Euro. Jnl of Applied Mathematics (2002), *vol.* 13, *pp.* 205–224. © 2002 Cambridge University Press
DOI: 10.1017/S0956792501004776 Printed in the United Kingdom

Model of a viscous layer deformation by thermocapillary forces

V. V. PUKHNACHOV

Lavrentyev Institute of Hydrodynamics, Lavrentyev Prospect 15, Novosibirsk, 630090 Russia

(*Received 17 January 2001; revised 6 August 2001*)

Three-dimensional nonstationary flow of a viscous incompressible liquid is investigated in a layer, driven by a nonuniform distribution of temperature on its free boundaries. If the temperature given on the layer boundaries is quadratically dependent on horizontal coordinates, external mass forces are absent, and the motion starts from rest then the free boundary problem for the Navier–Stokes equations has an 'exact' solution in terms of two independent variables. Here the free boundaries of the layer remain parallel planes and the distance between them must be also determined. In present paper, we formulate conditions for both the unique solvability of the reduced problem globally in time and the collapse of the solution in finite time. We further study qualitative properties of the solution such as its behaviour for large time (in the case of global solvability of the problem), and the asymptotics of the solution near the collapse moment in the opposite case.

1 Statement of problem

We consider thermocapillary motion of a viscous incompressible liquid bounded entirely by free surfaces. The domain occupied by liquid is denoted by Ω_t, and its boundary is denoted by Γ_t. The liquid density ρ and kinematic viscosity ν are taken to be constant, and the surface tension σ is taken to be a linear function of temperature θ:

$$\sigma = \sigma_0 - \kappa(\theta - \theta_0), \tag{1.1}$$

where σ_0, κ and θ_0 are positive constants. We will suppose further that the motion starts from rest, and that external mass forces do not act on the liquid. Moreover, we will assume that the temperature at free surface $\theta_\Gamma(\vec{x}, t)$ is a known function of the coordinates $\vec{x} = (x, y, z)$ and time t. Hence, the mathematical statement of the problem is reduced to determination of the domain Ω_t, $0 < t < T$ and the solution $\vec{v}(\vec{x}, t) = (u, v, w)$, $p(\vec{x}, t)$ of the Navier–Stokes equations

$$\vec{v}_t + \vec{v} \cdot \nabla\vec{v} = -\rho^{-1}\nabla p + \nu\Delta\vec{v},\ \nabla \cdot \vec{v} = 0 \tag{1.2}$$

in this domain, satisfying the initial conditions

$$\Omega_0 \text{ is given},\ \vec{v}(\vec{x}, 0) = 0,\ \vec{x} \in \Omega_0 \tag{1.3}$$

and the conditions on the free surfaces

$$-p\vec{n} + 2\rho\nu D \cdot \vec{n} = -2K\sigma\vec{n} + \nabla_\Gamma \sigma, \tag{1.4}$$

$$\vec{v} \cdot \vec{n} = V_n,\ \vec{x} \in \Gamma_t,\ 0 < t < T. \tag{1.5}$$

The following notation is used in (1.4), (1.5): $\vec{n}$ is the unit external normal to the surface Γ_t, $D = [\nabla\vec{v} + (\nabla\vec{v})^*]/2$ is the strain velocity tensor, K is the mean curvature of the surface Γ_t, $\nabla_\Gamma = \nabla - \vec{n}(\vec{n} \cdot \nabla)$ is the surface gradient, V_n is the velocity of displacement of the surface Γ_t in the direction of $\vec{n}$. After substitution of the expression for σ in the form of (1.1) with $\theta = \theta_\Gamma(\vec{x}, t)$ into (1.4) we obtain a closed statement of the free boundary problem for the Navier–Stokes equations.

The solvability conditions for the initial boundary value problem (1.2)–(1.5) are derived in Mogilevskii & Solonnikov [10]. Investigated in Andreev & Pukhnachov [3] are the invariance properties of this problem; the group classification of this problem relative to an 'arbitrary element' $\theta_\Gamma(\vec{x}, t)$ is satisfied there too. Examples of exact solutions of the equations of thermocapillary motion are presented in Birikh [4], Napolitano [11], Gupalo & Ryazanov [7], Andreev & Adamev [1] and Andreev *et al.* [2, Ch, 7] (see also the references therein). It must be noted that the majority of these exact solutions describe stationary flows determined by a system of ordinary differential equations. A solution of plane nonstationary flow for system (1.2) describing thermocapillary flow in a strip is given in Andreev & Pukhnachov [3]. It assumes the dependence $\theta_\Gamma = \theta^* + l(t)x^2$, where $\theta^* = const$ and l is an arbitrary function of t. This solution is derived via a system of equations with two independent variables. The possibility of the decrease of order of the problem considered in Andreev & Pukhnachov [3] results from the fact that its solution is a partially invariant solution [12] of the plane analogue of (1.2). The solution studied in the present paper is a natural generalization of the previous solution for the case of thermocapillary motion in a layer. It corresponds to the temperature distribution on the boundaries of the layer

$$\theta_\Gamma = \theta^* + l(t)x^2/2 + m(t)y^2/2, \tag{1.6}$$

where l and m are arbitrary functions of t. The further considerations are based on the following statement, which can be checked directly. If

$$u = (f+g)x,\ v = (f-g)y,\ w = -2\int_0^z f(\varsigma, t)d\varsigma, \tag{1.7}$$

$$p/\rho = \nu w_z(z,t) - \int_0^z w_t(\varsigma, t)d\varsigma - \frac{1}{2}w^2(z,t) + \chi(t),$$

where $f(z,t)$, $g(z,t)$ are the solutions of the system of equations

$$f_t + f^2 + g^2 - 2f_z\int_0^z f(\varsigma,t)d\varsigma = \nu f_{zz}, \tag{1.8}$$

$$g_t + 2fg - 2g_z\int_0^z f(\varsigma,t)d\varsigma = \nu g_{zz}$$

and χ is an arbitrary function of t, then the functions $\vec{v} = (u, v, w)$, p satisfy the Navier–Stokes equations (1.2). (Note that the solution (1.7) of the system (1.2) can be determined as usual as a partially invariant solution with rank two and defect two relative to the four-parameter Lie group generated by translations and Galilean translations along the x- and y-axes [9].)

Let us show that the solution (1.7) can be interpreted as a solution describing thermocapillary motion in the layer $|z| < s(t)$ where the temperature distribution is prescribed on its boundaries (1.6). In fact, in this case $K = 0$, $\nabla_\Gamma \sigma = (-\kappa x l(t), -\kappa y m(t))$ and the condition (1.4) will be satisfied at $z = s(t)$ if the functions f and g satisfy

$$f_z(s(t),t) = -k[l(t)+m(t)],\ 0 < t < T, \tag{1.9}$$

$$g_z(s(t),t) = -k[l(t)-m(t)],\ 0 < t < T,$$

where $k = \kappa/\rho\nu = const > 0$ and the function $\chi(t)$ is chosen in the form

$$\chi = \nu w_z(s(t),t) + \int_0^{s(t)} w_t(\varsigma,t)d\varsigma + \frac{1}{2}w^2(s(t),t).$$

Further, we assume that

$$f_z(0,t) = g_z(0,t) = 0,\ 0 < t < T \tag{1.10}$$

and continue the functions f, g (determined initially for $0 < z < s(t)$, $0 < t < T$) to the domain $-s(t) < z < 0$ in an even way. Then condition (1.4) will be satisfied at the lower boundary of the layer $z = -s(t)$ too. If we demand the condition

$$\frac{ds}{dt} = -2\int_0^{s(t)} f(z,t)dz,\ 0 < t < T \tag{1.11}$$

then we can satisfy the condition (1.5) on both boundaries of the layer. Finally, we assume that

$$s(0) = a > 0 \tag{1.12}$$

(which corresponds to the definition of the initial position of the layer) and

$$f(z,0) = g(z,0) = 0,\ 0 \leqslant z \leqslant a. \tag{1.13}$$

Then the initial conditions (1.3) will be satisfied.

