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An origami of genus 3 with arithmetic Kontsevich–Zorich monodromy

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Abstract

In this we exploit the arithmeticity criterion of Oh and Benoist–Miquel to exhibit an origami in the principal stratum of the moduli space of translation surfaces of genus three whose Kontsevich–Zorich monodromy is not thin in the sense of Sarnak.

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1. *Introduction*

The dynamics of the action of $SL(2, \mathbb{R})$ on moduli spaces of translation surfaces is driven by the Kontsevich–Zorich monodromy consisting of the matrices encoding changes of basis in absolute homology of translation surfaces along $SL(2, \mathbb{R})$-orbits.

The nature of the Kontsevich–Zorich monodromy depends heavily on the support of the ergodic $SL(2, \mathbb{R})$-invariant probability measure, and Sarnak asked how often a Kontsevich–Zorich monodromy is arithmetic or thin[1] in his sense (compare with [**Sa**, section 3·2]).

In the case of Masur–Veech measures (of connected components of the strata of moduli spaces of translation surfaces), the corresponding Kontsevich–Zorich monodromies contain the Rauzy–Veech groups[2], and, as it turns out, the arithmeticity of Rauzy–Veech groups was recently established in [**AMY**] and [**Gu**]. In particular, the Kontsevich–Zorich monodromies associated to Masur–Veech measures are always arithmetic.

In this paper, we focus on the Kontsevich–Zorich monodromies of the natural measures supported on Teichmüller curves[3]. Any Teichmüller curve is known to be defined over a

[1]Recall that a subgroup $\Gamma \subset GL_n(\mathbb{Z})$ with Zariski closure G is called arithmetic, resp. thin, when the index of Γ in the subgroup $G(\mathbb{Z})$ (of integral points of G) is finite, resp. infinite.

[2]Coming from a combinatorial process called Rauzy–Veech algorithm.

[3]Closed $SL(2, \mathbb{R})$-orbits in moduli spaces of translation surfaces.

totally real number field, and we say that a Teichmüller curve is arithmetic if and only if it is defined over $\mathbb{Q}$. Equivalently, a Teichmüller curve is arithmetic if and only if it contains an origami / square-tiled surface[4].

In the moduli space of translation surfaces of genus 2, it can be shown that the Kontsevich–Zorich monodromy of non-arithmetic, resp. arithmetic, Teichmüller curves are thin, resp. arithmetic (cf. [**Sa**, section 3·2]).[5]

In the moduli space of translation surfaces of genus 3, the main theorem of this note ensures the existence of arithmetic Kontsevich–Zorich monodromies associated to an arithmetic Teichmüller curves.

THEOREM 1·1. *The non-tautological part*[6] *of the Kontsevich–Zorich monodromy associated to a certain Teichmüller curve $\mathcal{C}$ generated by a certain origami $\mathcal{O}_1$ of genus* 3 *is arithmetic.*

Remark 1·2. It would be interesting to know whether the "majority" of non-tautological parts of Kontsevich–Zorich monodromies of origamis of genus 3 is arithmetic: for instance, is it true that the Kontsevich–Zorich monodromies of all but finitely many origamis in the minimal stratum $\mathcal{H}(4)$ of the moduli space of translation surfaces of genus 3 are arithmetic?

Closing this short introduction, let us describe the organisation of this paper. In Section 2, we describe the origami $\mathcal{O}_1$ and its Teichmüller curve $\mathcal{C}$. In Section 3, we compute the Kontsevich–Zorich monodromy of $\mathcal{C}$: in particular, we describe two 4×4 matrices (called $\rho(a)$ and $\rho(b)$ below) generating the non-tautological part of the Kontsevich–Zorich monodromy of $\mathcal{C}$. Finally, we rephrase Theorem 1·1 as Theorem 4·3 below (for the sake of convenience), and we show that the desired arithmeticity statement can be deduced from a recent theorem of Benoist–Miquel [**BM**] after some computations with certain powers of the two 4×4 matrices introduced above.

Remark 1·3. In this paper, we assume some familiarity with the basic features of origamis. In particular, the reader is invited to consult [**FM**, section 8 and Appendix C] for more details about the representation of origamis via permutations, the Veech and affine groups of origamis, etc.

[4]I.e., a translation surface obtained from a finite collection of squares of fixed sizes by gluing by translations pairs of parallel sides.

[5]On the other hand, to the best of our knowledge, it seems that there are no available results concerning the arithmeticity or thinness of the Kontsevich–Zorich monodromy of Teichmüller curves in moduli spaces of translation surfaces of genus $g \geqslant 3$.

[6]Here, the non-tautological part of the Kontsevich–Zorich monodromy of an origami X means the following. The absolute homology of an origami X admits a decomposition defined over $\mathbb{Z}$ into the direct sum of a tautological plane $H_1^{st}(X)$ and its symplectic orthogonal $H_1^{(0)}(X)$ (with respect to the intersection form). The Kontsevich–Zorich monodromy respects this decomposition and the non-tautological part of the Kontsevich–Zorich monodromy is its restriction to $H_1^{(0)}$. In particular, the non-tautological part of the Kontsevich–Zorich monodromy is a subgroup of $Sp(H_1^{(0)}(X)) \simeq Sp(2g-2, \mathbb{Z})$, where g is the genus of X.

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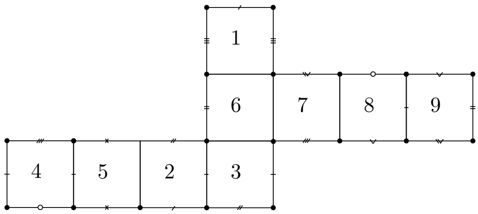

Fig. 1. Flat geometry of $\mathcal{O}_1$.

