of vulnerable American put options, which are a major type of financial derivatives traded in the over-the-counter markets.

As shown by Merton [19], it is difficult to price American put options analytically because American-style options can be exercised at any time before maturity with a positive probability. To price finite-lived American-style options, a variety of numerical methods have been developed. However, numerical methods are too time-consuming and lack of intuitive meaning which an explicit formula could provide. Geske and Johnson [8] argue that arbitrary accuracy can be achieved by considering put options that can only be exercised at a few discrete dates. Employing the three-point Richardson extrapolation method, they derive an approximate analytical formula of an American put. Bunch and Johnson [2] modify the Geske–Johnson formula and provide a two-point Richardson method for valuing American puts analytically. The Geske–Johnson technique is also expanded to stochastic interest rate economy for American option pricing (Ho, Stapleton, and Subrahmanyam [9]; Chung [4]). All these studies acquiesce in the case of exchange-listed option markets, where counterparty risk is not factored.

The option exposed to credit risk, termed as a 'vulnerable option', is first considered by Johnson and Stulz [12]. They assume that the option is the sole liability of the counterparty. Default happens if the value of the option is greater than the assets of the counterparty. Since their pioneer work, there has been a series of literature on vulnerable option pricing. Klein [13] allows the option writer to have other equally ranking liabilities besides the option itself. The correlation between the underlying asset and the assets of the counterparty is also considered. Taking stochastic interest rate into consideration, Klein and Inglis [14] employ partial differential equation approaches to derive a closed-form pricing formula of a vulnerable option. Klein and Inglis [15] incorporate the potential liability of the written option into the default barrier. Hui, Lo, and Lee [10] price vulnerable options with a dynamic default barrier. Hung and Liu [11] price vulnerable options under both deterministic and stochastic interest rates in an incomplete market. Tian et al. [21] obtain a closed-from pricing formula of vulnerable European options when the dynamics of asset prices follow jump–diffusion processes. Recently, several papers investigate vulnerable options under stochastic volatility models (see, e.g., Yang et al. [23]; Lee, Yang, and Kim [17]; Wang [22]). Extending the previous works on vulnerable European options to American type, Chang and Hung [3] provide analytical formulae of vulnerable American call and put options based on the two-point Geske and Johnson method. Klein and Yang [16] further investigate the properties of vulnerable American options.

In the studies of vulnerable American options mentioned above, the underlying asset and the assets of the counterparty are assumed to follow geometric Brownian motions. As pointed out in Merton [20], continuous assumptions on the dynamics of assets ignore sudden shocks to asset returns due to the arrival of important information. Financial crisis and significant business always result in sudden changes in firm values, which cannot be captured by continuous sample paths. Many empirical findings also show that jump is identifiable in stock data (Eraker [6]). Compared with the existing literature, this paper has two main contributions. First, we incorporate jumps into both the underlying asset and the assets of the counterparty. Discontinuous changes in the dynamics of the assets are described by compound Poisson processes. Besides retaining the advantages of the frameworks of Klein [13] and Chang and Hung [3], the proposed model can capture rare shocks and explain how jumps on the underlying asset and the assets of the counterparty impact option prices, respectively. Second, we extend two-point Geske–Johnson methods to the jump case, and then provide an approximate analytical pricing formula of vulnerable American put options. In addition, analytical pricing formulae of vulnerable European puts and vulnerable twice-exercisable European puts are also derived.

The remainder of the paper is structured as follows. In Section 2, a jump–diffusion model is proposed for the valuation of vulnerable American put options. In Section 3, we derive an approximate analytical formula of vulnerable American put options. Numerical simulations are presented in Section 4. The concluding remarks are given in Section 5.

2. THE MODEL

In this section, we describe the framework for valuing vulnerable American put options. The dynamics of the underlying asset and the assets of the counterparty are modeled directly under the risk-neutral measure Q. We assume that the default-free term structure is flat with an instantaneous riskless rate r and that the dynamics of the underlying asset S is governed by a jump–diffusion process under Q,

$$\frac{dS_t}{S_{t-}} = (r - k_S\lambda_S)dt + \sigma_S dW_t^{(1)} + (e^{Z_{t-}^{(1)}} - 1)dN_t^{(1)}, \tag{2.1}$$

where σ_S is the volatility of the underlying asset and $W_t^{(1)}$ is a standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, Q)$. Sudden changes in the underlying asset prices are captured by a compound Poisson process with intensity λ_S. In addition, if one shock happens at time t, the jump amplitude is controlled by $Z_t^{(1)}$. For any time $t \neq s$, we assume that $Z_t^{(1)}$ and $Z_s^{(1)}$ are independently and identically distributed. Specially, as in Merton [20], the size of a jump is assumed to be drawn from a lognormal distribution, and in this situation, $Z_t^{(1)}$ is normally distributed with mean μ_1 and standard deviation $\sigma_1 > 0$. In this case, the mean percentage jump of the price k_S is equal to $e^{\mu_1 + \frac{1}{2}\sigma_1^2} - 1$. Moreover, the situation where the underlying asset price evolves continuously is modeled by setting $\lambda_S = 0$.

Now we consider counterparty risk with the structural approach as in Klein [13]. A credit loss occurs if the market value of the assets of the counterparty, V_T, is less than some amount D^*. This default barrier is not set to the value of the option but corresponds to the amount of claims D outstanding at exercise time T. Once default happens at exercise time T, the notional claim on the option will be recovered at a ratio of $\frac{(1-\alpha)V_T}{D}$. Here α represents the deadweight costs due to the bankruptcy or reorganization, and the notional claim is the option value without default. In a nutshell, the outstanding claims are exogenous and all of the claims are equally ranking. Taking jump risk into consideration, we also assume that V is driven by the following jump–diffusion process:

$$\frac{dV_t}{V_{t-}} = (r - k_V\lambda_V)dt + \sigma_V dW_t^{(2)} + (e^{Z_{t-}^{(2)}} - 1)dN_t^{(2)}, \tag{2.2}$$

where σ_V is the volatility of the assets of the counterparty and $W_t^{(2)}$ is a standard Brownian motion on $(\Omega, \mathcal{F}, Q)$. Similarly, jumps are modeled by the last term with $dN_t^{(2)}$, which is a Poisson process with intensity λ_V. If the jump occurs at time t, the jump amplitude is controlled by $Z_t^{(2)}$. We assume that for $t \neq s$, $Z_t^{(2)}$ and $Z_s^{(2)}$ are independent and normally distributed with mean μ_2 and standard deviation $\sigma_2 > 0$. Given the jump arrival, the mean percentage jump is $k_V = \mathbb{E}[e^{Z_t^{(2)}}] - 1 = e^{\mu_2 + \frac{1}{2}\sigma_2^2} - 1$.

To describe the correlation between the assets of the counterparty and the underlying asset, $W_t^{(1)}$ and $W_t^{(2)}$ have a correlation coefficient ρ as in Klein [13]. For the discontinuous part, we assume that rare shocks on the underlying asset and the assets of the counterparty are not relevant to each other. Hence $(W_t^{(1)}, W_t^{(2)})$, $Z_t^{(1)}$, $N_t^{(1)}$, $Z_t^{(2)}$ and $N_t^{(2)}$ are mutually independent.

https://doi.org/10.1017/S0269964816000486 Published online by Cambridge University Press

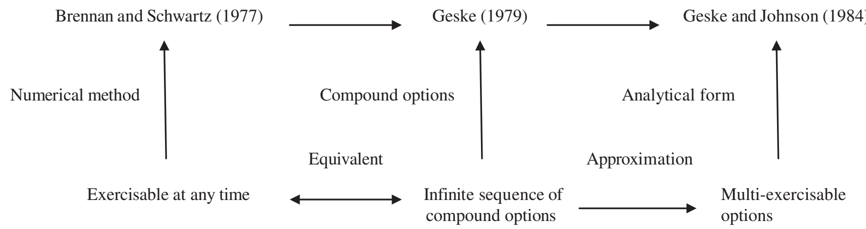

FIGURE 1. Method comparison.

At any instant, American put options might be exercised, which is the difficulty in deriving an explicit formula. Many numerical methods have been developed, however, they are time-consuming and lack of intuition which the comparative statics of an explicit expression can provide. Geske and Johnson [8] adopt a three-point Richardson extrapolation method to price American put options and get an analytical form.