2 Conditions for existence and non-existence of solution

Here the solvability conditions for the problem (1.8)–(1.13) are formulated and the qualitative properties of its solution are determined. Note that we are interested only in classical solutions of the above-mentioned problem. The input data of the problem (i.e. the functions $l(t)$ and $m(t)$) must be subjected to some conditions of smoothness and compatibility to ensure the existence of such solutions. Further, we assume that these functions are defined for all $t > 0$, moreover

$$l(t), m(t) \in C^{(1+\alpha)/2}[0,\infty),\ 0 < \alpha < 1, \tag{2.1}$$

$$l(0) = m(0) = 0, \tag{2.2}$$

where $C^{(1+\alpha)/2}[0,\infty)$ denotes the space of functions continuous on the semiaxis $t \geqslant 0$ and satisfying the Hölder conditions with exponent $(1+\alpha)/2$ on any compact set. The following notation is used below: S_T is the domain $\{z,t : 0 < z < s(t), 0 < t < T\}$, $C^{2+\alpha,1+\alpha/2}(\bar{S}_T)$ is the Hölder class used in the theory of parabolic equations (its definition can be found in [10]).

Proposition 1 Let the conditions (2.1), (2.2) be satisfied. Then one can find $T > 0$ such that the problem (1.8)–(1.13) has the unique solution $f(z,t)$, $g(z,t)$, $s(t)$; moreover $f, g \in C^{2+\alpha,1+\alpha/2}(\bar{S}_T)$, $s \in C^{2+\alpha/2}[0,T]$.

The proof of this proposition has a purely technical character. It is based on the transition from the Eulerian coordinate z to the Lagrangian coordinate ς in the problem (1.8)–(1.13). The connection between the Lagrangian and Eulerian coordinates is determined in terms of the solution of the Cauchy problem

$$z_t = -2\int_0^z f(\xi,t)d\xi \quad \text{when} \quad t > 0,$$

$$z = \varsigma \quad \text{when} \quad t = 0.$$

Here the domain S_T maps into the rectangle $\Pi = \{\varsigma, t : 0 < \varsigma < a, 0 < t < T\}$ and the equations (1.8) turns into the following equations for the functions $F(\varsigma,t) = f[z(\varsigma,t),t]$, $G(\varsigma,t) = g[z(\varsigma,t),t]$:

$$F_t + F^2 + G^2 = v \exp\left[2\int_0^t F(\varsigma,\tau)d\tau\right]\left\{\exp\left[2\int_0^t F(\varsigma,\tau)d\tau\right]F_\varsigma\right\}_\varsigma, \tag{2.3}$$

$$G_t + 2FG = v \exp\left[2\int_0^t F(\varsigma,\tau)d\tau\right]\left\{\exp\left[2\int_0^t F(\varsigma,\tau)d\tau\right]G_\varsigma\right\}_\varsigma.$$

The equality

$$z_\varsigma = \exp\left[-2\int_0^t F(\varsigma,\tau)d\tau\right]$$

was used in the derivation of (2.3). The above-mentioned solution of the Cauchy problem satisfies this equality. Then the boundary conditions (1.9) are rewritten in the form

$$F_\varsigma(a,t) = -k[l(t) + m(t)]\exp\left[2\int_0^z F(a,\tau)d\tau\right], \; 0 < t < T, \tag{2.4}$$

$$G_\varsigma(a,t) = -k[l(t) - m(t)]\exp\left[2\int_0^z F(a,\tau)d\tau\right], \; 0 < t < T.$$

The conditions (1.10), (1.13) give the following conditions for the functions F and G:

$$F_\varsigma(0,t) = G_\varsigma(0,t) = 0, \; 0 < t < T, \tag{2.5}$$

$$F(\varsigma,0) = G(\varsigma,0) = 0, \; 0 \leqslant \varsigma \leqslant a. \tag{2.6}$$

As a result we obtain the initial boundary value problem (2.4)–(2.6)in a fixed domain for the system of quasilinear integro-differential parabolic equations of the second order (2.3). Its local unique solvability in Hölder classes follows from general results of the theory of parabolic equations [8] and can be determined, for example, by the method of successive approximations; the convergence of this method is guaranteed for sufficiently small T. If the function $F(\varsigma,t)$ is known then the function $s(t)$ defining the position of the free boundary in the plane z,t is given by the formula

$$s(t) = \int_0^a \exp\left[-2\int_0^t F(\varsigma,\tau)d\tau\right]d\varsigma.$$

Hence the kinematic condition on the free boundary (1.11) is satisfied automatically.

So solvability of problem (1.8)–(1.13) on a small time interval demands that the functions $l(t)$, $m(t)$ only satisfy the smoothness condition (2.1) and compatibility condition (2.2). As will be shown below, these conditions are insufficient for the solvability of the global problem.

Proposition 2 Assume that

$$l(t) + m(t) \geqslant 0 \quad \text{for} \quad t \geqslant 0. \tag{2.7}$$

Moreover the inequality (2.7) is strict on some interval $(0, \tau)$. Then the 'life span' t_* of solution of the problem (1.8)–(1.13) is finite.

Proof Let us consider the functions

$$\bar{f}(t) = \frac{1}{s(t)} \int_0^{s(t)} f(z,t)dz, \ h = f - \bar{f}$$

so that quantity $\bar{f}$ is the mean value of the function $f(z,t)$ for any fixed t in the interval $[0, s(t)]$ and the mean value of the function $h(z,t)$ is equal to zero on this interval for any $t > 0$. The relation (1.11) will take the form

$$\frac{ds}{dt} = -2\bar{f}s, \tag{2.8}$$

so that knowledge of the function $\bar{f}$ determines completely the evolution of the free boundary in the problem (1.8)–(1.13).

We obtain the identity

$$\frac{d\bar{f}}{dt} = -\bar{f}^2 - \frac{1}{s}\int_0^s (g^2 + 3h^2)dz - \frac{vk(l+m)}{s} \tag{2.9}$$

after integration of the first equation (1.8) with respect to z over the interval $[0, s(t)]$ and taking into account the conditions (1.9)–(1.11). Further we may suppose without loss of generality that the number τ used in the formulation of Proposition 2 is less then the life span t_* of the solution of the studied problem. As follows from (2.9) and the conditions of Proposition 2, the function $\bar{f}$ decreases monotonically on the interval $[0, t_*)$ and (1.13) gives $\bar{f}(0) = 0$. So using (2.8) one can conclude that $s(t) \geqslant a$ when $0 \leqslant t < t_*$.

Integration of the identity (2.9) over the interval $(0, \tau)$ and elimination of necessarily nonpositive terms from the right-hand part of the resulting equality lead to the chain of inequalities

$$0 > \bar{f}(\tau) \geqslant -vk \int_0^\tau \frac{l(t)+m(t)}{s(t)}dt \geqslant -\frac{vk}{a}\int_0^\tau [l(t)+m(t)]dt = -\gamma,$$

where $\gamma = const > 0$, in accordance with the condition of Proposition 2. (The sharpness of the left inequality is guaranteed by this condition too.) The estimate $\bar{f}(t) \leqslant (1 + \gamma\tau - \gamma t)^{-1}\bar{f}(\tau)$ follows from this fact and inequality $d\bar{f}/dt \leqslant -\bar{f}^2$ which follows from (2.9). So far as $\bar{f}(\tau) < 0$, this estimate means that the solution of the problem (1.8)–(1.13) is destroyed at finite period of time $t_* \leqslant \gamma^{-1} + \tau$. □

Actually, Proposition 2 contains the necessary condition for global solvability of the problem (1.8)–(1.13). The determination of sufficient conditions for the existence of its

https://doi.org/10.1017/S0956792501004776 Published online by Cambridge University Press

solution for all $t > 0$ demands more effort. The main point here is obtaining the estimate for the maximum modulus of the functions f and g in the domain S_T for all $T > 0$. In the case when such an estimate is obtained, the proof of solvability of the problem (1.8)–(1.13) can be achieved globally by the proof scheme in Andreev *et al.* [2, Ch. 7, Theorem 1], using the method developed by Ladyzhenskaya *et al.* [8].

The specific character of our free boundary problem lies in the fact that its solution can cease to exist as t grows for two reasons. The first reason is demonstrated in Proposition 2. The existence of the function $\bar{f}(t)$ obtained in the process of its proof and equations (2.8) imply that $s \to \infty$ when $t \nearrow t_*$. The vanishing of the function $s(t)$ at finite time t^* is the other reason. This possibility explains the conditional character of Proposition 3 formulated below. Henceforth, generic positive quantities (generally speaking, depending on T) are denoted by C_k $(k = 1, 2, \ldots)$.