2. *An arithmetic Teichmüller curve $\mathcal{C}$ with a single cusp*

2·1. *The origami $\mathcal{O}_1$*

Consider the square-tiled surface $\mathcal{O}_1$ associated to the pair of permutations

$$h_{\mathcal{O}_1} = (1)(2, 3, 4, 5)(6, 7, 8, 9), \quad v_{\mathcal{O}_1} = (1, 2, 3, 6)(4, 7, 9, 8)(5).$$

The commutator $[h_{\mathcal{O}_1}, v_{\mathcal{O}_1}] := v_{\mathcal{O}_1} h_{\mathcal{O}_1} v_{\mathcal{O}_1}^{-1} h_{\mathcal{O}_1}^{-1}$ is

$$[h_{\mathcal{O}_1}, v_{\mathcal{O}_1}] = (1, 9)(2, 3)(4, 6)(5, 8)(7),$$

so that $\mathcal{O}_1 \in \mathcal{H}(1, 1, 1, 1)$ is a genus 3 square-tiled surface. See Fig. 1 above.

The $SL(2, \mathbb{Z})$-orbit of $\mathcal{O}_1$ consists of four elements. Indeed, this fact can be checked as follows. We recall that:

(i) the generators $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ of $SL(2, \mathbb{Z})$ act on pairs of permutations (h, v) by the rules $T(h, v) = (h, vh^{-1})$ and $S(h, v) = (hv^{-1}, v)$;

(ii) the pairs of permutations (h, v) and $(\phi h \phi^{-1}, \phi v \phi^{-1})$ give rise to the same square-tiled surface.

Therefore, the T-orbit of $\mathcal{O}_1$ is $\{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4\}$ where $\mathcal{O}_k := T^k(\mathcal{O}_1)$ is given by the pair of permutations $(h_{\mathcal{O}_1}, v_{\mathcal{O}_k})$ with

$$v_{\mathcal{O}_2} = (1, 2, 5, 7)(3)(4, 6, 8, 9), \quad v_{\mathcal{O}_3} = (1, 2, 7, 8)(3, 5, 6, 4)(9),$$

$$v_{\mathcal{O}_4} = (1, 2, 6, 9)(3, 7, 4, 5)(8).$$

As it turns out, the T-orbit of $\mathcal{O}_1$ accounts for its entire $SL(2, \mathbb{Z})$-orbit because

$$S(\mathcal{O}_1) = (\phi_4^{-1} h_{\mathcal{O}_1} \phi_4, \phi_4^{-1} v_{\mathcal{O}_4} \phi_4) \simeq \mathcal{O}_4, \quad S^2(\mathcal{O}_1) = (\phi_3^{-1} h_{\mathcal{O}_1} \phi_3, \phi_3^{-1} v_{\mathcal{O}_3} \phi_3) \simeq \mathcal{O}_3,$$

$$S^3(\mathcal{O}_1) = (\phi_2^{-1} h_{\mathcal{O}_1} \phi_2, \phi_2^{-1} v_{\mathcal{O}_2} \phi_2) \simeq \mathcal{O}_2,$$

where

$$\phi_4 = (1, 6, 2, 9, 4, 3)(5, 8)(7), \quad \phi_3 = (1, 5, 9, 8)(2, 6, 3, 4)(7)$$

and

$$\phi_2 = (1, 9)(2, 4, 5, 3, 6, 8)(7).$$

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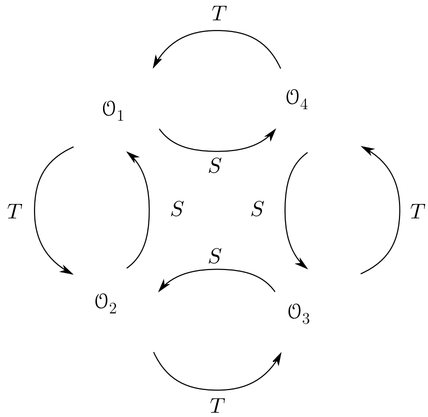

Fig. 2. $SL(2,\mathbb{Z})$-orbit of $\mathcal{O}_1$.

Remark 2·1. For later use, observe that the matrix $-\mathrm{Id}$ acts on pairs of permutations by $-\mathrm{Id}(h, v) = (h^{-1}, v^{-1})$. In particular, the action of $-\mathrm{Id}$ on $SL(2,\mathbb{Z})\cdot\mathcal{O}_1$ is completely described by the formulas

$$-\mathrm{Id}(\mathcal{O}_1) = (\psi_3^{-1} h_{\mathcal{O}_1}\psi_3, \psi_3^{-1} v_{\mathcal{O}_3}\psi_3) \simeq \mathcal{O}_3, \quad -\mathrm{Id}(\mathcal{O}_2) = (\psi_4^{-1} h_{\mathcal{O}_1}\psi_4, \psi_4^{-1} v_{\mathcal{O}_4}\psi_4) \simeq \mathcal{O}_4$$

where $\psi_3 := (1)(2, 8, 4, 6)(3, 7, 5, 9)$ and $\psi_4 := (1)(2, 9, 4, 7)(3, 8, 5, 6)$.

In summary, the $SL(2,\mathbb{Z})$-orbit of $\mathcal{O}_1$ can be depicted as in Figure 2 below.

It follows from this discussion that $SL(2,\mathbb{R})\cdot\mathcal{O}_1$ has a single cusp (i.e., single T-orbit in $SL(2,\mathbb{Z})\cdot\mathcal{O}_1$).

Remark 2·2. The homological dimension of $SL(2,\mathbb{R})\cdot\mathcal{O}_1$ in the sense of Forni [**Fo**] is three. Thus, by the results in [**Fo**], the Lyapunov spectrum of the Kontsevich–Zorich cocycle over $SL(2,\mathbb{R})\cdot\mathcal{O}_1$ with respect to the Haar measure has the form

$$1 = \lambda_1 > \lambda_2 \geqslant \lambda_3 > 0 > -\lambda_3 \geqslant -\lambda_2 > -\lambda_1 = -1$$

Moreover, the Eskin–Kontsevich–Zorich formula [**EKZ**] for the sum of non-negative Lyapunov exponents of the Kontsevich–Zorich cocycle implies that $1 + \lambda_2 + \lambda_3 = 2$, i.e.,

$$\lambda_2 + \lambda_3 = 1.$$

Finally, some *numerical* experiments indicate that $\lambda_2 \simeq 0.57...$ and $\lambda_3 \simeq 0.43...$

2·2. *The stabiliser of* $\mathcal{C}$

The group $\mathrm{Aff}(\mathcal{O}_1)$ of affine homeomorphisms of $\mathcal{O}_1$ is the stabilizer of $\mathcal{C} := SL(2,\mathbb{R})\cdot\mathcal{O}_1$ in the moduli space of translation surfaces.