Figure 1 summarizes the techniques adopted in the literature. Brennan and Schwartz [1] develop numerical solutions for the value of an American put option. Being exercisable at any time before maturity, an American option is just equivalent to an infinite sequence of compound European options. Compound options (options on options) are first formulated by Geske [7]. Then an analytical formula for American puts is given in Geske and Johnson [8]. Bunch and Johnson [2] propose a two-point Richardson method, which is more numerically accurate than Geske and Johnson [8]. Chang and Hung [3] adopt the method in Bunch and Johnson [2] (i.e., two-point Geske and Johnson method) to price vulnerable American options. This paper adopts two-point Geske and Johnson methods to price vulnerable American put options under jump–diffusions. Specifically, the value of a vulnerable American put can be approximated by a vulnerable European put, $P(T, K, S_0, V_0)$ with time to maturity T, initial price S_0, strike price K and the initial price of the counterparty's assets V_0, and a vulnerable twice-exercisable European put option, P^* (default may occur at half maturity $\frac{T}{2}$ and maturity T). Suggested by Bunch and Johnson [2], the value of a vulnerable American put at time zero, denoted by VAP, is expressed as,

$$VAP = P^* + (P^* - P(T, K, S_0, V_0)). \tag{2.3}$$

In the coming section, we derive the closed-form solutions for $P(T, K, S_0, V_0)$ and P^* in the proposed framework, and hence obtain the expression of VAP in (2.3).

3. VALUATION OF VULNERABLE AMERICAN PUT OPTIONS

The topic of this section is to give the approximate explicit pricing formula of vulnerable American put options under jump–diffusions. As in Merton [20], we assume that $Z_t^{(i)}, i = 1, 2$ are normally distributed with mean $\mu_i, i = 1, 2$ and standard deviation $\sigma_i > 0, i = 1, 2$. In contrast to non-vulnerable options, the expected payoff of a vulnerable American put option is comprised of two parts. If there is no default, it is the payoff on an American put. Otherwise, only a proportion of the payoff on the American put can be recovered where the proportion depends on the assets and the liabilities of the counterparty. According to (2.3), we show the pricing formulae of twice-exercisable vulnerable European put options, and then give the explicit formula for vulnerable American put options. As shown in Tian et al. [21], the explicit pricing formula of $P(T, K, S_0, V_0)$ is represented as [We denote the cumulative distribution function of k-dimensional normal distribution as $N_k(\xi_1, \xi_2, \ldots, \xi_k, \Upsilon)$, where

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$\xi_1, \xi_2, \ldots, \xi_k$ are normally distributed random variables, and Υ is the correlation coefficient matrix.],

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{(\lambda_S T)^{n_1}}{n_1!}\frac{(\lambda_V T)^{n_2}}{n_2!}e^{-\lambda_S T-\lambda_V T}\Big(Ke^{-rT}N_2(-b_1(n_1,T),b_2(n_2,T),-\bar{\rho}(n_1,n_2,T))$$
$$- S_0 e^{-k_S\lambda_S T+n_1\mu_1+\frac{1}{2}n_1\sigma_1^2}N_2(-a_1(n_1,T),a_2(n_2,T),-\bar{\rho}(n_1,n_2,T))$$
$$+ \frac{1-\alpha}{D}KV_0 e^{-k_V\lambda_V T+n_2\mu_2+\frac{1}{2}n_2\sigma_2^2}N_2(-d_1(n_1,T),d_2(n_2,T),\bar{\rho}(n_1,n_2,T))$$
$$- \frac{1-\alpha}{D}S_0V_0 e^{rT-k_S\lambda_S T-k_V\lambda_V T+n_1\mu_1+n_2\mu_2+\frac{1}{2}n_1\sigma_1^2+\frac{1}{2}n_2\sigma_2^2+\rho\sigma_S\sigma_V T}$$
$$\times N_2(-c_1(n_1,T),c_2(n_2,T),\bar{\rho}(n_1,n_2,T))\Big), \quad \textbf{(3.1)}$$

where the parameters are represented as follows:

$$a_1(n_1,t) = \frac{\ln\frac{S_0}{K}+(r+\frac{1}{2}\sigma_S^2-k_S\lambda_S)t+n_1\mu_1+n_1\sigma_1^2}{\sqrt{\sigma_S^2 t+n_1\sigma_1^2}},$$
$$a_2(n_2,t) = \frac{\ln\frac{V_0}{D^*}+(r-\frac{1}{2}\sigma_V^2-k_V\lambda_V)t+n_2\mu_2+\rho\sigma_S\sigma_V t}{\sqrt{\sigma_V^2 T+n_2\sigma_2^2}},$$
$$b_1(n_1,t) = \frac{\ln\frac{S_0}{K}+(r-\frac{1}{2}\sigma_S^2-k_S\lambda_S)t+n_1\mu_1}{\sqrt{\sigma_S^2 t+n_1\sigma_1^2}},$$
$$b_2(n_2,t) = \frac{\ln\frac{V_0}{D^*}+(r-\frac{1}{2}\sigma_V^2-k_V\lambda_V)t+n_2\mu_2}{\sqrt{\sigma_V^2 t+n_2\sigma_2^2}},$$
$$c_1(n_1,t) = \frac{\ln\frac{S_0}{K}+(r+\frac{1}{2}\sigma_S^2-k_S\lambda_S)t+n_1\mu_1+n_1\sigma_1^2+\rho\sigma_S\sigma_V t}{\sqrt{\sigma_S^2 t+n_1\sigma_1^2}},$$
$$c_2(n_2,t) = -\frac{\ln\frac{V_0}{D^*}+(r+\frac{1}{2}\sigma_V^2-k_V\lambda_V)t+n_2\mu_2+n_2\sigma_2^2+\rho\sigma_S\sigma_V t}{\sqrt{\sigma_V^2 t+n_2\sigma_2^2}},$$
$$d_1(n_1,t) = \frac{\ln\frac{S_0}{K}+(r-\frac{1}{2}\sigma_S^2-k_S\lambda_S)t+n_1\mu_1+\rho\sigma_S\sigma_V t}{\sqrt{\sigma_S^2 t+n_1\sigma_1^2}},$$
$$d_2(n_2,t) = -\frac{\ln\frac{V_0}{D^*}+(r+\frac{1}{2}\sigma_V^2-k_V\lambda_V)t+n_2\mu_2+n_2\sigma_2^2}{\sqrt{\sigma_V^2 t+n_2\sigma_2^2}},$$
$$\bar{\rho}(n_1,n_2,t) = \frac{\sigma_S\sigma_V t}{\sqrt{\sigma_S^2 t+n_1\sigma_1^2}\sqrt{\sigma_V^2 t+n_2\sigma_2^2}}\rho. \quad \textbf{(3.2)}$$

Based on Cox and Ross [5], an American put can be priced as the discounted expected value of all future cash flows. In the two-point Geske and Johnson framework, the American put can be exercised at half maturity $\frac{T}{2}$ and maturity T. At $\frac{T}{2}$, the put will be exercised when the payoff at half maturity exceeds the value of the option if not exercised. The critical stock price exercisable at $\frac{T}{2}$ is determined by the free boundary condition which the American put satisfies,

$$P\left(\frac{T}{2},K,S_{T/2},V_{T/2}\right) \geq \max(K-S_{T/2},0). \quad \textbf{(3.3)}$$

For vulnerable American put options, if default does not occur at half maturity, the critical stock price $S^*_{T/2}$ is some stock price S which satisfies

$$K - S = P\left(\frac{T}{2}, K, S, V_0 e^{\frac{rT}{2}}\right).$$

Here we approximately use $V_0 e^{\frac{rT}{2}}$ instead of $V_{T/2}$ to obtain a constant stock price $S^*_{T/2}$, which helps us obtain an explicit formula of vulnerable two-exercisable put options. Theoretically, we should use $V_{T/2}$ instead of $V_0 e^{\frac{rT}{2}}$ in the above equation. However, $V_{T/2}$ is unknown at initial time and the critical price $S^*_{T/2}$ is pre-determined at that time. From the viewpoint of the risk-neutral measure, we select $V_0 e^{\frac{rT}{2}}$ as the approximate value. The value of a vulnerable twice-exercisable put option denoted by P^* can be represented as follows:

$$\begin{aligned} P^* = {} & \mathbb{E}\Big[e^{-rT/2}(K - S_{T/2})\mathbf{1}(S_{T/2} \le S^*_{T/2}, V_{T/2} \ge D^*)\Big] \\ & + \mathbb{E}\Big[e^{-rT/2}(K - S_{T/2})\frac{(1-\alpha)V_{T/2}}{D}\mathbf{1}(S_{T/2} \le K, V_{T/2} < D^*)\Big] \\ & + \mathbb{E}\Big[e^{-rT}(K - S_T)\mathbf{1}(S_{T/2} > S^*_{T/2}, V_{T/2} \ge D^*, S_T \le K, V_T \ge D^*)\Big] \\ & + \mathbb{E}\Big[e^{-rT}(K - S_T)\frac{(1-\alpha)V_T}{D}\mathbf{1}(S_{T/2} > S^*_{T/2}, V_{T/2} \ge D^*, S_T \le K, V_T < D^*)\Big]. \end{aligned}$$