Proposition 3 Let the following inequalities be satisfied:

$$l(t) \leqslant m(t) \leqslant 0 \quad \text{for} \quad t \geqslant 0. \tag{2.10}$$

Then either

(a) One can find $t^* < \infty$ such that $s(t) > 0$ for $0 \leqslant t < t^*$ and $s \to 0$ when $t \nearrow t^*$. In this case, the estimates

$$|f(z,t)| \leqslant C_1, \ |g(z,t)| \leqslant C_2 \quad \text{when} \quad (z,t) \in \bar{S}_T \tag{2.11}$$

are valid ($T > 0$ is an arbitrary number less then t^*);

(b) The inequality $s(t) > 0$ is satisfied for any finite $t > 0$. Then the estimates (2.11) are valid in the domain $\bar{S}_T$ for any $T > 0$.

Proof Let us introduce the functions $\lambda = f + g$, $\mu = f - g$. It follows from (1.8) that these functions satisfy the equations

$$\lambda_t + \lambda^2 - 2\lambda_z \int_0^z f(\varsigma, t)d\varsigma = \nu\lambda_{zz}, \tag{2.12}$$

$$\mu_t + \mu^2 - 2\mu_z \int_0^z f(\varsigma, t)d\varsigma = \nu\mu_{zz}$$

in the domain S_T. Initial and boundary conditions for the system (2.12) are obtained from (1.9), (1.10), (1.13), and have the form

$$\lambda_z(s(t), t) = -2kl(t), \ \mu_z(s(t), t) = -2km(t), \tag{2.13}$$

$$\lambda_z(0, t) = \mu_z(0, t) = 0, \ 0 < t < T, \tag{2.14}$$

$$\lambda(z, 0) = \mu(z, 0) = 0. \tag{2.15}$$

If we note that the first and the second equations (2.12) are linear in to the functions λ and μ, we may apply the maximum principle [5] to the solution of the initial-boundary value problems (2.13)–(2.15) for these equations. In accordance with this principle, the non-negativity of the right-hand sides of the conditions (2.13) provided by the inequalities (2.10) and the homogeneity of the conditions (2.14), (2.15) imply the non-negativity of the functions λ and μ in the domain S_T, where $T < t^*$ in case (a) and T is an arbitrary

https://doi.org/10.1017/S0956792501004776 Published online by Cambridge University Press

positive number in case (b). This means that

$$f \geqslant 0 \text{ and } |g| \leqslant f \text{ for } (z,t) \in \bar{S}_T. \tag{2.16}$$

Hence, the proof of the first inequality (2.11) will imply the proof of the second. Moreover, the function $s(t)$ decreases monotonically for $t > 0$ by virtue of (2.8), (2.16), so that this fact, together with (2.12), implies the estimate

$$s(t) \leqslant a \text{ if } t \in [0, T]. \tag{2.17}$$

Finding uniform pointwise estimates of functions f_z, g_z in the domain $\bar{S}_T$ is the next step of the proof. It is evident that it is sufficient for this purpose to obtain similar estimates for the functions $\xi = \lambda_z$, $\eta = \mu_z$. As follows from (2.12)–(2.15), these functions are the solutions of the first initial-boundary value problems for the linear parabolic equations

$$\xi_t - 2\xi_z \int_0^z f(\varsigma, t)d\varsigma + 2g\xi = \nu\xi_{zz}, \tag{2.18}$$

$$\eta_t - 2\eta_z \int_0^z f(\varsigma, t)d\varsigma + 2g\eta = \nu\eta_{zz},$$

$$\xi(s(t), t) = -2kl(t), \; \eta(s(t), t) = -2km(t), \tag{2.19}$$

$$\xi(0, t) = \eta(0, t) = 0, \; 0 < t < T, \tag{2.20}$$

$$\xi(z, 0) = \eta(z, 0) = 0, \; 0 \leqslant z \leqslant a. \tag{2.21}$$

Now we note that the function g is non-negative in the domain $\bar{S}_T$. Indeed, this function satisfies a linear uniform parabolic equation (the second equation (1.8)) and conditions (1.9), (1.10), (1.13). Due to the condition (2.10), the right-hand side of the second equality (1.9) is non-negative for $0 < t < T$ that involves the non-negativeness of g on the base of the maximum principle. The estimates below are also valid by applying the maximum principle to the solutions of the problems (2.18)–(2.21) and inequalities (2.10) and $g \geqslant 0$ in $\bar{S}_T$:

$$0 \leqslant \xi = \lambda_z \leqslant C_3 = \max_{0 \leqslant t \leqslant T} [-2kl(t)],$$

$$0 \leqslant \eta = \mu_z \leqslant C_4 = \max_{0 \leqslant t \leqslant T} [-2km(t)].$$

So we conclude from these estimates and the definition of λ and μ that

$$0 \leqslant f_z \leqslant C_5 \text{ and } |g_z| \leqslant f_z \text{ for } (z,t) \in S_T \tag{2.22}$$

with $C_5 = C_3 + C_4$.

From the inequalities (2.22), the maximal value of the function $f(z,t)$ at some fixed t is achieved at the point $z = s(t)$ belonging to the free boundary of the domain S_T. Hence, one must obtain an estimate from above of the function $f(s(t), t)$ for the completion of the proof of Proposition 3. With this aim, let us consider the obvious representation

$$f(s(t), t) = \bar{f}(t) + \frac{1}{s(t)} \int_0^{s(t)} z f_z(z,t)dz. \tag{2.23}$$

The second term of the right-hand side is estimated from the inequalities (2.17), (2.22):

$$\frac{1}{s(t)} \int_0^{s(t)} z f_z(z,t)dz \leqslant \frac{aC_5}{2}. \tag{2.24}$$

https://doi.org/10.1017/S0956792501004776 Published online by Cambridge University Press

An upper estimate of $\bar{f}(t)$ is based on the inequality

$$\frac{d\bar{f}}{dt} \leqslant -\frac{\nu k(l+m)}{s},$$

following from (2.9). Integration of this inequality from zero to $t \leqslant T$ with $\bar{f}(0) = 0$, and replacing the functions $-l(t)$, $-m(t)$ with their maximal values on the interval $[0, T]$, implies

$$\bar{f}(t) \leqslant \frac{\nu C_5}{2} \int_0^t \frac{d\tau}{s(\tau)} \quad \text{for } t \in [0, T]. \tag{2.25}$$

Using estimates (2.24), (2.25) and representations (2.23), we conclude that

$$f(s(t), t) \leqslant \frac{C_5}{2} \left[\int_0^t \frac{d\tau}{s(\tau)} + a \right] \quad \text{if } 0 \leqslant t \leqslant T.$$

Here, (a) $T < t^*$, and (b) $T > 0$ is arbitrary. □

Remark Proposition 3 holds if condition (2.10) is replaced by

$$m(t) \leqslant l(t) \leqslant 0 \quad \text{for} \quad t \geqslant 0. \tag{2.26}$$

This follows from the invariance of the Navier–Stokes equations with respect to the transform $x' = -y,\ y' = x,\ u' = -v,\ v = u$.

3 Qualitative properties of solutions

It will be shown below that all the hypothetical possibilities considered in Proposition 3 can be realized. So as not to overload the paper, we consider two simple cases of the behaviour of the functions $l(t)$ and $m(t)$ defining the 'destiny' of the solution of our problem.

Proposition 4 Let the solution of the problem (1.8)–(1.13) be determined in some domain S_T. Suppose that the conditions (2.10) are satisfied, and moreover,

$$l(t) = m(t) = 0 \quad \text{when } t \geqslant \tau \tag{3.1}$$

and $l + m \not\equiv 0$ when $0 \leqslant t \leqslant \tau$. Then the problem (1.8)–(1.13) is solvable in the domain S_T for any $T > 0$, and the following estimates are valid: either

$$s = C_6 t^{-2} + \mathrm{O}(t^{-3}) \text{ when } t \to \infty, \tag{3.2}$$

$$f = t^{-1} + \mathrm{O}(t^{-2}),\ g = \mathrm{O}(t^{-2}) \text{ when } 0 \leqslant z \leqslant s(t), \tag{3.3}$$

or

$$s = C_7 t^{-1} + \mathrm{O}(t^{-2}) \text{ when } t \to \infty, \tag{3.4}$$

$$f = g = t^{-1}/2 + \mathrm{O}(t^{-2}) \text{ when } 0 \leqslant z \leqslant s(t).$$

The last situation is possible only in the case $l = 0$ or $m = 0$ for all $t \geqslant 0$.

https://doi.org/10.1017/S0956792501004776 Published online by Cambridge University Press

Proof First, note that Proposition 1 implies that in any case, one can find the existence time τ of the solution of problems (1.8)–(1.13). So far as the conditions of Proposition 3 are satisfied, the inequalities

$$\lambda = f + g \geqslant 0, \mu = f - g \geqslant 0 \tag{3.5}$$

are valid in the domain $\bar{S_T}$. Moreover, at least one of the functions λ, μ is not identically equal to zero on the upper boundary of this domain, i.e. at $t = \tau$, $0 \leqslant z \leqslant s(\tau)$; otherwise, we arrive at a contradiction with the condition $l + m \not\equiv 0$, $0 \leqslant t \leqslant \tau$ (the functions λ and μ satisfy this condition as the solutions of the problem (2.12)–(2.15) by virtue of the strict maximum principle [5]). So we can conclude from this and (3.5) that $\bar{f}(\tau) > 0$.