It is not hard to see that the subgroup $\mathrm{Aut}(\mathcal{O}_1) \subset \mathrm{Aff}(\mathcal{O}_1)$ of automorphisms of $\mathcal{O}_1$ is trivial. It follows that the elements of $\mathrm{Aff}(\mathcal{O}_1)$ are determined by their linear parts in $SL(2,\mathbb{R})$, that is, the natural map

$$\mathrm{Aff}(\mathcal{O}_1) \longrightarrow SL(2,\mathbb{R})$$

is injective. Hence, $\mathrm{Aff}(\mathcal{O}_1)$ is isomorphic to its image $SL(\mathcal{O}_1)$ under this map.

The group $SL(\mathcal{O}_1)$ is the finite-index subgroup of $SL(2, \mathbb{Z})$ consisting of all elements of $SL(2, \mathbb{R})$ stabilising $\mathcal{O}_1$: in the literature, $SL(\mathcal{O}_1)$ is called the Veech group of $\mathcal{O}_1$.

From Figure 2 above, we see that $SL(\mathcal{O}_1)$ is an index four subgroup of $SL(2, \mathbb{Z})$. Furthermore, $SL(\mathcal{O}_1)$ is a congruence subgroup of level 4, and the Teichmüller curve $\mathcal{C} = SL(2, \mathbb{R})/SL(\mathcal{O}_1)$ has genus zero. Thus, $SL(\mathcal{O}_1)$ is generated by elliptic and parabolic elements: indeed, one can check that $SL(\mathcal{O}_1)$ is generated by the following two elliptic matrices

$$a := \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad b := \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix}$$

of orders 3.

The group structure of $SL(\mathcal{O}_1)$ is provided by the following lemma:

LEMMA 2·3. *$SL(\mathcal{O}_1)$ is the free product*

$$SL(\mathcal{O}_1) = \langle a \rangle * \langle b \rangle \simeq \mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}.$$

Proof. Consider the twelve cones $C_k \subset \mathbb{R}^2 - \{(0, 0)\}$ defined by the following properties:

(i) $C_{6+l} = -C_l$ for each $l = 1, \dots, 6$;
(ii) each C_l, $l = 1, \dots, 6$, consists of the convex combinations of positive multiples of the vectors v_l and v_{l+1}, where $v_1 := (1, 0)$, $v_2 := (2, 1)$, $v_3 := (1, 1)$, $v_4 := (1, 2)$, $v_5 := (0, 1)$, $v_6 := (-1, 1)$ and $v_7 := (-1, 0)$.

A simple calculation shows that

(i) $a(v_l) = v_{l+4}$ for each $k = 1, \dots, 6$;
(ii) $b(v_1) = v_3$, $b(v_2) = v_7$, $b(v_3) = -v_2$, $b(v_4) = (-5, -3) \in C_8$, $b(v_5) = (-3, -2) \in C_8$, $b(v_6) = (-4, -3) \in C_8$ and $b(v_7) = -v_3$.

It follows that $\{a, a^2\} = \langle a \rangle \setminus \{\text{Id}\}$ and $\{b, b^2\} = \langle b \rangle \setminus \{\text{Id}\}$ play ping-pong with the tables

$$X := (C_1 \cup C_2) \cup (C_7 \cup C_8)$$

and

$$Y := C_3 \cup C_4 \cup C_5 \cup C_6 \cup C_9 \cup C_{10} \cup C_{11} \cup C_{12}$$

in the sense that X and Y are disjoint subsets of $\mathbb{R}^2$ such that

(i) $a(X) = (C_5 \cup C_6) \cup (C_{11} \cup C_{12}) \subset Y$, $a^2(X) = (C_9 \cup C_{10}) \cup (C_3 \cup C_4) \subset Y$;
(ii) $b(Y) \subset C_2 \cup C_8 \subset X$, $b^2(Y) \subset C_1 \cup C_7 \subset X$.

By the ping-pong lemma[7], we conclude that $SL(\mathcal{O}_1) = \langle a \rangle * \langle b \rangle$.

Remark 2·4. The construction of these cones was inspired by Brav–Thomas paper [**BT**].

[7] Here, we are using the version of the ping-pong lemma stated as [**BT**, Theorem 2·1].

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3. *The Kontsevich–Zorich monodromy of* $\mathcal{C}$

The representation $\alpha : \mathrm{Aff}(\mathcal{O}_1) \to \mathrm{Sp}(H_1(\mathcal{O}_1, \mathbb{Z}))$ is called Kontsevich-Zorich cocycle over $\mathcal{C}$. In the sequel, we will compute the image under α of the generators a and b of $SL(\mathcal{O}_1) \simeq \mathrm{Aff}(\mathcal{O}_1)$.

3·1. *The relative homology groups of* $\mathcal{O}_k$, $k = 1, \ldots, 4$

Given $\mathcal{O}_k \in SL(2, \mathbb{Z}) \cdot \mathcal{O}_1$, $k = 1, \ldots, 4$, let us denote by $\sigma_g^{(k)}$, resp., $\zeta_g^{(k)}$ the relative cycles on $\mathcal{O}_k$ consisting of the bottommost horizontal and leftmost vertical sides of the square numbered $g \in \{1, \ldots, 9\}$.

Note that each square g of $\mathcal{O}_k$ gives a relation $\sigma_g^{(k)} + \zeta_{h_{\mathcal{O}_1}(g)}^{(k)} = \zeta_g^{(k)} + \sigma_{v_{\mathcal{O}_k}(g)}^{(k)}$, that is.