In what follows, we divide P^* into two parts and show the analytical forms, respectively. The add of the first two terms of P^*, denoted by $P_1(S^*_{T/2})$, can be seemed as the value of a vulnerable European put option with maturity $T/2$ and strike price $S^*_{T/2}$ without default and K when default occurs. Furthermore, the method used in Tian et al. [21] implies the explicit formula of $P_1(S^*_{T/2})$ as follows:

$$\begin{aligned} P_1(S^*_{T/2}) = {} & \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{(\lambda_S T/2)^{n_1}}{n_1!}\frac{(\lambda_V T/2)^{n_2}}{n_2!}e^{-\lambda_S T/2 - \lambda_V T/2} \\ & \times \left(Ke^{-rT/2}N_2\left(-b_1^*\left(n_1, \frac{T}{2}\right), b_2\left(n_2, \frac{T}{2}\right), -\bar{\rho}\left(n_1, n_2, \frac{T}{2}\right)\right)\right. \\ & - S_0 e^{-k_S\lambda_S T/2 + n_1\mu_1 + \frac{1}{2}n_1\sigma_1^2}N_2\left(-a_1^*\left(n_1, \frac{T}{2}\right), a_2\left(n_2, \frac{T}{2}\right), -\bar{\rho}\left(n_1, n_2, \frac{T}{2}\right)\right) \\ & + \frac{1-\alpha}{D}KV_0 e^{-k_V\lambda_V T/2 + n_2\mu_2 + \frac{1}{2}n_2\sigma_2^2} \\ & \times N_2\left(-d_1\left(n_1, \frac{T}{2}\right), d_2\left(n_2, \frac{T}{2}\right), \bar{\rho}\left(n_1, n_2, \frac{T}{2}\right)\right) \\ & - \frac{1-\alpha}{D}S_0 V_0 e^{rT/2 - k_S\lambda_S T/2 - k_V\lambda_V T/2 + n_1\mu_1 + n_2\mu_2 + \frac{1}{2}n_1\sigma_1^2 + \frac{1}{2}n_2\sigma_2^2 + \rho\sigma_S\sigma_V T/2} \\ & \left. \times N_2\left(-c_1\left(n_1, \frac{T}{2}\right), c_2\left(n_2, \frac{T}{2}\right), \bar{\rho}\left(n_1, n_2, \frac{T}{2}\right)\right)\right), \end{aligned} \quad \textbf{(3.4)}$$

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where

$$a_1^*\left(n_1,\frac{T}{2}\right)=\frac{\ln\frac{S_0}{S^*_{T/2}}+(r+\frac{1}{2}\sigma_S^2-k_S\lambda_S)T/2+n_1\mu_1+n_1\sigma_1^2}{\sqrt{\sigma_S^2T/2+n_1\sigma_1^2}},$$

$$b_1^*\left(n_1,\frac{T}{2}\right)=\frac{\ln\frac{S_0}{S^*_{T/2}}+(r-\frac{1}{2}\sigma_S^2-k_S\lambda_S)T/2+n_1\mu_1}{\sqrt{\sigma_S^2T/2+n_1\sigma_1^2}},\tag{3.5}$$

and other parameters are listed in (3.2).

Next we will derive the explicit expression for the last two terms of P^*, which is denoted by $P_2(S^*_{T/2})$,

$$\begin{aligned}P_2(S^*_{T/2})&=\mathbb{E}\Big[e^{-rT}(K-S_T)\mathbf{1}(S_{T/2}>S^*_{T/2},V_{T/2}\geq D^*,S_T\leq K,V_T\geq D^*)\Big]\\&\quad+\mathbb{E}\Big[e^{-rT}(K-S_T)\frac{(1-\alpha)V_T}{D}\mathbf{1}(S_{T/2}>S^*_{T/2},V_{T/2}\geq D^*,S_T\leq K,V_T< D^*)\Big].\end{aligned}\tag{3.6}$$

Itô formula implies the following equations:

$$\begin{aligned}\ln S_{T/2}&=\ln S_0+\left(r-\frac{1}{2}\sigma_S^2-k_S\lambda_S\right)T/2+\sigma_S W^{(1)}_{T/2}+\sum_{k=1}^{N^{(1)}_{T/2}}Z^{(1)}_{\tau^{(1)}_k},\\ \ln V_{T/2}&=\ln V_0+\left(r-\frac{1}{2}\sigma_V^2-k_V\lambda_V\right)T/2+\sigma_V W^{(2)}_{T/2}+\sum_{k=1}^{N^{(2)}_{T/2}}Z^{(2)}_{\tau^{(2)}_k},\\ \ln S_T&=\ln S_0+\left(r-\frac{1}{2}\sigma_S^2-k_S\lambda_S\right)T+\sigma_S W^{(1)}_{T}+\sum_{k=1}^{N^{(1)}_{T}}Z^{(1)}_{\tau^{(1)}_k},\\ \ln V_T&=\ln V_0+\left(r-\frac{1}{2}\sigma_V^2-k_V\lambda_V\right)T+\sigma_V W^{(2)}_{T}+\sum_{k=1}^{N^{(2)}_{T}}Z^{(2)}_{\tau^{(2)}_k},\end{aligned}\tag{3.7}$$

where $\tau^{(i)}_k, i=1,2$ denote the k-th jump time of $N^{(i)}_t, i=1,2$, respectively. Then, conditional on $\mathcal{G}_T^{(m_1,m_2,n_1,n_2)}:=\{N^{(1)}_{T/2}=m_1,N^{(2)}_{T/2}=m_2,N^{(1)}_T=n_1,N^{(2)}_T=n_2\}$, it is clear that

$$\left(\ln\frac{S_{T/2}}{S_0},\ln\frac{V_{T/2}}{V_0},\ln\frac{S_T}{S_0},\ln\frac{V_T}{V_0}\right)$$

are normally distributed random variables in R^4.

For further calculation, define

$$\ln S_{T/2,m_1}:=\ln S_0+\left(r-\frac{1}{2}\sigma_S^2-k_S\lambda_S\right)T/2+\sigma_S W^{(1)}_{T/2}+\sum_{k=1}^{m_1}\xi^{(1)}_k,\tag{3.8}$$

$$\ln V_{T/2,m_2}:=\ln V_0+\left(r-\frac{1}{2}\sigma_V^2-k_V\lambda_V\right)T/2+\sigma_V W^{(2)}_{T/2}+\sum_{k=1}^{m_2}\xi^{(2)}_k,\tag{3.9}$$

$$\ln S_{T,n_1} := \ln S_0 + \left(r - \frac{1}{2}\sigma_S^2 - k_S\lambda_S\right)T + \sigma_S W_T^{(1)} + \sum_{k=1}^{n_1}\xi_k^{(1)}, \tag{3.10}$$

$$\ln V_{T,n_2} := \ln V_0 + \left(r - \frac{1}{2}\sigma_V^2 - k_V\lambda_V\right)T + \sigma_V W_T^{(2)} + \sum_{k=1}^{n_2}\xi_k^{(2)}, \tag{3.11}$$

where $\xi_k^{(i)}, i = 1, 2$ are independent normally distributed with mean $\mu_i, i = 1, 2$ and standard deviation $\sigma_i > 0, i = 1, 2$. In fact, these notations represent the stock prices conditional on the numbers of Poisson jumps. Noting that the whole probability space $\Omega = \bigcup_{m_1=0}^{\infty}\bigcup_{m_2=0}^{\infty}\bigcup_{n_1=m_1}^{\infty}\bigcup_{n_2=m_2}^{\infty}\mathcal{G}_T^{(m_1,m_2,n_1,n_2)}$ and $\mathcal{G}_T^{(k_1,k_2,k_3,k_4)}\bigcap\mathcal{G}_T^{(j_1,j_2,j_3,j_4)} = \emptyset$ for any $k_1 \neq j_1$, $k_2 \neq j_2$, $k_3 \neq j_3$ or $k_4 \neq j_4$, we can rewrite $P_2(S_{T/2}^*)$ as follows:

$$\begin{aligned}
P_2(S_{T/2}^*) &= \mathbb{E}\Big[e^{-rT}(K - S_T)\mathbf{1}(S_{T/2} > S_{T/2}^*, V_{T/2} \geq D^*, S_T \leq K, V_T \geq D^*)\Big] \\
&\quad + \mathbb{E}\left[e^{-rT}(K - S_T)\frac{(1-\alpha)V_T}{D}\mathbf{1}(S_{T/2} > S_{T/2}^*, V_{T/2} \geq D^*, S_T \leq K, V_T < D^*)\right] \\
&= \mathbb{E}\Bigg[\mathbb{E}\Bigg[e^{-rT}(K - S_T)\mathbf{1}(S_{T/2} > S_{T/2}^*, V_{T/2} \geq D^*, S_T \leq K, V_T \geq D^*) \\
&\quad + e^{-rT}(K - S_T)\frac{(1-\alpha)V_T}{D}\mathbf{1}(S_{T/2} > S_{T/2}^*, V_{T/2} \geq D^*, S_T \leq K, V_T < D^*)\Bigg| \\
&\quad \times N_{T/2}^{(1)}, N_{T/2}^{(2)}, N_T^{(1)}, N_T^{(2)}\Bigg]\Bigg] \\
&= \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\sum_{n_1=m_1}^{\infty}\sum_{n_2=m_2}^{\infty} Q(N_{T/2}^{(1)} = m_1, N_{T/2}^{(2)} = m_2, N_T^{(1)} = n_1, N_T^{(2)} = n_2) \\
&\quad \times P_{m_1,m_2,n_1,n_2},
\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}
&P_{m_1,m_2,n_1,n_2} \\
&\quad = e^{-rT}\Bigg\{\mathbb{E}\Big[(K - S_{T,n_1})\mathbf{1}(S_{T/2,m_1} > S_{T/2}^*, V_{T/2,m_2} \geq D^*, S_{T,n_1} \leq K, V_{T,n_2} \geq D^*)\Big] \\
&\quad + \mathbb{E}\Bigg[(K - S_{T,n_1})\frac{(1-\alpha)V_{T,n_2}}{D} \\
&\quad \times \mathbf{1}(S_{T/2,m_1} > S_{T/2}^*, V_{T/2,m_2} \geq D^*, S_{T,n_1} \leq K, V_{T,n_2} < D^*)\Bigg]\Bigg\}.
\end{aligned} \tag{3.13}$$

Since $N^{(1)}$ and $N^{(2)}$ are independent and their increments are also independent, one gets that

$$\begin{aligned}
&Q(N_{T/2}^{(1)} = m_1, N_{T/2}^{(2)} = m_2, N_T^{(1)} = n_1, N_T^{(2)} = n_2) \\
&\quad = Q(N_{T/2}^{(1)} = m_1, N_{T/2}^{(2)} = m_2, N_T^{(1)} - N_{T/2}^{(1)} = n_1 - m_1, N_T^{(2)} - N_{T/2}^{(2)} = n_2 - m_2)
\end{aligned}$$

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$$
\begin{aligned}
&= Q(N_{T/2}^{(1)} = n_0, N_T^{(1)} - N_{T/2}^{(1)} = n_1 - m_1)Q(N_{T/2}^{(2)} = m_2, N_T^{(2)} - N_{T/2}^{(2)} = n_2 - m_2) \\
&= Q(N_{T/2}^{(1)} = n_0)Q(N_T^{(1)} - N_{T/2}^{(1)} = n_1 - m_1)Q(N_{T/2}^{(2)} = m_2)Q(N_T^{(2)} - N_{T/2}^{(2)} = n_2 - m_2) \\
&= \frac{(\lambda_S T/2)^{m_1}}{m_1!} e^{-\lambda_S T/2} \frac{(\lambda_S T/2)^{n_1-m_1}}{(n_1-m_1)!} e^{-\lambda_S T/2} \frac{(\lambda_V T/2)^{m_2}}{m_2!} \\
&\quad \times e^{-\lambda_V T/2} \frac{(\lambda_V T/2)^{n_2-m_2}}{(n_2-m_2)!} e^{-\lambda_V T/2} \\
&= \frac{(\lambda_S T/2)^{n_1}}{m_1!(n_1-m_1)!} \frac{(\lambda_V T/2)^{n_2}}{m_2!(n_2-m_2)!} e^{-\lambda_S T - \lambda_V T}. \qquad \textbf{(3.14)}
\end{aligned}
$$

Next we will show the explicit formula of P_{m_1,m_2,n_1,n_2} so that we can get the explicit expression of $P_2(S_{T/2}^*)$ and P^*. To this end, we divide P_{m_1,m_2,n_1,n_2} into four parts as follows:

$$
P_{m_1,m_2,n_1,n_2} = e^{-rT}[A_1 + A_2 + A_3 + A_4], \qquad \textbf{(3.15)}
$$

where A_1, A_2, A_3 and A_4 are given by

$$
A_1 = K\mathbb{E}\Big[\mathbf{1}(S_{T/2,m_1} > S_{T/2}^*, V_{T/2,m_2} \geq D^*, S_{T,n_1} \leq K, V_{T,n_2} \geq D^*)\Big], \qquad \textbf{(3.16)}
$$

$$
A_2 = -\mathbb{E}\Big[S_{T,n_1}\mathbf{1}(S_{T/2,m_1} > S_{T/2}^*, V_{T/2,m_2} \geq D^*, S_{T,n_1} \leq K, V_{T,n_2} \geq D^*)\Big], \qquad \textbf{(3.17)}
$$

$$
A_3 = K\mathbb{E}\Big[\frac{(1-\alpha)V_{T,n_2}}{D}\mathbf{1}(S_{T/2,m_1} > S_{T/2}^*, V_{T/2,m_2} \geq D^*, S_{T,n_1} \leq K, V_{T,n_2} < D^*)\Big], \qquad \textbf{(3.18)}
$$

$$
A_4 = -\mathbb{E}\Big[S_{T,n_1}\frac{(1-\alpha)V_{T,n_2}}{D}\mathbf{1}(S_{T/2,m_1} > S_{T/2}^*, V_{T/2,m_2} \geq D^*, S_{T,n_1} \leq K, V_{T,n_2} < D^*)\Big]. \qquad \textbf{(3.19)}
$$

Then we can get the closed forms of A_1, A_2, A_3 and A_4, respectively. The explicit expressions are shown in (A.1)–(A.4) and the detailed proofs are given in the Appendix. Combining with (3.4), (3.12) and (3.15), we have obtained the explicit formula of vulnerable two-exercisable put options. To sum up, we have got the value of vulnerable American put options shown in (2.3),

$$
\begin{aligned}
VAP &= 2P^* - P(T, K, S_0, V_0) \\
&= 2P_1(S_{T/2}^*) + 2P_2(S_{T/2}^*) - P(T, K, S_0, V_0), \qquad \textbf{(3.20)}
\end{aligned}
$$

where their respective explicit expressions are in (3.4), (3.12) and (3.1).

We can also employ the two-point Geske–Johnson method to price vulnerable American call options. Denote C^* as the price of a vulnerable twice-exercisable European call option and $C(T, K, S_0, V_0)$ as the price of a vulnerable European call with expiry date T. Similarly, for vulnerable twice-exercisable European call options, the critical stock price $S_{T/2}^*$ is some stock price S that satisfies $S - K = C(\frac{T}{2}, K, S, V_0 e^{\frac{rT}{2}})$. The value of a vulnerable European call option is derived by Tian et al. [21]. The value of a vulnerable twice-exercisable call

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option C^* is represented as

$$C^* = \mathbb{E}\Big[e^{-rT/2}(S_{T/2}-K)\mathbf{1}(S_{T/2} \geq S^*_{T/2}, V_{T/2} \geq D^*)\Big]$$
$$+\,\mathbb{E}\Big[e^{-rT/2}(S_{T/2}-K)\frac{(1-\alpha)V_{T/2}}{D}\mathbf{1}(S_{T/2} \geq K, V_{T/2} < D^*)\Big]$$
$$+\,\mathbb{E}\Big[e^{-rT}(S_T-K)\mathbf{1}(S_{T/2} < S^*_{T/2}, V_{T/2} \geq D^*, S_T \geq K, V_T \geq D^*)\Big]$$
$$+\,\mathbb{E}\Big[e^{-rT}(S_T-K)\frac{(1-\alpha)V_T}{D}\mathbf{1}(S_{T/2} < S^*_{T/2}, V_{T/2} \geq D^*, S_T \geq K, V_T < D^*)\Big].$$

Then the value of a vulnerable American call option at time zero, denoted by VAC, can be expressed as

$$VAC = 2C^* - C(T, K, S_0, V_0).$$

The calculation of VAC is similar to that of VAP, hence we choose not to show it here.

4. NUMERICAL ANALYSIS

In this section, numerical simulations are performed to illustrate the impact of the parameters on a vulnerable American put option.