Now we can use the identity (2.9), where the last right-hand term is absent for $t \geqslant \tau$, as follows from (3.1). The inequality $d\bar{f}/dt \leqslant -\bar{f}^2$ follows from this identity and integration of this inequality from $t = \tau$, taking account of the positiveness of $\bar{f}(\tau)$, implies the estimate

$$\bar{f}(t) \leqslant \frac{\bar{f}(\tau)}{1 + (t - \tau)\bar{f}(\tau)} \text{ when } t \geqslant \tau. \tag{3.6}$$

In accordance with (2.8), the upper estimate of $\bar{f}(t)$ implies a lower estimate of the function $s(t)$. Thus, (2.8), (3.6) give

$$s(t) \geqslant \frac{s(\tau)}{[1 + (t - \tau)\bar{f}(\tau)]^2} \text{ when } t \geqslant \tau. \tag{3.7}$$

The global existence theorem is valid for the problem (1.8)–(1.13) on the ground of Proposition 3 and inequality (3.7).

Now let us obtain the asymptotic representations (3.2)–(3.4). With this aim, we use the formulation of the problem (1.8)–(1.13) in Lagrangian coordinates (2.3)–(2.6), where the boundary condition (2.4) is homogeneous for $t \geqslant \tau$ by virtue of the assumption (3.1). We introduce the functions $\Lambda(\varsigma, t) = \lambda(z, t)$, $\mathrm{M}(\varsigma, t) = \mu(z, t)$ and obtain the initial-boundary value problem

$$\Lambda_t + \Lambda^2 = \nu \exp\left[\int_0^t (\Lambda + \mathrm{M})dt\right] \left\{\exp\left[\int_0^t (\Lambda + \mathrm{M})dt\right] \Lambda_\varsigma\right\}_\varsigma, \tag{3.8}$$

$$\mathrm{M}_t + \mathrm{M}^2 = \nu \exp\left[\int_0^t (\Lambda + \mathrm{M})dt\right] \left\{exp\left[\int_0^t (\Lambda + \mathrm{M})dt\right] \mathrm{M}_\varsigma\right\}_\varsigma$$

in the semistrip $\Sigma_\tau = \{\varsigma, t : 0 < \varsigma < a, t > \tau\}$,

$$\Lambda_\varsigma(a, t) = \mathrm{M}_\varsigma(a, t) = 0, \ t > \tau, \tag{3.9}$$

$$\Lambda_\varsigma(0, t) = \mathrm{M}_\varsigma(0, t) = 0, \ t > \tau, \tag{3.10}$$

$$\Lambda(\varsigma, \tau) = \Lambda_0(\varsigma), \ \mathrm{M}(\varsigma, \tau) = \mathrm{M}_0(\varsigma), \ 0 \leqslant \varsigma \leqslant a. \tag{3.11}$$

Here the functions Λ_0, M_0 are defined by the equalities

$$\Lambda_0(\varsigma) = \lambda[z(\varsigma, \tau), \tau], \ \mathrm{M}_0(\varsigma) = \mu[z(\varsigma, \tau), \tau], \tag{3.12}$$

where τ is a parameter, and the connection between the Lagrangian coordinate ς and the

Eulerian coordinate z is given by the formula

$$z(\varsigma,\tau)=\int_0^{\varsigma}\exp\left[-2\int_0^{t}F(\rho,\sigma)d\sigma\right]d\rho,\ 0\leqslant\varsigma\leqslant a,\ t\geqslant 0$$

The boundedness of $F(\varsigma,t)=f(z,t)$ for $\varsigma\in[0,a]$ and any finite $t\geqslant 0$ is guaranteed by the solvability of the problem (1.8)–(1.13) globally. This provides the mutual uniqueness of correspondence between the variables ς and z.

The existence of solutions of system (3.8) not depending on ς is the remarkable peculiarity of this system. Such solutions are compatible with the boundary conditions (3.9), (3.10). This circumstance permits us to use them as barrier functions for the solution of the problem (3.8)–(3.10). We choose these functions in the following form:

$$\Lambda^-(t)=\frac{\lambda_{min}}{1+\lambda_{min}(t-\tau)},\quad \Lambda^+(t)=\frac{\lambda_{max}}{1+\lambda_{max}(t-\tau)},$$

$$\mathrm{M}^-(t)=\frac{\mu_{min}}{1+\mu_{min}(t-\tau)},\quad \mathrm{M}^+(t)=\frac{\mu_{max}}{1+\mu_{max}(t-\tau)},$$

where λ_{min} (λ_{max}) and μ_{min} (μ_{max}) are the minimal (maximal) values of functions $\lambda(z,\tau)$ and $\mu(z,\tau)$ in the interval $0\leqslant z\leqslant s(\tau)$.

Here we use the condition of Proposition 4, $l(t)+m(t)\not\equiv 0$ for $0\leqslant\ t\leqslant\tau$. We may assume without loss of generality that one can find an interval (t_1,t_2), $0\leqslant t_1<t_2\leqslant\tau$ such that the strict inequality

$$l(t)<0\quad\text{when}\quad t_1<t<t_2 \tag{3.13}$$

is satisfied. At the same time, the function $m(t)$ can vanish identically (note that both functions l and m are nonpositive for $t\geqslant 0$ in accordance with condition (2.10)). The case when $l=0$ for all $t\geqslant 0$ and the inequality analogous to (3.13) is satisfied for the function $m(t)$ is considered in a similar way.

First, let us consider the special case $m=0$ for all $t\geqslant 0$. Then the second condition (2.13) is homogeneous and implies the equality $\mu=0$ in the domain $\bar{S}_T$ for any $T>0$ from the uniqueness theorem for the solution of the initial boundary value problem (2.12)–(2.15) for the function μ. This means that the functions f and g coincide for all $z\in[0,s(t)]$, $t\geqslant 0$.

On the other hand, the values of the function λ are strictly positive on the upper boundary $t=\tau$, $0\leqslant z\leqslant s(\tau)$ of the domain S_T in consequence of inequality (3.13) and the strict maximum principle applied to the solution of the problem (2.12)–(2.15) for the function λ. So $\lambda_{min}=min\lambda(z,\tau)>0$. Now let us consider the function $P^-=\Lambda-\Lambda^-$. By virtue of (3.8)–(3.10), it is the solution of the following problem:

$$P^-_t+(\Lambda+\Lambda^-)P^-=v\exp\left(\int_0^t\Lambda dt'\right)\left[\exp\left(\int_0^t\Lambda dt'\right)P^-_{\varsigma}\right]_{\varsigma},\ (\varsigma,t)\in\Sigma_T,$$

$$P^-_{\varsigma}(0,t)=P^-_{\varsigma}(a,t)=0,\ t>\tau,$$

$$P^-(\varsigma,\tau)=\Lambda_0(\varsigma)-\lambda_{min},\ 0\leqslant\varsigma\leqslant a$$

(here we take into account that $\lambda[z(\varsigma,\tau),\tau]=\Lambda_0(\varsigma)$ in accordance with (3.12)). It follows from the maximum principle that $P^-(\varsigma,\tau)\geqslant 0$ in the semistrip $\bar{\Sigma}_T$; this fact implies the estimate

$$\Lambda(\varsigma, \tau) \geqslant \frac{\lambda_{min}}{1 + \lambda_{min}(t - \tau)} \text{ for } (\varsigma, t) \in \bar{\Sigma}_T$$

by virtue of definition of this function.

