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# The combined semi-classical and relaxation limit in a quantum hydrodynamic semiconductor model

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We discuss the combined semi-classical and relaxation limit of a one-dimensional isentropic quantum hydrodynamical model for semiconductors. The quantum hydrodynamic equations consist of the isentropic Euler equations for the particle density and current density, including the quantum potential and a momentum relaxation term. The momentum equation is highly nonlinear and contains a dispersive term with third-order derivatives. The equations are self-consistently coupled to the Poisson equation for the electrostatic potential. With the help of the Maxwell-type iteration, we prove that, as the relaxation time and Planck constant tend to zero, periodic initial-value problems of a scaled one-dimensional isentropic quantum hydrodynamic model have unique smooth solutions existing in the time interval where the classical drift-diffusion model has smooth solutions. Meanwhile, we justify a formal derivation of the classical drift-diffusion model from the quantum hydrodynamic model.

## 1. Introduction

Quantum hydrodynamic models are important and necessary in order to model and simulate electron transport, affected by extremely high electric fields, in ultra-small (sub-micron) semiconductor devices such as resonant tunnelling diodes, where quantum effects (like particle tunnelling through potential barriers and build-up in quantum wells [5]) take place and dominate the process. Such quantum mechanical phenomena cannot be simulated by classical hydrodynamic models. The advantage of macroscopic quantum hydrodynamic models relies on the fact that they not only are able to describe directly the dynamics of the physically observable and simulate the main characters of quantum effects, but are also numerically less expensive than microscopic models like the Schrödinger and Wigner–Boltzmann equations. The quantum hydrodynamic model is a moment model, derived from the quantum Wigner–Boltzmannn equation using a velocity-moment method or from the Schrödinger equation. (details are given in [5, 7, 9]). The isentropic quantum hydrodynamic system governs the evolution of the electron density, the electron current density and the electrostatic potential. In this paper, we study an isentropic quantum hydrodynamic model for semiconductors. More specifically, with appropriate scaling, our model can be represented as

$$\left.\begin{aligned} n_t + \frac{1}{\epsilon}\partial_x(nu) = 0, \\ \partial_t(nu) + \frac{1}{\epsilon}\partial_x(nu^2 + p(n)) = \frac{n\partial_x\phi}{\epsilon} + \frac{h^2}{4\epsilon}\partial_x(n\partial_{xx}(\log n)) - \frac{nu}{\epsilon^2}, \\ \lambda^2\partial_{xx}\phi = n - b(x). \end{aligned}\right\} \tag{1.1}$$

The variables are the electron density $n$, the mean velocity $u$ and the electrostatic potential $\phi$. $p(n)$ is the given strictly increasing function and denotes the pressure. The function $b(x)$ denotes the prescribed density of positively charged background ions (doping profile). The parameters are the (scaled) Planck constant $h$, the momentum relaxation time $\epsilon$ and the Debye length $\lambda$. In the real semiconductor device, the rescaled parameters $\lambda$, $\epsilon$, $h$ take the form

$$h^2 = \frac{\hbar^2}{2mk_{\mathrm{B}}T_0L^2}, \qquad \lambda^2 = \frac{\epsilon k_{\mathrm{B}}T_0}{q^2C_mL^2}, \qquad \epsilon^2 = \frac{k_{\mathrm{B}}T_0\tau_0^2}{mL^2},$$

which may be sufficiently small or $O(1)$, depending on the different real situations. Note that the scaling $t = \epsilon\tilde{t}$ converts (1.1) back into the original quantum isentropic model in [5, 9] with $\tilde{t}$ as its time variable. The scaled-time variable $t$ was first introduced in [21] in order to study the relation between the classical hydrodynamical and drift-diffusion models.

Since we are interested in the small relaxation time and small Planck constant analysis, we can take $\lambda = 1$, $h = \epsilon$. Moreover, we introduce the enthalpy $f(n)$ for $n > 0$, which satisfies

$$f'(n) = \frac{p'(n)}{n}.$$

Since $p(n)$ is strictly increasing, $f(n)$ is also strictly increasing. As in [11, 18], it is convenient to make use of the variable transformation $n = w^2$. With the simplifications above, we can rewrite the model (1.1) as

$$\left.\begin{aligned} w_t + \frac{1}{2\epsilon}wu_x + \frac{1}{\epsilon}uw_x = 0, \\ u_t + \frac{1}{\epsilon}uu_x + \frac{1}{\epsilon}f(w^2)_x = \frac{1}{\epsilon}\phi_x + \frac{\epsilon}{2}\left(\frac{w_{xx}}{w}\right)_x - \frac{u}{\epsilon^2}, \\ \phi_{xx} = w^2 - b(x). \end{aligned}\right\} \tag{1.2}$$

Recently, many efforts have been made to solve the quantum hydrodynamic (Euler–Poisson) system. The existence and uniqueness of thermal equilibrium and non-thermal equilibrium steady-state classical solutions for one-dimensional and high-dimensional quantum models have been studied in [1, 3, 4, 6, 23, 26]. For the time-dependent system, the local- and global-in-time existence of the classical solutions was obtained in the bounded domain [14] (subject to boundary conditions on the density and the electrostatic potential) and on the real line [11, 18].

Relaxation limits in the classical hydrodynamic equations have been performed extensively. In one spatial dimension, the relaxation limit problem for isentropic

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and non-isentropic hydrodynamic models has been investigated in the compactness frameworks for non-smooth solutions of conservation laws [8, 10, 12, 21]. In [16, 17], Lattanzio and Marcati considered the multidimensional isentropic unipolar hydrodynamic model and the corresponding bipolar model with $x$ in a bounded domain, assumed the existence of $L^\infty$-solutions in a $\tau$-independent time interval, and justified the relaxation limit in a compactness framework for non-smooth solutions. In [19, 25], Li and Yong, respectively, studied the diffusive relaxation of multidimensional isentropic and non-isentropic hydrodynamical models for semiconductors by the Maxwell iteration. Investigations of the relaxation limit and semi-classical and relaxation limits for the unipolar and bipolar quantum hydrodynamic models by compactness arguments were made in [15, 27].