The effects of some basic variables on vulnerable option prices are illustrated in Figures 2–5, including outstanding claims, jump intensity, mean jump size and standard deviation of the jump size. Preference parameters listed in Table 1 represent a typical business situation. In the base case, the vulnerable American put option is at the money, and written by a highly leveraged firm. Time to maturity is assumed to be one year. The market value of the option writer's assets and the underlying asset are correlated with an instantaneous correlation coefficient $\rho = 0.5$. Shocks on stock prices happen once a year. In the following tables and figures, we change one of the parameter values to investigate the impact on the vulnerable option price with other variables adopted in Table 1.

Convergence analysis is reported in Table 2. We first test the convergence speed of critical stock prices. The sum of the first 30 terms of the sequence is close enough to the

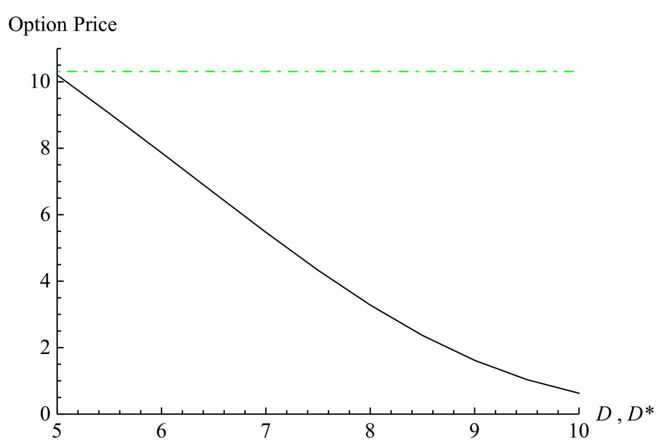

FIGURE 2. Option price against default barrier. The solid and dot-dashed lines correspond to the proposed model and the two-point Geske–Johnson model with jumps, respectively.

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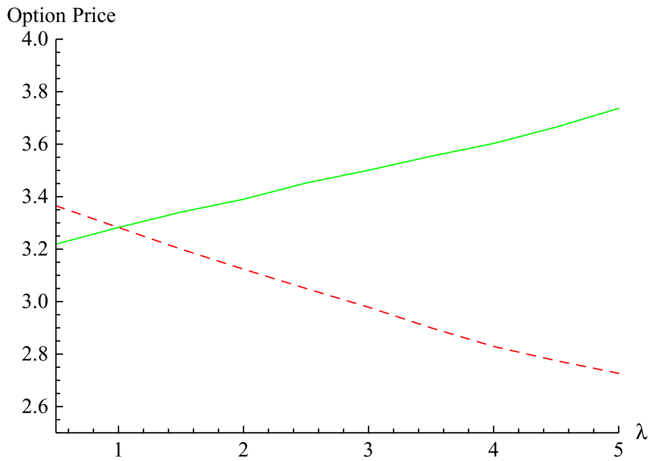

FIGURE 3. Option price against jump intensities of the underlying asset and the assets of the counterparty. The solid and the dashed lines correspond to λ_S and to λ_V, respectively.

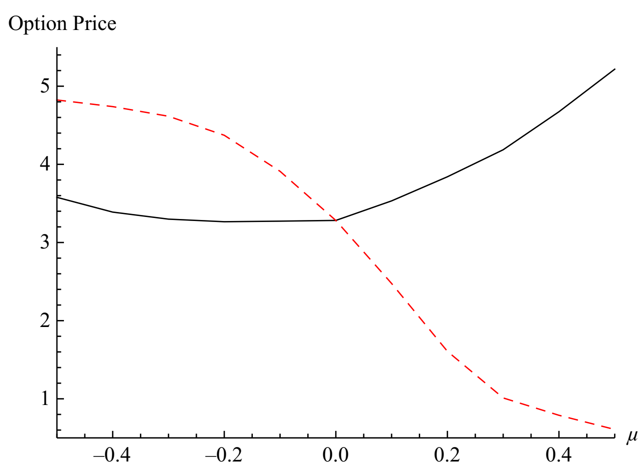

FIGURE 4. Option price against mean jump size in the proposed model. The solid and dashed lines correspond to μ_1 and μ_2, respectively.

true value. We take the sum of m_1, m_2 from 0 to 30 as the approximate critical stock price. Turning to option prices, it is also accurate enough for the first 30 terms of the series. For $P_1(S^*_{T/2})$ and $P(T, K, S_0, V_0)$, the sum of n_1, n_2 from 0 to 30 is taken. For P_2, we take the sum of m_1, $n_1 - m_1$, m_2, $n_2 - m_2$ from 0 to 30 in (3.20) in the following numerical analysis. Besides, we compared our results with the prices calculated by least-square Monte Carlo (LSMC) simulations in Longstaff and Schwartz [18]. Each vulnerable American option price is the average of 20 LSMC simulation estimates. Each LSMC simulation is based on 20,000 (10,000 plus 10,000 antithetic) sample paths with 50 exercise points and 1,000 time steps per year. The error in approximation is defined as the difference between a LSMC price and our approximate analytical solution divided by the LSMC simulation price. The mean and standard deviation of LSMC simulation prices are also reported in Table 2. The values calculated by the derived pricing formula are a little lower than those calculated by Longstaff and Schwartz [18].

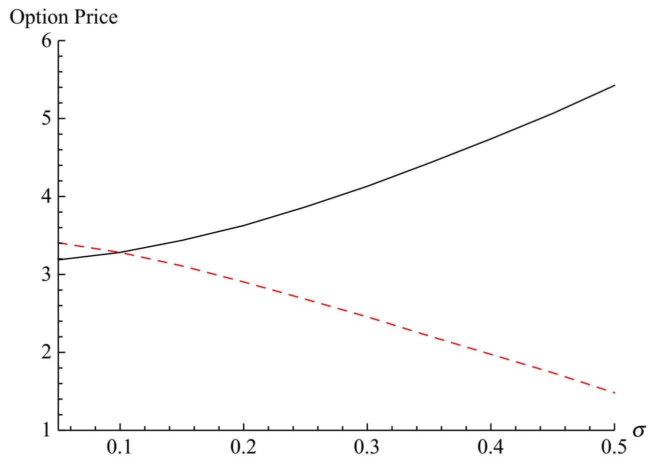

FIGURE 5. Option price against standard deviation of the jump size in the proposed model. The solid and dashed lines correspond to σ_1 and σ_2, respectively.

TABLE 1. Parameter values in the base case.

Parameter	Value	Parameter	Value
Volatility	$\sigma_S = 0.3$	Volatility	$\sigma_V = 0.3$
Initial price	$S_0 = 10$	Initial price	$V_0 = 10$
Mean jump size of S	$\mu_1 = 0$	Mean jump size of V	$\mu_2 = 0$
Annual jump intensity	$\lambda_S = 1$	Annual jump intensity	$\lambda_V = 1$
Standard deviation of the jump size	$\sigma_1 = 0.1$	Spot rate	$r = 0.02$
Standard deviation of the jump size	$\sigma_2 = 0.1$	Correlation coefficient	$\rho = 0.5$
Time to maturity	$T = 1.0$	Strike price	$K = 10$
Default barrier	$D^* = 8$	Outstanding claims	$D = 8$
Deadweight cost associated with bankruptcy	$\alpha = 0.5$		

TABLE 2. Convergence analysis.

	m_1, $n_1 - m_1$, m_2, $n_2 - m_2 =$	5	10	20	30	LSMC	Std. of LSMC	Error in (%) approximation
Critical price $S^*_{T/2}$	Base	8.6193	8.6192	8.6192	8.6192			
	$\lambda_S = 5$	8.4956	8.3081	8.3077	8.3077			
	$\lambda_V = 5$	8.8544	8.7415	8.7414	8.7414			
Option price	Base	3.2822	3.2824	3.2824	3.2824	3.3257	0.043	1.30
	$\lambda_S = 5$	3.4520	3.7364	3.7365	3.7365	3.8196	0.052	2.18
	$\lambda_V = 5$	2.5405	2.7268	2.7269	2.7269	2.7783	0.048	1.85

Figure 2 presents the prices varying with outstanding claims and default barriers. Obviously, the price of two-point jump–diffusions should be constant since there is no default risk in this model. In Figure 2, we assume that the counterparty cannot continue in operation if the assets at expiration date is less than the outstanding claims, that is, $D = D^*$. When $D = D^* = 5$, which is half of the assets of the counterparty, the difference between the values of the two-point Geske–Johnson jump–diffusion model and the proposed model

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is 0.112. In this case, default seldom happens. Theoretically, when $D \to \infty$, the prices converge to zero. The counterparty shall encounter greater credit risk as the default threshold turns higher.