The inequality

$$\Lambda(\varsigma, t) \leqslant \frac{\lambda_{max}}{1 + \lambda_{max}(t - \tau)} \text{ for } (\varsigma, t) \in \bar{\Sigma}_T$$

is obtained in a similar way. Then we rewrite this inequality in terms of $\lambda(z, t)$, and take into account that $\lambda = 2f$, $f = g$ by virtue of $\mu = 0$, so that

$$\frac{\lambda_{min}}{1 + \lambda_{min}(t - \tau)} \leqslant 2f(z, t) \leqslant \frac{\lambda_{max}}{1 + \lambda_{max}(t - \tau)} \text{ when } 0 \leqslant z \leqslant s(t),\ t \geqslant \tau.$$

The correctness of asymptotics (3.4) for the functions f and g is obtained; consequently the asymptotics for the function s follows immediately from (2.8).

Let us pass to the analysis of general case where the inequality (3.13) is satisfied in parallel with

$$m(t) < 0 \text{ when } t_3 < t < t_4, \tag{3.14}$$

where t_3 and t_4 are some numbers from the interval $[0, \tau]$. First of all we see that the inequality (3.14) implies $\mu_{min} = \min \mu(z, \tau) > 0$. This fact permits us to prove the non-negativity of the functions $Q^- = \mathrm{M} - \mathrm{M}^-$, $Q^+ = \mathrm{M}^+ - \mathrm{M}$ in the domain $\bar{\Sigma}_T$, and to obtain the estimates

$$\frac{\mu_{min}}{1 + \mu_{min}(t - \tau)} \leqslant \mathrm{M}(\varsigma, t) \leqslant \frac{\mu_{max}}{1 + \mu_{max}(t - \tau)} \text{ for } (\varsigma, t) \in \bar{\Sigma}_T.$$

The last inequalities take the following form in terms of the functions f and g ($\mu = f - g$):

$$\frac{\mu_{min}}{1 + \mu_{min}(t - \tau)} \leqslant f(z, t) - g(z, t) \leqslant \frac{\mu_{max}}{1 + \mu_{max}(t - \tau)} \text{ when } 0 \leqslant z \leqslant s(t),\ t \geqslant \tau.$$

These upper and lower estimates $\Lambda(\varsigma, t)$ imply the following inequalities for the function $\lambda = f + g$:

$$\frac{\lambda_{min}}{1 + \lambda_{min}(t - \tau)} \leqslant f(z, t) + g(z, t) \leqslant \frac{\lambda_{max}}{1 + \lambda_{max}(t - \tau)} \text{ when } 0 \leqslant z \leqslant s(t),\ t \geqslant \tau.$$

As a result, we come to the relations $f + g = t^{-1} + \mathrm{O}(t^{-2})$, $f - g = t^{-1} + \mathrm{O}(t^{-2})$ when $t \to \infty$, $0 \leqslant z \leqslant s(t)$. This fact proves the correctness of the asymptotic representations (3.3) for the general case, when the both functions $l(t)$ and $m(t)$ take negative values even at some part over the interval $(0, \tau)$. The use of (2.8) and (3.3) gives the required asymptotics (3.2) of function $s(t)$. □

Proposition 5 Let us suppose that solution of problems (1.8)–(1.13) is defined in the domain S_T. If the inequality (2.10) and the condition

$$l + m = -A/vk = const < 0 \text{ when } t \geqslant \tau \tag{3.15}$$

are satisfied then one can find such finite $t^* > 0$ that $s(t) > 0$ for $0 \leqslant t < t^*$ and $s \to 0$ when $t \nearrow t^*$.

https://doi.org/10.1017/S0956792501004776 Published online by Cambridge University Press

Proof Let us use the identity (2.9), and rewrite it in terms of the functions f and g:

$$\frac{d}{dt}\int_0^s f dz + \int_0^s (3f^2 + g^2)dz = -\nu k(l+m). \tag{3.16}$$

We introduce the functions

$$U(t) = \int_0^{s(t)} f(z,t)dz, \ V(t) = \int_0^{s(t)} g(z,t)dz. \tag{3.17}$$

Using the Cauchy–Bunyakovsky inequality and condition (3.15), we obtain from (3.16) a differential inequality for the function U:

$$\frac{dU}{dt} \leqslant -\frac{3U^2}{s} + A \text{ when } t \geqslant \tau. \tag{3.18}$$

Note that the inequality $U(\tau) = U_0 > 0$ is valid by virtue of the conditions of Proposition 5. Moreover, the function $U(t)$ is non-negative for $t \geqslant \tau$ and cannot vanish as long as $s(t) > 0$. These statements follow from the strict maximum principle applied to the functions $f+g$, $f-g$ (see the beginning of the Proof of Proposition 4). Then it follows from (2.8), (3.17) that $ds/dt < 0$ for these values of t. This fact permits us to rewrite the inequality (3.18) in more convenient form by introducing the function $U^2 = W(s)$:

$$\frac{dW}{ds} - \frac{3W}{s} \geqslant -A \ s \leqslant s(\tau) = s_0.$$

Integration of the last inequality leads to the result

$$W(s) \geqslant \frac{sA}{2}\left(1 - \frac{s^2}{s_0^2} + \frac{2U_0^2 s^3}{As_0^3}\right) \equiv \frac{sA}{2}R^2(s),$$

and, moreover, $R(s) \geqslant C_8 > 0$ for $s \in [0, s_0]$. The estimate below is obtained from this fact and the relation $ds/dt = -2W^{1/2}(s)$ following from (2.8), (3.17) and the definition of function W:

$$\int_s^{s_0} \frac{dr}{R(r)\sqrt{2Ar}} \geqslant t - \tau.$$

The integral entering this estimate converges when $s \to 0$. This fact guarantees finiteness of the value of t^* corresponding to the vanishing-time of the function s, and we obtain the following estimate of t^*:

$$t^* \leqslant \frac{1}{\sqrt{2A}}\int_0^{s_0} \frac{ds}{R(s)\sqrt{s}} + \tau.$$

□

The interest in investigating the behaviour of solution of the problem (1.8)–(1.13) near the moment t^* follows from this proposition. The simplest solution of this question can be found in the case when both functions $l(t)$ and $m(t)$ take constant values beginning with some τ.

Proposition 6 Let the conditions of Proposition 5 be satisfied. Moreover, assume that

$$l - m = -B/\nu k = const \text{ when } t \geqslant \tau \tag{3.19}$$

https://doi.org/10.1017/S0956792501004776 Published online by Cambridge University Press

with $|B| \leqslant A$. Then the following relations are valid as $t \nearrow t^*$:

$$\frac{s}{(t^*-t)^2} \to 4A\left(1+\sqrt{1-8\beta^2/9}\right), \tag{3.20}$$

$$\frac{U}{t^*-t} \to A\left(1+\sqrt{1-8\beta^2/9}\right), \tag{3.21}$$

$$\frac{V}{t^*-t} \to \frac{4B}{3} \tag{3.22}$$

where $\beta = B/A$ ($|\beta| \leqslant 1$) and the functions $U(t)$, $V(t)$ are defined by (3.17).

Proof The identity (3.15) and the analogous identity

$$\frac{d}{dt}\int_0^s g dz + 4\int_0^s f g dz = -\nu k(l-m), \tag{3.23}$$

obtained by integration of the second equation (1.8) by z over the interval $[0, s(t)]$ with the use of relations (1.9)–(1.11) are the basis of the proof. Both identities are considered for the values $t \geqslant \tau$, when their right-hand sides are constant by virtue of conditions (3.15), (3.19).

Let us denote the mean value of function $g(z,t)$ on the interval $0 \leqslant z \leqslant s(t)$ as $\bar{g}(t)$ and put $j(z,t) = g - \bar{g}$. The identities (3.16), (3.23) can be rewritten in the form

$$\frac{dU}{dt} + \frac{3U^2+V^2}{s} + \int_0^s (3h^2 + j^2) dz = A, \tag{3.24}$$

$$\frac{dV}{dt} + \frac{4UV}{s} + \int_0^s hj dz = B \text{ when } t \in [\tau, t^*)$$

with the help of these functions and the functions $\bar{f}(t)$, $h(z,t) = f - \bar{f}$ introduced before. (Here we use the evident equalities $U = s\bar{f}$, $V = s\bar{g}$ following from (3.17) and definition of functions $\bar{f}$ and $\bar{g}$.)