Finally, the semi-classical limits from the stationary and non-stationary quantum hydrodynamical model have been partly discussed (see [1, 2, 6, 27]). To the best of our knowledge, no results for the combined semi-classical and relaxation limit of the time-dependent quantum hydrodynamic model (1.2) have been found. In this paper, we investigate the combined relaxation and semi-classical limit in (1.2). We expect the quantum hydrodynamic model and the classical drift-diffusion model to give similar results when $\epsilon$ is small, which can be seen formally as follows. Applying the Maxwell-type iteration to the momentum equations in (1.2) gives

$$\begin{aligned} u &= -\epsilon\partial_x f(w^2) + \epsilon\partial_x\phi - \epsilon u\partial_x u + \epsilon^3\left(\frac{w_{xx}}{w}\right)_x - \epsilon^2\partial_t u \\ &= -\epsilon\partial_x f(w^2) + \epsilon\partial_x\phi + O(\epsilon^2). \end{aligned}$$

Substituting the truncation $u = -\epsilon\partial_x f(w^2) + \epsilon\partial_x\phi$ into the mass equations in (1.2), we arrive at the quantum drift-diffusion model [13, 22]

$$\left.\begin{aligned} 2w\partial_t w - \partial_x(p(w^2)_x - w^2\phi_x) = 0, \\ \phi_{xx} = w^2 - b(x), \end{aligned}\right\} \tag{1.3}$$

which is a parabolic–elliptic-type system, provided that $p'(w^2) > 0$. We state our main results in the following.

THEOREM 1.1. *Suppose that $p \in C^4(0,\infty)$, $p'(w^2) > 0$, $b(x) \in H^4(\Omega)$ and that the classical drift-diffusion model (1.3) with initial data*

$$w(x,0) = w_0(x), \quad x \in (0,1](=: \Omega), \quad \int_0^1 (w_0^2(x) - b(x))\,\mathrm{d}x = 0$$

*has a solution $w \in C([0,T_*], H^6(\Omega)) \cap C^1([0,T_*], H^5(\Omega))$ with a positive lower bound and $\phi \in C([0,T_*], H^4(\Omega))$.*

*Then, for $\epsilon$ sufficiently small, the quantum isentropic model (1.2) with periodic initial data*

$$w(x,0) = w_0(x), \qquad u(x,0) = \epsilon\phi_x(x,0) - \frac{\epsilon p(w_0^2(x))}{w_0^2(x)} \tag{1.4}$$

*has a unique solution*

$$(w^\epsilon, u^\epsilon, \phi^\epsilon) \in C([0,T_*], H^4(\Omega)) \times C([0,T_*], H^3(\Omega)) \times C([0,T_*], H^4(\Omega)),$$

*and there exists a constant $K > 0$, independent of $\epsilon$ but dependent on $T_* < \infty$, such that*

$$\sup_{t\in[0,T_*]}(\|w^\epsilon - w_\epsilon\|_2 + \epsilon\|(w^\epsilon - w_\epsilon)_{xxx}\|_1 + \|u^\epsilon - u_\epsilon\|_3 + \|\phi^\epsilon - \phi_\epsilon\|_4) \leqslant K\epsilon^2, \quad (1.5)$$

*where*

$$w_\epsilon = w, \qquad u_\epsilon = \epsilon\partial_x\phi - \epsilon\partial_x f(w^2), \qquad \phi_\epsilon = \phi.$$

To prove our results, we shall adopt and modify the arguments in [24, 25]. However, we have to face up to two difficulties here. The first one is from the potential, which is a third-order dispersive term. Our strategy is to reformulate the law of conservation of mass in the quantum hydrodynamic equation as a fourth-order wave equation as in [11, 27]. The second difficulty is that we cannot directly make use of the previous convergence-stability lemma (which was first formulated in [24]), since it only fits the symmetrizable hyperbolic system with a relaxation term. Thus, we must establish the corresponding convergence-stability lemma for our quantum hydrodynamic model with the third-order dispersive term.

Remark 1.2. As far as we know, it is the first time that the convergence-stability lemma of the symmetrizable hyperbolic system has been extended to that of the non-hyperbolic case, and we apply it in order to investigate the combined semi-classical and relaxation limit for the quantum isentropic hydrodynamic model for semiconductors.

Remark 1.3. Our conclusion implies that if the classical drift-diffusion model (1.3) has a global smooth solution on $[0, \infty]$ with $w$ having a positive lower bound, then there exists $\epsilon_0 > 0$ such that the quantum isentropic hydrodynamical model (1.2) has a unique smooth solution up to time $\infty$, when $\epsilon < \epsilon_0$, and when $T < \infty$ (1.5) also holds. Moreover, employing similar arguments, we can investigate the combined semi-classical and relaxation limit for the bipolar quantum hydrodynamic model for semiconductors. Moreover, on the basis of the convergence-stability lemma, we can use the matched-expansion method to discuss the combined semi-classical and relaxation limit for the quantum hydrodynamic semiconductor models with more general initial data.

Remark 1.4. In [11, 27], the global existence of smooth (or strong) solutions is initially established for corresponding initial-value problems (IVPs), and then the relaxation limit and semi-classical limit are investigated by compactness arguments. However, we use proper scaling and assume the existence of the drift-diffusion equations; we then construct the solutions of the one-dimensional quantum hydrodynamic model with appropriate initial data. Moreover, we also obtain the convergence rate. Finally, due to the combined semi-classical and relaxation limit, the estimate (1.5) is different from that obtained in [20].

This paper is organized as follows. In § 2 we establish the convergence-stability lemma and construct the formal approximations. Section 3 is devoted to validating the formal approximation and the existence of the solution to $(w^\epsilon, u^\epsilon, \phi^\epsilon)$ in the time interval where $w$ is well defined.

## 2. The convergence-stability lemma and formal approximations

In this section, we establish the convergence-stability lemma and construct formal approximations. Initially, for

$$\left.\begin{aligned} w_t + \frac{1}{2\epsilon} w u_x + \frac{1}{\epsilon} u w_x &= 0, \\ u_t + \frac{1}{\epsilon} u u_x + \frac{1}{\epsilon} f(w^2)_x &= \frac{1}{\epsilon}\phi_x + \frac{\epsilon}{2}\left(\frac{w_{xx}}{w}\right)_x - \frac{u}{\epsilon^2}, \\ \phi_{xx} &= w^2 - b(x), \end{aligned}\right\} \tag{2.1}$$

we have the following lemma.