Figure 3 shows the effect of jump intensities λ_S and λ_V. When λ_S increases from 0.5 to 5, the range of vulnerable option prices is (3.219, 3.736). In contrast, the vulnerable option price decreases from 3.365 to 2.727 as λ_V increases from 0.5 to 5. A stronger jump intensity of the underlying asset corresponds to a higher price while the price decreases with the jump intensity of the assets of the counterparty. The difference between the prices with $\lambda_S = 5$ and $\lambda_S = 0.5$ is 0.517. By contrast, the price declines slightly fast (-0.538) as λ_V increases. The jump intensity of the underlying asset has a positive impact on the expected payoff of the option. The default probability, or discounted rate of the vulnerable option price, shall be affected by the jump intensity of the assets of the counterparty.

Figure 4 plots option prices of the proposed model against mean jump sizes of S_t and V_t. When the mean jump size of the underlying asset changes from -0.5 to 0.5, vulnerable option prices perform a U-shaped curve. Different from other variables, monotonicity does not exist. Option prices arrive at the minimum point when the mean jump size of the underlying asset is -0.2. Jump risk enhances option values. Thus, the larger the absolute value of μ_1 is, the higher option prices become. By contrast, vulnerable option prices decrease monotonically when the mean jump size of the assets of the counterparty changes from -0.5 to 0.5. A positive jump size of the assets of the counterparty corresponds to a negative compensation term. Therefore, option values decrease with the mean jump size of the assets of the counterparty. Figure 5 shows option values against the standard deviation of the jump sizes of S_t and V_t. When σ_1 and σ_2 change from 0.05 to 0.5, respectively, the ranges of option values are (3.187, 5.424) and (3.406, 1.482).

5. CONCLUSION

We investigate the vulnerable American put option pricing where the dynamics is governed by a jump–diffusion model. Compared with the existing models for vulnerable option pricing, the main contribution is that we take jump risk into consideration. Shocks on both of the underlying asset and the assets of the counterparty are modeled by compound Poisson processes. Employing the two-point Geske–Johnson method, a vulnerable American put is approximated by a series of vulnerable European puts and twice-exercisable vulnerable European puts. Based on Bunch and Johnson [2], the approximate closed-form pricing formula for vulnerable American puts is derived. We also present and discuss numerical simulations of the pricing formula.

In the numerical illustrations, the proposed model is compared with the two-point Geske–Johnson jump–diffusion model. We further examine the performance of the proposed model under different parameters assumptions. We find that jump risk on the underlying asset has a positive effect on the price, while the impact of shocks from the assets of the counterparty on the price is negative.

Acknowledgments

The authors thank the anonymous referee and the editor for their helpful comments and valuable suggestions that led to several important improvements.

The authors acknowledge financial support from National Natural Science Foundation of China (grant numbers 71320107003, 71532009 and 71603049), Core Projects in Tianjin Education Bureaus Social Science Program (grant number 2014ZD13), Beijing Planning Office of Philosophy and Social Science (grant number 15JGB128), Youth Social Science Foundation of Beijing Federation of Social Science Circles (Grant No. QNRC201617), and the Fundamental Research Funds for the Central Universities in PPSUC grant number 2016JKF01207.

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APPENDIX

Recall that

$$\left(\ln\frac{S_{T/2}}{S_0},\ln\frac{V_{T/2}}{V_0},\ln\frac{S_T}{S_0},\ln\frac{V_T}{V_0}\right)$$

are normally distributed random variables in R^4. Let $(\xi_1,\xi_2,\xi_3,\xi_4)$ be standard normal random variables in R^4 with the same correlation matrix Ψ as

$$\left(\ln\frac{S_{T/2}}{S_0},\ln\frac{V_{T/2}}{V_0},\ln\frac{S_T}{S_0},\ln\frac{V_T}{V_0}\right).$$

In addition, the (i,j)-element ρ_{ij} of the correlation matrix Ψ can be obtained from (3.8)–(3.11). For example, the value of ρ_{12} is given by

$$\rho_{12} = \frac{\mathrm{Cov}\left(\ln \frac{S_{T/2}}{S_0}, \ln \frac{V_{T/2}}{V_0}\right)}{\sqrt{\mathrm{Var}\left(\ln \frac{S_{T/2}}{S_0}\right) \mathrm{Var}\left(\ln \frac{V_{T/2}}{V_0}\right)}} = \frac{\rho\sigma_S\sigma_V T/2}{\sqrt{(\sigma_S^2 T/2 + m_1\sigma_1^2)(\sigma_V^2 T/2 + m_2\sigma_2^2)}}.$$

Now we can rewrite

$$\left(\ln \frac{S_{T/2}}{S_0}, \ln \frac{V_{T/2}}{V_0}, \ln \frac{S_T}{S_0}, \ln \frac{V_T}{V_0}\right)$$

in (3.8)–(3.11) as follows:

$$\begin{aligned}
\ln \frac{S_{T/2,m_1}}{S_0} &= \left(r - \frac{1}{2}\sigma_S^2 - k_S\lambda_S\right) T/2 + m_1\mu_1 + \sqrt{\sigma_S^2 T/2 + m_1\sigma_1^2}\,\xi_1, \\
\ln \frac{V_{T/2,m_2}}{V_0} &= \left(r - \frac{1}{2}\sigma_V^2 - k_V\lambda_V\right) T/2 + m_2\mu_2 + \sqrt{\sigma_V^2 T/2 + m_2\sigma_2^2}\,\xi_2, \\
\ln \frac{S_{T,n_1}}{S_0} &= \left(r - \frac{1}{2}\sigma_S^2 - k_S\lambda_S\right) T + n_1\mu_1 + \sqrt{\sigma_S^2 T + n_1\sigma_1^2}\,\xi_3, \\
\ln \frac{V_{T,n_2}}{V_0} &= \left(r - \frac{1}{2}\sigma_V^2 - k_V\lambda_V\right) T + n_2\mu_2 + \sqrt{\sigma_V^2 T + n_2\sigma_2^2}\,\xi_4.
\end{aligned}$$

To simplify the calculations, we use the following notations:

$$\begin{aligned}
M_1(m_1) &:= \mathbb{E}\left[\ln \frac{S_{T/2}}{S_0}\right] = \left(r - \frac{1}{2}\sigma_S^2 - k_S\lambda_S\right) T/2 + m_1\mu_1, \\
M_2(m_2) &:= \mathbb{E}\left[\ln \frac{V_{T/2}}{V_0}\right] = \left(r - \frac{1}{2}\sigma_V^2 - k_V\lambda_V\right) T/2 + m_2\mu_2, \\
M_3(n_1) &:= \mathbb{E}\left[\ln \frac{S_T}{S_0}\right] = \left(r - \frac{1}{2}\sigma_S^2 - k_S\lambda_S\right) T + n_1\mu_1, \\
M_4(n_2) &:= \mathbb{E}\left[\ln \frac{V_T}{V_0}\right] = \left(r - \frac{1}{2}\sigma_V^2 - k_V\lambda_V\right) T + n_2\mu_2.
\end{aligned}$$

Moreover, recall the notations $b_1(n_1,t)$ and $b_2(n_2,t)$ in (3.2) and $b_1^*(n_1,t)$ in (3.5),

$$\begin{aligned}
b_1(n_1,t) &= \frac{\ln \frac{S_0}{K} + (r - \frac{1}{2}\sigma_S^2 - k_S\lambda_S)t + n_1\mu_1}{\sqrt{\sigma_S^2 t + n_1\sigma_1^2}}, \\
b_1^*(n_1,t) &= \frac{\ln \frac{S_0}{S_{T/2}^*} + (r - \frac{1}{2}\sigma_S^2 - k_S\lambda_S)t + n_1\mu_1}{\sqrt{\sigma_S^2 t + n_1\sigma_1^2}}, \\
b_2(n_2,t) &= \frac{\ln \frac{V_0}{D^*} + (r - \frac{1}{2}\sigma_V^2 - k_V\lambda_V)t + n_2\mu_2}{\sqrt{\sigma_V^2 t + n_2\sigma_2^2}}.
\end{aligned}$$