(i) $\sigma_1^{(1)} = \sigma_2^{(1)}$, $\sigma_2^{(1)} + \zeta_3^{(1)} = \zeta_2^{(1)} + \sigma_3^{(1)}$, $\sigma_3^{(1)} + \zeta_4^{(1)} = \zeta_3^{(1)} + \sigma_6^{(1)}$, $\sigma_4^{(1)} + \zeta_5^{(1)} = \zeta_4^{(1)} + \sigma_7^{(1)}$, $\zeta_2^{(1)} = \zeta_5^{(1)}$, $\sigma_6^{(1)} + \zeta_7^{(1)} = \zeta_6^{(1)} + \sigma_1^{(1)}$, $\sigma_7^{(1)} + \zeta_8^{(1)} = \zeta_7^{(1)} + \sigma_9^{(1)}$, $\sigma_8^{(1)} + \zeta_9^{(1)} = \zeta_8^{(1)} + \sigma_4^{(1)}$, $\sigma_9^{(1)} + \zeta_6^{(1)} = \zeta_9^{(1)} + \sigma_8^{(1)}$;

(ii) $\sigma_1^{(2)} = \sigma_2^{(2)}$, $\sigma_2^{(2)} + \zeta_3^{(2)} = \zeta_2^{(2)} + \sigma_5^{(2)}$, $\zeta_3^{(2)} = \zeta_4^{(2)}$, $\sigma_4^{(2)} + \zeta_5^{(2)} = \zeta_4^{(2)} + \sigma_6^{(2)}$, $\sigma_5^{(2)} + \zeta_2^{(2)} = \zeta_5^{(2)} + \sigma_7^{(2)}$, $\sigma_6^{(2)} + \zeta_7^{(2)} = \zeta_6^{(2)} + \sigma_8^{(2)}$, $\sigma_7^{(2)} + \zeta_8^{(2)} = \zeta_7^{(2)} + \sigma_1^{(2)}$, $\sigma_8^{(2)} + \zeta_9^{(2)} = \zeta_8^{(2)} + \sigma_9^{(2)}$, $\sigma_9^{(2)} + \zeta_6^{(2)} = \zeta_9^{(2)} + \sigma_4^{(2)}$;

(iii) $\sigma_1^{(3)} = \sigma_2^{(3)}$, $\sigma_2^{(3)} + \zeta_3^{(3)} = \zeta_2^{(3)} + \sigma_7^{(3)}$, $\sigma_3^{(3)} + \zeta_4^{(3)} = \zeta_3^{(3)} + \sigma_5^{(3)}$, $\sigma_4^{(3)} + \zeta_5^{(3)} = \zeta_4^{(3)} + \sigma_3^{(3)}$, $\sigma_5^{(3)} + \zeta_2^{(3)} = \zeta_5^{(3)} + \sigma_6^{(3)}$, $\sigma_6^{(3)} + \zeta_7^{(3)} = \zeta_6^{(3)} + \sigma_4^{(3)}$, $\sigma_7^{(3)} + \zeta_8^{(3)} = \zeta_7^{(3)} + \sigma_8^{(3)}$, $\sigma_8^{(3)} + \zeta_9^{(3)} = \zeta_8^{(3)} + \sigma_1^{(3)}$, $\zeta_6^{(3)} = \zeta_9^{(3)}$;

(iv) $\sigma_1^{(4)} = \sigma_2^{(4)}$, $\sigma_2^{(4)} + \zeta_3^{(4)} = \zeta_2^{(4)} + \sigma_6^{(4)}$, $\sigma_3^{(4)} + \zeta_4^{(4)} = \zeta_3^{(4)} + \sigma_7^{(4)}$, $\sigma_4^{(4)} + \zeta_5^{(4)} = \zeta_4^{(4)} + \sigma_5^{(4)}$, $\sigma_5^{(4)} + \zeta_2^{(4)} = \zeta_5^{(4)} + \sigma_3^{(4)}$, $\sigma_6^{(4)} + \zeta_7^{(4)} = \zeta_6^{(4)} + \sigma_9^{(4)}$, $\sigma_7^{(4)} + \zeta_8^{(4)} = \zeta_7^{(4)} + \sigma_4^{(4)}$, $\zeta_8^{(4)} = \zeta_9^{(4)}$, $\sigma_9^{(4)} + \zeta_6^{(4)} = \zeta_9^{(4)} + \sigma_1^{(4)}$.

3·2. *The action of* $SL(2, \mathbb{Z})$ *on the relative homology groups*

The matrix $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ takes $\mathcal{O}_k$ to $\mathcal{O}_{k+1}$, and it acts on the corresponding relative homology groups by the matrix $T_{k,k+1}$ such that

$$T_{k,k+1}(\sigma_g^{(k)}) = \sigma_g^{(k+1)}, \quad T_{k,k+1}(\zeta_g^{(k)}) = \sigma_g^{(k+1)} + \zeta_{h_{\mathcal{O}_1}(g)}^{(k+1)}.$$

Similarly, the matrix $S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ takes $\mathcal{O}_k$ to $\mathcal{O}_{k-1}$, and it acts on the corresponding relative homology groups by the matrix $S_{k+1,k}$ such that

$$S_{k+1,k}(\sigma_g^{(k+1)}) = \zeta_{\phi_k(g)}^{(k)} + \sigma_{v_{\mathcal{O}_k}(\phi_k(g))}^{(k)}, \quad S_{k+1,k}(\zeta_g^{(k+1)}) = \zeta_{\phi_k(g)}^{(k)}.$$

