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Amorphous molecular beam epitaxy: global solutions and absorbing sets

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The parabolic equation

$$u_t + u_{xxxx} + u_{xx} = -(|u_x|^\alpha)_{xx}, \qquad \alpha > 1,$$

is studied under the boundary conditions $u_x|_{\partial\Omega} = u_{xxx}|_{\partial\Omega} = 0$ in a bounded real interval Ω. Solutions from two different regularity classes are considered: It is shown that unique *mild solutions* exist locally in time for any $\alpha > 1$ and initial data $u_0 \in W^{1,q}(\Omega)$ $(q > \alpha)$, and that they are global if $\alpha \leqslant \frac{5}{3}$. Furthermore, from a semidiscrete approximation scheme global *weak solutions* are constructed for $\alpha < \frac{10}{3}$, and for suitable transforms of such solutions the existence of a bounded absorbing set in $L^1(\Omega)$ is proved for $\alpha \in [2, \frac{10}{3})$. The article closes with some numerical examples which do not only document the roughening and coarsening phenomena expected for thin film growth, but also illustrate our results about absorbing sets.

1 Introduction and main results

We consider the nonlinear parabolic problem

$$\begin{aligned} u_t + u_{xxxx} + u_{xx} &= -(|u_x|^\alpha)_{xx} \qquad \text{in } \Omega \times (0, T), \\ u_x|_{\partial\Omega} = u_{xxx}|_{\partial\Omega} &= 0, \\ u|_{t=0} &= u_0, \end{aligned} \tag{1.1}$$

where $\Omega = (0, L) \subset \mathbb{R}$ is a bounded real interval, $\alpha > 1$ is a parameter, and $u_0 \in L^2(\Omega)$. Since the PDE and the lateral boundary conditions in (1.1) are invariant under addition of constants, we may and will assume throughout that

$$\int_\Omega u_0 = 0.$$

Note that the PDE in (1.1) is equivalent to the more general equation

$$v_t + Av_{xxxx} + Bv_{xx} = -C(|v_x|^\alpha)_{xx} \qquad \text{in } (0, L' \times (0, T'), \tag{1.2}$$

which can be seen upon the substitution $v(x,t) = au(bx, ct)$ with

$$a := A^{\frac{\alpha}{2(\alpha-1)}} B^{\frac{2-\alpha}{2(\alpha-1)}} C^{-\frac{1}{\alpha-1}}, \qquad b := \sqrt{\frac{B}{A}} \qquad \text{and} \qquad c := \frac{B^2}{A}.$$

In this case, $L = \sqrt{\frac{B}{A}} L'$ and $T = \frac{B^2}{A} T'$.

The fourth-order equation in (1.2) arises in the modelling of the epitaxial growth of thin films of certain metallic glasses (for instance, $Zr_{65}Al_{7.5}Cu_{27.5}$), where usually $\alpha = 2$. In fact, in molecular beam epitaxy the particles are deposited through a directed beam rather than through vapour deposition (for such a model see Ortiz *et al.* [16] and King *et al.* [10]), so that evaporation can be assumed to be negligible, and the growing film relaxes entirely by surface diffusion. Surface diffusion is modelled by the fourth order term Av_{xxxx} in (1.2) [7, 12], whereas the Edwards–Wilkinson term Bv_{xx} should actually be absent, since it usually models diffusion by evaporation-condensation [5]. As claimed in Siegert & Plischke [18], however, the same term takes account of so-called Schwoebel barriers which model an uphill current of particles at step edges of the film profile. Finally, for $\alpha = 2$ the so-called conserved KPZ term $-C(|v_x|^\alpha)_{xx}$ [8] models driven Arrhenius dynamics, i.e. the activation of a particle needed to cross an energy barrier before diffusion. Note that in amorphous molecular beam epitaxy no layers form in the substrate. For more details on physical background and relevance we refer to Blömker & Gugg [2] and the references therein [18, 11, 17].

As to the mathematical treatment, only little seems to be known about (1.1). In Blömker & Gugg [2, 3], the authors study the PDE in (1.1) for $\alpha = 2$ with an additional space-time white noise and prove existence of global (rather weak) 'martingale solutions' of the resulting stochastic equation under periodic boundary conditions.

As a first step towards a satisfactory understanding of the properties of (1.1) *with* noise, we find it adequate to investigate (1.1) *without* any sources. Especially the numerical part (see § 5) will show that indeed the dynamics of the deterministic problem is quite rich and rather comparable to that of the stochastic model [3]. In the slightly different setting of epitaxial growth by vapour deposition, the same effect was observed in Ortiz *et al.* [16] and King *et al.* [10].

Also note that in this article we focus our attention to an equation in one space dimension. For possible analogues in two dimensions describing the actual physical process, in the presence of the nonlinear term no appropriate a priori estimates seem to be available. It is then not clear how to obtain solutions being global in time. This problem was observed for other thin film equations, too, for example, in Blömker & Gugg [3].

The nonlinearity in (1.1), formally being expanded according to

$$(|u_x|^\alpha)_{xx} = \alpha |u_x|^{\alpha-2} u_x u_{xxx} + \alpha(\alpha-1)|u_x|^{\alpha-2} u_{xx}^2, \tag{1.3}$$

is by no means trivial to handle: albeit it is of third (and therefore of 'lower') order, at the same time it exhibits some type of superlinear growth with respect to u. In such a situation one usually cannot expect global solutions to exist for arbitrary initial data, unless the *structure* of the nonlinear term is 'nice' in some sense. Although this is partly true for (1.3) – see (3.22), (4.13) – for large α we will only be able to construct solutions which are local in time. Also, note that we cannot expect classical solutions to exist in the case $\alpha < 2$, because then the right hand side in (1.3) may become singular at the points where u_x vanishes, even when u is smooth.

We first consider *mild* solutions (see Definition 3.1) and prove in § 3 the following:

- For any $\alpha > 1$, (1.1) has a unique local-in-time mild solution, provided that $u_0 \in W^{1,q}(\Omega)$ for some $q > \alpha$ (Theorem 3.2).
- If $\alpha \in (1, \frac{5}{3}]$ and $u_0 \in W^{1,2}(\Omega)$ then the mild solution is global in time (Theorem 3.7).

We next investigate *weak* solutions and shall obtain in § 4 that global weak solutions exist for a larger class of α (covering the case $\alpha = 2$ that is important in the applications). Apart from that, we will see that for $\alpha \geqslant 2$ such solutions are uniformly attracted by some set which is bounded in a certain sense:

- If $\alpha \in (1, \frac{10}{3})$ then for any $u_0 \in L^2(\Omega)$, problem (1.1) has at least one global weak solution (Theorem 4.3).
- Given $\alpha \in [2, \frac{10}{3})$, there exists a diffeomorphism $\Phi : \mathbb{R} \to (0, \infty)$ such that the transformed function $v(x,t) := \Phi(u(x,t))$ satisfies

$$\int_\Omega v(t) \leqslant C_0 + \left(\int_\Omega v(0)\right) \cdot e^{-\nu t} \qquad \text{for a.e. } t > 0 \tag{1.4}$$

  with certain positive constants C_0 and ν. Here, u is the weak solution constructed above (Theorem 4.5).

In the case $\alpha > 2$, we may choose $\Phi(s) = e^{\beta s}$ for sufficiently small $\beta > 0$, so that (1.4) more conveniently reads

$$\int_\Omega e^{\beta u(t)} \leqslant C_0 + \left(\int_\Omega e^{\beta u_0}\right) \cdot e^{-\nu t} \qquad \text{for a.e. } t > 0;$$

if $\alpha = 2$ we may choose $\Phi(s) \sim s^p$ for large $s > 0$ and $\Phi(s) \sim e^{\beta s}$ as $s \to -\infty$, where $p > 0$ is arbitrarily large and $\beta \in (0, \frac{8}{3})$ (see Lemma 4.4).

From a mathematical point of view, this absorption property (1.4) is a little surprising since in case of the linear equation

$$u_t + u_{xxxx} + u_{xx} = 0, \tag{1.5}$$

for instance, one has the explicit solutions $u_k(x,t) = e^{\lambda_k t} \cos(\frac{k\pi x}{L})$, $k \in \mathbb{N}$, where the eigenvalue $\lambda_k = (\frac{k\pi}{L})^2((\frac{k\pi}{L})^2 - 1)$ is positive for $k = 1$ if Ω is large enough such that $L > \pi$. Thus, the existence of an absorbing set is strongly due to the presence of the nonlinearity in (1.1), and therefore it is not surprising that this can be proved only for α not too small (such that the damping term will be strong enough). Absorption effects of a similar flavour are known to be true for several other types of nonlinearities $N(u)$ added on the right hand side of (1.5), for example in the Kuramoto–Sivashinsky equation (where $N(u) = -u_x^2$ – see Nicolaenko *et al.* [13, 14]), the Cahn–Hilliard equation (where $N(u) = (u^3)_{xx}$ – cf. Novick-Cohen [15] and Nicolaenko *et al.* [14]), or also in an equation proposed by Ortiz *et al.* [16] to describe epitaxial growth of thin $YBa_2Cu_3O_{7-\delta}$ films

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(with $N(u) = (u_x^3)_x$ – see [16] [16] and King *et al.* [10]). The latter two equations have the favourable advantage that they give rise to certain Lyapunov functionals, which however seem to be absent (or at least hard to find) for both the Kuramoto–Sivashinsky equation *and* (1.1). We point out that also the *second* order Sivashinsky equation, as discussed in Karlin & Makhviladze [9], exhibits a similar absorption effect.

In § 5 we illustrate our results by some numerical simulations, based on the approximation scheme from § 4. It turns out that for a wide range of α typical phenomena in thin film growth like roughening and coarsening are also present in the computed solutions of (1.1). Moreover, as the above absorption estimates suggest, the computed solution functions are least regular at their smallest values. Long term simulations for several values of α illustrate the existence of an absorbing set for $\alpha \geqslant 2$, whereas no absorption is observed for $1 < \alpha < 2$. Moreover, all computed steady states look qualitatively very similar which indicates that absorbing sets might be rather small. § 6 gives some final remarks about the resemblance of our numerical solutions to those of the second order Sivashinsky equation from Karlin & Makhviladze [9].

2 Preliminaries

Given $q \in (1, \infty)$, the operator $Lu := u_{xxxx} + u_{xx}$ in $C_0^\infty(\Omega)$ extends to the two closed operators $A_q^{(N)}$ and $A_q^{(D)}$ in $L^q(\Omega)$ with their respective domains $D(A_q^{(N)}) := \{u \in W^{4,q}(\Omega) \mid u_x|_{\partial\Omega} = u_{xxx}|_{\partial\Omega} = 0\}$ and $D(A_q^{(D)}) := \{u \in W^{4,q}(\Omega) \mid u|_{\partial\Omega} = u_{xx}|_{\partial\Omega} = 0\}$. Let us collect some useful properties valid for both $A_q^{(N)}$ and $A_q^{(D)}$. Here and in the sequel we suppress the index q when there is no danger of confusion. In the next lines we also drop the superscripts '(N)' and '(D)'.

First, it is well-known (see Theorem 8.2 in Denk *et al.* [4], for instance) that there exists $\mu > 0$ such that $\tilde{A} := A + \mu\, id$ is a sectorial operator in $L^q(\Omega)$, where, clearly, μ can be chosen large enough so as to work for both $A^{(N)}$ and $A^{(D)}$. Consequently, both A and $\tilde{A}$ generate analytic semigroups $(e^{-tA})_{t\geqslant 0}$ and $(e^{-t\tilde{A}})_{t\geqslant 0}$, respectively, which are linked via the relation $e^{-t\tilde{A}} = e^{-\mu t} e^{-tA}$ ([6]). Moreover, $\tilde{A}$ possesses densely defined, closed fractional powers $\tilde{A}^\beta$ for any $\beta \in [0, 1]$, with $\tilde{A}^0 = id$ and $\tilde{A}^1 = \tilde{A}$. Since zero is contained in the resolvent set of $\tilde{A}$, we have

$$\|u\|_{W^{4,q}(\Omega)} \leqslant c\|\tilde{A}u\|_{L^q(\Omega)} \qquad \forall\, u \in D(A_q), \tag{2.1}$$

where – as throughout this work – c is a generic constant that may vary from line to line.

Furthermore, we have the smoothing estimate [6]

$$\|\tilde{A}^\beta e^{-t\tilde{A}} u\|_{L^q(\Omega)} \leqslant ct^{-\beta}\|u\|_{L^q(\Omega)} \qquad \forall\, u \in L^q(\Omega), \quad \forall\, t > 0, \quad \forall\, \beta \in [0, 1]. \tag{2.2}$$

An important role will be played by the Gagliardo–Nirenberg interpolation inequality (Theorem I.10.1 in Friedman [6])

$$\|D^j u\|_{L^p(\Omega)} \leqslant c\|u\|^a_{W^{m,r}(\Omega)}\|u\|^{1-a}_{L^q(\Omega)} \qquad \forall\, u \in W^{m,r}(\Omega), \tag{2.3}$$

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valid for $1 < p, q, r < \infty$, integers $m \geqslant 1$ and $j \in \{0, \dots, m-1\}$ with $j \geqslant \frac{1}{p} - \frac{1}{q}$, and

$$a = \frac{j + \frac{1}{q} - \frac{1}{p}}{m + \frac{1}{q} - \frac{1}{r}} \in [0, 1).$$

(Here, $D^0 u := u$ and $D^{j+1} u := (D^j u)_x$ for $j \geqslant 0$.)

As an application of (2.1)–(2.3), we immediately obtain the following 'L^p-L^q-estimate'.