LEMMA 2.1 (Huang *et al.* [11]). *Assume $p(w^2) \in C^4(0,+\infty)$, $p'(w^2) > 0$, $b(x) \in H^4(\Omega)$, $b(x) > 0$ and*

$$w(x,0), u(x,0) \in H^4(\Omega) \times H^3(\Omega), \quad w(x,0) > 0, \quad \int_0^1 (w(x,0)^2 - b(x))\,\mathrm{d}x = 0.$$

*Then the unique classical solution $(w,u,\phi)$ of (2.1) with initial data $w(x,0)$, $u(x,0)$ exists for $t \in [0,T_*]$, satisfying $w > 0$ in $\Omega \times [0,T_*]$ and*

$$\begin{aligned} &w \in C^i([0,T_*]; H^{4-2i}(\Omega)), \quad i = 0,1,2, \\ &u \in C^j([0,T_*]; H^{3-2j}(\Omega)), \quad \phi \in C^j([0,T_*]; H^{4-2j}(\Omega)), \quad j = 0,1. \end{aligned}$$

Fix $\epsilon$ according to lemma 2.1. There is a time interval $[0,T]$ such that (1.2) with the initial data has a unique $H^4(\Omega) \times H^3(\Omega) \times H^4(\Omega)$ solution

$$(w^\epsilon, u^\epsilon, \phi^\epsilon) \in C([0,T], H^4(\Omega)) \times C([0,T], H^3(\Omega)) \times C([0,T], H^4(\Omega)).$$

Define

$$T_\epsilon = \sup\{T > 0 : w^\epsilon > 0,\ (w^\epsilon, u^\epsilon, \phi^\epsilon) \in H^4(\Omega) \times H^3(\Omega) \times H^4(\Omega)\}; \tag{2.2}$$

namely, $[0,T_\epsilon)$ is the maximal time interval of $H^4 \times H^3 \times H^4$-existence. Note that $T_\epsilon$ may tend to 0 as $\epsilon$ goes to a certain singular point, say 0.

In order to show that $\lim_{\epsilon\to 0} T_\epsilon > 0$, we need the convergence-stability lemma as in [24, 25]. First, we make the following assumption.

ASSUMPTION 2.2 (convergence assumption). There exist $T_* > 0$ and $(w_\epsilon, u_\epsilon, \phi_\epsilon) \in H^4(\Omega) \times H^3(\Omega) \times H^4(\Omega)$ for each $\epsilon$, satisfying $w_\epsilon(t,x) > 0$ such that, for $t \in [0, \min\{T_*, T_\epsilon\})$,

$$\sup_{x,t} |(w^\epsilon, u^\epsilon, \phi^\epsilon)(x,t) - (w_\epsilon, u_\epsilon, \phi_\epsilon)(x,t)| = o(1),$$

$$\sup_t (\|w^\epsilon(x,t) - w_\epsilon(x,t)\|_4 + \|u^\epsilon(x,t) - u_\epsilon(x,t)\|_3 + \|\phi^\epsilon(x,t) - \phi_\epsilon(x,t)\|_4) = O(1)$$

as $\epsilon$ tends to the singular point.

With such a convergence assumption, we are in a position to establish the convergence-stability lemma.

LEMMA 2.3. *Assume that* $(\bar{w},\bar{u})(x,\epsilon)\in H^4(\Omega)\times H^3(\Omega)$, $\bar{w}(x,\epsilon)>0$,

$$\int_0^1 (\bar{w}^2(x,\epsilon)-b(x))\,\mathrm{d}x = 0 \quad \textit{for all } (x,\epsilon),$$

*and that the convergence assumption holds. Let* $[0,T_\epsilon)$ *be the maximal time interval such that (1.2) with the initial data* $(\bar{w},\bar{u})(x,\epsilon)$ *has a unique solution*

$$(w^\epsilon,u^\epsilon,\phi^\epsilon)\in C([0,T_\epsilon],H^4(\Omega))\times C([0,T_\epsilon],H^3(\Omega))\times C([0,T_\epsilon],H^4(\Omega)).$$

*Then*

$$T_\epsilon > T_*$$

*for all* $\epsilon$ *in a neighbourhood of the singular point.*

*Proof.* By contradiction, there exists a sequence $\{\epsilon_k\}_{k\geqslant 1}$ such that $\lim_{k\to\infty}\epsilon_k=0$ and $T_{\epsilon_k}\leqslant T_*$. Owing to the convergence assumption, there exists a $k$ such that $w^{\epsilon_k}>0$. Next, from $(1.2)_1$ and

$$\int_0^1 (\bar{w}^2(x,\epsilon)-b(x))\,\mathrm{d}x = 0,$$

we have

$$\int_0^1 (w^{\epsilon_k 2}(x,t)-b(x))\,\mathrm{d}x = 0$$

for all $(x,t)\in\Omega\times[0,T_{\epsilon_k})$. On the other hand, we deduce from

$$\begin{aligned}\|w^{\epsilon_k}\|_4+\|u^{\epsilon_k}\|_3+\|\phi^{\epsilon_k}\|_4 \leqslant{}& \|w^{\epsilon_k}-w_{\epsilon_k}\|_4+\|u^{\epsilon_k}-u_{\epsilon_k}\|_3\\ &+\|\phi^{\epsilon_k}-\phi_{\epsilon_k}\|_4+\|w_{\epsilon_k}\|_4+\|u_{\epsilon_k}\|_3+\|\phi_{\epsilon_k}\|_4,\end{aligned}$$

and the convergence assumption that $\|w^{\epsilon_k}\|_4+\|u^{\epsilon_k}\|_3+\|\phi^{\epsilon_k}\|_4$ is bounded uniformly with respect to $t\in[0,T_{\epsilon_k})$. Now we apply lemma 2.1, beginning at a time $t<T_{\epsilon_k}$, to continue this solution beyond $T_{\epsilon_k}$. This contradicts the definition of $T_\epsilon$ in (2.2) and completes the proof. □

Due to lemma 2.3, our task is reduced to finding a $(w_\epsilon,u_\epsilon,\phi_\epsilon)(x,t)$ such that the convergence assumption holds. We propose a construction of the approximation $(w_\epsilon,u_\epsilon,\phi_\epsilon)$ in assumption 2.2 for the quantum isentropic hydrodynamic model (1.2). Let $w$ solve the IVP of the classical drift-diffusion model:

$$\left.\begin{aligned} 2w\partial_t w-\partial_x(p(w^2)_x-w^2\phi_x)=0,\\ \phi_{xx}=w^2-b(x),\\ w(x,0)=w_0(x),\qquad \int_0^1 (w_0^2(x)-b(x))\,\mathrm{d}x=0.\end{aligned}\right\} \tag{2.3}$$

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Inspired by the Maxwell iteration, we take $(w_\epsilon, u_\epsilon, \phi_\epsilon) = (w, \epsilon\phi_x - \epsilon f(w^2)_x, \phi)$ and define

$$\begin{aligned} R &= \frac{1}{\epsilon}\left(\partial_t u_\epsilon + \frac{u_\epsilon u_{\epsilon x}}{\epsilon} - \frac{\epsilon}{2}\left(\frac{w_{\epsilon xx}}{w_\epsilon}\right)_x\right) \\ &= \partial_t(\phi_x - f(w_\epsilon^2)_x) + (\phi_x - f(w_\epsilon^2)_x)(\phi_x - f(w_\epsilon^2)_x)_x - \frac{1}{2}\left(\frac{w_{\epsilon xx}}{w_\epsilon}\right)_x. \end{aligned} \tag{2.4}$$

Then we have

$$\left.\begin{aligned} w_{\epsilon t} + \frac{1}{2\epsilon} w_\epsilon u_{\epsilon x} + \frac{1}{\epsilon} u_\epsilon w_{\epsilon x} &= 0, \\ u_{\epsilon t} + \frac{1}{\epsilon} u_\epsilon u_{\epsilon x} + \frac{1}{\epsilon} f(w_\epsilon^2)_x &= \frac{1}{\epsilon}\phi_{\epsilon x} + \frac{\epsilon}{2}\left(\frac{w_{\epsilon xx}}{w_\epsilon}\right)_x - \frac{u_\epsilon}{\epsilon^2} + \epsilon R, \\ \phi_{\epsilon xx} &= w_\epsilon^2 - b(x). \end{aligned}\right\} \tag{2.5}$$

Regarding $(w_\epsilon, u_\epsilon, \phi_\epsilon)$, we have the following regularity result.