Finally, $-\mathrm{Id}$ exchange $\mathcal{O}_1$ and $\mathcal{O}_3$, resp. $\mathcal{O}_2$ and $\mathcal{O}_4$, and it acts on the corresponding relative homology groups by the matrices $(-\mathrm{Id})_{1,3} = (-\mathrm{Id})_{3,1}^{-1}$ and $(-\mathrm{Id})_{2,4} = (-\mathrm{Id})_{4,2}^{-1}$ such that

$$(-\mathrm{Id})_{1,3}(\sigma_g^{(1)}) = -\sigma_{v_{\mathcal{O}_3}(\psi_3(g))}^{(3)}, \quad (-\mathrm{Id})_{1,3}(\zeta_g^{(1)}) = -\zeta_{h_{\mathcal{O}_3}(\psi_3(g))}^{(3)},$$

and

$$(-\mathrm{Id})_{2,4}(\sigma_g^{(2)}) = -\sigma_{v_{\mathcal{O}_4}(\psi_4(g))}^{(4)}, \quad (-\mathrm{Id})_{2,4}(\zeta_g^{(2)}) = -\zeta_{h_{\mathcal{O}_4}(\psi_4(g))}^{(4)}.$$

3·3. *The absolute homology groups of $\mathcal{O}_k$, $k = 1, \ldots, 4$*

The absolute homology group $H_1(\mathcal{O}_1, \mathbb{Q})$ has a basis $\mathcal{B}_k := \{\Sigma_0^{(k)}, Z_0^{(k)}, \Sigma_1^{(k)}, \Sigma_2^{(k)}, Z_1^{(k)}, Z_2^{(k)}\}$ where

$$\Sigma_0^{(k)} := \sum_{g=1}^{9} \sigma_g^{(k)}, \quad Z_0^{(k)} := \sum_{g=1}^{9} \zeta_g^{(k)},$$
$$\Sigma_1^{(k)} := \sum_{j=1}^{4} \sigma^{(k)}_{h^j_{\mathcal{O}_1}(2)} - 4\sigma_1^{(k)}, \; \Sigma_2^{(k)} := \sum_{j=1}^{4} \sigma^{(k)}_{h^j_{\mathcal{O}_1}(6)} - 4\sigma_1^{(k)}$$

and

$$Z_1^{(1)} := \sum_{j=1}^{4} \zeta^{(1)}_{v^j_{\mathcal{O}_1}(1)} - 4\zeta_5^{(1)}, \quad Z_2^{(1)} := \sum_{j=1}^{4} \zeta^{(1)}_{v^j_{\mathcal{O}_1}(4)} - 4\zeta_5^{(1)},$$
$$Z_1^{(2)} := \sum_{j=1}^{4} \zeta^{(2)}_{v^j_{\mathcal{O}_1}(1)} - 4\zeta_3^{(2)}, \quad Z_2^{(2)} := \sum_{j=1}^{4} \zeta^{(1)}_{v^j_{\mathcal{O}_1}(4)} - 4\zeta_3^{(1)},$$
$$Z_1^{(3)} := \sum_{j=1}^{4} \zeta^{(1)}_{v^j_{\mathcal{O}_1}(1)} - 4\zeta_9^{(1)}, \quad Z_2^{(3)} := \sum_{j=1}^{4} \zeta^{(1)}_{v^j_{\mathcal{O}_1}(4)} - 4\zeta_9^{(1)},$$
$$Z_1^{(4)} := \sum_{j=1}^{4} \zeta^{(1)}_{v^j_{\mathcal{O}_1}(1)} - 4\zeta_8^{(1)}, \quad Z_2^{(4)} := \sum_{j=1}^{4} \zeta^{(1)}_{v^j_{\mathcal{O}_1}(4)} - 4\zeta_8^{(1)}.$$

Note that this basis is adapted to the decomposition $H_1(\mathcal{O}_k, \mathbb{Q}) = H_1^{st}(\mathcal{O}_k, \mathbb{Q}) \oplus H_1^{(0)}(\mathcal{O}_k, \mathbb{Q})$ in the sense that this decomposition corresponds to the partition $\mathcal{B}_k = \mathcal{B}_k^{st} \cup \mathcal{B}_k^{(0)}$ where $\mathcal{B}_k^{st} = \{\Sigma_0^{(k)}, Z_0^{(k)}\}$ and $\mathcal{B}_k^{(0)} = \mathcal{B}_k \setminus \mathcal{B}_k^{st}$, i.e.,

$$H_1^{st}(\mathcal{O}_k, \mathbb{Q}) = \mathbb{Q}\Sigma_0^{(k)} \oplus \mathbb{Q}Z_0^{(k)}$$

and

$$H_1^{(0)}(\mathcal{O}_k, \mathbb{Q}) = \mathbb{Q}\Sigma_1^{(k)} \oplus \mathbb{Q}Z_1^{(k)} \oplus \mathbb{Q}\Sigma_2^{(k)} \oplus \mathbb{Q}Z_2^{(k)}.$$

Moreover, it is worth pointing out that the matrix of the restriction to $H_1^{(0)}(\mathcal{O}_1, \mathbb{Z})$ of the intersection form Ω in the basis $\mathcal{B}_1^{(0)}$ is

$$\Omega = \begin{pmatrix} 0 & 0 & -6 & -3 \\ 0 & 0 & -3 & 3 \\ 6 & 3 & 0 & 0 \\ 3 & -3 & 0 & 0 \end{pmatrix}.$$

3·4. *The action of Aff($\mathcal{O}_1$) on the absolute homology group*

The formulae from the previous two subsections say that the matrices of $T_{k,k+1}$, $S_{k+1,k}$ and $-(\mathrm{Id})_{k,k+2}$ with respect to the bases $\mathcal{B}_l$ are