Lemma 2.1 *Suppose $\beta \in [0,1]$ and $1 < p \leqslant q < \infty$. Then we have*

$$\|\tilde{A}^\beta e^{-t\tilde{A}} u\|_{L^q(\Omega)} \leqslant c t^{-\beta - \frac{1}{4}(\frac{1}{p} - \frac{1}{q})} \|u\|_{L^p(\Omega)} \qquad \forall\, u \in L^p(\Omega), \quad \forall\, t > 0. \tag{2.4}$$

Proof We successively apply (2.2), (2.3), (2.1) and then (2.2) to the left-hand side of (2.4) to see that with $a = \frac{1}{4}(\frac{1}{p} - \frac{1}{q})$ we have

$$\begin{aligned}
\|\tilde{A}^\beta e^{-t\tilde{A}} u\|_{L^q(\Omega)} &= \|\tilde{A}^\beta e^{-\frac{t}{2}\tilde{A}} e^{-\frac{t}{2}\tilde{A}} u\|_{L^q(\Omega)} \\
&\leqslant c \left(\frac{t}{2}\right)^{-\beta} \|e^{-\frac{t}{2}\tilde{A}} u\|_{L^q(\Omega)} \\
&\leqslant c t^{-\beta} \|e^{-\frac{t}{2}\tilde{A}} u\|^a_{W^{4,p}(\Omega)} \|e^{-\frac{t}{2}\tilde{A}} u\|^{1-a}_{L^p(\Omega)} \\
&\leqslant c t^{-\beta} \|\tilde{A} e^{-\frac{t}{2}\tilde{A}} u\|^a_{L^p(\Omega)} \|e^{-\frac{t}{2}\tilde{A}} u\|^{1-a}_{L^p(\Omega)} \\
&\leqslant c t^{-\beta} \left(\frac{t}{2}\right)^{-a} \|u\|_{L^p(\Omega)},
\end{aligned}$$

which yields the claim. □

Our interest in fractional powers is motivated by the following

Lemma 2.2 *Suppose $1 < q < \infty$ and $k \in \{1, 2, 3\}$. Then for all sufficiently small $\varepsilon > 0$ there exists a constant $c_\varepsilon > 0$ such that*

$$\|u\|_{W^{k,q}(\Omega)} \leqslant c_\varepsilon \|\tilde{A}^{\frac{k}{4}+\varepsilon} u\|_{L^q(\Omega)} \qquad \forall\, u \in D(A_q^{(N)}) \quad (\textit{resp. } D(A_q^{(D)})).$$

Proof In view of (2.1) and the Gagliardo–Nirenberg inequality (2.3), we have

$$\begin{aligned}
\|u\|_{W^{k,q}(\Omega)} &\leqslant c \|u\|^a_{W^{4,q}(\Omega)} \|u\|^{1-a}_{L^q(\Omega)} \\
&\leqslant c \|\tilde{A} u\|^a_{L^q(\Omega)} \|u\|^{1-a}_{L^q(\Omega)}
\end{aligned}$$

with $a = \frac{k}{4}$. Therefore Lemma II.17.1 in Friedman [6] yields

$$\|u\|_{W^{k,q}(\Omega)} \leqslant c_\varepsilon \|\tilde{A}^{a+\varepsilon} u\|_{L^q(\Omega)}$$

for $\varepsilon > 0$ small enough. □

An important connection between $A^{(N)}$ and $A^{(D)}$ is provided by the identities

$$\begin{aligned}
(e^{-tA^{(N)}} u)_x &= e^{-tA^{(D)}} u_x \qquad \text{and} \\
(e^{-t\tilde{A}^{(N)}} u)_x &= e^{-t\tilde{A}^{(D)}} u_x \qquad \forall\, u \in W^{1,q}(\Omega),
\end{aligned} \tag{2.5}$$

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which can easily be seen by differentiating the Neumann problem $v_t + A^{(N)}v = 0,\ v|_{t=0} = u$, with respect to x.

We also need the following auxiliary lemma, whose elementary proof is left to the reader.

Lemma 2.3 (i) *Let $\beta > 0$ and $\delta \in (0,1)$. Then*

$$s^{-\beta} - t^{-\beta} \leqslant s^{-\beta-\delta}(t-s)^{\delta} \qquad \forall\, 0 < s < t < \infty.$$

(ii) *For all $\gamma, \delta \in (0,1)$ there exists a constant $c > 0$ such that*

$$\int_0^t \sigma^{-\gamma}(t-\sigma)^{-\delta}\, d\sigma \leqslant c t^{1-\gamma-\delta} \qquad \forall\, t > 0.$$

(iii) *Given $\alpha > 1$ there exists $c > 0$ such that*

$$\big||z|^{\alpha} - |y|^{\alpha}\big| \leqslant c(|y|^{\alpha-1} + |z|^{\alpha-1}) \cdot |z-y| \qquad \forall\, y, z \in \mathbb{R}$$

holds.

Apart from that, we will frequently employ the Poincaré inequality in its following, easily proved, version:

Lemma 2.4 *Suppose $q \in [1,\infty)$ and $u \in W^{1,q}(\Omega)$ has a zero in $\bar{\Omega}$. Then*

$$\|u\|_{L^q(\Omega)} \leqslant q^{-\frac{1}{q}} |\Omega| \cdot \|u_x\|_{L^q(\Omega)}. \tag{2.6}$$

3 Mild solutions

For simplicity of notation, throughout this section A stands for the Neumann operator $A_q^{(N)}$ with some $q > 1$ being clear from the context.

Definition 3.1 *Let $\alpha > 1$, $0 < T \leqslant \infty$ and $q > \alpha$. We say that a function $u \in C^0([0,T); W^{1,q}(\Omega))$ is a* mild solution *of (1.1) if the identity*

$$u(t) = e^{-tA}u_0 - \int_0^t D^2 e^{-(t-s)A}|u_x|^{\alpha}(s)\, ds \qquad \forall\, t \in (0,T) \tag{3.1}$$

holds.

Remark. (i) The condition $u \in C^0([0,T); W^{1,q}(\Omega))$ ensures that the integral in (3.1) exists and that $u(0) = u_0$. In particular, it is implicitly required that $u_0 \in W^{1,q}(\Omega)$.
(ii) It can easily be checked that smooth mild solutions are classical if $\alpha \geqslant 2$. In view of (1.3), we believe that the same is in general false for $\alpha < 2$.

3.1 Local existence and uniqueness of mild solutions

Theorem 3.2 *Let $1 < \alpha < q$. Then for all $M > 0$ there exists some $T > 0$ depending on M only such that for any $u_0 \in W^{1,q}(\Omega)$ with $\|u_0\|_{W^{1,q}(\Omega)} \leqslant M$, (1.1) has a unique mild solution $u \in C^0([0,T]; W^{1,q}(\Omega))$.*

Proof With $T \in (0,1)$ and $R > 0$ to be fixed later, we let $X := C^0([0,T]; W^{1,q}(\Omega))$ and consider the closed ball $B := \{u \in X \mid \|u\|_X \leqslant R\}$. We define the nonlinear mapping $F : B \to X$ by

$$(Fu)(t) := e^{-tA}u_0 - \int_0^t D^2 e^{-(t-s)A}|u_x|^\alpha(s)\,ds, \qquad t \in [0,T].$$

Then for $u \in B$ and small $\varepsilon > 0$ we have, using Lemmas 2.1 and 2.2,

$$\begin{aligned}
\left\| D^2 e^{-(t-s)A}|u_x|^\alpha(s) \right\|_{W^{1,q}(\Omega)} &\leqslant \left\| e^{-(t-s)A}|u_x|^\alpha(s) \right\|_{W^{3,q}(\Omega)} \\
&\leqslant c e^{\mu(t-s)} \left\| e^{-(t-s)\tilde{A}}|u_x|^\alpha(s) \right\|_{W^{3,q}(\Omega)} \\
&\leqslant c e^{\mu} \left\| \tilde{A}^{\frac{3}{4}+\varepsilon} e^{-(t-s)\tilde{A}}|u_x|^\alpha(s) \right\|_{L^q(\Omega)} \\
&\leqslant c(t-s)^{-(\frac{3}{4}+\varepsilon)-\frac{1}{4}(\frac{\alpha}{q}-\frac{1}{q})} \left\| |u_x|^\alpha(s) \right\|_{L^{\frac{q}{\alpha}}(\Omega)} \\
&\leqslant c(t-s)^{-(\frac{3}{4}+\varepsilon)-\frac{\alpha-1}{4q}} \|u\|_X^\alpha.
\end{aligned}$$

Since $q > \alpha - 1$, it is possible to choose $\varepsilon > 0$ sufficiently small so that $\frac{3}{4} + \varepsilon + \frac{\alpha-1}{4q} < 1$, whence an integration and the inclusion $u \in X$ imply

$$\left\| \int_0^t D^2 e^{-(t-s)A}|u_x|^\alpha(s) \right\|_{W^{1,q}(\Omega)} \leqslant c_1 T^{\frac{1}{4}-\varepsilon-\frac{\alpha-1}{4q}} R^\alpha \qquad \forall\, t \in [0,T]$$

holds with some $c_1 > 0$. Since, by (2.6), (2.5) and (2.4),

$$\begin{aligned}
\|e^{-tA}u_0\|_{W^{1,q}(\Omega)} &= e^{\mu t}\|e^{-t\tilde{A}}u_0\|_{W^{1,q}(\Omega)} \\
&\leqslant c e^{\mu}\|(e^{-t\tilde{A}}u_0)_x\|_{L^q(\Omega)} \\
&= c\|e^{-t\tilde{A}^{(D)}}u_{0x}\|_{L^q(\Omega)} \\
&\leqslant c_2\|u_{0x}\|_{L^q(\Omega)},
\end{aligned}$$

we infer that

$$\begin{aligned}
\|Fu\|_X &\leqslant c_2\|u_{0x}\|_{L^q(\Omega)} + c_1 T^{\frac{1}{4}-\varepsilon-\frac{\alpha-1}{4q}} R^\alpha \\
&\leqslant R \qquad \forall\, u \in B,
\end{aligned}$$

provided that $R \geqslant c_2\|u_{0x}\|_{L^q(\Omega)} + 1$ and T is sufficiently small. It thus follows that F maps B into B for such choices of R and T.

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Now if u and v are in B, then similarly

$$\begin{aligned}\left\|D^2e^{-(t-s)A}\Big(|u_x|^\alpha(s)-|v_x|^\alpha(s)\Big)\right\|_{W^{1,q}(\Omega)} &\leqslant c(t-s)^{-(\frac{3}{4}+\varepsilon)-\frac{\alpha-1}{4q}}\left\||u_x|^\alpha(s)-|v_x|^\alpha(s)\right\|_{L^{\frac{q}{\alpha}}(\Omega)}\\ &\leqslant c(t-s)^{-(\frac{3}{4}+\varepsilon)-\frac{\alpha-1}{4q}}\|u_x-v_x\|_{L^q(\Omega)}\Big(\|u_x\|^{\alpha-1}_{L^q(\Omega)}+\|v_x\|^{\alpha-1}_{L^q(\Omega)}\Big)\\ &\leqslant cR^{\alpha-1}(t-s)^{-(\frac{3}{4}+\varepsilon)-\frac{\alpha-1}{4q}}\|u-v\|_X,\end{aligned}$$

where we have used that

$$\Big||u_x|^\alpha-|v_x|^\alpha\Big|\leqslant c(\alpha)\,|u_x-v_x|\cdot\Big(|u_x|^{\alpha-1}+|v_x|^{\alpha-1}\Big)$$

holds at almost every point in $\Omega\times(0,T)$—cf. Lemma 2.3(iii) above.

Consequently, after possibly diminishing T once more, we obtain

$$\begin{aligned}\|Fu-Fv\|_X &\leqslant cT^{\frac{1}{4}-\varepsilon-\frac{\alpha-1}{4q}}\|u-v\|_X\\ &\leqslant \frac{1}{2}\|u-v\|_X \qquad \forall\, u,v\in B,\end{aligned}$$

whereby F is seen to be a contraction on B. Now existence and uniqueness of a mild solution on $(0,T)$ result from Banach's fixed point theorem. □

We now obtain some further regularity properties of mild solutions.

Lemma 3.3 *Let $1<\alpha<q$. Then for all $p\in[q,\infty)$, $M>0$, $T>0$ and any small $\varepsilon>0$, there is a constant $c=c(p,M,T,\varepsilon)$ such that if u is a mild solution on $(0,T)$ with $\|u\|_{C^0([0,T];W^{1,q}(\Omega))}\leqslant M$ then*

$$\|u(t)\|_{W^{k,p}(\Omega)}\leqslant ct^{-\frac{1}{4}(\frac{1}{q}-\frac{1}{p}+k-1+\varepsilon)} \qquad \forall\, t\in(0,T) \tag{3.2}$$

holds for $k=1,2,3$ if $\alpha\in(1,2)$ and for $k=1,2,3,4$ if $\alpha\geqslant 2$. In the case $k=1$ we may set $\varepsilon=0$ in (3.2).

Proof We represent u in the form

$$u(t)=e^{-tA}u_0-\int_0^t D^2e^{-(t-s)A}f(s)\,ds, \qquad t\in(0,T), \tag{3.3}$$

with $f(t):=|u_x|^\alpha(t)$.

Starting with the case $k=1$, we observe that the hypothesis implies that

$$\|f(t)\|_{L^{\frac{q}{\alpha}}(\Omega)}\leqslant M^\alpha \qquad \forall\, t\in[0,T]. \tag{3.4}$$

Using this, (2.5), Lemma 2.2 and (2.4), we obtain

$$\begin{aligned}\|u(t)\|_{W^{1,p}(\Omega)} &\leqslant \|e^{-tA}u_0\|_{W^{1,p}(\Omega)}+\left\|\int_0^t D^2e^{-(t-s)A}f(s)\,ds\right\|_{W^{1,p}(\Omega)}\\ &\leqslant c\left(\|e^{-t\tilde{A}^{(D)}}u_{0x}\|_{L^p(\Omega)}+\int_0^t\left\|\tilde{A}^{\frac{3}{4}+\varepsilon}e^{-(t-s)\tilde{A}}f(s)\right\|_{L^p(\Omega)}ds\right)\end{aligned}$$

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$$\leqslant c\left(t^{-\frac{1}{4}(\frac{1}{q}-\frac{1}{p})}\|u_{0x}\|_{L^q(\Omega)}+\int_0^t (t-s)^{-(\frac{3}{4}+\varepsilon)-\frac{1}{4}(\frac{\alpha}{q}-\frac{1}{p})}\|f(s)\|^{\alpha}_{L^{\frac{q}{\alpha}}(\Omega)}\,ds\right)$$
$$\leqslant c\Big(t^{-\frac{1}{4}(\frac{1}{q}-\frac{1}{p})}\|u_{0x}\|_{L^q(\Omega)}+t^{\frac{1}{4}-\varepsilon-\frac{1}{4}(\frac{\alpha}{q}-\frac{1}{p})}M^{\alpha}\Big)$$

for sufficiently small $\varepsilon > 0$, where, as in the rest of the proof, c only depends on the quantities listed in the formulation of the lemma. Since $q > \alpha - 1$ implies $\frac{1}{4}(\frac{1}{q}-\frac{1}{p}) > -\frac{1}{4}+\varepsilon+\frac{1}{4}(\frac{\alpha}{q}-\frac{1}{p})$ for small ε, this proves (3.2) for $k = 1$.