LEMMA 2.4. *Assume that $p(\cdot) \in C^4(0,\infty)$ for $w > 0$ and that $p'(w^2) > 0$. If $w \in C([0,T_*], H^4(\Omega)) \cap C^1([0,T_*], H^3(\Omega))$ has positive lower bound and, moreover, if $b(x) \in H^2(\Omega)$, then $u_\epsilon \in C([0,T_*], H^3(\Omega)) \cap C^1([0,T_*], H^2(\Omega))$ and $R \in C([0,T_*], H^1(\Omega))$.*

Lemma 2.4 can easily be proved using the well-known calculus inequalities in Sobolev spaces, and thus we omit the details here.

## 3. Proof of theorem 1.1

Having constructed the formal approximation $(w_\epsilon, u_\epsilon, \phi_\epsilon)(x,t)$ for the periodic IVP of the isentropic quantum hydrodynamical model (1.2), we prove here the validity of the approximation under some regularity assumptions on the given data and an existence result for (2.1) in this section. Due to lemma 2.3, it suffices to prove the error estimate in (1.5) for $t \in [0, T_*)$.

First, in order to avoid the dispersive third-order term, we reformulate $(1.2)_1$ of the quantum hydrodynamic equation as a fourth-order wave equation. To do this, by differentiating $(1.2)_1$ with respect to time $t$ and substituting this into $(1.2)_2$, we have

$$\begin{aligned} w_{tt} + \frac{1}{w} w_t^2 + \frac{1}{\epsilon^2} w_t - \frac{1}{2\epsilon^2 w}(p(w^2) + w^2u^2)_{xx} \\ + \frac{1}{2\epsilon^2 w}(w^2\phi_x)_x + \tfrac{1}{4} w_{xxxx} - \frac{1}{4w} w_{xx}^2 = 0. \end{aligned} \tag{3.1}$$

Similarly, from (2.5), we can derive the wave equation satisfied by $w_\epsilon$ as follows:

$$\begin{aligned} w_{\epsilon tt} + \frac{1}{w_\epsilon} w_{\epsilon t}^2 + \frac{1}{\epsilon^2} w_{\epsilon t} - \frac{1}{2\epsilon^2 w_\epsilon}(p(w_\epsilon^2) + w_\epsilon^2 u_\epsilon^2)_{xx} \\ + \frac{1}{2\epsilon^2 w_\epsilon}(w_\epsilon^2 \phi_{\epsilon x})_x + \frac{1}{4} w_{\epsilon xxxx} - \frac{1}{4w_\epsilon} w_{\epsilon xx}^2 = -\frac{1}{2} R_x. \end{aligned} \tag{3.2}$$

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Let us introduce the perturbed variable $(\psi,\eta,e)$, which is defined by

$$(\psi,\eta,e)^* = (w_\epsilon - w^\epsilon, u_\epsilon - u^\epsilon, \phi_\epsilon - \phi^\epsilon)^*.$$

Then, from $(1.2)_{2,3}$, $(2.5)_{2,3}$, (3.1) and (3.2), it follows that the error $(\psi,\eta,e)$ satisfies

$$\begin{aligned}\psi_{tt} &+ \frac{1}{\epsilon^2}\psi_t + \frac{1}{4}\psi_{xxxx} + \frac{1}{w_\epsilon}w_{\epsilon t}^2 - \frac{1}{w^\epsilon}w_t^{\epsilon 2}\\ &- \frac{1}{2\epsilon^2 w_\epsilon}(p(w_\epsilon^2) + w_\epsilon^2 u_\epsilon^2)_{xx} + \frac{1}{2\epsilon^2 w^\epsilon}(p(w^{\epsilon 2}) + w^{\epsilon 2}u^{\epsilon 2})_{xx}\\ &+ \frac{1}{2\epsilon^2 w_\epsilon}(w_\epsilon^2\phi_{\epsilon x})_x - \frac{1}{2\epsilon^2 w^\epsilon}(w^{\epsilon 2}\phi_x^\epsilon)_x - \frac{1}{4w_\epsilon}w_{\epsilon xx}^2 + \frac{1}{4w^\epsilon}w_{xx}^{\epsilon 2} = -\frac{1}{2}R_x, \end{aligned} \tag{3.3}$$

$$\eta_t + \frac{1}{\epsilon^2}\eta + \frac{1}{\epsilon}(f(w_\epsilon^2) + u_\epsilon u_{\epsilon x} - f(w^{\epsilon 2}) - u^\epsilon u_x^\epsilon)_x = \frac{1}{\epsilon}e_x + \frac{\epsilon}{2}\left(\frac{w_{\epsilon xx}}{w_\epsilon} - \frac{w_{xx}^\epsilon}{w^\epsilon}\right)_x + \epsilon R, \tag{3.4}$$

and

$$e_{xx} = (w_\epsilon + w^\epsilon)\psi. \tag{3.5}$$

Moreover, from $(1.2)_1$, $(2.5)_1$, we obtain

$$\psi_t + \frac{1}{2\epsilon}w^\epsilon\eta_x + \frac{1}{2\epsilon}\eta u_{\epsilon x} + \frac{1}{\epsilon}u^\epsilon\psi_x + \frac{1}{\epsilon}\eta w_{\epsilon x} = 0. \tag{3.6}$$

For the sake of clarity, we divide the necessary arguments into the following lemmas.