The system (3.24) is not closed for the functions U and V; however, this fact does not prevent us from finding the asymptotics of its solution near the moment t^* when $s(t^*) = 0$. The point is that the integral terms of (3.24) tend to zero quickly when $t \nearrow t^*$. The proof depends upon the representations

$$f(z,t) = \bar{f}(t) + \int_{b(t)}^z f_z(\varsigma, t) d\varsigma, \tag{3.25}$$

$$g(z,t) = \bar{g}(t) + \int_{c(t)}^z g_z(\varsigma, t) d\varsigma,$$

where $b(t)$ and $c(t)$ are points from the interval $[0, s(t)]$ where the function $f(z,t)\,(g(z,t))$ takes its mean value as a function of z. Using the uniform estimates (2.22) of the functions $|f_z|$, $|g_z|$ which are valid by virtue of (2.10), and remembering the definitions of the functions h, j, we arrive at the inequalities

$$|h| \leqslant C_5 s, \ |j| \leqslant C_5 s \quad \text{when } 0 \leqslant z \leqslant s(t),\ \tau \leqslant t \leqslant t^*.$$

Now if the system (3.24) is rewritten in the form

$$\frac{dU}{dt} + \frac{3U^2 + V^2}{s} + \Phi(t) = A, \qquad (3.26)$$

$$\frac{dV}{dt} + \frac{4UV}{s} + \Psi(t) = B,$$

where

$$\Phi = \int_0^{s(t)} [3h^2(z,t) + j^2(z,t)]dz, \; \Psi = 4\int_0^{s(t)} h(z,t)j(z,t)dz$$

then the estimates

$$|\Phi| \leqslant C_9 s^3, \; |\Psi| \leqslant C_9 s^3 \text{ when } \tau \leqslant t \leqslant t^* \qquad (3.27)$$

with $C_9 = 4C_5^2/3 = const$ will be valid for the functions $\Phi(t)$, $\Psi(t)$.

It follows from the strict monotonicity of function s on the interval $[\tau, t^*)$ obtained in Proposition 5 that we may convert the dependence on s to t, and consider Φ and Ψ as functions of the variable s; $\Phi[t(s)] = \phi(s)$, $\Psi[t(s)] = \psi(s)$, where $0 \leqslant s \leqslant s_0 = s(\tau)$. The further reasoning is based on the transformation of (3.26) into a third-order system with the help of the change of variables

$$U = (As)^{1/2}q(\rho), \; V = (As)^{1/2}r(\rho), \; \rho = \ln(1/s). \qquad (3.28)$$

Substitution of (3.28) into (3.26) and recalling that $ds/dt = -2U$ leads to the system of equations

$$2q\frac{dq}{d\rho} + 2q^2 + r^2 = 1 - \frac{\varphi(s)}{A},$$

$$2q\frac{dr}{d\rho} + 3qr = \beta - \frac{\psi(s)}{A}, \qquad (3.29)$$

$$\frac{ds}{d\rho} = -s, \; \rho \geqslant \rho_0,$$

where $\rho_0 = \ln(1/s_0)$, $\beta = B/A = \text{const}$, $|\beta| \leqslant 1$ by virtue of the conditions of Proposition 6. Our aim is to investigate the behaviour of the solution of the Cauchy problem

$$q(\rho_0) = q_0, \; r(\rho_0) = r_0, \; s = \exp(-\rho_0) = s_0 \qquad (3.30)$$

for the system (3.29) when $\rho \to \infty$, where

$$q_0 = (As_0)^{-1/2}\int_0^{s_0} f(z,t)dz, \; r_0 = (As_0)^{-1/2}\int_0^{s_0} g(z,\tau)dz.$$

Moreover, it is assumed that the functions f and g are already defined in the domain S_τ so that $|r_0| \leqslant q_0$, $q_0 > 0$ on the basis of (2.16), (3.17) and (3.28) (note that conditions of Proposition 6 guarantee the satisfaction of the inequalities (2.10), providing estimates (2.16)). The inequalities

$$q(\rho) > 0, \; |r(\rho)| \leqslant q(\rho) \qquad (3.31)$$

for any finite $\rho \geqslant \rho_0$ also follow from these relations, but here they play the role of *a priori* estimates for solution of the Cauchy problem (3.29), (3.30).

First, note that the trajectory of the dynamical system (3.29) emerging from the point (q_0, r_0, s_0) cannot leave the limits of the cylindrical sector $K_N = \{q, r, s : 0 <$

$q^2 + r^2 < N^2$, $|r| < q$, $0 < s < s_0\}$ of phase space $\mathscr{R}^3$ (where N is sufficiently large) when $\rho \geqslant \rho_0$. It is sufficient for the proof of this statement to check that no outgoing points of the system (3.29) are situated on the boundary of the domain K_N. In fact, the rectangles $q = r$, $0 \leqslant s \leqslant s_0$ and $q = -r$, $0 \leqslant s \leqslant s_0$ cannot contain outgoing points in view of inequalities (3.31). The upper segment of K_N, i.e. the circular sector $0 \leqslant q \leqslant N$, $|r| \leqslant q$, $s = s_0$, consists of ingoing points in accordance with the third equation of system (3.29). The lower segment of K_N cannot contain outgoing points, since it corresponds to the value $\rho = \infty$.

Now one must check the absence of exit points on the cylindrical part of the boundary K_N, i.e. on the set $H_N = \{q, r, s\ :\ q^2 + r^2 = N^2, |r| \leqslant q, 0 \leqslant s \leqslant s_0\}$. The field of directions of the dynamical system (3.29) is characterized by the vector $\vec{l}$ with components $(2q)^{-1}[1 - 2q^2 - r^2 - A^{-1}\varphi(s)]$, $(2q)^{-1}[\beta - 3qr - A^{-1}\psi(s)]$, $-s$. The scalar product $\vec{l} \cdot \vec{n}$ of $\vec{l}$ and the unit external normal $\vec{n} = (\cos\omega, \sin\omega, 0)$ to the surface H_N, where $\omega = \operatorname{arctg}(r/q)$, gives

$$\vec{l} \cdot \vec{n} = -N(1 + \sin^2\omega) - \frac{[A^{-1}\varphi(s) - 1]\cos\omega + [A^{-1}\psi(s) - \beta]\sin\omega}{(2N\cos\omega)}.$$

Since $|\omega| \leqslant \pi/4$ on the surface H_N, $|\varphi| \leqslant C_9 s_0^3$, $|\psi| \leqslant C_9 s_0^3$ by virtue of (3.27), and $|\beta| \leqslant 1$, one can obtain the inequality $\vec{l} \cdot \vec{n} < 0$ on the surface H_N by choosing N larger than $\max\,[(1 + A^{-1}C_9 s_0^3)^{1/2}, (q_0^2 + r_0^2)^{1/2}]$, and that the point (q_0, r_0, s_0) belongs to the set $\bar{K}_N$. Then both inequalities (3.31) and the *a priori* estimate

$$q \leqslant N \ \text{when}\ \rho \geqslant \rho_0 \tag{3.32}$$

are valid for the solution of the Cauchy problem (3.29), (3.30).

Let us return to the system (3.29). It has the unique equilibrium point

$$q = q^* \equiv 0.5\left(1 + \sqrt{1 - 8\beta^2/9}\right)^{1/2}, \tag{3.33}$$

$$r = r^* \equiv 2^{-1/2}\operatorname{Sgn}\beta\left(1 + \sqrt{1 - 8\beta^2/9}\right)^{1/2}, \ s = 0$$

in the domain K_N. Linearization of (3.29) near the equilibrium point leads to the system

$$\frac{dQ}{d\rho} = -2Q - \frac{r^*}{q^*}R,$$

$$\frac{dR}{d\rho} = -\frac{3r^*}{2q^*}Q - \frac{3}{2}R, \ \frac{dS}{d\rho} = -S.$$

The eigenvalues of the matrix of this system are

$$\lambda_{1,2} = \frac{-7q^* \pm \sqrt{(q^*)^2 + 24(r^*)^2}}{2q^*}, \ \lambda_3 = -1.$$

Since $q^* > 0$ and $|r^*| \leqslant q^*$, and $|\beta| \leqslant 1$ all eigenvalues $\lambda_i\,(i = 1, 2, 3)$ are negative. The equilibrium point $(q^*, r^*, 0)$ of system (3.29) is stable in accordance with the Lyapunov theorem.

The proof of the fact that trajectory of the dynamic system (3.29) starting from the point (q_0, r_0, s_0) at the 'moment' $\rho = \rho_0$ finishes at $\rho \to \infty$ at the equilibrium point $(q^*, r^*, 0)$ will complete the proof of Proposition 6. From this fact, (3.20)–(3.22) are derived without any problems. By virtue of the third equation (3.29), the projection of the desired trajectory on the plane $s = 0$ approaches asymptotically (by virtue of the third equation (3.29)) as $\rho \to \infty$ the trajectory of two-dimensional dynamical system

$$\frac{dq}{d\rho} = (2q)^{-1}(1 - 2q^2 - r^2), \tag{3.34}$$

$$\frac{dr}{d\rho} = (2q)^{-1}(\beta - 3qr)$$

emerging from the point (q_0, r_0) at $\rho = \rho_0$; denote this trajectory by L. As was proved before, the curve L is contained in a circular sector $\bar{D}_N = \{q, r : q^2 + r^2 \leqslant N^2, |r| \leqslant q\}$.