We next consider $k = 2$. Using the result for $k = 1$, we can now generalise (3.4) to obtain

$$\|f(t)\|_{L^p(\Omega)} \leqslant ct^{-\frac{1}{4}(\frac{\alpha}{q}-\frac{1}{p})}, \tag{3.5}$$

To prove (3.2) for $k = 2$ we proceed in three steps.

Step 1. We first claim that for all $p \geqslant q$ we have

$$\|u_x(t)-u_x(s)\|_{L^p(\Omega)} \leqslant cs^{-\frac{1}{4}(\frac{\alpha}{q}-\frac{1}{p})}(t-s)^{\frac{\alpha-1}{4q}} \qquad \forall 0 < s < t \leqslant T. \tag{3.6}$$

To this end, we differentiate (3.3) with respect to x to obtain

$$\begin{aligned}\|u_x(t)-u_x(s)\|_{L^p(\Omega)} &\leqslant \left\|\Big((e^{-tA}-e^{-sA})u_0\Big)_x\right\|_{L^p(\Omega)}\\ &\quad+\left\|\int_0^t D^3e^{-(t-\sigma)A}f(\sigma)\,d\sigma-\int_0^s D^3e^{-(s-\sigma)A}f(\sigma)\,d\sigma\right\|_{L^p(\Omega)}\\ &=: I_1+I_2.\end{aligned} \tag{3.7}$$

Here, by (2.5), (2.4) and Lemma 2.3(i),

$$\begin{aligned} I_1 &= \left\|\Big(e^{-tA^{(D)}}-e^{-sA^{(D)}}\Big)u_{0x}\right\|_{L^p(\Omega)}\\ &= \left\|\int_s^t \frac{d}{d\sigma}e^{-\sigma A^{(D)}}u_{0x}\,d\sigma\right\|_{L^p(\Omega)}\\ &= \left\|-\int_s^t A^{(D)}e^{-\sigma A^{(D)}}u_{0x}\,d\sigma\right\|_{L^p(\Omega)}\\ &\leqslant c\left\|\int_s^t (\tilde{A}^{(D)}-\mu)e^{-\sigma\tilde{A}^{(D)}}u_{0x}\,d\sigma\right\|_{L^p(\Omega)}\\ &\leqslant c\int_s^t\Big(\sigma^{-1-\frac{1}{4}(\frac{1}{q}-\frac{1}{p})}+\mu\sigma^{-\frac{1}{4}(\frac{1}{q}-\frac{1}{p})}\Big)\,d\sigma\cdot\|u_{0x}\|_{L^q(\Omega)}\\ &\leqslant c\int_s^t \sigma^{-1-\frac{1}{4}(\frac{1}{q}-\frac{1}{p})}\,d\sigma\cdot\|u_{0x}\|_{L^q(\Omega)}\\ &= c\Big(s^{-\frac{1}{4}(\frac{1}{q}-\frac{1}{p})}-t^{-\frac{1}{4}(\frac{1}{q}-\frac{1}{p})}\Big)\\ &\leqslant cs^{-\frac{1}{4}(\frac{1}{q}-\frac{1}{p})-\delta_1}(t-s)^{\delta_1}\end{aligned} \tag{3.8}$$

for any $\delta_1 > 0$. As to I_2, we find

$$\begin{aligned} I_2 &\leqslant \left\|\int_s^t D^3e^{-(t-\sigma)A}f(\sigma)\,d\sigma\right\|_{L^p(\Omega)}+\left\|\int_0^s D^3[e^{-(t-\sigma)A}-e^{-(s-\sigma)A}]f(\sigma)\,d\sigma\right\|_{L^p(\Omega)}\\ &=: I_{21}+I_{22},\end{aligned} \tag{3.9}$$

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where

$$
\begin{aligned}
I_{21} &\leqslant c \int_s^t \|\tilde{A}^{\frac{3}{4}+\varepsilon} e^{-(t-\sigma)\tilde{A}} f(\sigma)\|_{L^p(\Omega)} \, d\sigma \\
&\leqslant c \int_s^t (t-\sigma)^{-(\frac{3}{4}+\varepsilon)} \sigma^{-\frac{1}{4}(\frac{\alpha}{q}-\frac{1}{p})} \, d\sigma \\
&\leqslant c s^{-\frac{1}{4}(\frac{\alpha}{q}-\frac{1}{p})} \cdot (t-s)^{\frac{1}{4}-\varepsilon} \qquad (3.10)
\end{aligned}
$$

for any small $\varepsilon > 0$, due to Lemma 2.2, (2.2) and (3.5). Moreover, Lemma 2.2, (3.5) and Lemma 2.3(i) and (ii) yield

$$
\begin{aligned}
I_{22} &\leqslant c \int_0^s \left\| \tilde{A}^{\frac{3}{4}+\varepsilon} [e^{-(t-\sigma)\tilde{A}} - e^{-(s-\sigma)\tilde{A}}] f(\sigma) \right\|_{L^p(\Omega)} d\sigma \\
&= c \int_0^s \left\| -\tilde{A}^{\frac{3}{4}+\varepsilon} \int_{s-\sigma}^{t-\sigma} \tilde{A} e^{-\tau\tilde{A}} \, d\tau f(\sigma) \right\|_{L^p(\Omega)} d\sigma \\
&\leqslant c \int_0^s \left(\int_{s-\sigma}^{t-\sigma} \tau^{-(\frac{7}{4}+\varepsilon)} \, d\tau \right) \cdot \sigma^{-\frac{1}{4}(\frac{\alpha}{q}-\frac{1}{p})} \, d\sigma \\
&= c \int_0^s \left((s-\sigma)^{-(\frac{3}{4}+\varepsilon)} - (t-\sigma)^{-(\frac{3}{4}+\varepsilon)} \right) \cdot \sigma^{-\frac{1}{4}(\frac{\alpha}{q}-\frac{1}{p})} \, d\sigma \\
&\leqslant c \int_0^s (s-\sigma)^{-(\frac{3}{4}+\varepsilon+\delta_2)} \sigma^{-\frac{1}{4}(\frac{\alpha}{q}-\frac{1}{p})} \, d\sigma \cdot (t-s)^{\delta_2} \\
&\leqslant c s^{\frac{1}{4}-(\varepsilon+\delta_2)-\frac{1}{4}(\frac{\alpha}{q}-\frac{1}{p})} (t-s)^{\delta_2} \qquad (3.11)
\end{aligned}
$$

for small ε and arbitrary $\delta_2 > 0$. Upon fixing any $\varepsilon < \frac{1}{4} - \frac{\alpha-1}{4q}$ and choosing $\delta_1 := \frac{\alpha-1}{4q}$ and $\delta_2 := \frac{1}{4} - \varepsilon$, we now immediately derive (3.6) from (3.7)–(3.11).

Step 2. We next deduce the Hölder estimate

$$
\|f(t) - f(s)\|_{L^p(\Omega)} \leqslant c s^{-\frac{1}{4}(\frac{2\alpha-1}{q}-\frac{1}{p})} (t-s)^{\frac{\alpha-1}{4q}} \qquad \forall\, 0 < s < t \leqslant T. \qquad (3.12)
$$

Indeed, from Lemma 2.3(iii), we have

$$
\left| |u_x|^\alpha(x,t) - |u_x|^\alpha(x,s) \right|^p \leqslant c \Big(|u_x|^{p(\alpha-1)}(x,t) + |u_x|^{p(\alpha-1)}(x,s) \Big) \cdot |u_x(x,t) - u_x(x,s)|^p
$$

for a.e. $x \in \Omega$ and hence Hölder's inequality (applied to any pair of numbers $r, r' \in (1,\infty)$ with $\frac{1}{r} + \frac{1}{r'} = 1$) implies

$$
\begin{aligned}
\left\| |u_x|^\alpha(t) - |u_x|^\alpha(s) \right\|_{L^p(\Omega)} &\leqslant c \left(\int_\Omega \left(|u_x|^{p(\alpha-1)}(t) - |u_x|^{p(\alpha-1)}(s) \right) \cdot |u_x(t) - u_x(s)|^p \right)^{\frac{1}{p}} \\
&\leqslant c \left[\left(\int_\Omega |u_x|^{p(\alpha-1)r'}(t) \right)^{\frac{1}{pr'}} + \left(\int_\Omega |u_x|^{p(\alpha-1)r'}(s) \right)^{\frac{1}{pr'}} \right] \\
&\quad \times \left(\int_\Omega |u_x(t) - u_x(s)|^{pr} \right)^{\frac{1}{pr}} \\
&= c \Big(\|u_x(t)\|^{\alpha-1}_{L^{p(\alpha-1)r'}(\Omega)} + \|u_x(s)\|^{\alpha-1}_{L^{p(\alpha-1)r'}(\Omega)} \Big) \cdot \|u_x(t) - u_x(s)\|_{L^{pr}(\Omega)}
\end{aligned}
$$

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$$\leqslant cs^{-\frac{\alpha-1}{4}(\frac{1}{q}-\frac{1}{p(\alpha-1)r'})} \cdot s^{-\frac{1}{4}(\frac{\alpha}{q}-\frac{1}{pr})} \cdot (t-s)^{\frac{\alpha-1}{4q}}$$
$$= cs^{-\frac{1}{4}(\frac{\alpha-1}{q}-\frac{1}{pr'})-\frac{1}{4}(\frac{\alpha}{q}-\frac{1}{pr})} \cdot (t-s)^{\frac{\alpha-1}{4q}},$$

from which (3.12) follows.

Step 3. We now assert that

$$\|u_{xx}(t)\|_{L^p(\Omega)} \leqslant ct^{-\frac{1}{4}(\frac{1}{q}-\frac{1}{p}+1+\varepsilon)} \qquad \forall\, t \in (0,T) \tag{3.13}$$

holds for all $p \geqslant q$ and any small $\varepsilon > 0$. In combination with (3.2) for $k=1$ this will complete the proof of (3.2) in the case $k=2$. By (3.3),

$$\begin{aligned}\|u_{xx}(t)\|_{L^p(\Omega)} &\leqslant \|(e^{-tA}u_0)_{xx}\|_{L^p(\Omega)} + \left\|\int_0^t D^4 e^{-(t-s)A} f(s)\,ds\right\|_{L^p(\Omega)} \\ &=: J_1 + J_2,\end{aligned} \tag{3.14}$$

where due to

$$\begin{aligned}J_1 &\leqslant c\|(e^{-t\tilde{A}^{(D)}}u_{0x})_x\|_{L^p(\Omega)} \\ &\leqslant c\|(\tilde{A}^{(D)})^{\frac{1}{4}+\varepsilon}e^{-t\tilde{A}^{(D)}}u_{0x}\|_{L^p(\Omega)} \\ &\leqslant ct^{-(\frac{1}{4}+\varepsilon)-\frac{1}{4}(\frac{1}{q}-\frac{1}{p})}\|u_{0x}\|_{L^q(\Omega)}\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}J_2 &\leqslant c\left\|\int_0^t \tilde{A}e^{-(t-s)\tilde{A}}(f(t)-f(s))\,ds\right\|_{L^p(\Omega)} + \left\|\int_0^t \tilde{A}e^{-(t-s)\tilde{A}}f(t)\,ds\right\|_{L^p(\Omega)} \\ &=: J_{21} + J_{22}.\end{aligned} \tag{3.16}$$

Now the estimate gained in Step 2, (2.2) and Lemma 2.3(ii) yield

$$\begin{aligned}J_{21} &\leqslant c\int_0^t (t-s)^{-1+\frac{\alpha-1}{4q}} \cdot s^{-\frac{1}{4}(\frac{2\alpha-1}{q}-\frac{1}{p})}\,ds \\ &\leqslant ct^{1-(1-\frac{\alpha-1}{4q})-\frac{1}{4}(\frac{2\alpha-1}{q}-\frac{1}{p})} \\ &= ct^{-\frac{1}{4}(\frac{\alpha}{q}-\frac{1}{p})},\end{aligned} \tag{3.17}$$

while (3.5) gives

$$\begin{aligned}J_{22} &= \|(1-e^{-t\tilde{A}})f(t)\|_{L^p(\Omega)} \\ &\leqslant c\|f(t)\|_{L^p(\Omega)} \\ &\leqslant ct^{-\frac{1}{4}(\frac{\alpha}{q}-\frac{1}{p})}.\end{aligned} \tag{3.18}$$

Since $q > \alpha - 1$, the terms on the right hand sides of (3.17) and (3.18) are dominated by the term on the right of (3.15) – if u_0 is not trivial, of course – and thereby (3.13) has been established.

In the cases $k = 3$ and $k = 4$ the proof is very similar, each of these parts making use of the part concerning $k - 1$. For example, when $k = 3$ the first step consists of proving

$$\|u_{xx}(t) - u_{xx}(s)\|_{L^p(\Omega)} \leqslant cs^{-\frac{1}{4}(\frac{\alpha}{q}-\frac{1}{p}+1+\varepsilon)}(t-s)^{\frac{\alpha-1}{4q}} \qquad \forall\, 0 < s < t \leqslant T \tag{3.19}$$

for small $\varepsilon > 0$. In doing this, we use that in the expression parallelling (3.7), we may write

$$\begin{aligned} &\int_0^t D^4 e^{-(t-\sigma)A} f(\sigma)\, d\sigma - \int_0^s D^4 e^{-(s-\sigma)A} f(\sigma)\, d\sigma \\ &\quad = \int_0^t D^3 e^{-(t-\sigma)A^{(D)}} f_x(\sigma)\, d\sigma - \int_0^s D^3 e^{-(s-\sigma)A^{(D)}} f_x(\sigma)\, d\sigma, \end{aligned} \tag{3.20}$$

for instance. To estimate this (and thereby prove (3.19) analogously as (3.6) – cf. (3.7)–(3.11)), using the result for $k = 2$ we derive from Hölder's inequality the following counterpart of (3.5),

$$\begin{aligned} \|f_x(t)\|_{L^p(\Omega)} &= c\left(\int_\Omega |u_x|^{p(\alpha-1)}(t)|u_{xx}|^p(t)\right)^{\frac{1}{p}} \\ &\leqslant c\left(\int_\Omega |u_x|^{p(\alpha-1)r'}(t)\right)^{\frac{1}{pr'}} \cdot \left(\int_\Omega |u_{xx}|^{pr}(t)\right)^{\frac{1}{pr}} \\ &= c\|u_x(t)\|^{\alpha-1}_{L^{p(\alpha-1)r'}(\Omega)} \cdot \|u_{xx}(t)\|_{L^{pr}(\Omega)} \\ &\leqslant ct^{-\frac{\alpha-1}{4}(\frac{1}{q}-\frac{1}{p(\alpha-1)r'})} \cdot t^{-\frac{1}{4}(\frac{1}{q}-\frac{1}{pr}+1+\varepsilon)} \\ &= ct^{-\frac{1}{4}(\frac{\alpha}{q}-\frac{1}{p}+1+\varepsilon)}, \end{aligned}$$

where $\frac{1}{r} + \frac{1}{r'} = 1$. (The appearance of the (arbitrarily small) $\varepsilon > 0$ in the latter inequality explains why, contrary to (3.6), an ε shows up in (3.19).)