Lemma 3.1. *Set*

$$D = D(t) = \frac{\|\psi\|_2 + \|\eta\|_3 + \epsilon\|\partial_{xxx}\psi\|_1 + \epsilon\|\psi_t\|_2 + \|\phi\|_4}{\epsilon}.$$

*Then we have*

$$|u^\epsilon|, |u_x^\epsilon|, |u_{xx}^\epsilon| \leqslant C\epsilon + C\epsilon D, \qquad |w_x^\epsilon| \leqslant C + C\epsilon D,$$
$$|w_t^\epsilon|, |w_{tx}^\epsilon|, |w_{xx}^\epsilon|, |w_{xxx}^\epsilon| \leqslant C + CD, \qquad |\phi_x^\epsilon|, |\phi_{xx}^\epsilon|, |\phi_{xxx}^\epsilon| \leqslant C + C\epsilon D.$$

*Proof.* It is obvious from the Sobolev's inequality and lemma 2.4 that

$$|u^\epsilon| \leqslant |u^\epsilon - u_\epsilon| + |u_\epsilon| \leqslant C\epsilon + C\epsilon D.$$

□

We can prove other estimates similarly.

Lemma 3.2. *From (3.5), we have*

$$\|e\|^2 + \|e_x\|^2 + \|e_{xx}\|_2^2 \leqslant C\|\psi\|^2. \tag{3.7}$$

*Proof.* Since

$$e = \int_0^x e_x \,\mathrm{d}x,$$

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we have

$$|e|^2 \leqslant \int_0^x |e_x|^2 \,\mathrm{d}x \leqslant \int_0^1 |e_x|^2 \,\mathrm{d}x,$$

which implies that

$$\|e\|^2 \leqslant \|e_x\|^2.$$

Furthermore, from (3.5) we have

$$\|e_x\|^2 \leqslant C\|\psi\|^2$$

with the help of Poincaré's inequality. Finally, it follows from (3.5) and the standard theory of elliptic partial differential equations that

$$\|e_{xx}\|^2 \leqslant C\|\psi\|^2, \qquad \|e_{xx}\|_2^2 \leqslant C\|\psi\|_2^2.$$

This completes the proof of lemma 3.2. □

LEMMA 3.3. *Under the assumptions of theorem 1.1, it follows that, for* $0 \leqslant \alpha \leqslant 2$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (\partial_x^\alpha \eta)^2 \,\mathrm{d}x + \frac{15}{16\epsilon^2}\int (\partial_x^\alpha \eta)^2 \,\mathrm{d}x \leqslant \tfrac{1}{2}\|\partial_x^\alpha \psi_t\|^2 + C(\epsilon^4 + (1+D^8)(\|\psi\|_{\alpha+1}^2 + \|\eta\|_\alpha^2 + \|\partial_x^\alpha e_x\|^2 + \epsilon^2\|\partial_x^\alpha \psi_{xx}\|^2)). \tag{3.8}$$

*Proof.* We multiply (3.4) by $\eta$ and integrate the resultant equation over $\Omega$. Then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \eta^2 \,\mathrm{d}x + \frac{1}{\epsilon^2}\int \eta^2 \,\mathrm{d}x + \frac{1}{\epsilon}\int (f(w_\epsilon^2) - f(w^{\epsilon 2}) + u_\epsilon u_{\epsilon x} - u^\epsilon u_x^\epsilon)_x \eta \,\mathrm{d}x = \int \left(\frac{1}{\epsilon}e_x + \frac{\epsilon}{2}\left(\frac{w_{\epsilon xx}}{w_\epsilon} - \frac{w_{xx}^\epsilon}{w^\epsilon}\right)_x + \epsilon R\right)\eta \,\mathrm{d}x. \tag{3.9}$$

From the Cauchy–Schwarz and Young inequalities, we have

$$\frac{1}{\epsilon}\int e_x \eta \,\mathrm{d}x + \int \epsilon R \eta \,\mathrm{d}x \leqslant \frac{\|\eta\|^2}{16\epsilon^2} + C\|e_x\|^2 + C\epsilon^4,$$

and

$$\begin{aligned}\frac{1}{\epsilon}\int (u_\epsilon u_{\epsilon x} - u^\epsilon u_x^\epsilon)_x \eta \,\mathrm{d}x + \frac{\epsilon}{2}\int \left(\frac{w_{\epsilon xx}}{w_\epsilon} - \frac{w_{xx}^\epsilon}{w^\epsilon}\right)_x \eta \,\mathrm{d}x \\ = -\frac{1}{\epsilon}\int (u_\epsilon(u_\epsilon - u^\epsilon)_x - u_x^\epsilon(u_\epsilon - u^\epsilon))\eta_x \,\mathrm{d}x - \frac{\epsilon}{2}\int \left(\frac{w^\epsilon \psi_{xx} - \psi w_{\epsilon xx}}{w_\epsilon w^\epsilon}\right)\eta_x \,\mathrm{d}x \\ \leqslant \frac{\|\eta_x\|^2}{16\epsilon^2} + C(1+D)(\|\psi\|^2 + \epsilon^2\|\psi_{xx}\|^2 + \|\eta\|^2 + \|\eta_x\|^2),\end{aligned}$$

with the help of integration by parts. Moreover, noting that

$$f(w_\epsilon^2) - f(w^{\epsilon 2}) = (w_\epsilon^2 - w^{\epsilon 2})\int_0^1 f'(w_\epsilon^2 + \sigma(w_\epsilon^2 - w^{\epsilon 2}))\,\mathrm{d}\sigma,$$

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we have

$$\begin{aligned}\|f(w_\epsilon^2)-f(w^{\epsilon 2})\| &\leqslant C\|\psi\|\|\int_0^1 f'(w_\epsilon^2+\sigma(w_\epsilon^2-w^{\epsilon 2}))\,\mathrm{d}\sigma\|_4\\ &\leqslant C\|\psi\|\int_0^1 \|f'(w_\epsilon^2+\sigma(w_\epsilon^2-w^{\epsilon 2}))\|_4\,\mathrm{d}\sigma[0]\\ &\leqslant C\|\psi\|\int_0^1 (1+\|\sigma(w_\epsilon^2-w^{\epsilon 2})\|_4)\,\mathrm{d}\sigma \leqslant C(1+D^4)\|\psi\|.\end{aligned}$$

Furthermore, we can obtain

$$\frac{1}{\epsilon}\int (f(w_\epsilon^2)-f(w^{\epsilon 2}))_x\eta\,\mathrm{d}x \leqslant \frac{\|\eta_x\|^2}{16\epsilon^2}+C(1+D^8)\|\psi\|^2. \tag{3.10}$$

Inserting the above inequalities into (3.9) yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \eta^2\,\mathrm{d}x+\frac{15}{16\epsilon^2}\int \eta^2\,\mathrm{d}x\\ &\qquad\leqslant \frac{\|\eta_x\|^2}{8\epsilon^2}+C\epsilon^4+C\|e_x\|^2+C(1+D^8)(\|\psi\|^2+\epsilon^2\|\psi_{xx}\|^2+\|\eta\|_1^2).\end{aligned} \tag{3.11}$$