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$$T_{1,2} = \begin{pmatrix} 1&1&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&1&0\\ 0&0&0&1&0&1\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}, \quad T_{2,3} = \begin{pmatrix} 1&1&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&-1&-1\\ 0&0&0&1&0&1\\ 0&0&0&0&1&0\\ 0&0&0&0&-1&-1 \end{pmatrix},$$

$$T_{3,4} = \begin{pmatrix} 1&1&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&1\\ 0&0&0&1&1&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}, \quad T_{4,1} = \begin{pmatrix} 1&1&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&1\\ 0&0&0&1&-1&-1\\ 0&0&0&0&1&0\\ 0&0&0&0&-1&-1 \end{pmatrix},$$

$$S_{1,4} = \begin{pmatrix} 1&0&0&0&0&0\\ 1&1&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&0&1&0&1&0\\ 0&0&0&1&0&1 \end{pmatrix}, \quad S_{4,3} = \begin{pmatrix} 1&0&0&0&0&0\\ 1&1&0&0&0&0\\ 0&0&-1&-1&0&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&1\\ 0&0&-1&-1&1&0 \end{pmatrix},$$

$$S_{3,2} = \begin{pmatrix} 1&0&0&0&0&0\\ 1&1&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&1&0\\ 0&0&1&0&0&1 \end{pmatrix}, \quad S_{2,1} = \begin{pmatrix} 1&0&0&0&0&0\\ 1&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&-1&-1&0&0\\ 0&0&0&1&0&1\\ 0&0&-1&-1&1&0 \end{pmatrix},$$

$$(-\mathrm{Id})_{1,3} = \begin{pmatrix} -1&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&0&-1&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&0&-1&0\\ 0&0&0&0&0&-1 \end{pmatrix} = (-\mathrm{Id})_{2,4}.$$

This allows us to compute the images $\alpha(a)$ and $\alpha(b)$ of the generators a and b of $SL(\mathcal{O}_1) \simeq \mathrm{Aff}(\mathcal{O}_1)$ under the KZ cocycle $\alpha : \mathrm{Aff}(\mathcal{O}_1) \to \mathrm{Sp}(H_1(\mathcal{O}_1, \mathbb{Z}))$. Indeed,

$$a = \begin{pmatrix} 0 & -1\\ 1 & -1 \end{pmatrix} = (-\mathrm{Id})TS^{-1}, \quad b = \begin{pmatrix} 1 & -3\\ 1 & -2 \end{pmatrix} = ST^{-3},$$

so that

$$\alpha(a) = (-\mathrm{Id})_{3,1} T_{2,3} S_{2,1}^{-1}, \quad \alpha(b) = S_{2,1} T_{2,3}^{-1} T_{3,4}^{-1} T_{4,1}^{-1}.$$

For later use, we observe that these formulae give that the non-tautological subrepresentation $\rho : \mathrm{Aff}(\mathcal{O}_1) \to \mathrm{Sp}(H_1^{(0)}(\mathcal{O}_1, \mathbb{Z}))$ of α takes values

$$\rho(a) = \begin{pmatrix} 0&0&-1&0\\ 0&0&1&1\\ 0&1&0&-1\\ 1&0&1&1 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} 1&0&3&3\\ -1&-1&-2&-1\\ 0&1&-1&-1\\ -1&-1&-1&-1 \end{pmatrix}$$

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(with respect to the basis $\mathcal{B}_1^{(0)}$ of $H_1^{(0)}(\mathcal{O}_1, \mathbb{Z})$) at the two generators a and b of $SL(\mathcal{O}_1)$. Moreover, if we denote by $p_1 = ST^{-4}S^{-1}T^4$, $p_2 = ST^{-4}ST^6 \in SL(\mathcal{O}_1)$, then the characteristic polynomials $\chi_{p_1}(x)$ and $\chi_{p_2}(x)$ of the matrices $\rho(p_1)$ and $\rho(p_2)$ are

$$\chi_{p_1}(x) = x^4 - 11x^3 + 29x^2 - 11x + 1$$

and

$$\chi_{p_2}(x) = x^4 - 2x^3 - 16x^2 - 2x + 1.$$

4. *Arithmeticity of the Kontsevich–Zorich group associated to $\mathcal{C}$*

This section is devoted to the study of the image of the representation $\rho : \mathrm{Aff}(\mathcal{O}_1) \to \mathrm{Sp}(H_1^{(0)}(\mathcal{O}_1, \mathbb{Z}))$ describing the non-tautological part of the Kontsevich–Zorich cocycle.

4·1. *Zariski density of $\rho(\mathrm{Aff}(\mathcal{O}_1))$ in $\mathrm{Sp}(H_1^{(0)}(\mathcal{O}_1, \mathbb{R}))$*

The matrices $\rho(p_1)$ and $\rho(p_2)$ are Galois-pinching[8] in the sense of the paper [**MMY**] and the splitting fields of their characteristic polynomials are disjoint.

Indeed, these facts follow from the analysis of the discriminants

$$\Delta_1(\chi_{p_1}) = (-11)^2 - 4 \times (29 - 2) = 13, \quad \Delta_1(\chi_{p_2}) = (-2)^2 - 4 \times (-16 - 2) = 2^2 \times 19$$

and

$$\Delta_2(\chi_{p_1}) = (29 + 2)^2 - 4 \times (-11)^2 = 3^2 \times 53, \quad \Delta_2(\chi_{p_2}) = (-16 + 2)^2 - 4 \times (-2)^2 = 6^2 \times 5$$

(cf. [**MMY**, section 6·7]).

By the Zariski density criterion of Prasad–Rapinchuk [**PR**, theorem 9·10] (see also [**Ri**, Theorem 1·5]), we have that $\rho(\mathrm{Aff}(\mathcal{O}_1))$ is Zariski-dense in $\mathrm{Sp}(H_1^{(0)}(\mathcal{O}_1, \mathbb{R}))$.

Remark 4·1. The Zariski-denseness of $\rho(\mathrm{Aff}(\mathcal{O}_1))$ allows to apply the main result of [**EM**] in order to deduce that the Lyapunov spectrum of $\mathcal{C}$ is simple, i.e.,

$$1 = \lambda_1 > \lambda_2 > \lambda_3 > -\lambda_3 > -\lambda_2 > -\lambda_1 = -1$$

4·2. *Arithmeticity of $\rho(SL(\mathcal{O}_1))$ in $\mathrm{Sp}(H_1^{(0)}(\mathcal{O}_1, \mathbb{R}))$*

Denote by

$$\Theta = \begin{pmatrix} 1 & 1 & 1 & -1 \\ -1 & 0 & 0 & 1 \\ -1 & -1 & 0 & -1 \\ 0 & 1 & -1 & 1 \end{pmatrix}.$$

After using Θ to change the basis $\mathcal{B}_1^{(0)}$, we obtain the matrices

$$A := \Theta^{-1}\rho(a)\Theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad B := \Theta^{-1}\rho(b)\Theta = \begin{pmatrix} -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

[8]Recall that a matrix $A \in Sp(2d, \mathbb{Z})$ is Galois-pinching whenever its eigenvalues are real and its characteristic polynomial is an irreducible polynomial over $\mathbb{Q}$ with largest possible Galois group (of order $2^d d!$).