Using (3.19), one proceeds to show

$$\|f_x(t) - f_x(s)\|_{L^p(\Omega)} \leqslant cs^{-\frac{1}{4}(\frac{2\alpha-1}{q}-\frac{1}{p}+1+\varepsilon)}(t-s)^{\frac{\alpha-1}{4q}} \qquad \forall 0 < s < t \leqslant T, \tag{3.21}$$

which is obtained in the same way as (3.12). This enables us to prove, in the third step,

$$\|u_{xxx}(t)\|_{L^p(\Omega)} \leqslant ct^{-\frac{1}{4}(\frac{1}{q}-\frac{1}{p}+2+\varepsilon)} \qquad \forall\, t \in (0, T),$$

which yields (3.2) for $k = 3$.

Since the details of the proof for $k = 3$ (and also for $k = 4$) are almost identical to those presented in case of $k = 2$, we confine ourselves with a concluding remark on the question why we require $\alpha \geqslant 2$ when $k = 4$: Then, namely, one has a formula similar to (3.20), with f_x replaced by f_{xx}. But (cf. (1.3))

$$f_{xx} = \alpha(\alpha-1)|u_x|^{\alpha-2}u_{xx}^2 + \alpha|u_x|^{\alpha-2}u_x u_{xxx},$$

which may become singular at zeroes of u_x if $\alpha < 2$, even if u is smooth. Therefore, to control f_{xx} we assume that $\alpha \geqslant 2$. □

The above lemma and its proof also provide some regularity with respect to time.

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Corollary 3.4 *Under the assumptions of Lemma 3.3,*

$$u,\ u_x,\ u_{xx},\ |u_x|^\alpha, \quad \text{and} \quad (|u_x|^\alpha)_x$$

all are Hölder continuous on $(0, T]$ *as* $L^p(\Omega)$*-valued functions. Moreover,* u_{xxx} *is continuous on* $(0, T]$ *with respect to the weak topology in* $L^p(\Omega)$.

Proof The part concerning Hölder continuity follows from (3.6), (3.12), (3.19), (3.21) and the Poincaré inequality (2.6).

To see weak continuity of u_{xxx}, fix $t_0 \in (0, T]$ and $(0, T] \ni t_j \to t_0$. Then $\|u(t_j)\|_{W^{3,p}(\Omega)} \leqslant c(t_0)$ for all j and hence $u(t_{j_m}) \rightharpoonup v$ in $W^{3,p}(\Omega)$ for a subsequence $t_{j_m} \to t_0$. But $u(t_j) \to u(t_0)$ in $W^{2,p}(\Omega)$ by the above result, and hence $v = u(t_0)$. From a standard argument it now follows that $u(t_j) \rightharpoonup u(t_0)$ for the whole sequence and thus $u(t) \rightharpoonup u(t_0)$ in $W^{3,p}(\Omega)$. □

3.2 Global mild solutions for $\alpha \leqslant \frac{5}{3}$

Let us first prove an *a priori* estimate for $\|u\|_{L^\infty((0,T);L^2(\Omega))}$ and $\|u\|_{L^2((0,T);W^{2,2}(\Omega))}$ that is quite the same as the one achievable for the linear equation (1.5).

Lemma 3.5 *Let* $\alpha > 1$, $T \in (0, \infty]$ *and* u *be a mild solution of (1.1) on* $\Omega \times (0, T)$*. Then*

$$\int_\Omega u^2(t) + \int_0^t \int_\Omega u_{xx}^2 \leqslant \left(\int_\Omega u_0^2 \right) \cdot e^t \qquad \forall\, t \in (0, T). \tag{3.22}$$

Proof Our goal is to show that $t \mapsto \int_\Omega u^2(t)$ is differentiable on $(0, T)$ with

$$\frac{1}{2} \frac{d}{dt} \int_\Omega u^2 = -\int_\Omega u_{xx}^2 - \int_\Omega u u_{xx} - \int_\Omega |u_x|^\alpha u_{xx}. \tag{3.23}$$

Since the last term vanishes due to the boundary condition $u_x|_{\partial\Omega} = 0$, and since

$$-\int_\Omega u u_{xx} \leqslant \frac{1}{2} \int_\Omega u_{xx}^2 + \frac{1}{2} \int_\Omega u^2$$

by Young's inequality, an integration of

$$\frac{d}{dt} \int_\Omega u^2 + \int_\Omega u_{xx}^2 \leqslant \int_\Omega u^2 \tag{3.24}$$

then will yield the assertion.

For the proof of (3.23), in view of Corollary 3.4 we may assume without loss of generality that u, u_x, u_{xx} and $|u_x|^\alpha$ are $L^2(\Omega)$-valued Hölder continuous on $[0, T)$ – if necessary we shift the time axis so as to replace $t = 0$ with $t = \tau$ for arbitrarily small $\tau > 0$.

Then for $t \in (0, T)$ and small $h \neq 0$ we have, using integration by parts,

$$\begin{aligned} \frac{1}{h} \left(\int_\Omega u^2(t+h) - \int_\Omega u^2(t) \right) &= \int_\Omega \frac{u(t+h) - u(t)}{h} \cdot (u(t+h) + u(t)) \\ &= \int_\Omega \left(\frac{e^{-(t+h)A} - e^{-tA}}{h} u_0 \right) \cdot (u(t+h) + u(t)) \end{aligned}$$

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$$
\begin{aligned}
&- \int_\Omega \frac{1}{h} \bigg(\int_0^{t+h} e^{-(t+h-s)A} |u_x|^\alpha(s)\, ds \\
&\qquad - \int_0^t e^{-(t-s)A} |u_x|^\alpha(s)\, ds \bigg) \cdot (u_{xx}(t+h) + u_{xx}(t)) \\
&=: I_1(h) + I_2(h). \qquad (3.25)
\end{aligned}
$$

Since $\frac{e^{-(t+h)A} - e^{-tA}}{h} u_0 \to -Ae^{-tA}u_0$ in $L^2(\Omega)$ and $u(t+h) + u(t) \to 2u(t)$ in $L^2(\Omega)$, we get

$$
I_1(h) \to -2 \int_\Omega Ae^{-tA}u_0 \cdot u(t) \qquad \text{as } h \to 0. \qquad (3.26)
$$

As to $I_2(h)$, in Chapter 7 in Friedman [6] it is shown that $\varphi(t) := \int_0^t e^{-(t-s)A}\rho(s)\, ds$ is differentiable on $(0, T)$ with $\varphi'(t) = -A\int_0^t e^{-(t-s)A}\rho(s)\, ds + \rho(t)$, provided that ρ is Hölder continuous on $[0, T)$ with values in $L^2(\Omega)$. Consequently, since $u_{xx}(t+h) + u_{xx}(t) \to 2u_{xx}(t)$ in $L^2(\Omega)$,

$$
\begin{aligned}
I_2(h) \to\ & 2 \int_\Omega \left(A \int_0^t e^{-(t-s)A} |u_x|^\alpha(s)\, ds \right) \cdot u_{xx}(t) \\
& - 2 \int_\Omega |u_x|^\alpha(t) \cdot u_{xx}(t). \qquad (3.27)
\end{aligned}
$$

Collecting (3.25)–(3.27), we obtain that $t \mapsto \int_\Omega u^2(t)$ indeed is differentiable with

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 &= -\int_\Omega Au \cdot u - \int_\Omega |u_x|^\alpha u_{xx} \\
&= -\int_\Omega (u_{xx} + u) \cdot u_{xx} - \int_\Omega |u_x|^\alpha u_{xx},
\end{aligned}
$$

as claimed. □

Unfortunately, so as to extend a given local mild solution for all times, the above *a priori* estimate appears to be not yet sufficient. Namely, if we wish to apply Theorem 3.2, we should exclude finite-time blow-up of $u(t)$ in $W^{1,q}(\Omega)$ for some $q > \alpha$. This, however, is not directly accomplished by Lemma 3.5. Nevertheless we can use it to obtain global solutions for α not too large: In the proof of Theorem 3.7 we shall first establish a differential inequality similar to (3.24) for $\int_\Omega u_x^2$, but containing an expression involving the nonlinearity (see (3.30)). This term, however, can be estimated using (3.22) and some interpolation arguments, provided α is small.

We need the following simple lemma.

Lemma 3.6 *Suppose $1 \leqslant p < \infty$, $T > 0$ and $y \in C^1([0, T)) \cap L^p((0, T))$ is nonnegative with*

$$
y'(t) \leqslant Cy^{p+1}(t) \qquad \forall\, t \in (0, T) \qquad (3.28)
$$

for some constant C. Then $y \in L^\infty((0, T))$.

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Proof If the claim were false, there would exist a sequence $t_k \nearrow T$ such that $y(t_k) \to +\infty$ as $k \to \infty$. Integrating (3.28), we obtain $\frac{y^{-p}(t_k)-y^{-p}(t)}{-p} \leqslant C(t_k - t)$ for $0 < t < t_k$, that is, $y(t) \geqslant c(t_k - t)^{-\frac{1}{p}}$ for such t. Letting $k \to \infty$ and then $t \nearrow T$, we obtain a contradiction. □

This enables us to prove the following theorem.

Theorem 3.7 *Suppose $\alpha \in (1, \frac{5}{3}]$ and $u_0 \in W^{1,q}(\Omega)$ for some $q > \alpha$. Then the unique mild solution of (1.1) is global in time; that is, it can be continued for all times.*

Proof Suppose that there exists some $T \in (0,\infty)$ such that u exists for $t \in (0,T)$ but cannot be extended beyond T. Again by Corollary 3.4, we may assume that $u, u_x, u_{xx}, |u_x|^\alpha$ and $(|u_x|^\alpha)_x$ are Hölder continuous and u_{xxx} is weakly continuous on $[0,T)$ with values in $L^2(\Omega)$. Then, in particular, Theorem 3.2 says that $\|u(t)\|_{W^{1,2}(\Omega)}$ must blow up at $t = T$, that is,

$$\limsup_{t\to T} \|u(t)\|_{W^{1,2}(\Omega)} = \infty. \tag{3.29}$$

To derive a contradiction from this, we first prove that $t \mapsto \int_\Omega u_x^2(t)$ is differentiable on $(0,T)$ with

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_x^2 = -\int_\Omega u_{xxx}^2 - \int_\Omega u_x u_{xxx} - \int_\Omega (|u_x|^\alpha)_x u_{xxx}. \tag{3.30}$$

The proof of (3.30) is quite the same as that of (3.23), using the identity

$$\begin{aligned}
\frac{1}{h}\left(\int_\Omega u_x^2(t+h) - \int_\Omega u_x^2(t)\right) &= -\int_\Omega \frac{u(t+h)-u(t)}{h} \cdot (u_{xx}(t+h) + u_{xx}(t)) \\
&= -\int_\Omega \frac{e^{-(t+h)A} - e^{-tA}}{h} u_0 \cdot (u_{xx}(t+h) + u_{xx}(t)) \\
&\quad + \int_\Omega \frac{1}{h}\left(\int_0^{t+h} e^{-(t+h-s)A^{(D)}}(|u_x|^\alpha)_x(s)\,ds - \int_0^t e^{-(t-s)A^{(D)}}(|u_x|^\alpha)_x(s)\,ds\right) \\
&\quad \times (u_{xxx}(t+h) + u_{xxx}(t))
\end{aligned}$$

and $L^2(\Omega)$-valued Hölder continuity of u_{xx} and $(|u_x|^\alpha)_x$ as well as weak continuity of u_{xxx} this time in taking $h \to 0$.