Now let us suppose that $|\beta| < 1$. Then

$$\vec{m}(q, r) = ((2q)^{-1}(1 - 2q^2 - r^2), (2q)^{-1}(\beta - 3qr))$$

does not vanish on the boundary of the domain $D_{N,\epsilon} = \{q, r : \epsilon^2 < q^2 + r^2 < N^2, |r| < q\}$, where the number $\epsilon > 0$ is chosen less than $q_0 > 0$. (Note that for $|\beta| < 1$ and sufficiently small ϵ all points of the 'arch' $q^2 + r^2 = \epsilon^2$, $|r| \leqslant q$ are ingoing points for system (3.34), so that the trajectory L does not fall outside the limits of not only the sector $\bar{D}_N$ but also the domain $\bar{D}_{N,\epsilon}$ for $\rho \geqslant \rho_0$.) Since $\vec{m} \neq 0$ on the boundary of the domain $D_{N,\epsilon}$ we may calculate the rotation of the vector field $\vec{m}(q, r)$ on this boundary. Simple calculations show that this rotation is equal to unity. Since $q = q^*$, $r = r^*$ is the unique singular point of the field $\vec{m}$ in the domain $\bar{D}_{N,\epsilon}$ and the index of this point is equal to unity, then system (3.34) has no limit cycles in the domain $\bar{D}_{N,\epsilon}$. So it follows that point (q^*, r^*) is the limit point of curve L when $\rho \to \infty$.

Now let $\beta = 1$ (case $\beta = -1$ is considered in a similar way). Here the vector $\vec{m}$ vanishes on the boundary of the domain $D_{N,\epsilon} : \vec{m}(1/\sqrt{3}, 1/\sqrt{3}) = 0$. If at the same time $q_0 = r_0 = 1/\sqrt{3}$, then the trajectory L consists of one point. If $q_0 = r_0 \neq 1/\sqrt{3}$, $\epsilon \leqslant q_0 \leqslant N$ then the line L is a part of a segment of the straight line $q = r$. In this case, the dependence $q(\rho)$ is defined from solution of the Cauchy problem

$$\frac{dq}{d\rho} = (2q)^{-1}(1 - 3q^2) \text{ when } \rho > \rho_0,\ q(\rho_0) = q_0.$$

It is evident that $q \to q^* = 1/\sqrt{3}$ when $\rho \to \infty$. If the point (q_0, r_0) lies strictly inside the domain $\bar{D}_{N,\epsilon}$ then we can narrow down a little the opening angle of this domain and achieve the situation when vector field $\vec{m}$ has no zeros on the boundary of the domain $\epsilon^2 < q^2 + r^2 < N^2$, $|r| < (1 - \delta)q$ containing the point (q_0, r_0) (the last fact can be ascertained for small enough $\delta > 0$). Now the above reasoning about the rotation of the field $\vec{m}$ can be repeated almost literally.

So now we have shown that the relations

$$q \to q^*,\ r \to r^* \text{ when } \rho \to \infty. \tag{3.35}$$

hold under the conditions of Proposition 6. Concerning $\rho = \ln(1/s)$, this follows from (3.35) and definition of q^* and r^* (3.33), with account of equations (3.28), that

$$\frac{U}{(As)^{1/2}} \to 0.5\left(1 + \sqrt{1 - 8\beta^2/9}\right)^{1/2}, \tag{3.36}$$

$$\frac{V}{(As)^{1/2}} \to [0.5\left(1-\sqrt{1-8\beta^2/9}\right)]^{1/2} \text{ when } s \to 0.$$

The first of this relations means that $ds/dt \to -\left[As\left(1+\sqrt{1-8\beta^2/9}\right)\right]^{1/2}$ when $t \nearrow t^*$ by virtue of the equation $ds/dt = -2U$. The limit equality (3.20) follows from this fact and then the relations (3.21), (3.22) can be obtained from (3.36). □

This proposition deserves some comment. The reason for introducing the functions $U(t), V(t)$ is the following: these functions remain bounded in the limit $t \nearrow t^*$ in contrast to the functions $\bar{f}(t), \bar{g}(t)$; moreover, $U = \mathrm{O}(t^* - t)$, $V = \mathrm{O}(t^* - t)$ when $t \nearrow t^*$. On the other hand, the equalities below follow from the formulae $U = \bar{f}s$, $V = \bar{g}s$, relations (3.20)–(3.22) and representations (3.25) by virtue of equations (2.22):

$$f(z,t) = \frac{1}{4(t^*-t)} + \mathrm{O}(t^*-t)^2, \tag{3.37}$$

$$g(z,t) = \frac{3}{8\beta(t^*-t)}\left(1-\sqrt{1-8\beta^2/9}\right) + \mathrm{O}(t^*-t)^2 \text{ when } t \nearrow t^*,\ 0 \leqslant z \leqslant s(t).$$

Thus, formulae (3.20), (3.37) imply that smooth joining of free boundaries of the layer takes place in the moment t^*, although the longitudinal components of the liquid velocity grow infinitely when $t \nearrow t^*$.

4 Discussion and conclusion

(a) The system (1.8) admits solutions with $f = g$, $f = -g$ and $g = 0$. In accordance with (1.7), the first two cases describe plane flows. These cases can be realized if one puts $m = 0$ and $l = 0$ in (1.6). The case $g = 0$ corresponds to the equality $l = m$ for all $t \geqslant 0$. In this case, the solution of the problem (1.8)–(1.13) describes axisymmetric motion.

(b) Proposition 2 gives sufficient conditions for blow-up of the solution of problems (1.8)–(1.19) in finite time t_*. Note that this phenomenon has a purely inertial character; viscous forces cannot prevent it, although these forces guarantee the space regularity of the solution.

The question about the structure of the solution singularity near the moment t_* is still open. This question is studied in detail in Galaktionov & Vazquez [6] for the plane analogue of the discussed problem. More precisely, (1.8) with $g = f$ is considered in Galaktionov & Vazquez [6] ; there the modified problem with free boundary (1.9)–(1.13) is investigated – one puts $l = m = 0$ in the condition (1.9) and substitutes condition (1.13) for f by the following:

$$f(z,0) = f_0(z), \quad 0 \leqslant z \leqslant a.$$

(We shall call this problem P.)

Let the function f_0 satisfy the natural smoothness and compatibility conditions and also the inequality $f_0 \leqslant 0$ for $z \in [0,a]$ and some 'steepness condition' [6]. Then the solution f, s of problem P has the asymptotics

$$s(t) \sim \frac{\pi}{2\sqrt{\alpha(t_* - t)}}, \quad f(z,t) \sim -\frac{\cos^2\left(z\sqrt{\alpha(t_* - t)}\right)}{t_* - t}$$

when $t \nearrow t_*$, $0 \leqslant z < s(t)$, where $\alpha = \text{const} > 0$.

(c) Sufficient conditions for solvability of problem P for all $t \geqslant 0$ are determined in Pukhnachov [13]. Also constructed in this paper is a class of its exact solutions of the form

$$f = a(t) + b(t)\cos[\pi nz/s(t)], \tag{4.1}$$

where n is a natural number and functions a, b, s form the solution of a dynamical system. The value of the solutions (4.1) consists of the fact that these solutions represent the leading terms of both the blow-up of solutions of problem P when $t \nearrow t_*$ and its regular solutions when $t \to \infty$.

(d) Let us consider the problem (1.8)–(1.13) for the case $l + m = 0$ where the function $l(t)$ is non-negative and $l > 0$ at some interval $0 < t_1 < t < t_2 < \infty$. Then the statement of Proposition 2 is valid, although its proof requires some small modification. The case $l = -m$ is interesting from the physical point of view in so far as tangential stress applied to the free boundary in the x-direction has the same magnitude but opposite sign to that applied in the y-direction. In this case, the analysis of problems (1.8)–(1.13) shows that the layer thickness $2s(t)$ is a monotonically increasing function of time, and there exist some $t_* < \infty$ such that $s \to \infty$ when $t \nearrow t_*$.