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Remark 4·2. Since the matrices $\rho(a)$ and $\rho(b)$ preserve the symplectic form induced by Ω in Subsection 3·3 above, we have that $\Theta^{-1}\rho(a)\Theta$ and $\Theta^{-1}\rho(b)\Theta$ are symplectic matrices with respect to

$$\Theta^t\Omega\Theta = \begin{pmatrix} 0 & -9 & 0 & 0 \\ 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & -9 & 0 \end{pmatrix}.$$

At this point, the proof of Theorem 1·1 is reduced to:

THEOREM 4·3. *$\rho(\mathrm{Aff}(\mathcal{O}_1))$ has finite index in $\mathrm{Sp}(H_1^{(0)}(\mathcal{O}_1), \mathbb{Z})$.*

Proof. Let us consider the permutation matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

exchanging the second and fourth basis vectors and let us show that the conjugate

$$P \cdot \langle A, B\rangle \cdot P$$

of $\rho(\mathrm{Aff}(\mathcal{O}_1)) = \langle A, B\rangle$ is arithmetic, i.e., it has finite-index in $\mathrm{Sp}(4, \mathbb{Z})$.

We found[9] that the matrices $x = P(A^2B)^2(AB^2)^2P$, $y = PABA^2BA(AB^2)^2P$ and $z = PA^2BA^2(B^2A)^2BP$ are interesting because

$$[y, x] = yxy^{-1}x^{-1} = \begin{pmatrix} 1 & 0 & 0 & 18 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad x^6[y, x] = \begin{pmatrix} 1 & 0 & 18 & 0 \\ 0 & 1 & 0 & 18 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$y^6[y, x]^{-1} = \begin{pmatrix} 1 & 18 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -18 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad z^6(x^6[y, x])^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -18 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

generate the positive root groups of $\mathrm{Sp}(4, \mathbb{R})$ and, thus, $P \cdot \langle A, B\rangle \cdot P$ intersects the subgroup $U(\mathbb{Z})$ of unipotent upper triangular matrices of $\mathrm{Sp}(4, \mathbb{Z})$ in a finite-index subgroup[10].

Since we know that $\langle A, B\rangle$ is Zariski-dense, we can apply the arithmeticity criterion of Oh [**Oh**] and Benoist–Miquel [**BM**] saying that Zariski dense subgroups of $\mathrm{Sp}(4, \mathbb{Z})$ containing a finite-index subgroup of $U(\mathbb{Z})$ are arithmetic to get the desired conclusion.

[9]For this sake, we asked Sage to look words on A, B, A^2 and B^2 of size $\leqslant 10$ fixing the first basis vector.

[10]This argument was inspired by [**SV**, Section 2]. Note that if we want to generate a finite-index subgroup of the unipotent *radical* of the parabolic subgroup associated to the flag $\mathbb{Q}e_1 \subset \mathbb{Q}e_1 \oplus \mathbb{Q}e_2 \oplus \mathbb{Q}e_3 \subset \mathbb{Q}^4$, then it suffices to use the matrices $[y, x]$, $x^6[y, x]$ and $y^6[y, x]^{-1}$.

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Remark 4·4. S. Kohl pointed out[11] to us that ρ is not faithful: indeed,

$$(ABA^{-1}BA^{-1}BAB^{-1})^3 = \mathrm{Id},$$

so that $(aba^{-1}ba^{-1}bab^{-1})^3 = \begin{pmatrix} -24587 & 42408 \\ 15048 & -25955 \end{pmatrix}$ lies in $\ker(\rho)$.

This is coherent with the arithmeticity statement in Theorem 4·3: if ρ were faithful, then $\mathrm{Sp}(H_1^{(0)}(\mathcal{O}_1), \mathbb{Z})$ would contain a finite-index subgroup isomorphic to a free group[12] on five generators, namely $\rho(\Gamma(4)) \subset \rho(\mathrm{Aff}(\mathcal{O}_1))$. This is a contradiction because it is well known that $\mathrm{Sp}(4, \mathbb{Z})$ does not contain lattices isomorphic to free groups (thanks to Kazhdan property (T)).

4·3. *Final comments*

This paper grew from the following attempt to produce examples of origamis generating thin Kontsevich–Zorich monodromies.

By an argument in the spirit of Remark 4·4, if $\mathcal{O}$ is an origami of genus $g \geqslant 3$ such that the representation $\rho : \mathrm{Aff}(\mathcal{O}) \to \mathrm{Sp}(H_1^{(0)}(\mathcal{O}, \mathbb{Z}))$ is faithful and $\rho(\mathrm{Aff}(\mathcal{O}))$ is Zariski-dense in $\mathrm{Sp}(H_1^{(0)}(\mathcal{O}, \mathbb{R}))$, then $\mathcal{O}$ has thin Kontsevich–Zorich monodromy. On the other hand, if $\mathcal{O}$ has some direction with homological dimension one (i.e., whose cylinders have waist curves spanning a one-dimensional subspace of $H_1(\mathcal{O}, \mathbb{R})$), then it is not hard to check that a Dehn multitwist along this direction would belong to the kernel of ρ. Hence, it is natural to try to detect origamis with thin Kontsevich–Zorich monodromies among the origamis without directions of homological dimension one.

Remark 4·5. A related strategy towards the same goal would be to show that $\rho(\mathrm{Aff}(\mathcal{O}))$ fits the assumptions of the ping-pong lemma (compare with the proof of Lemma 2·3). Nevertheless, it is not easy to implement this idea in general because the construction of "ping-pong subsets" might be somewhat tricky (see [**FR**, page 5387 and Subsections 3·2 and 3·3]).