To digest the superlinear nonlinearity in (3.30), we first use Young's and Hölder's inequalities (with exponents 2 and $\frac{1}{\alpha-1}$, respectively) in estimating

$$\begin{aligned}
I &:= \left|\int_\Omega (|u_x|^\alpha)_x \cdot u_{xxx}\right| \\
&= \left|\alpha \int_\Omega |u_x|^{\alpha-2} u_x u_{xx} u_{xxx}\right| \\
&\leqslant \frac{1}{8}\int_\Omega u_{xxx}^2 + c\int_\Omega |u_x|^{2\alpha-2} u_{xx}^2 \\
&\leqslant \frac{1}{8}\int_\Omega u_{xxx}^2 + c\left(\int_\Omega u_x^2\right)^{\alpha-1} \cdot \left(\int_\Omega |u_{xx}|^{\frac{2}{2-\alpha}}\right)^{2-\alpha}.
\end{aligned}$$

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By the Gagliardo–Nirenberg and the Poincaré inequality,

$$\begin{aligned}\left(\int_\Omega |u_{xx}|^{\frac{2}{2-\alpha}}\right)^{2-\alpha} &= \|u_{xx}\|_{L^{\frac{2}{2-\alpha}}(\Omega)}\\ &\leqslant c\|u_x\|_{W^{2,2}(\Omega)}^{\frac{\alpha+1}{2}}\cdot\|u_x\|_{L^2(\Omega)}^{\frac{3-\alpha}{2}}\\ &\leqslant c\|u_{xxx}\|_{L^2(\Omega)}^{\frac{\alpha+1}{2}}\cdot\|u_x\|_{L^2(\Omega)}^{\frac{3-\alpha}{2}},\end{aligned}$$

because $u_x|_{\partial\Omega}=0$ implies that for all t, both $u_x(t)$ and $u_{xx}(t)$ have at least one zero in $\bar{\Omega}$. Hence

$$\begin{aligned}I &\leqslant \frac{1}{8}\int_\Omega u_{xxx}^2 + c\left(\int_\Omega u_{xxx}^2\right)^{\frac{\alpha+1}{4}}\cdot\left(\int_\Omega u_x^2\right)^{\frac{3\alpha-1}{4}}\\ &\leqslant \frac{1}{4}\int_\Omega u_{xxx}^2 + c\left(\int_\Omega u_x^2\right)^{\frac{3\alpha-1}{3-\alpha}},\end{aligned}$$

where we have once more used Young's inequality with exponent $\frac{4}{\alpha+1}$. Altogether, (3.30) yields

$$\frac{d}{dt}\int_\Omega u_x^2 \leqslant c\left(\int_\Omega u_x^2 + \left(\int_\Omega u_x^2\right)^{\frac{3\alpha-1}{3-\alpha}}\right),$$

which in view of Lemma 3.6 implies that in contrast to our assumption $\int_\Omega u_x^2(t)$ remains bounded as $t\to T$, provided that $\int_0^T(\int_\Omega u_x^2)^p<\infty$ holds for $p:=\frac{3\alpha-1}{3-\alpha}-1=\frac{4\alpha-4}{3-\alpha}$. Applying the Gagliardo–Nirenberg and the Poincaré inequality and using (3.23), however, we infer from $p\leqslant 2$ (which is equivalent to $\alpha\leqslant\frac{5}{3}$) that

$$\begin{aligned}\int_0^T\left(\int_\Omega u_x^2\right)^p &= \int_0^T \|u_x(t)\|_{L^2(\Omega)}^{2p}\\ &\leqslant c\int_0^T \|u_{xx}(t)\|_{L^2(\Omega)}^p\cdot\|u(t)\|_{L^2(\Omega)}^p\\ &\leqslant cT^{\frac{2-p}{2}}\cdot\|u\|_{L^\infty((0,T);L^2(\Omega))}^p\cdot\left(\int_0^T\int_\Omega u_{xx}^2\right)^{\frac{p}{2}}\\ &\leqslant cT^{\frac{2-p}{2}}e^{pT},\end{aligned}$$

which yields the desired contradiction. □

4 Weak solutions

In this section we weaken our solution concept to exploit the *a priori* estimate (3.22) (resp. (4.12) and (4.13) below) more effectively and thereby obtain global 'solutions' for a larger class of α.

Another object of our interest is to present and, of course, to prove convergence of, a semi-discrete nonlinear approximation scheme which will be used in § 5.

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To be more precise, let us abbreviate

$$W_N^{2,p}(\Omega) := \left\{ u \in W^{2,p}(\Omega) \mid u_x|_{\partial\Omega} = 0 \;\wedge\; \int_\Omega u = 0 \right\}$$

for $1 \leqslant p < \infty$ and introduce the following definition.

Definition 4.1 *Let $0 < T \leqslant \infty$. By a weak solution of (1.1) we mean a function*

$$u \in L^1_{loc}(\bar{\Omega} \times [0, T)) \cap L^1_{loc}([0, T); W_N^{2,1}(\Omega))$$

such that

$$-\int_\Omega u_0 \varphi(0) - \int_0^T \int_\Omega u\varphi_t + \int_0^T \int_\Omega u_{xx}\varphi_{xx} + \int_0^T \int_\Omega u\varphi_{xx} = -\int_0^T \int_\Omega |u_x|^\alpha \varphi_{xx} \quad (4.1)$$

holds for all $\varphi \in C^2(\bar{\Omega} \times [0, T))$ with compact support in $\bar{\Omega} \times [0, T)$ and $\varphi_x|_{\partial\Omega} = 0$.

In the case $T = \infty$ we call u a global weak solution.

For the convergence proof (but also later on in Theorem 4.5) we shall need a discrete Gronwall lemma in its following version.

Lemma 4.2 *Suppose that A and B are real numbers with $A \neq 0$, that $N \in \mathbb{N}$ and that $(a_k)_{k=0,\ldots,N}$ is a finite sequence of real numbers satisfying*

$$\frac{a_k - a_{k-1}}{\tau} \leqslant Aa_k + B \qquad \forall\, k = 1, \ldots, N \quad (4.2)$$

with some positive τ subject to

$$\tau A < 1. \quad (4.3)$$

Then

$$a_k \leqslant \left(a_0 + \frac{B}{A}\right)(1 - \tau A)^{-k} - \frac{B}{A} \qquad \forall\, k = 0, \ldots, N. \quad (4.4)$$

We are now ready to prove global existence of weak solutions for $\alpha < \frac{10}{3}$.

Theorem 4.3 *Suppose $\alpha \in (1, \frac{10}{3})$ and $u_0 \in L^2(\Omega)$ fulfils $\int_\Omega u_0 = 0$. Then (1.1) has at least one global weak solution.*

Proof We shall employ Rothe's method: Let us fix a small time step size $\tau \in (0, \frac{1}{2})$ and consider the sequence of solutions $u_k^{(\tau)} \in W_N^{2,2}(\Omega), k = 1, 2, 3, \ldots,$ of the nonlinear elliptic problems

$$\int_\Omega \frac{u_k^{(\tau)} - u_{k-1}^{(\tau)}}{\tau} \cdot \varphi + \int_\Omega u_{kxx}^{(\tau)}\varphi_{xx} + \int_\Omega u_{kxx}^{(\tau)}\varphi = -\int_\Omega |u_{kx}^{(\tau)}|^\alpha \varphi_{xx} \qquad \forall\, \varphi \in W_N^{2,2}(\Omega), \quad (4.5)$$

where we set $u_0^{(\tau)} := u_0$. To see that $u_1^{(\tau)}, u_2^{(\tau)}, \ldots$ indeed exist, we proceed as follows: For

$\tau < \frac{1}{2}$, the operator $L : W_N^{2,2}(\Omega) \to (W_N^{2,2}(\Omega))^\star$ defined by

$$(Lu)(\varphi) := \int_\Omega u_{xx}\varphi_{xx} + \int_\Omega u_{xx}\varphi + \frac{1}{\tau}\int_\Omega u\varphi, \qquad \varphi \in W_N^{2,2}(\Omega),$$

is a linear isomorphism due to the Lax-Milgram lemma, because

$$\begin{aligned}(Lu)(u) &= \int_\Omega u_{xx}^2 + \int_\Omega uu_{xx} + \frac{1}{\tau}u^2\\ &= \frac{1}{3}\left(\int_\Omega u_{xx}^2 + \int_\Omega u_x^2 + \int_\Omega u^2\right) + \frac{2}{3}\int_\Omega u_{xx}^2 + \frac{4}{3}\int_\Omega u_x^2 + \left(\frac{1}{\tau} - \frac{1}{3}\right)\int_\Omega u^2\\ &\geqslant \frac{1}{3}\|u\|_{W^{2,2}(\Omega)}^2 + \left(\frac{1}{\tau} - \frac{1}{3} - \frac{2}{3}\right)\int_\Omega u^2\\ &\geqslant \frac{1}{3}\|u\|_{W^{2,2}(\Omega)}^2 \qquad \forall\, u \in W_N^{2,2}(\Omega),\end{aligned}$$

where we have used Young's inequality. Furthermore, for fixed $v \in L^2(\Omega)$,

$$(N_v u)(\varphi) := \frac{1}{\tau}\int_\Omega v\varphi - \int_\Omega |u_x|^\alpha \varphi_{xx}, \qquad \varphi \in W_N^{2,2}(\Omega),$$

defines a continuous (nonlinear) mapping from $W_N^{2,2}(\Omega)$ to $(W_N^{2,2}(\Omega))^\star$ since we have $W_N^{2,2}(\Omega) \hookrightarrow W^{1,\alpha}(\Omega)$ by the Sobolev embedding theorem.

Now solving (4.5) for $k \geqslant 1$ is equivalent to finding a fixed point $u_k^{(\tau)} \in W_N^{2,2}(\Omega)$ of $L^{-1}N_v$ with $v = u_{k-1}^{(\tau)}$. To accomplish this, we first claim that for any $v \in L^2(\Omega)$, $L^{-1}N_v$ is a compact operator from $W_N^{2,2}(\Omega)$ into itself. In fact, suppose $\|u\|_{W_N^{2,2}(\Omega)} \leqslant R$ for some $R > 0$ and let $w := L^{-1}N_v u \in W_N^{2,2}(\Omega)$. Clearly, $\|w\|_{W_N^{2,2}(\Omega)} \leqslant c_0(\|v\|_{L^2(\Omega)} + R^\alpha)$ holds with some c_0. Writing $h(x) := \int_0^x \frac{v(y)-w(y)}{\tau}dy$, $y \in \Omega$, we then have $h|_{\partial\Omega} = 0$. Thus,

$$\begin{aligned}\int_\Omega w_{xx}\varphi_{xx} &= -\int_\Omega w_{xx}\varphi + \int_\Omega \frac{v-w}{\tau}\varphi - \int_\Omega |u_x|^\alpha\varphi_{xx}\\ &= \int_\Omega w_x\varphi_x + \int_\Omega h_x\varphi - \alpha\int_\Omega |u_x|^{\alpha-2}u_xu_{xx}\varphi_x\\ &= \int_\Omega (w_x - h - \alpha|u_x|^{\alpha-2}u_xu_{xx})\varphi_x\end{aligned}$$

holds for all $\varphi \in W_N^{2,2}(\Omega)$ and hence also for any $\varphi \in W^{2,2}(\Omega)$ satisfying $\varphi_x|_{\partial\Omega} = 0$. Due to the embedding $W^{2,2}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$, the function $H := w_x - h - \alpha|u_x|^{\alpha-2}u_xu_{xx}$ is in $L^2(\Omega)$ with

$$\begin{aligned}\|H\|_{L^2(\Omega)} &\leqslant c_1(\|w\|_{W^{1,2}(\Omega)} + \|v\|_{L^2(\Omega)} + R^\alpha)\\ &\leqslant c_2(\|v\|_{L^2(\Omega)} + R^\alpha)\end{aligned} \tag{4.6}$$

for some positive c_1 and c_2. Therefore $\bar{w} := w_x$ fulfils

$$\int_\Omega \bar{w}_x\varphi_{xx} = \int_\Omega H\varphi_x \qquad \forall\,\varphi \in W^{2,2}(\Omega) \text{ with } \varphi_x|_{\partial\Omega} = 0 \tag{4.7}$$

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and thus

$$\int_\Omega \bar{w}_x \psi_x = \int_\Omega H\psi \qquad \forall\, \psi \in W_0^{1,2}(\Omega), \tag{4.8}$$

because any such ψ is the derivative of $\varphi(x) := \int_0^x \psi(y)dy$ which is admissible in (4.7). Since evidently $\bar{w}$ belongs to $W_0^{1,2}(\Omega)$, it follows from (4.6), (4.8) and elliptic regularity theory (Theorem 17.2 in Friedman [6]) that $\bar{w} \in W^{2,2}(\Omega)$ and hence $w \in W^{3,2}(\Omega)$ with

$$\|w\|_{W^{3,2}(\Omega)} \leqslant c_3(\|v\|_{L^2(\Omega)} + R^\alpha). \tag{4.9}$$

Together with the compactness of the embedding $W^{3,2}(\Omega) \hookrightarrow W^{2,2}(\Omega)$ this shows that for fixed v, the operator $L^{-1}N_v$ maps bounded sets of $W_N^{2,2}(\Omega)$ into compact sets in $W_N^{2,2}(\Omega)$, as claimed.

In view of the Leray–Schauder fixed point theorem, in order to prove the existence of a fixed point of $L^{-1}N_v$ it is sufficient to show that there is $R > 0$ such that if some $u \in W_N^{2,2}(\Omega)$ satisfies $u = \lambda L^{-1}N_v u$ for some $\lambda \in (0,1)$ then $\|u\|_{W_N^{2,2}(\Omega)} \leqslant R$. But $Lu = \lambda N_v u$ implies

$$\begin{aligned}\int_\Omega u_{xx}^2 + \int_\Omega u u_{xx} + \frac{1}{\tau}\int_\Omega u^2 &= \frac{\lambda}{\tau}\int_\Omega uv - \lambda \int_\Omega |u_x|^\alpha u_{xx} \\ &\leqslant \frac{1}{2\tau}\int_\Omega u^2 + \frac{\lambda^2}{2\tau}\int_\Omega v^2,\end{aligned}$$

since $\int_\Omega |u_x|^\alpha u_{xx} = \frac{1}{\alpha+1}\int_\Omega (|u_x|^\alpha u_x)_x = 0$. Therefore, we indeed find a λ-independent *a priori* estimate $\|u\|_{W^{2,2}(\Omega)} \leqslant R = R(\tau, \|v\|_{L^2(\Omega)})$, and thereby the proof of solvability of (4.5) is complete.