Similarly, taking $\partial_x^\alpha$, $\alpha \geqslant 1$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (\partial_x^\alpha\eta)^2\,\mathrm{d}x+\frac{15}{16\epsilon^2}\int (\partial_x^\alpha\eta)^2\,\mathrm{d}x\\ &\qquad\leqslant \frac{\|\partial_x^\alpha\eta_x\|^2}{8\epsilon^2}+C\epsilon^4+C\|\partial_x^\alpha e_x\|^2+C(1+D^8)(\|\psi\|_\alpha^2+\epsilon^2\|\partial_x^\alpha\psi_{xx}\|^2+\|\eta\|_{\alpha+1}^2).\end{aligned} \tag{3.12}$$

Moreover, from (3.6), we may deduce that

$$\|\eta_x\|^2 \leqslant 4\epsilon^2\|\psi_t\|+C\epsilon^2\|\eta\|^2+\epsilon^2(1+D^2)\|\psi_x\|^2+C\|\eta\|^2 \tag{3.13}$$

and

$$\|\partial_x^\alpha\eta_x\|^2 \leqslant 4\epsilon^2\|\partial_x^\alpha\psi_t\|+C\epsilon^2\|\eta\|_\alpha^2+\epsilon^2(1+D^2)\|\psi_x\|_\alpha^2+C\|\eta\|_\alpha^2. \tag{3.14}$$

Therefore, by combining (3.11) and (3.14), we can establish (3.8). This completes the proof. □

Lemma 3.4. *Under the assumptions of theorem 1.1, we have*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\frac{1}{2\epsilon^2}(\partial_x^\alpha\psi)^2+\partial_x^\alpha\psi\partial_x^\alpha\psi_t+(\partial_x^\alpha\psi_t)^2+\tfrac{1}{4}(\partial_x^\alpha\psi_{xx})^2\right)\mathrm{d}x+\frac{1}{4}\int(\partial_x^\alpha\psi_{xx})^2\,\mathrm{d}x\\ &\quad+\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{p'(w_\epsilon^2+\sigma(w_\epsilon^2-w^{\epsilon 2}))(w_\epsilon+w^\epsilon)}{2\epsilon^2 w_\epsilon}\partial_x^\alpha\psi_x^2\,\mathrm{d}x\\ &\quad+\left(\frac{27}{16\epsilon^2}-\frac{17}{16}\right)\int(\partial_x^\alpha\psi_t)^2\,\mathrm{d}x+\frac{1}{2\epsilon^2}\int\frac{p'(w_\epsilon^2+\sigma(w_\epsilon^2-w^{\epsilon 2}))(w_\epsilon+w^\epsilon)}{w_\epsilon}\partial_x^\alpha\psi_x^2\,\mathrm{d}x\\ &\qquad\leqslant C(1+D^3)(\|\partial_x^\alpha\psi_{xx}\|^2+\|\partial_x^\alpha\psi_t\|^2)+\frac{C(1+D^8)}{\epsilon^2}(\|\psi\|_{1+\alpha}^2+\|\eta\|_{1+\alpha}^2)+C\epsilon^2.\end{aligned} \tag{3.15}$$

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*Proof.* Multiplying (3.3) by $\psi$ and integrating the resultant equation over $\Omega$, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\psi\psi_t+\frac{1}{2\epsilon^2}\psi^2\right)\mathrm{d}x-\int\psi_t^2\,\mathrm{d}x+\frac{1}{4}\int\psi_{xx}^2\,\mathrm{d}x\\
&\quad+\int\left(\frac{1}{w_\epsilon}w_{\epsilon t}^2-\frac{1}{w^\epsilon}w_t^{\epsilon 2}-\frac{1}{2\epsilon^2 w_\epsilon}(p(w_\epsilon^2)+w_\epsilon^2u_\epsilon^2)_{xx}+\frac{1}{2\epsilon^2 w^\epsilon}(p(w^{\epsilon 2})+w^{\epsilon 2}u^{\epsilon 2})_{xx}\right.\\
&\qquad\left.+\frac{1}{2\epsilon^2 w_\epsilon}(w_\epsilon^2\phi_{\epsilon x})_x-\frac{1}{2\epsilon^2 w^\epsilon}(w^{\epsilon 2}\phi_x^\epsilon)_x-\frac{1}{4w_\epsilon}w_{\epsilon xx}^2+\frac{1}{4w^\epsilon}w_{xx}^{\epsilon 2}\right)\psi\,\mathrm{d}x\\
&\qquad\qquad=-\frac{1}{2}\int R_x\psi\,\mathrm{d}x.
\end{aligned}\tag{3.16}$$

In the following we deal with the terms in (3.16) one by one. First, using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&\int\left(\frac{1}{w_\epsilon}w_{\epsilon t}^2-\frac{1}{w^\epsilon}w_t^{\epsilon 2}\right)\psi\,\mathrm{d}x+\frac{1}{2}\int R_x\psi\,\mathrm{d}x\\
&\quad=\int\frac{w^\epsilon(w_\epsilon+w^\epsilon)_t\psi_t+(w_t^\epsilon)^2\psi}{w_\epsilon w^\epsilon}\psi\,\mathrm{d}x+\frac{1}{2}\int R_x\psi\,\mathrm{d}x\\
&\quad\leqslant\tfrac{1}{16}\|\psi_t\|^2+C(1+D^2)\|\psi\|^2+\frac{C}{\epsilon^2}\|\psi\|^2+C\epsilon^2,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2\epsilon^2}\int\left(\frac{(w_\epsilon^2\phi_{\epsilon x})_x}{w_\epsilon}-\frac{(w^{\epsilon 2}\phi_x^\epsilon)_x}{w^\epsilon}\right)\psi\,\mathrm{d}x-\frac{1}{4}\int\left(\frac{w_{\epsilon xx}^2}{w_\epsilon}-\frac{w_{xx}^{\epsilon 2}}{w^\epsilon}\right)\psi\,\mathrm{d}x\\
&\quad=\frac{1}{2\epsilon^2}\int\frac{w^\epsilon(2w_\epsilon w_{\epsilon x}\phi_{\epsilon x}+w_\epsilon^2\phi_{\epsilon xx})-w_\epsilon(2w^\epsilon w_x^\epsilon\phi_x^\epsilon+w^{\epsilon 2}\phi_{xx}^\epsilon)}{w_\epsilon w^\epsilon}\psi\,\mathrm{d}x\\
&\qquad-\frac{1}{4}\int\left(\frac{-w_{\epsilon xx}^2\psi+w_\epsilon(w_{\epsilon xx}+w_{xx}^\epsilon)\psi_{xx}}{w_\epsilon w^\epsilon}\right)\psi\,\mathrm{d}x\\
&\quad\leqslant\frac{C(1+D^2)}{\epsilon^2}(\|\psi\|^2+\|\psi_x\|^2+\|e_x\|^2+\|e_{xx}\|^2)+C(1+D^2)(\|\psi\|^2+\|\psi_{xx}\|^2).
\end{aligned}$$