As it turns out, the origami $\mathcal{O}_1$ is one of the smallest examples of origamis of genus 3 having no direction with homological dimension one (compare with Remark 2·2) and this explains our interest on its Kontsevich–Zorich monodromy.

Anyhow, once we detect a good candidate origami $\mathcal{O}$, the first step is the computation of its Kontsevich–Zorich monodromy, i.e., the Zariski closure of $\rho(\mathrm{Aff}(\mathcal{O}))$ (compare with Subsection 4·1). Here, the criterion of Prasad–Rapinchuk [**PR**, theorem 9·10] (see also [**Ri**, theorem 1·5]) informally says that the Zariski closure is "often" a symplectic group Sp or a product of SL_2's. Moreover, the techniques in [**MMY**] indicate that the Zariski closure tends to be a symplectic group in many situations including $\mathcal{H}(4)$, but this must be taken with a grain of salt because the case of products of SL_2 happens in nature: for instance, Eskin–Kontsevich–Zorich [**EKZ2**] noted that the so-called "stairs" origamis in $\mathcal{H}(2g-2)$ and $\mathcal{H}(g-1, g-1)$ are covered by special "square-tiled cyclic covers" and this information can

[11] Actually he computed with GAP the words on A, B, A^{-1}, B^{-1} of sizes $1, 2, \ldots$, and he noticed that the set of words of length 12 has size $< 2^{12+1}$. This led him to the nontrivial relation of length $2 \cdot 12 = 24$ above.

[12] Alternatively, Lemma 2·3 could be directly used to show that $\rho(\mathrm{Aff}(\mathcal{O}_1))$ would contain a finite-index subgroup isomorphic to a free group if ρ were faithful.

be used to show that the Kontsevich–Zorich monodromy of a "stairs" origami is contained[13] in a product of SL_2's.

Finally, even if $\rho(\mathrm{Aff}(\mathcal{O}))$ is Zariski-dense in $\mathrm{Sp}(H_1^{(0)}(\mathcal{O}, \mathbb{R}))$, it is certainly a challenging problem to obtain the faithfulness of ρ. Here, the case of arithmetic Teichmüller curves of genus zero might be a good starting point of investigation (because the corresponding Veech groups are generated by elliptic and parabolic elements of $SL(2, \mathbb{Z})$), but our discussion of $\mathcal{O}_1$ in the previous subsection shows that this situation is not always favourable towards the construction of thin Kontsevich–Zorich monodromies.

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REFERENCES

[AMY] A. AVILA, C. MATHEUS and J.–C. YOCCOZ. Zorich conjecture for hyperelliptic Rauzy–Veech groups. *Math. Ann.* **370** (2018), 785–809.

[BM] Y. BENOIST and S. MIQUEL. Arithmeticity of discrete subgroups containing horospherical lattices, Preprint (2018) available at arXiv:1805.00045.

[BT] C. BRAV and H. THOMAS. Thin monodromy in Sp(4). *Compositio Math.* **150** (2014), 333–343.

[EKZ] A. ESKIN, M. KONTSEVICH and A. ZORICH. Sum of Lyapunov exponents of the Hodge bundle with respect to the Teichmüller geodesic flow. *Publ. Math. Inst. Hautes Études Sci.* **120** (2014), 207–333.

[EKZ2] A. ESKIN, M. KONTSEVICH and A. ZORICH. Lyapunov spectrum of square-tiled cyclic covers. *J. Mod. Dyn.* **5** (2011), no. 2, 319–353.

[EM] A. ESKIN and C. MATHEUS. A coding-free simplicity criterion for the Lyapunov exponents of Teichmüller curves. *Geom. Dedicata* **179** (2015), 45–67.

[Fo] G. FORNI. A geometric criterion for the nonuniform hyperbolicity of the Kontsevich–Zorich cocycle. Appendix by Carlos *Matheus. J. Mod. Dyn.* **5** (2011), 355–395.

[FM] G. FORNI and C. MATHEUS. Introduction to Teichmüller theory and its applications to dynamics of interval exchange transformations, flows on surfaces and billiards. *J. Mod. Dyn.* **8** (2014), 271–436.

[FR] E. FUCHS and I. RIVIN. Generic thinness in finitely generated subgroups of $SL_n(\mathbb{Z})$. *Int. Math. Res. Not. IMRN* (2017), no. 17, 5385–5414.

[Gu] R. GUTIÉRREZ–ROMO. Classification of Rauzy–Veech groups: proof of the Zorich conjecture. *Invent. Math.* (2018).

[MMY] C. MATHEUS. M. MÖLLER and J.–C. YOCCOZ. A criterion for the simplicity of the Lyapunov spectrum of square-tiled surfaces. *Invent. Math.* **202** (2015), 333–425.

[Oh] H. OH. On discrete subgroups containing a lattice in a horospherical subgroup. *Israel J. Math.* **110** (1999), 333–340.

[PR] G. PRASAD and A. RAPINCHUK. *Generic elements in Zariski-dense subgroups and isospectral locally symmetric spaces*. Thin groups and superstrong approximation. Math. Sci. Res. Inst. Publ. **61**, (Cambridge University Press, Cambridge, 2014), 211–252.

[Ri] I. RIVIN. Large Galois groups with applications to Zariski density. Preprint (2015) available at arxiv.1312.3009.

[Sa] P. SARNAK. Notes on thin matrix groups. Thin groups and superstrong approximation. Math. Sci. Res. Inst. Publ., **61** (Cambridge University Press, Cambridge), (2014), 343–362.

[SV] S. SINGH and T. N. VENKATARAMANA. Arithmeticity of certain symplectic hypergeometric groups. *Duke Math. J.* **163** (2014), 591–617.

[13] Actually, we did some computations with the *first few* stairs origamis and their Kontsevich–Zorich monodromies turned out to be *equal* to products of SL_2.

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