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Based on the above notations, we can derive A_1 as follows:

$$
\begin{aligned}
A_1 &= K\mathbb{E}\left[\mathbf{1}(S_{T/2,m_1} > S^*_{T/2}, V_{T/2,m_2} \geq D^*, S_{T,n_1} \leq K, V_{T,n_2} \geq D^*)\right] \\
&= K\mathbb{E}\left[\mathbf{1}\left(\ln\frac{S_{T/2,m_1}}{S_0} > \ln\frac{S^*_{T/2}}{S_0}, \ln\frac{V_{T/2,m_2}}{V_0} > \ln\frac{D^*}{V_0}, \ln\frac{S_{T,n_1}}{S_0} \leq \ln\frac{K}{S_0}, \ln\frac{V_{T,n_2}}{V_0} \geq \ln\frac{D^*}{V_0}\right)\right] \\
&= K\mathbb{E}\left[\mathbf{1}\left(\frac{\ln\frac{S_{T/2,m_1}}{S_0} - M_1(m_1)}{\sqrt{\sigma_S^2 T/2 + m_1\sigma_1^2}} > -b_1^*\left(m_1, \frac{T}{2}\right), \frac{\ln\frac{V_{T/2,m_2}}{V_0} - M_2(m_2)}{\sqrt{\sigma_V^2 T/2 + m_2\sigma_2^2}} > -b_2\left(m_2, \frac{T}{2}\right),\right.\right. \\
&\quad \times \left.\left.\frac{\ln\frac{S_{T,n_1}}{S_0} - M_3(n_1)}{\sqrt{\sigma_S^2 T + n_1\sigma_1^2}} \leq -b_1(n_1, T), \frac{\ln\frac{V_{T,n_2}}{V_0} - M_4(n_2)}{\sqrt{\sigma_V^2 T + n_2\sigma_2^2}} \geq -b_2(n_2, T)\right)\right] \\
&= K\mathbb{E}\left[\mathbf{1}\left(-\xi_1 \leq b_1^*\left(m_1, \frac{T}{2}\right), -\xi_2 \leq b_2\left(m_2, \frac{T}{2}\right), \xi_3 \leq -b_1(n_1, T), -\xi_4 \leq b_2(n_2, T)\right)\right] \\
&= KN_4\left(b_1^*\left(m_1, \frac{T}{2}\right), b_2\left(m_2, \frac{T}{2}\right), -b_1(n_1, T), b_2(n_2, T), \Psi_1\right), \qquad \textbf{(A.1)}
\end{aligned}
$$

where Ψ_1 is the correlation matrix of $(-\xi_1, -\xi_2, \xi_3, -\xi_4)$ under Q.

To derive A_2, we define a new probability measure Q_2 as follows:

$$
\left.\frac{dQ_2}{dQ}\right|_{\mathcal{F}_T} = e^{\sqrt{\sigma_S^2 T + n_1\sigma_1^2}\xi_3 - \frac{1}{2}(\sigma_S^2 T + n_1\sigma_1^2)}.
$$

Then, we have that under Q_2, $(\xi_1 - \rho_{13}\sqrt{\sigma_S^2 T + n_1\sigma_1^2}, \xi_2 - \rho_{23}\sqrt{\sigma_S^2 T + n_1\sigma_1^2}, \xi_3 - \sqrt{\sigma_S^2 T + n_1\sigma_1^2}, \xi_4 - \rho_{34}\sqrt{\sigma_S^2 T + n_1\sigma_1^2})$ are standard normal random variables in R^4 with the correlation matrix Ψ. Therefore, the closed form for A_2 is derived similarly,

$$
\begin{aligned}
A_2 &= -\mathbb{E}\left[S_{T,n_1}\mathbf{1}(S_{T/2,m_1} > S^*_{T/2}, V_{T/2,m_2} \geq D^*, S_{T,n_1} \leq K, V_{T,n_2} \geq D^*)\right] \\
&= -\mathbb{E}\left[S_{T,n_1}\mathbf{1}\left(-\xi_1 \leq b_1^*\left(m_1, \frac{T}{2}\right), -\xi_2 \leq b_2\left(m_2, \frac{T}{2}\right), \xi_3 \leq -b_1(n_1, T), -\xi_4 \leq b_2(n_2, T)\right)\right] \\
&= -S_0 e^{M_3(n_1) + \frac{1}{2}(\sigma_S^2 T + n_1\sigma_1^2)}\mathbb{E}\left[e^{\sqrt{\sigma_S^2 T + n_1\sigma_1^2}\xi_3 - \frac{1}{2}(\sigma_S^2 T + n_1\sigma_1^2)}\right. \\
&\quad \times \left.\mathbf{1}\left(-\xi_1 \leq b_1^*\left(m_1, \frac{T}{2}\right), -\xi_2 \leq b_2\left(m_2, \frac{T}{2}\right), \xi_3 \leq -b_1(n_1, T), -\xi_4 \leq b_2(n_2, T)\right)\right] \\
&= -S_0 e^{M_3(n_1) + \frac{1}{2}(\sigma_S^2 T + n_1\sigma_1^2)} \\
&\quad \times \mathbb{E}^{Q_2}\left[\mathbf{1}\left(-\xi_1 \leq b_1^*\left(m_1, \frac{T}{2}\right), -\xi_2 \leq b_2\left(m_2, \frac{T}{2}\right), \xi_3 \leq -b_1(n_1, T), -\xi_4 \leq b_2(n_2, T)\right)\right] \\
&= -S_0 e^{M_3(n_1) + \frac{1}{2}(\sigma_S^2 T + n_1\sigma_1^2)}\mathbb{E}^{Q_2}\left[\mathbf{1}\left(-\xi_1 + \rho_{13}\sqrt{\sigma_S^2 T + n_1\sigma_1^2} \leq b_1^*\left(m_1, \frac{T}{2}\right)\right.\right. \\
&\quad + \rho_{13}\sqrt{\sigma_S^2 T + n_1\sigma_1^2}, -\xi_2 + \rho_{23}\sqrt{\sigma_S^2 T + n_1\sigma_1^2} \leq b_2\left(m_2, \frac{T}{2}\right) + \rho_{23}\sqrt{\sigma_S^2 T + n_1\sigma_1^2}, \\
&\quad \xi_3 - \sqrt{\sigma_S^2 T + n_1\sigma_1^2} \leq -b_1(n_1, T) - \sqrt{\sigma_S^2 T + n_1\sigma_1^2}, \\
&\quad \left.\left. -\xi_4 + \rho_{34}\sqrt{\sigma_S^2 T + n_1\sigma_1^2} \leq b_2(n_2, T) + \rho_{34}\sqrt{\sigma_S^2 T + n_1\sigma_1^2}\right)\right]
\end{aligned}
$$

https://doi.org/10.1017/S0269964816000486 Published online by Cambridge University Press

$$= -S_0 e^{M_3(n_1)+\frac{1}{2}(\sigma_S^2 T+n_1\sigma_1^2)} N_4\Bigg(b_1^*\left(m_1,\frac{T}{2}\right)+\rho_{13}\sqrt{\sigma_S^2T+n_1\sigma_1^2}, b_2\left(m_2,\frac{T}{2}\right)$$
$$+\rho_{23}\sqrt{\sigma_S^2T+n_1\sigma_1^2}, -b_1(n_1,T)-\sqrt{\sigma_S^2T+n_1\sigma_1^2}, b_2(n_2,T)+\rho_{34}\sqrt{\sigma_S^2T+n_1\sigma_1^2}, \Psi_1\Bigg), \tag{A.2}$$

where Ψ_1 is the correlation matrix of $(-\xi_1, -\xi_2, \xi_3, -\xi_4)$ under Q_2 (Q). Analogously, we can obtain the explicit expression for A_3 by defining another probability measure Q_3 below:

$$\frac{dQ_3}{dQ}\bigg|_{\mathcal{F}_T} = e^{\sqrt{\sigma_V^2 T+n_2\sigma_2^2}\xi_4-\frac{1}{2}(\sigma_V^2T+n_2\sigma_2^2)}.$$