Similarly to (3.10), we have

$$\begin{aligned}
&-\frac{1}{2\epsilon^2}\int\left(\frac{p(w_\epsilon^2)_{xx}}{w_\epsilon}-\frac{p(w^{\epsilon 2})_{xx}}{w^\epsilon}\right)\psi\,\mathrm{d}x\\
&\quad=-\frac{1}{2\epsilon^2}\int\left(\frac{1}{w^\epsilon}\left(p(w_\epsilon^2)-p(w^{\epsilon 2})\right)_{xx}+p(w_\epsilon^2)_{xx}\left(\frac{1}{w_\epsilon}-\frac{1}{w^\epsilon}\right)\right)\psi\,\mathrm{d}x\\
&\quad=\frac{1}{2\epsilon^2}\int\left(\frac{\psi}{w^\epsilon}\right)_x(p(w_\epsilon^2)-p(w^{\epsilon 2}))_x\,\mathrm{d}x+\frac{1}{2\epsilon^2}\int\frac{p(w_\epsilon^2)_{xx}}{w_\epsilon w^\epsilon}\psi^2\,\mathrm{d}x\\
&\quad\leqslant\int\frac{p'(w_\epsilon^2+\sigma(w_\epsilon^2-w^{\epsilon 2}))(w_\epsilon+w^\epsilon)}{2\epsilon^2 w_\epsilon}\psi_x^2\,\mathrm{d}x+\frac{C(1+D^2)}{\epsilon^2}(\|\psi\|^2+\|\psi_x\|^2).
\end{aligned}$$

Moreover, with the aid of integration by parts, we have

$$
\begin{aligned}
-\frac{1}{2\epsilon^2}\int\bigg(\frac{(w_\epsilon^2u_\epsilon^2)_{xx}}{w_\epsilon}-\frac{(w^{\epsilon2}u^{\epsilon2})_{xx}}{w^\epsilon}\bigg)\psi\,\mathrm{d}x&\\
=-\frac{1}{2\epsilon^2}\int\bigg(&\frac{(w_\epsilon^2)_{xx}u_\epsilon^2+2(w_\epsilon^2)_x(u_\epsilon^2)_x+w_\epsilon^2(u_\epsilon^2)_{xx}}{w_\epsilon}\\
&-\frac{(w^{\epsilon2})_{xx}u^{\epsilon2}+2(w^{\epsilon2})_x(u^{\epsilon2})_x+w^{\epsilon2}(u^{\epsilon2})_{xx}}{w^\epsilon}\bigg)\psi\,\mathrm{d}x\\
\leqslant\frac{C(1+D^3)}{\epsilon^2}&(\|\psi\|^2+\|\psi_x\|^2+\epsilon^2\|\psi_{xx}\|^2+\|\eta\|^2+\|\eta_x\|^2).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int\bigg(\psi\psi_t+\frac{1}{2\epsilon^2}\psi^2\bigg)\,\mathrm{d}x-\frac{17}{16}\int\psi_t^2\,\mathrm{d}x+\frac{1}{4}\int\psi_{xx}^2\,\mathrm{d}x&\\
+\int\frac{p'(w_\epsilon^2+\sigma(w_\epsilon^2-w^{\epsilon2}))(w_\epsilon+w^\epsilon)}{2\epsilon^2w_\epsilon}\psi_x^2\,\mathrm{d}x&\\
\leqslant\frac{C(1+D^8)}{\epsilon^2}(\|\psi\|_1^2+\|\eta\|_1^2)+C(1+D^3)\|\psi_{xx}\|^2+C\epsilon^2.&
\end{aligned}\tag{3.17}
$$

Next, multiplying (3.3) by $2\psi_t$ and integrating the resultant equation over $\Omega$, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int(\psi_t^2+\tfrac{1}{4}\psi_{xx}^2)\,\mathrm{d}x+\frac{2}{\epsilon^2}\int\psi_t^2\,\mathrm{d}x&\\
+2\int\bigg(\frac{1}{w_\epsilon}w_{\epsilon t}^2-\frac{1}{w^\epsilon}w_t^{\epsilon2}-\frac{1}{2\epsilon^2w_\epsilon}(p(w_\epsilon^2)+w_\epsilon^2u_\epsilon^2)_{xx}+\frac{1}{2\epsilon^2w^\epsilon}(p(w^{\epsilon2})+w^{\epsilon2}u^{\epsilon2})_{xx}&\\
+\frac{1}{2\epsilon^2w_\epsilon}(w_\epsilon^2\phi_{\epsilon x})_x-\frac{1}{2\epsilon^2w^\epsilon}(w^{\epsilon2}\phi_x^\epsilon)_x-\frac{1}{4w_\epsilon}w_{\epsilon xx}^2+\frac{1}{4w^\epsilon}w_{xx}^{\epsilon2}\bigg)\psi_t\,\mathrm{d}x&\\
=-\int R_x\psi_t\,\mathrm{d}x.&
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\int\bigg(2\bigg(\frac{1}{w_\epsilon}w_{\epsilon t}^2-\frac{1}{w^\epsilon}w_t^{\epsilon2}\bigg)+R_x\bigg)\psi_t\,\mathrm{d}x&\\
=\int\bigg(\frac{2w^\epsilon(w_\epsilon+w^\epsilon)_t\psi_t-2(w_t^\epsilon)^2\psi}{w_\epsilon w^\epsilon}+R_x\bigg)\psi_t\,\mathrm{d}x&\\
\leqslant C(1+D^2)\|(\psi,\psi_t)\|^2+\frac{1}{16\epsilon^2}\|\psi_t\|^2+C\epsilon^2&
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{\epsilon^2}\int\bigg(\frac{(w_\epsilon^2\phi_{\epsilon x})_x}{w_\epsilon}-\frac{(w^{\epsilon2}\phi_x^\epsilon)_x}{w^\epsilon}\bigg)\psi_t\,\mathrm{d}x-\frac{1}{2}\int\bigg(\frac{w_{\epsilon xx}^2}{w_\epsilon}-\frac{w_{xx}^{\epsilon2}}{w^\epsilon}\bigg)\psi_t\,\mathrm{d}x&\\
=\frac{1}{\epsilon^2}\int\bigg(\frac{(w_\epsilon^2)_x\phi_{\epsilon x}+w_\epsilon^2\phi_{\epsilon xx}}{w_\epsilon}-\frac{(w^{\epsilon2})_x\phi_x^\epsilon+w^{\epsilon2}\phi_{xx}^\epsilon}{w^\epsilon}\bigg)\psi_t\,\mathrm{d}x&\\
-\frac{1}{2}\int\frac{-w_{\epsilon xx}^2\psi+w_\epsilon(w_{\epsilon xx}+w_{xx}^\epsilon)\psi_{xx}}{w_\epsilon w^\epsilon}\psi_t\,\mathrm{d}x&
\end{aligned}
$$