We next define the Rothe functions

$$\begin{aligned} u^{(\tau)}(x,t) &:= \frac{k\tau - t}{\tau}\, u_{k-1}^{(\tau)}(x) + \frac{t-(k-1)\tau}{\tau}\, u_k^{(\tau)}(x) \qquad \text{and} \\ \bar{u}^{(\tau)}(x,t) &:= u_k^{(\tau)}(x) \end{aligned} \tag{4.10}$$

for $x \in \Omega$ and $t \in [(k-1)\tau, k\tau)$, $k = 1,2,3,\ldots$. To establish τ-independent estimates, we let $\varphi := u_k^{(\tau)}$ in (4.5) and see that

$$\begin{aligned}\frac{1}{\tau}\int_\Omega |u_k^{(\tau)}|^2 + \int_\Omega |u_{kxx}^{(\tau)}|^2 &= -\int_\Omega u_k^{(\tau)} u_{kxx}^{(\tau)} + \frac{1}{\tau}\int_\Omega u_{k-1}^{(\tau)} u_k^{(\tau)} - \int_\Omega |u_{kx}^{(\tau)}|^\alpha u_{kxx}^{(\tau)} \\ &\leqslant \frac{1}{2}\int_\Omega |u_k^{(\tau)}|^2 + \frac{1}{2}\int_\Omega |u_{kxx}^{(\tau)}|^2 + \frac{1}{2\tau}\int_\Omega |u_k^{(\tau)}|^2 + \frac{1}{2\tau}\int_\Omega |u_{k-1}^{(\tau)}|^2,\end{aligned}$$

whence

$$\frac{1}{\tau}\left(\int_\Omega |u_k^{(\tau)}|^2 - \int_\Omega |u_{k-1}^{(\tau)}|^2\right) + \int_\Omega |u_{kxx}^{(\tau)}|^2 \leqslant \int_\Omega |u_k^{(\tau)}|^2 \qquad \text{for } k = 1,2,3,\ldots \tag{4.11}$$

An application of the discrete Gronwall inequality (4.4) shows that

$$\begin{aligned}\int_\Omega |u_k^{(\tau)}|^2 &\leqslant \left(\int_\Omega u_0^2\right)\cdot (1-\tau)^{-k} \\ &\leqslant \left(\int_\Omega u_0^2\right)\cdot e^{2k\tau} \qquad \text{for } k = 1,2,3,\ldots,\end{aligned}$$

because $\tau < \frac{1}{2}$. Summing up (4.11) over $k = 1, \dots, N$, $N \in \mathbb{N}$, we infer from this that

$$\begin{aligned}\int_\Omega |u_N^{(\tau)}|^2 + \sum_{k=1}^N \tau \int_\Omega |u_{kxx}^{(\tau)}|^2 &\leqslant \int_\Omega u_0^2 + \sum_{k=1}^N \tau \int_\Omega |u_k^{(\tau)}|^2 \\ &\leqslant \left(\int_\Omega u_0^2\right) \cdot \left(1 + \tau \sum_{k=1}^N (1-\tau)^{-k}\right) \\ &= \left(\int_\Omega u_0^2\right) \cdot (1-\tau)^{-N} \\ &\leqslant \left(\int_\Omega u_0^2\right) \cdot e^{2N\tau} \qquad \text{for } N = 1, 2, 3, \dots\end{aligned}$$

As to $\bar{u}^{(\tau)}$, this means that

$$\|\bar{u}^{(\tau)}\|_{L^\infty((0,T);L^2(\Omega))} + \|\bar{u}^{(\tau)}\|_{L^2((0,T);W^{2,2}(\Omega))} \leqslant c\|u_0\|_{L^2(\Omega)} e^T \qquad \forall\, T > 0, \tag{4.12}$$

where we have used the Poincaré inequality (2.6); evidently, this also implies

$$\|u^{(\tau)}\|_{L^\infty((0,T);L^2(\Omega))} + \|u^{(\tau)}\|_{L^2((0,T);W^{2,2}(\Omega))} \leqslant c\|u_0\|_{L^2(\Omega)} e^T \qquad \forall\, T > 0. \tag{4.13}$$

Moreover, from (4.5) and a completion argument we have

$$\int_0^T \int_\Omega u_t^{(\tau)} \varphi + \int_0^T \int_\Omega \bar{u}_{xx}^{(\tau)} \varphi_{xx} + \int_0^T \int_\Omega \bar{u}_{xx}^{(\tau)} \varphi = -\int_0^T \int_\Omega |\bar{u}_x^{(\tau)}|^\alpha \varphi_{xx} \tag{4.14}$$

for all $\varphi \in L^p((0,T); W_N^{2,p}(\Omega))$, $p := \max\{2, \frac{10}{10-3\alpha}\}$, whence (4.12) gives

$$\begin{aligned}\left|\int_0^T \int_\Omega u_t^{(\tau)} \varphi\right| &\leqslant c\Big(\|\bar{u}^{(\tau)}\|_{L^2((0,T);W^{2,2}(\Omega))} \|\varphi_{xx}\|_{L^2(\Omega\times(0,T))} + \|\bar{u}_x^{(\tau)}\|^\alpha_{L^{p'}(\Omega\times(0,T))} \|\varphi_{xx}\|_{L^2(\Omega\times(0,T))}\Big) \\ &\leqslant c(T)\|\varphi\|_{L^p((0,T);W^{2,p}(\Omega))},\end{aligned} \tag{4.15}$$

where p' is defined via $\frac{1}{p} + \frac{1}{p'} = 1$. In deriving the last line in (4.15) we make use of the Gagliardo–Nirenberg and the Poincaré inequality in estimating

$$\begin{aligned}\int_0^T \int_\Omega |\bar{u}_x^{(\tau)}|^{\alpha p'} &= \int_0^T \|\bar{u}_x^{(\tau)}(t)\|^{\alpha p'}_{L^{\alpha p'}(\Omega)} \\ &\leqslant c\|\bar{u}^{(\tau)}\|^{\frac{\alpha p'+2}{4}}_{L^\infty((0,T);L^2(\Omega))} \cdot \int_0^T \|\bar{u}_{xx}^{(\tau)}(t)\|^{\frac{3\alpha p'-2}{4}}_{L^2(\Omega)} \\ &\leqslant c(T),\end{aligned} \tag{4.16}$$

observing that $p \geqslant \frac{10}{10-3\alpha}$ implies $\frac{3\alpha p'}{4} \leqslant \frac{3\cdot\frac{10}{3}-2}{4} = 2$. Consequently,

$$\|u_t^{(\tau)}\|_{L^{p'}((0,T);(W_N^{2,p}(\Omega))^\star)} \leqslant c(T), \tag{4.17}$$

and thus the Aubin-Lions lemma (see Theorem III.2.1 in Temam [19], for instance) ensures that

$$(u^{(\tau)})_{\tau\in(0,\frac{1}{2})} \quad \text{is strongly precompact in } L^2((0,T); W^{1,\infty}(\Omega)). \tag{4.18}$$

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Now from (4.12), (4.13), (4.16), (4.17), (4.18) and functional analytical standard arguments, we infer the existence of a sequence of numbers $\tau_n \searrow 0$ with the property that for any $T > 0$, we have

$$\begin{aligned} u^{(\tau_n)} \overset{\star}{\rightharpoonup} u \quad \text{and} \quad \bar{u}^{(\tau_n)} \overset{\star}{\rightharpoonup} \bar{u} &\ \text{ in } L^\infty((0,T);L^2(\Omega)),\\ u^{(\tau_n)} \rightharpoonup u \quad \text{and} \quad \bar{u}^{(\tau_n)} \rightharpoonup \bar{u} &\ \text{ in } L^2((0,T);W_N^{2,2}(\Omega)),\\ u_t^{(\tau_n)} \rightharpoonup u_t &\ \text{ in } L^{p'}((0,T);(W_N^{2,p}(\Omega))^\star),\\ |\bar{u}_x^{(\tau_n)}|^\alpha \rightharpoonup v &\ \text{ in } L^{p'}((\Omega\times(0,T)),\\ u^{(\tau_n)} \to u &\ \text{ in } L^2((0,T);W^{1,\infty}(\Omega)) \qquad \text{as well as}\\ u^{(\tau_n)} \to u \quad \text{and} \quad u_x^{(\tau_n)} \to u_x &\ \text{ a.e. in } \Omega\times(0,T). \end{aligned}$$

Since (4.18) and the definition of $u^{(\tau)}$ and $\bar{u}^{(\tau)}$ imply (see Lemma 2.3 in King *et al.* [10])

$$\|u^{(\tau)} - \bar{u}^{(\tau)}\|_{L^2((0,T);W^{1,\infty}(\Omega))} \to 0 \qquad \text{as } \tau \to 0,$$

we conclude that also

$$\bar{u}^{(\tau_n)} \to 0 \quad \text{and} \quad \bar{u}_x^{(\tau_n)} \to u_x \qquad \text{a.e. in } \Omega\times(0,T)$$

for a further subsequence and thus we can identify $\bar{u} = u$ and $v = |u_x|^\alpha$. Now from (4.14) we easily obtain that u in fact is a weak solution of (1.1). □

Remark (i) For later use (cf. the proof of Theorem 4.5), let us emphasise that the solutions $u_k^{(\tau)} \in W_N^{2,2}(\Omega)$ of (4.5) are actually in $W^{3,2}(\Omega)$ and thus in $C^2(\bar{\Omega})$. This results from the considerations leading to (4.12).

In the case $\alpha \geqslant 2$ one even obtains more regularity: then, namely, we infer that $(|u_{kx}^{(\tau)}|^\alpha)_{xx} = \alpha|u_{kx}^{(\tau)}|^{\alpha-2}u_{kx}^{(\tau)}u_{kxxx}^{(\tau)} + \alpha(\alpha-1)|u_{kx}^{(\tau)}|^{\alpha-2}|u_{kxx}^{(\tau)}|^2$ is an $L^2(\Omega)$-function by what we have just observed, so that standard elliptic regularity theory yields first $u_k^{(\tau)} \in W^{4,2}(\Omega)$ and finally $u_k^{(\tau)} \in C^4(\bar{\Omega})$.

(ii) We do not know whether weak solutions are unique.

(iii) Of course there are alternative methods to construct weak solutions: one would consist of approximating the term $(|u_x|^\alpha)_{xx}$ by, say, $(f_\varepsilon(u_x))_{xx}$ with a smooth and bounded approximation f_ε of $f(s) := |s|^\alpha$; also, one could introduce a higher order viscosity term of the form $\varepsilon(-1)^m D^{2m}u$ on the left of (1.1). Finally, weak solutions can be obtained from a suitable (spatially discrete) Galerkin approximation – this is done in Blömker & Gugg [2], for instance. Let us mention, however, that we are in doubt whether any of these three approaches provides weak solutions which have a controllable large time behaviour as asserted below in Theorem 4.5 for our solutions. Moreover, the proof of Theorem 4.5 will actually show that the claimed absorption estimate (4.20) is true even for the approximate Rothe functions.

For the proof of the absorption property (4.20) below, we need to introduce, given $\alpha \geqslant 2$, a class of functions $\Phi \in W_{loc}^{3,\infty}(\mathbb{R})$ such that

$$\Phi, \Phi', \Phi'' \text{ and } \Phi''' \quad \text{are positive on } \mathbb{R},$$

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and which satisfy the following hypothesis (H2) (in case $\alpha = 2$) resp. (Hα) (if $\alpha > 2$).

(H2) (i) There exists $m < \frac{8}{3}$ such that

$$\frac{\Phi'''(s)}{\Phi''(s)} \leqslant m \qquad \text{for a.e. } s \in \mathbb{R}.$$

(ii) There exist positive constants $C_1^{(2)}, C_2^{(2)}$ and $\mu^{(2)}$ such that the function $P(s) := \int_0^s (\Phi'''(\sigma))^{\frac{1}{4}}\, d\sigma$ satisfies

$$P^4(s) \geqslant \frac{|\Omega|^4}{2(\frac{2}{3} - \frac{m}{4})^2} \frac{(\Phi'')^2(s)}{\Phi'''(s)} - C_1 \qquad \text{and}$$
$$P^4(s) \geqslant \mu\Phi(s) - C_2 \qquad \forall\, s \in \mathbb{R}.$$

(Hα) There exist positive constants $C_1^{(\alpha)}, C_2^{(\alpha)}$ and $\mu^{(\alpha)}$ such that $P(s) := \int_0^s (\Phi'''(\sigma))^{\frac{1}{\alpha+2}}\, d\sigma$ fulfils

$$\Gamma_1 \cdot \frac{(\Phi''')^{\frac{2\alpha}{\alpha-2}}(s)}{(\Phi'')^{\frac{\alpha+2}{\alpha-2}}(s)} + \Gamma_2 \cdot \frac{(\Phi'')^{\frac{\alpha+2}{\alpha}}(s)}{(\Phi''')^{\frac{2}{\alpha}}(s)} \leqslant \frac{1}{4} \frac{\alpha(\alpha+2)}{(\alpha+1)|\Omega|^{\alpha+2}} |P(s)|^{\alpha+2} + C_1^{(\alpha)} \qquad \text{and}$$
$$|P(s)|^{\alpha+2} \geqslant \mu\Phi(s) - C_2^{(\alpha)} \qquad \text{for a.e. } s \in \mathbb{R}$$

with

$$\Gamma_1 := \frac{\alpha-2}{\alpha+2}\left(\frac{4(\alpha+1)}{\alpha(\alpha+2)}\right)^{\frac{4}{\alpha-2}} \quad \text{and} \quad \Gamma_2 := \frac{\alpha}{\alpha+2}\left(\frac{8(\alpha+1)}{\alpha(\alpha+2)}\right)^{\frac{2}{\alpha}}. \tag{4.19}$$

Let us make sure that these assumptions are met by some 'nice' functions.

Lemma 4.4 (i) *(Hα) is satisfied upon the choice*

$$\Phi(s) := e^{\beta s}, \qquad s \in \mathbb{R},$$

with sufficiently small $\beta > 0$.
(ii) *(H2) is fulfilled if we define* Φ *by the relations* $\lim_{s\to-\infty} \Phi(s) = \lim_{s\to-\infty} \Phi'(s) = 0$ *and*

$$\Phi''(s) = \begin{cases} e^{\beta s}, & s \leqslant s_0, \\ bs^p, & s > s_0, \end{cases}$$

with any

$$\beta \in \left(0, \frac{8}{3}\right), \quad p > 1, \quad s_0 \geqslant \frac{p}{\beta} \quad \text{and} \quad b = s_0^{-p} e^{\beta s_0},$$

that is, it we set

$$\Phi(s) := \begin{cases} \frac{1}{\beta^2} e^{\beta s}, & s \leqslant s_0, \\ \frac{b}{(p+1)(p+2)}(s^{p+2} - s_0^{p+2}) + \left(\frac{1}{\beta} e^{\beta s_0} - \frac{b s_0^{p+1}}{p+1}\right)(s - s_0) + \frac{1}{\beta^2} e^{\beta s_0}, & s > s_0. \end{cases}$$

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Proof (i) We calculate

$$P(s) = \tfrac{\alpha+2}{\beta}\left(e^{\frac{\beta}{\alpha+2}s} - 1\right), \qquad \text{and}$$

$$I_1(s) := \frac{(\Phi''')^{\frac{2\alpha}{\alpha-2}}(s)}{(\Phi'')^{\frac{\alpha+2}{\alpha-2}}(s)} = \beta^{4(\alpha-1)}\alpha - 2e^{\beta s}, \qquad I_2(s) := \frac{(\Phi'')^{\frac{\alpha+2}{\alpha}}(s)}{(\Phi''')^{\frac{2}{\alpha}}(s)} = \beta^{\frac{2(\alpha-1)}{\alpha}} e^{\beta s}.$$

Thus, if $\beta \leqslant 1$ we have $\Gamma_1 I_1(s) \leqslant c e^{\beta s}$ and $|P(s)|^{\alpha+2} \geqslant c_1 \beta^{-(\alpha+2)} e^{\beta s} - c_2(\beta)$, so that (H$\alpha$) is true for arbitrary $\mu > 0$, provided β is small enough.