Therefore, it holds that

$$A_3 = K\mathbb{E}\left[\frac{(1-\alpha)V_{T,n_2}}{D}\mathbf{1}(S_{T/2,m_1} > S^*_{T/2}, V_{T/2,m_2}\geq D^*, S_{T,n_1}\leq K, V_{T,n_2}< D^*)\right]$$
$$= \frac{(1-\alpha)K}{D}\mathbb{E}\Bigg[V_{T,n_2}\mathbf{1}\Bigg(-\xi_1\leq b_1^*\left(m_1,\frac{T}{2}\right), -\xi_2\leq b_2\left(m_2,\frac{T}{2}\right), \xi_3\leq -b_1(n_1,T),$$
$$\times\ \xi_4\leq -b_2(n_2,T)\Bigg)\Bigg]$$
$$= \frac{(1-\alpha)KV_0}{D}e^{M_4(n_2)+\frac{1}{2}(\sigma_V^2T+n_2\sigma_2^2)}\mathbb{E}^{Q_2}\Bigg[\mathbf{1}\Bigg(-\xi_1+\rho_{14}\sqrt{\sigma_V^2T+n_2\sigma_2^2}\leq b_1^*\left(m_1,\frac{T}{2}\right)$$
$$+\rho_{14}\sqrt{\sigma_V^2T+n_2\sigma_2^2}, -\xi_2+\rho_{24}\sqrt{\sigma_V^2T+n_2\sigma_2^2}\leq b_2\left(m_2,\frac{T}{2}\right)+\rho_{24}\sqrt{\sigma_V^2T+n_2\sigma_2^2},$$
$$\xi_3-\rho_{34}\sqrt{\sigma_V^2T+n_2\sigma_2^2}\leq -b_1(n_1,T)-\rho_{34}\sqrt{\sigma_V^2T+n_2\sigma_2^2},$$
$$\xi_4-\sqrt{\sigma_V^2T+n_2\sigma_2^2}\leq b_2(n_2,T)-\sqrt{\sigma_V^2T+n_2\sigma_2^2}\Bigg)\Bigg]$$
$$= \frac{(1-\alpha)KV_0}{D}e^{M_4(n_2)+\frac{1}{2}(\sigma_V^2T+n_2\sigma_2^2)}N_4\Bigg(b_1^*\left(m_1,\frac{T}{2}\right)+\rho_{14}\sqrt{\sigma_V^2T+n_2\sigma_2^2}, b_2\left(m_2,\frac{T}{2}\right)$$
$$+\rho_{24}\sqrt{\sigma_V^2T+n_2\sigma_2^2}, -b_1(n_1,T)-\rho_{34}\sqrt{\sigma_V^2T+n_2\sigma_2^2}, b_2(n_2,T)+\sqrt{\sigma_V^2T+n_2\sigma_2^2}, \Psi_3\Bigg), \tag{A.3}$$

where we have used the fact that $(\xi_1-\rho_{14}\sqrt{\sigma_V^2T+n_2\sigma_2^2}, \xi_2-\rho_{24}\sqrt{\sigma_V^2T+n_2\sigma_2^2}, \xi_3-\rho_{34}\sqrt{\sigma_V^2T+n_2\sigma_2^2}, \xi_4-\sqrt{\sigma_V^2T+n_2\sigma_2^2})$ are standard normal random variables in R^4 with the correlation matrix Ψ under Q_3. In addition, Ψ_3 is the correlation matrix of $(-\xi_1, -\xi_2, \xi_3, \xi_4)$ under Q_3 (Q).

In the following, we focus on the derivation of the term A_4. Similarly, we can work under a new probability measure Q_4 defined below:

$$\frac{dQ_4}{dQ}\bigg|_{\mathcal{F}_T} = e^{\sqrt{\sigma_S^2T+n_1\sigma_1^2}\xi_3+\sqrt{\sigma_V^2T+n_2\sigma_2^2}\xi_4-\frac{1}{2}(\sigma_S^2T+n_1\sigma_1^2)-\frac{1}{2}(\sigma_V^2T+n_2\sigma_2^2)-\rho_{34}\sqrt{(\sigma_S^2T+n_1\sigma_1^2)(\sigma_V^2T+n_2\sigma_2^2)}}.$$

https://doi.org/10.1017/S0269964816000486 Published online by Cambridge University Press

Based on the definition of Q_4, we have that under Q_4, $(\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3, \bar{\xi}_4)$ are standard normal random variables in R^4 with the correlation matrix Ψ, where

$$\bar{\xi}_1 := \xi_1 - \rho_{13}\sqrt{\sigma_S^2 T + n_1\sigma_1^2} - \rho_{14}\sqrt{\sigma_V^2 T + n_2\sigma_2^2},$$
$$\bar{\xi}_2 := \xi_2 - \rho_{23}\sqrt{\sigma_S^2 T + n_1\sigma_1^2} - \rho_{24}\sqrt{\sigma_V^2 T + n_2\sigma_2^2},$$
$$\bar{\xi}_3 := \xi_3 - \sqrt{\sigma_S^2 T + n_1\sigma_1^2} - \rho_{34}\sqrt{\sigma_V^2 T + n_2\sigma_2^2},$$
$$\bar{\xi}_4 := \xi_4 - \rho_{34}\sqrt{\sigma_S^2 T + n_1\sigma_1^2} - \sqrt{\sigma_V^2 T + n_2\sigma_2^2}.$$

To simplify the derivation of A_4, we introduce the following notations:

$$e^*(m_1, n_1, n_2, t) = b_1^*(m_1, t) + \rho_{13}\sqrt{\sigma_S^2 T + n_1\sigma_1^2} + \rho_{14}\sqrt{\sigma_V^2 T + n_2\sigma_2^2},$$
$$e(n_1, n_2, t) = b_1(n_1, t) - \sqrt{\sigma_S^2 T + n_1\sigma_1^2} - \rho_{34}\sqrt{\sigma_V^2 T + n_2\sigma_2^2},$$
$$f(m_2, n_1, n_2, t) = b_2(m_2, t) + \rho_{23}\sqrt{\sigma_S^2 T + n_1\sigma_1^2} + \rho_{24}\sqrt{\sigma_V^2 T + n_2\sigma_2^2},$$
$$g(n_1, n_2, t) = b_2(n_2, t) - \rho_{34}\sqrt{\sigma_S^2 T + n_1\sigma_1^2} - \sqrt{\sigma_V^2 T + n_2\sigma_2^2}.$$

Similar to A_3, one gets that

$$\begin{aligned}
A_4 &= -\mathbb{E}\left[S_{T,n_1}\frac{(1-\alpha)V_{T,n_2}}{D}\mathbf{1}\left(S_{T/2,m_1} > S^*_{T/2}, V_{T/2,m_2} \geq D^*, S_{T,n_1} \leq K, V_{T,n_2} < D^*\right)\right] \\
&= -\frac{(1-\alpha)}{D}\mathbb{E}\left[S_{T,n_1}V_{T,n_2}\mathbf{1}\left(-\xi_1 \leq b_1^*\left(m_1, \frac{T}{2}\right),\right.\right. \\
&\quad \left.\left. -\ \xi_2 \leq b_2\left(m_2, \frac{T}{2}\right), \xi_3 \leq -b_1(n_1, T), \xi_4 \leq -b_2(n_2, T)\right)\right] \\
&= -\frac{(1-\alpha)}{D}e^{M_3(n_1)+M_4(n_2)+\frac{1}{2}(\sigma_S^2 T+n_1\sigma_1^2)+\frac{1}{2}(\sigma_V^2 T+n_2\sigma_2^2)+\rho_{34}\sqrt{(\sigma_S^2 T+n_1\sigma_1^2)(\sigma_V^2 T+n_2\sigma_2^2)}} \\
&\quad \times \mathbb{E}^{Q_4}\left[\mathbf{1}\left(-\xi_1 \leq b_1^*\left(m_1, \frac{T}{2}\right), -\xi_2 \leq b_2\left(m_2, \frac{T}{2}\right), \xi_3 \leq -b_1(n_1, T), \xi_4 \leq -b_2(n_2, T)\right)\right] \\
&= -\frac{(1-\alpha)}{D}e^{M_3(n_1)+M_4(n_2)+\frac{1}{2}(\sigma_S^2 T+n_1\sigma_1^2)+\frac{1}{2}(\sigma_V^2 T+n_2\sigma_2^2)+\rho_{34}\sqrt{(\sigma_S^2 T+n_1\sigma_1^2)(\sigma_V^2 T+n_2\sigma_2^2)}} \\
&\quad \times \mathbb{E}^{Q_4}\left[\mathbf{1}\left(-\bar{\xi}_1 \leq e^*\left(m_1, n_1, n_2, \frac{T}{2}\right),\right.\right. \\
&\quad \left.\left. -\ \bar{\xi}_2 \leq f\left(m_2, n_1, n_2, \frac{T}{2}\right), \bar{\xi}_3 \leq -e(n_1, n_2, T), \bar{\xi}_4 \leq -g(n_1, n_2, T)\right)\right] \\
&= -\frac{(1-\alpha)}{D}S_0V_0e^{M_3(n_1)+M_4(n_2)+\frac{1}{2}(\sigma_S^2 T+n_1\sigma_1^2)+\frac{1}{2}(\sigma_V^2 T+n_2\sigma_2^2)+\rho_{34}\sqrt{(\sigma_S^2 T+n_1\sigma_1^2)(\sigma_V^2 T+n_2\sigma_2^2)}} \\
&\quad \times N_4\left(e^*\left(m_1, n_1, n_2, \frac{T}{2}\right), f\left(m_2, n_1, n_2, \frac{T}{2}\right), -e(n_1, n_2, T), -g(n_1, n_2, T), \Psi_3\right), \qquad \textbf{(A.4)}
\end{aligned}$$

where Ψ_3 is the correlation matrix of $(-\xi_1, -\xi_2, \xi_3, \xi_4)$ under Q_4 (Q).