Indeed, now the boundary condition (1.9) for the function f is uniform, while the right side of the same condition for the function g is strictly negative within the interval (t_1, t_2). On the basis of the maximum principle, we have $g^2 > 0$ when $t_1 < t < t_2$, $0 \leqslant z \leqslant s(t)$. At once, we consider the first equation (1.8) as a linear parabolic equation with coefficients f and $-2\int_0^z f\,d\zeta$ before f and f_z, respectively, and with a source $-g^2$. Remembering that the function $f(z,t)$ satisfies the uniform boundary and initial conditions (1.9), (1.13), we conclude that $f < 0$ when $t \in (t_1, t_2)$, $z \in [0, s(t)]$ in view of the maximum principle. It means that the mean value $\bar{f}(t)$ of the function f in the interval $[0, s(t)]$ is negative if $t \in (t_1, t_2)$. As was shown in the proof of Proposition 2, the inequality $\bar{f}(\tau) < 0$ implies the estimate $\bar{f}(t) \leqslant (1 + \gamma\tau - \gamma\, t)^{-1}\bar{f}(\tau)$ where $t > \tau \in (t_1, t_2)$, $\gamma = const > 0$. On account of (2.8), this estimate guarantees the required property $s \to \infty$ when $t \nearrow t_* \leqslant \tau + \gamma^{-1}$. We emphasize that this effect has a purely nonlinear character.

(e) Here we suggest some comments on Proposition 4. The exceptional case described by formulae (3.5) corresponds to plane motion. If the equality $l = 0$ or $m = 0$ is broken at some arbitrarily small interval of time then the solution of the problem (1.8)–(1.13) is symmetrized with growth of t, as follows from relations (3.4) (note that $g = 0$ for axisymmetric motion). The essential distinction between plane and three-dimensional regimes of thinning of the layer is demonstrated by the asymptotics of function s: in the first case $s \sim t^{-1}$ and in the second case $s \sim t^{-2}$ when $t \to \infty$.

(f) Let us consider the problem of thermocapillary motion of viscous liquid in a layer with linear dependence of the free boundary temperature on the space coordinates,

$$\theta_\Gamma = A(t)x + B(t)y.$$

It is not difficult to see that its solution can be obtained in the form

$$u = u(z,t),\ v = v(z,t),\ w = 0,\ p = 0,\ s = a = \text{const.}$$

The functions u, v are determined as solutions of the second initial boundary value problem for the linear equation of heat conduction (the details are omitted).

Thus, a linear distribution of temperature on the plane free boundary of a viscous layer does not lead to a thickness change. It is natural to suppose on the basis of this observation that the main change of the layer thickness under the action of thermocapillary forces takes place near the critical points of the temperature field on free surface. This concept can be considered as an additional motivation for the investigation of solutions of the Navier–Stokes equations of the form (1.7). The unboundedness of functions u and v when $x, y \to \infty$ is the evident defect of this solution. However, we can consider it as a solution describing the local behaviour of liquid in vicinity of critical points of the temperature field on the free boundary.

(g) The problem (1.1)–(1.6) can be referred to as a 'non-connected' problem of thermocapillary convection. It originates from the assumption that the temperature is a given function at the free boundary. A more complicated 'connected' problem deals with the boundary condition of second or third kind for the temperature. Let us suppose that the free boundary is thermo-insulated. Then the condition (1.6) should be replaced by

$$\theta_z(s(t), t) = 0, \quad 0 < t < T. \tag{4.2}$$

In this case, we have the temperature representation

$$\theta = \varphi x^2 + \psi y^2 + \omega$$

where $\varphi(z,t)$, $\psi(z,t)$, $\omega(z,t)$ are determined from the parabolic system coupled with the equations (1.8) via boundary conditions including (4.2). The plane analogue of this problem was studied numerically in Andreev & Pukhnachov [3].

(h) To identify a physical system that might correspond to the solution of the problem (1.1)–(1.6), let us consider a liquid film of mean initial thickness a and the diameter d suspended in a solid frame at the presence of gravity with the acceleration g. It is clear that following relations should be satisfied: $a \ll d$, $d \sim (2\sigma_0/\rho g)^{1/2} = L$, where L is the capillary constant and σ_0 is a characteristic value of the surface tension coefficient. This gives an upper estimate for a. The lower estimate limiting the applicability of our approach is $a \gg \lambda$, where λ is the thickness of the double diffusional electric layer. In this case, we can neglect the disjoining pressure in the film.

Now let us introduce the dimensionless parameter $q = \rho g \beta a^2/\kappa$ where β is the volumetric coefficient of thermal expansion and κ is the parameter in relation (1.1). If $q \ll 1$, we can ignore the contribution of the buoyancy in a formation of the velocity field and, therefore, eliminate from consideration the heat equation in the context of our problem.

As an example, we consider a pure water film at low gravity ($g = 1\,cm/s^2$) near the temperature $298\,K$; in this case, $L = 12\,cm$. If we choose $a = 0.1\,cm$, $d = 5\,cm$, $\lambda = 10^{-6}\,cm$ then inequalities $\lambda \ll a \ll d$ will be true as well as $q \ll 1$ (in fact, here $q = 1.6 \cdot 10^{-5}$). In addition, we note that the characteristic time of the problem is of order $[a\rho/\kappa(l_* + m_*)]^{1/2} = \tau$, where l_* and m_* are the maximum values of the functions l and m, respectively. Putting $l_* = m_* = 10^{-2}K/cm^2$, we obtain $\tau = 5.6\,s$.

Acknowledgements

This work was undertaken at the time of the author's visit to the Max Planck Institute for Mathematics in the Sciences (Leipzig). The author would like to express his sincere gratitude to the management and staff of Max Planck Institute, and personally to Professor E. Zeidler for the assignment of excellent conditions for scientific research. The author is also very thankful to the referees for fruitful comments.

References

[1] Andreev, V. K. & Admaev, O.V. (1992) Axisymmetric thermocapillary flow in cylinder and cylindrical layer. *Hydromechanics and Heat/Mass Transfer in Microgravity*, pp. 169–172.

[2] Andreev, V. K., Kaptsov, O. V., Pukhnachov, V. V. & Rodionov, A. A. (1998) *Applications of Group-Theoretical Methods in Hydrodynamics*. Kluwer Academic.

[3] Andreev, V. K. & Pukhnachov, V. V. (1983) Invariant solutions of the equations of thermocapillary motion. *Chislennye Metody Mechaniki Sploshnoi Sredy*, **14**(5), 3–23 (in Russian).

[4] Birikh, R. V. (1996) On thermocapillary convection in horisontal liquid layer. *Prikladnaya Mekhanika i Tekhnicheskaya Fizika*, **3**, 69–72 (in Russian).

[5] Friedman, A. (1964) *Partial Differential Equations of Parabolic Type*. Prentice-Hall.

[6] Galaktionov, V. A. & Vazquez, J. L. (1999) Blow-up for a class of solutions with free boundaries for the Navier–Stokes equations. *Adv. Diff. Eq.* **4**, 297–321.

[7] Gupalo, Ju. P. & Ryazantsev, Ju. S. (1988) On thermocapillary motion of a liquid with free surface at nonlinear dependence of surface tension on temperature. *Izvestiya AN SSSR, Mekhanika Zhidkosti i Gaza*, **5**, 132–137 (in Russian).

[8] Ladyzhenskaya, O. A., Solonnikov, V. A. & Uraltseva, N. N. (1968) *Linear and Quasilinear Equations of Parabolic Type*. AMS.

[9] Meleshko, S. V. & Pukhnachov, V. V. (1999) On one class of partially invariant solutions of the Navier–Stokes equations. *Prikladnaya Mekhanika i Tekhnicheskaya Fizika*, **40**(2), 24–33 (in Russian).

[10] Mogilevskii, I. Sh. & Solonnikov, V. A. (1992) On the solvability of an evolution free boundary problem for the Navier–Stokes equations in the Hölder spaces of functions. *Mathematical problems related to the Navier–Stokes equations. Series on Advances in Mathematics for Applied Sciences*, **11**, 105–181.

[11] Napolitano, L. G. (1980) Plane Marangoni–Poiseuille flow of two immiscible fluids. *Acta Astronautica*, —bf 7(4–5), 461–478.

[12] Ovsiannikov, L. V. (1982) *Group Analysis of Differential Equations*. Academic Press.

[13] Pukhnachov, V. V. (1999) On a problem of viscous strip deformation with a free boundary. *C. R. Acad. Sci. Paris*, **t.328**, Serie 1, 357–362.