(ii) As

$$\frac{\Phi'''(s)}{\Phi''(s)} = \begin{cases} \beta, & s \leqslant s_0, \\ \frac{p}{s}, & s > s_0, \end{cases}$$

(H2) (i) is satisfied with $m := \beta$. Since furthermore

$$\frac{(\Phi'')^2(s)}{\Phi'''(s)} = \begin{cases} \frac{1}{\beta} e^{\beta s}, & s \leqslant s_0, \\ \frac{b}{p} s^{p+1}, & s > s_0, \end{cases}$$

we see that both inequalities in (H2) ii) are, trivially, fulfilled (with any fixed $\mu > 0$) for $s \leqslant s_0$ and suitable constants $C_1^{(2)}$ and $C_2^{(2)}$. As to $s > s_0$, we observe that

$$P(s) = \frac{4(bp)^{\frac{1}{4}}}{p+3}\left(s^{\frac{p+3}{4}} - s_0^{\frac{p+3}{4}}\right) + 4\beta^{-\frac{3}{4}}\left(e^{\frac{s_0}{4}} - 1\right),$$

so that

$$P^4(s) \geqslant c_1 s^{p+3} - c_2 \qquad \forall\, s > s_0$$

holds with certain c_1, c_2. Since

$$\frac{(\Phi'')^2(s)}{\Phi'''(s)} \leqslant c_3 s^{p+1} + c_4 \qquad \text{and}$$
$$\Phi(s) \leqslant c_5 s^{p+2} + c_6 \qquad \forall\, s > s_0$$

with suitable $c_3, \ldots, c_6$, (H2) ii) is also valid for $s > s_0$. □

Let us finally prove that after the transformation $v = \Phi(u)$ all weak solutions (that can be approximated as above) are uniformly absorbed by a bounded set in $L^1(\Omega)$.

Theorem 4.5 *Let* $\alpha \in [2, \frac{10}{3})$ *and* Φ *be a function satisfying* (Hα) *resp.* (H2). *Then there exist positive constants* ν *and* C_0 *such that any weak solution as constructed in Theorem* 4.3 *obeys the estimate*

$$\int_\Omega \Phi(u(t)) \leqslant \left(\int_\Omega \Phi(u_0)\right) \cdot e^{-\nu t} + C_0 \qquad \textit{for a.e. } t > 0. \tag{4.20}$$

Remark The inequality (4.20) is trivially satisfied if $\int_\Omega \Phi(u_0) = \infty$.

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Proof We use the family $(\bar{u}^{(\tau)})_{\tau\in(0,\frac{1}{2})}$ of approximate solutions defined by (4.10) and (4.5), where henceforth we restrict ourselves to a suitable sequence of numbers $\tau_n \to 0$ along which

$$\bar{u}^{(\tau_n)} \to u \qquad \text{a.e. in } \Omega \times (0,\infty)$$

is valid, which implies that

$$\bar{u}^{(\tau_n)}(t) \to u(t) \qquad \text{a.e. in } \Omega \quad \forall\, t \in (0,\infty) \setminus N, \tag{4.21}$$

where N has measure zero.

Since Φ is a convex function, we have

$$\frac{1}{\tau}\Big(\Phi(u_k^{(\tau)}) - \Phi(u_{k-1}^{(\tau)})\Big) \leqslant \Phi'(u_k^{(\tau)}) \cdot \frac{u_k^{(\tau)} - u_{k-1}^{(\tau)}}{\tau} \qquad \text{in } \Omega \tag{4.22}$$

for all $k \in \mathbb{N}$. According to the remark following Theorem 4.3, $\Phi'(u_k^{(\tau)})$ belongs to $C^2(\bar{\Omega})$. As furthermore $\Phi''(u_k^{(\tau)})u_{kx}^{(\tau)}|_{\partial\Omega} = 0$, we may choose $\varphi := \Phi'(u_k^{(\tau)})$ as a test function in (4.5) to obtain from (4.22)

$$\begin{aligned}
\frac{1}{\tau}\left(\int_\Omega \Phi(u_k^{(\tau)}) - \int_\Omega \Phi(u_{k-1}^{(\tau)})\right) &\leqslant \int_\Omega \Phi'(u_k^{(\tau)}) \cdot \frac{u_k^{(\tau)} - u_{k-1}^{(\tau)}}{\tau} \\
&= -\int_\Omega \Big(\Phi'(u_k^{(\tau)})\Big)_{xx} \cdot u_{kxx}^{(\tau)} - \int_\Omega \Big(\Phi'(u_k^{(\tau)})\Big)_{xx} u_k^{(\tau)} \\
&\quad - \int_\Omega \Big(\Phi'(u_k^{(\tau)})\Big)_{xx} |u_{kx}^{(\tau)}|^\alpha \\
&=: I_1 + I_2 + I_3.
\end{aligned} \tag{4.23}$$

Using Young's inequality, we estimate

$$\begin{aligned}
I_1 &= -\int_\Omega \Phi''(u_k^{(\tau)})|u_{kxx}^{(\tau)}|^2 - \int_\Omega \Phi'''(u_k^{(\tau)})|u_{kx}^{(\tau)}|^2 u_{kxx}^{(\tau)} \\
&\leqslant \frac{1}{4}\int_\Omega \frac{(\Phi''')^2(u_k^{(\tau)})}{\Phi''(u_k^{(\tau)})}|u_{kx}^{(\tau)}|^4
\end{aligned} \tag{4.24}$$

and twice integrate by parts in calculating

$$\begin{aligned}
I_3 &= -\int_\Omega \Big(\Phi'(u_k^{(\tau)})\Big)_{xx} |u_{kx}^{(\tau)}|^\alpha \\
&= \alpha \int_\Omega \Phi''(u_k^{(\tau)})|u_{kx}^{(\tau)}|^\alpha\, u_{kxx}^{(\tau)} \\
&= -\frac{\alpha}{\alpha+1}\int_\Omega \Phi'''(u_k^{(\tau)})|u_{kx}^{(\tau)}|^{\alpha+2}.
\end{aligned} \tag{4.25}$$

Let us first consider separately the case $\alpha = 2$ which is most transparent: then, by (H2),

$$I_1 + I_3 \leqslant -\left(\frac{2}{3} - \frac{m}{4}\right)\int_\Omega \Phi'''(u_k^{(\tau)})|u_{kx}^{(\tau)}|^4$$

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and we can estimate

$$I_2 = \int_\Omega \Phi''(u_k^{(\tau)})|u_{kx}^{(\tau)}|^2 \leqslant \frac{1}{2}\left(\frac{2}{3}-\frac{m}{4}\right)\int_\Omega \Phi'''(u_k^{(\tau)})|u_{kx}^{(\tau)}|^4 + \frac{1}{2(\frac{2}{3}-\frac{m}{4})}\int_\Omega \frac{(\Phi'')^2(u_k^{(\tau)})}{\Phi'''(u_k^{(\tau)})},$$

so that (4.23) yields

$$\frac{1}{\tau}\left(\int_\Omega \Phi(u_k^{(\tau)}) - \int_\Omega \Phi(u_{k-1}^{(\tau)})\right) \leqslant -\frac{1}{2}\left(\frac{2}{3}-\frac{m}{4}\right)\int_\Omega \Phi'''(u_k^{(\tau)})(u_{kx}^{(\tau)})^4 + \frac{1}{2(\frac{2}{3}-\frac{m}{4})}\int_\Omega \frac{(\Phi'')^2(u_k^{(\tau)})}{\Phi'''(u_k^{(\tau)})}. \tag{4.26}$$

Now $\Phi'''(u_k^{(\tau)})|u_{kx}^{(\tau)}|^4 = |P(u_k^{(\tau)})_x|^4$ holds with $P(s) = \int_0^s (\Phi'''(\sigma))^{\frac{1}{4}}\,d\sigma$ as introduced in (H2), and since $P(u_k^{(\tau)})$ has a zero in Ω (because $u_k^{(\tau)}$ itself has one), the Poincaré inequality (2.6) together with (H2) gives

$$\begin{aligned}\frac{1}{\tau}\left(\int_\Omega \Phi(u_k^{(\tau)}) - \int_\Omega \Phi(u_{k-1}^{(\tau)})\right) &\leqslant -\frac{1}{2}\left(\frac{2}{3}-\frac{m}{4}\right)\cdot\frac{4}{|\Omega|^4}\int_\Omega |P(u_k^{(\tau)})|^4 + \frac{1}{2(\frac{2}{3}-\frac{m}{4})}\int_\Omega \frac{(\Phi'')^2(u_k^{(\tau)})}{\Phi'''(u_k^{(\tau)})}\\ &\leqslant -\frac{1}{4}\left(\frac{2}{3}-\frac{m}{4}\right)\cdot\frac{4}{|\Omega|^4}\int_\Omega |P(u_k^{(\tau)}|^4 + c\\ &\leqslant -\frac{(\frac{2}{3}-\frac{m}{4})\mu}{|\Omega|^4}\int_\Omega \Phi(u_k^{(\tau)}) + B\end{aligned}$$

with certain constants c and B. Upon an obvious choice of $v > 0$, this means that the numbers $a_k := \int_\Omega \Phi(u_k^{(\tau)})$, $k = 0, 1, 2, \ldots$, satisfy

$$\frac{a_k - a_{k-1}}{\tau} \leqslant -v a_k + B \qquad \forall\, k \geqslant 1$$

whence Lemma 4.2 shows that

$$\begin{aligned}\int_\Omega \Phi(u_k^{(\tau)}) &\leqslant \left(\int_\Omega \Phi(u_0) - \frac{B}{v}\right)\cdot(1+v\tau)^{-k} + \frac{B}{v}\\ &\leqslant \left(\int_\Omega \Phi(u_0)\right)\cdot(1+v\tau)^{-k} + \frac{B}{v}.\end{aligned} \tag{4.27}$$

To prove (4.20) for $t \in (0,\infty)\setminus N$, we fix $\varepsilon > 0$ and choose $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that $(1+\frac{vt}{k})^k \geqslant \frac{1}{1+\varepsilon}e^{vt}$ for all $k \geqslant k_0$. Then for any $\tau \in (\tau_n)_{n\in\mathbb{N}}$ with $\tau \leqslant \frac{t}{k_0}$, there exists a unique $k = k(\tau) \geqslant k_0$ such that $t \in [(k-1)\tau, k\tau)$, and with this k we obtain from (4.27)

$$\begin{aligned}\int_\Omega \Phi(\bar{u}^{(\tau)}(t)) &= \int_\Omega \Phi(u_k^{(\tau)})\\ &\leqslant \left(\int_\Omega \Phi(u_0)\right)\cdot\left(1+\frac{vt}{k}\right)^{-k} + \frac{B}{v}\\ &\leqslant (1+\varepsilon)\left(\int_\Omega \Phi(u_0)\right)\cdot e^{-vt} + \frac{B}{v}.\end{aligned}$$

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Letting τ tend to zero along the sequence $(\tau_n)_{n\in\mathbb{N}}$, we conclude after taking $\varepsilon \to 0$ that (4.20) is valid with $C_0 := \frac{B}{\nu}$, so that the proof is complete in the case $\alpha = 2$.

For $\alpha > 2$ the procedure is quite similar, employing (Hα) rather than (H2): Starting at (4.23), (4.24) and (4.25), this time we estimate I_1 and I_2 by Young's inequality (in the form $ab \leqslant \varepsilon a^p + \frac{p-1}{p}(p\varepsilon)^{-\frac{1}{p-1}} b^{\frac{p}{p-1}}$ for $a, b, \varepsilon > 0$ and $p > 1$) to get

$$I_1 \leqslant \frac{1}{4}\int_\Omega \frac{(\Phi''')^2(u_k^{(\tau)})}{\Phi''(u_k^{(\tau)})} \cdot |u_{kx}^{(\tau)}|^4$$
$$\leqslant \frac{1}{4}\cdot\frac{\alpha}{\alpha+1}\int_\Omega \Phi'''(u_k^{(\tau)})|u_{kx}^{(\tau)}|^{\alpha+2} + \Gamma_1 \int_\Omega \frac{(\Phi''')^{\frac{2\alpha}{\alpha-2}}(u_k^{(\tau)})}{(\Phi'')^{\frac{\alpha+2}{\alpha-2}}(u_k^{(\tau)})}$$

and

$$I_2 = \int_\Omega \Phi''(u_k^{(\tau)}) \cdot |u_{kx}^{(\tau)}|^2$$
$$\leqslant \frac{1}{4}\cdot\frac{\alpha}{\alpha+1}\int_\Omega \Phi'''(u_k^{(\tau)})|u_{kx}^{(\tau)}|^{\alpha+2} + \Gamma_2 \int_\Omega \frac{(\Phi'')^{\frac{\alpha+2}{\alpha}}(u_k^{(\tau)})}{(\Phi''')^{\frac{2}{\alpha}}(u_k^{(\tau)})}$$

with the constants Γ_1 and Γ_2 defined by (4.19). Hence, instead of (4.26) we now obtain

$$\frac{1}{\tau}\left(\int_\Omega \Phi(u_k^{(\tau)}) - \int_\Omega \Phi(u_{k-1}^{(\tau)})\right) \leqslant -\frac{1}{2}\frac{\alpha}{\alpha+1}\int_\Omega \Phi'''(u_k^{(\tau)})\cdot|u_{kx}^{(\tau)}|^{\alpha+2}$$
$$+ \int_\Omega \left(\Gamma_1 \frac{(\Phi''')^{\frac{2\alpha}{\alpha-2}}(u_k^{(\tau)})}{(\Phi'')^{\frac{\alpha+2}{\alpha-2}}(u_k^{(\tau)})} + \Gamma_2 \frac{(\Phi'')^{\frac{\alpha+2}{\alpha}}(u_k^{(\tau)})}{(\Phi''')^{\frac{2}{\alpha}}(u_k^{(\tau)})}\right). \tag{4.28}$$

Using $P(s) = \int_0^s (\Phi'''(\sigma))^{\frac{1}{\alpha+2}}\, d\sigma$, applying the Poincaré inequality and recalling (Hα), we infer that

$$\frac{1}{\tau}\left(\int_\Omega \Phi(u_k^{(\tau)}) - \int_\Omega \Phi(u_{k-1}^{(\tau)})\right) \leqslant -\frac{1}{2}\frac{\alpha}{\alpha+1}\cdot\frac{\alpha+2}{|\Omega|^{\alpha+2}}\int_\Omega |P(u_k^{(\tau)})|^{2+\alpha}$$
$$+\frac{1}{4}\frac{\alpha}{\alpha+1}\cdot\frac{\alpha+2}{|\Omega|^{\alpha+2}}\int_\Omega |P(u_k^{(\tau)})|^{2+\alpha} + c$$
$$\leqslant -\frac{1}{4}\frac{\alpha(\alpha+2)}{(\alpha+1)|\Omega|^{\alpha+2}}\mu\int_\Omega \Phi(u_k^{(\tau)}) + \tilde{B}$$

holds with constants c and $\tilde{B}$. As above, (4.20) results from this with $\nu = \frac{1}{4}\frac{\alpha(\alpha+2)}{(\alpha+1)|\Omega|^{\alpha+2}}$ and $C_0 = \frac{\tilde{B}}{\nu}$. □

Remark The convexity argument entailing (4.22) and (4.23) was introduced to justify the formal asymptotics

$$\Phi(v(t)) - \Phi(v(t-\tau)) = \int_{t-\tau}^t \Phi'(v(s))v_t(s)\,ds \approx \Phi'(v(t))\cdot(v(t)-v(t-\tau))$$

for $\tau \approx 0$. A similar reasoning was previously used in Lemma 1.5 in Alt & Luckhaus [1] in a slightly different setting.

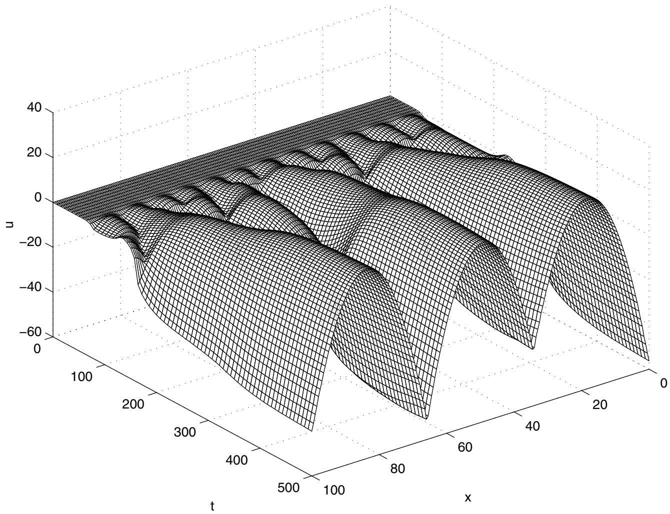

FIGURE 1. Film growth for $t \in [0, 500]$ and $\alpha = 2$.

5 Numerical examples

For the simulation of thin film growth governed by equation (1.1), we use a Galerkin approach based on the approximation scheme from § 4. We consider the domain $\Omega = (0, 100)$ in which randomly distributed imperfections of height 10^{-5} serve as initial conditions. We use 150 subdivisions of Ω for the finite element approximation. All computations were run under Matlab 5.3.

Figure 1 shows a simulation of the film growth up to time $T = 500$. The simulation correctly mimics certain important properties of epitaxial film growth which are known from experimental observations. First, the mean deviation of the film profile increases over time, known as roughening. After some initial period, islands form in the substrate, where the number of islands decreases over time, whereas their size increases, a process called coarsening.

Moreover, as the absorption estimate in Theorem 4.5 with Φ from Lemma 4.4 suggests, the computed solution functions are less regular at smaller values than at larger values: they are smooth except for kinks at local minimisers (cf. Figure 2). A look at film profiles at larger times reveals that numerically the solutions converge to a steady state. Figure 3 shows such a state, along with the development of minimal and maximal film heights as well as the L_∞—norm of gradients in Figure 4.

All our numerical simulations for $\alpha = 2$ exhibit a steady state of the form from Figure 3. This indicates that the uniformly absorbing set from Theorem 4.5 may be rather small.

For $\alpha > 2$ the numerically observed solution behaviour is qualitatively similar to the case $\alpha = 2$, apart from the fact that the curvature at local maximisers of steady states increases for growing α. Figure 5 shows such a steady state for $\alpha = 4$, and Figure 6 reports the corresponding growth behaviour of u and u_x.

For $\alpha < 2$ the numerics indicate that the curvature at local maximisers of solutions quickly tends to zero for α approaching one. For $1 < \alpha < 2$, however, also the qualitative

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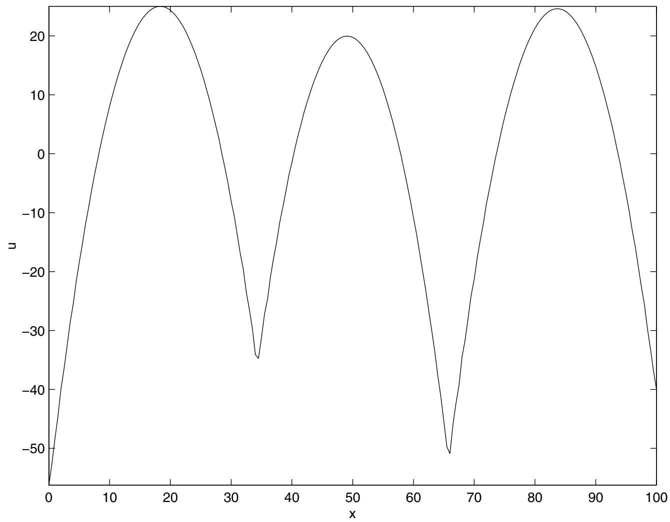

FIGURE 2. Film profile for $t = 500$ and $\alpha = 2$.

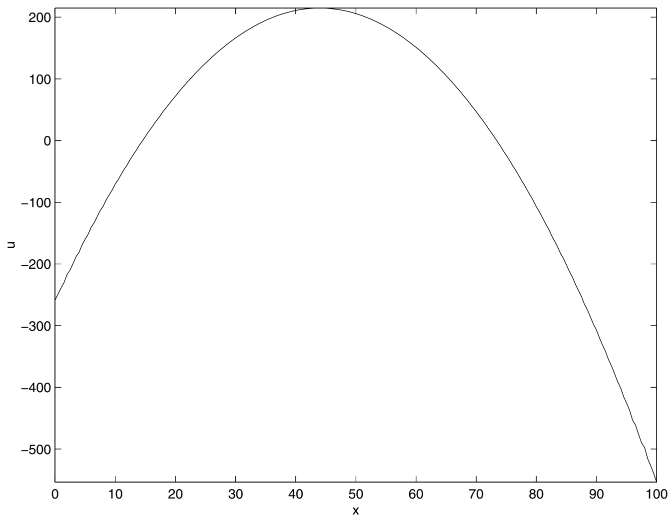

FIGURE 3. Film profile for $t = 10^4$ and $\alpha = 2$.

growth behaviour changes. Whereas still a single island is formed in the long run (cf. Figure 7), this solution is *not* a steady state. In fact, Figure 8 documents that the island keeps growing, thereby developing steep gradients (close to the boundaries of Ω).

We emphasise that this result is in accordance with Theorem 4.5, since we showed the existence of an absorbing set only for $\alpha \geqslant 2$. The numerical simulations for $1 < \alpha < 2$ support our conjecture that for small α the damping effect of the nonlinear term in (1.1) is too weak to create an absorbing set.

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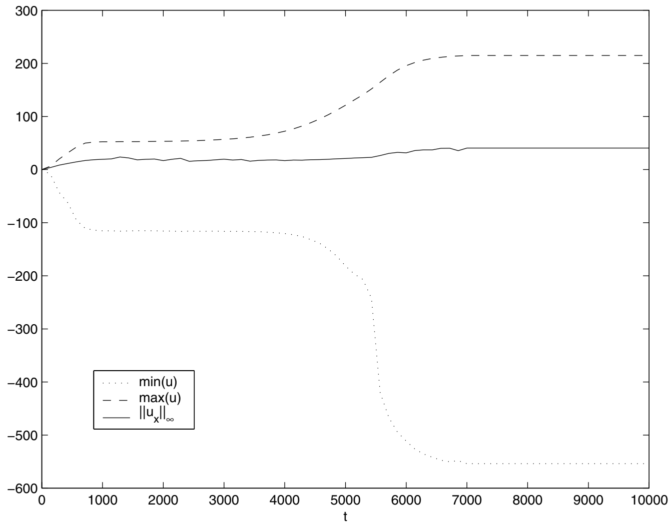

FIGURE 4. min(u), max(u), and $||u_x||_\infty$ for $t \in [0, 10^4]$ and $\alpha = 2$.

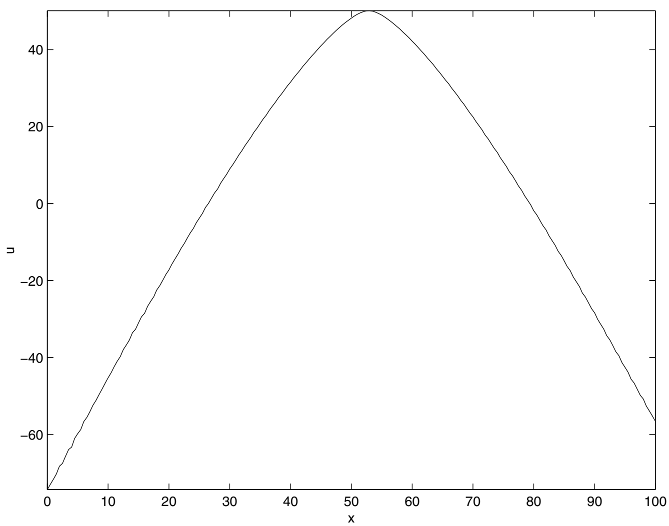

FIGURE 5. Film profile for $t = 10^4$ and $\alpha = 4$.

6 Final remarks

A comparison of our numerical results to those from [9] shows a close resemblance between the numerical solutions of (1.1) with $\alpha = 2$ and of the second order Sivashinsky equation

$$u_t - u_{xx} = \frac{1}{2}(u_x)^2 + \frac{\gamma}{2}(\mathscr{H}(u))_x \qquad \text{in } \Omega \times (0, T),$$
$$u|_{t=0} = u_0,$$

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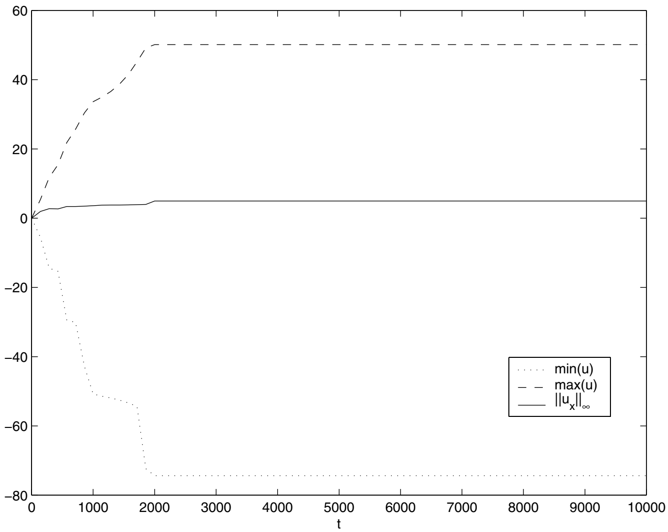

FIGURE 6. min(u), max(u), and $||u_x||_\infty$ for $t \in [0, 10^4]$ and $\alpha = 4$.

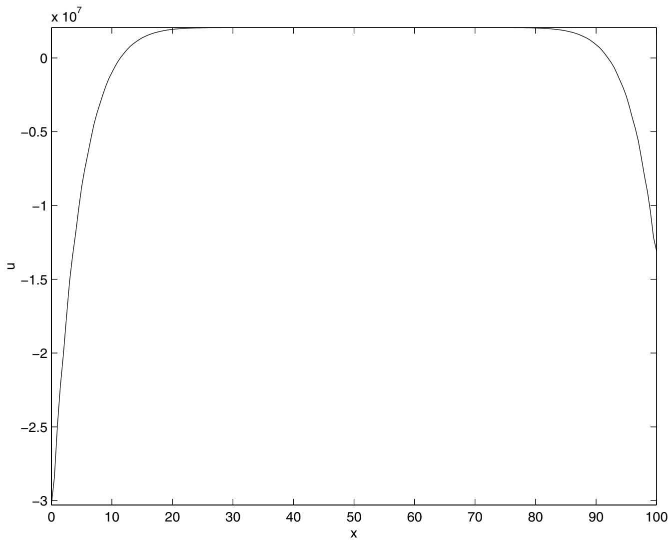

FIGURE 7. Film profile for $t = 10^6$ and $\alpha = 1.1$.

with $\Omega = (0, L)$ and periodic boundary conditions, where γ is some gas expansion parameter and $\mathscr{H}$ stands for the Hilbert transform. In particular, for moderately sized $L > 0$ the numerical solutions from Karlin & Makhviladze [9] also show roughening and coarsening, they are smooth except for kinks at local minimisers, and numerics indicate that they tend to a steady state. There is, in particular, a remarkable resemblance between our Figure 1 and Figure 4 in Karlin & Makhviladze [9]. Possible explanations of this connection between the fourth order equation (1.1) for amorphous molecular beam

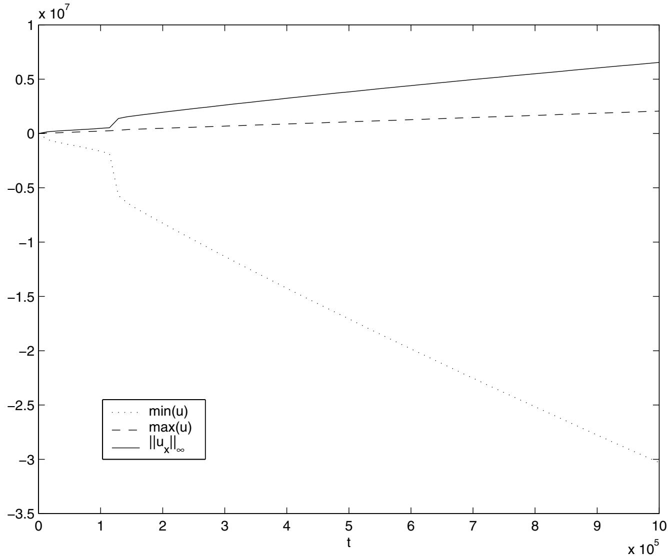

FIGURE 8. min(u), max(u), and $||u_x||_\infty$ for $t \in [0, 10^6]$ and $\alpha = 1.1$.

epitaxy and the second order Sivashinsky equation for the instability of plane flame fronts will be subject of future research.

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