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$$\leqslant \frac{C(1+D^2)}{\epsilon^2}(\|\psi_x\|^2+\|e_x\|^2+\|e_{xx}\|^2)+\frac{1}{16\epsilon^2}\|\psi_t\|^2 + C(1+D^2)(\|\psi\|^2+\|\psi_t\|^2+\|\psi_{xx}\|^2).$$

Moreover, with the aid of integration by parts, we have

$$\begin{aligned}
&-\frac{1}{\epsilon^2}\int\left(\frac{p(w_\epsilon^2)_{xx}}{w_\epsilon}-\frac{p(w^{\epsilon 2})_{xx}}{w^\epsilon}\right)\psi_t\,\mathrm{d}x\\
&\quad=\frac{1}{\epsilon^2}\int\left(\left(\frac{1}{w_\epsilon}\right)_x(p(w_\epsilon^2)-p(w^\epsilon 2))_x\psi_t+\frac{1}{w_\epsilon}p'(w_\epsilon^2+\sigma(w_\epsilon^2-w^{\epsilon 2})\right)\\
&\qquad\times(w_\epsilon+w^\epsilon)(\psi_x^2)_t)\,\mathrm{d}x\\
&\quad\geqslant\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{p'(w_\epsilon^2+\sigma(w_\epsilon^2-w^{\epsilon 2}))(w_\epsilon+w^\epsilon)}{2\epsilon^2 w_\epsilon}\psi_x^2\,\mathrm{d}x-\frac{C(1+D^8)}{\epsilon^2}\|\psi\|_1^2-\frac{1}{16\epsilon^2}\|\psi_t\|^2
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{1}{\epsilon^2}\int\left(\frac{(w_\epsilon^2u_\epsilon^2)_{xx}}{w_\epsilon}-\frac{(w^{\epsilon 2}u^{\epsilon 2})_{xx}}{w^\epsilon}\right)\psi_t\,\mathrm{d}x\\
&\quad=-\frac{1}{\epsilon^2}\int\left(\frac{(w_\epsilon^2)_{xx}u_\epsilon^2+2(w_\epsilon^2)_x(u_\epsilon^2)_x+w_\epsilon^2(u_\epsilon^2)_{xx}}{w_\epsilon}\right.\\
&\qquad\qquad\left.-\frac{(w^{\epsilon 2})_{xx}u^{\epsilon 2}+2(w^{\epsilon 2})_x(u^{\epsilon 2})_x+w^{\epsilon 2}(u^{\epsilon 2})_{xx}}{w^\epsilon}\right)\psi_t\,\mathrm{d}x\\
&\quad\leqslant-\frac{2}{\epsilon^2}\int\frac{w_\epsilon^2w^\epsilon u_\epsilon u_{\epsilon xx}-w^{\epsilon 2}w_\epsilon u^\epsilon u^\epsilon_{xx}}{w_\epsilon w^\epsilon}\psi_t\,\mathrm{d}x+\frac{1}{16\epsilon^2}\|\psi_t\|^2\\
&\qquad+\frac{C(1+D^3)}{\epsilon^2}(\|\psi\|^2+\|\psi_x\|^2+\epsilon^2\|\psi_{xx}\|^2+\|\eta\|^2+\|\eta_x\|^2)\\
&\quad\leqslant-\frac{2}{\epsilon^2}\int\frac{-u_\epsilon u_{\epsilon xx}w_\epsilon^2\psi+w_\epsilon^2w^\epsilon u_{\epsilon xx}\eta+w_\epsilon u_{\epsilon xx}u^\epsilon(w_\epsilon^2-w^{\epsilon 2})+u^\epsilon w_\epsilon w^{\epsilon 2}\eta_{xx}}{w_\epsilon w^\epsilon}\psi_t\,\mathrm{d}x\\
&\qquad+\frac{1}{16\epsilon^2}\|\psi_t\|^2+\frac{C(1+D^3)}{\epsilon^2}(\|\psi\|^2+\|\psi_x\|^2+\epsilon^2\|\psi_{xx}\|^2+\|\eta\|^2+\|\eta_x\|^2)\\
&\quad\leqslant\frac{4}{\epsilon}\int u^\epsilon w^\epsilon\left(\frac{1}{w^\epsilon}\left(\psi_t+\frac{1}{2\epsilon}\eta u_{\epsilon x}+\frac{1}{\epsilon}u^\epsilon\psi_x+\frac{1}{\epsilon}\eta w_{\epsilon x}\right)_x\right)\psi_t\,\mathrm{d}x\\
&\qquad+\frac{1}{16\epsilon^2}\|\psi_t\|^2+\frac{C(1+D^3)}{\epsilon^2}(\|\psi\|^2+\|\psi_x\|^2+\epsilon^2\|\psi_{xx}\|^2+\|\eta\|^2+\|\eta_x\|^2)\\
&\quad\leqslant C(1+D^3)\|\psi_t\|^2+\frac{1}{8\epsilon^2}\|\psi_t\|^2+\frac{C(1+D^3)}{\epsilon^2}(\|\psi\|_1^2+\epsilon^2\|\psi_{xx}\|^2+\|\eta\|_1^2).
\end{aligned}$$

Furthermore, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\psi_t^2+\frac{p'(w_\epsilon^2+\sigma(w_\epsilon^2-w^{\epsilon 2}))(w_\epsilon+w^\epsilon)}{2\epsilon^2w_\epsilon}\psi_x^2+\tfrac{1}{4}\psi_{xx}^2\right)\mathrm{d}x+\frac{27}{16\epsilon^2}\int\psi_t^2\,\mathrm{d}x \leqslant\frac{C(1+D^8)}{\epsilon^2}(\|\psi\|_1^2+\|\eta\|_1^2+\epsilon^2\|\psi_{xx}\|^2)+C\epsilon^2.$$

Therefore, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\frac{1}{2\epsilon^2}\psi^2+\psi\psi_t+\psi_t^2+\frac{p'(w_\epsilon^2+\sigma(w_\epsilon^2-w^{\epsilon 2}))(w_\epsilon+w^\epsilon)}{2\epsilon^2 w_\epsilon}\psi_x^2+\tfrac{1}{4}\psi_{xx}^2\right)\mathrm{d}x$$
$$+\left(\frac{27}{16\epsilon^2}-\frac{17}{16}\right)\int\psi_t^2\,\mathrm{d}x+\frac{1}{4}\int\psi_{xx}^2\,\mathrm{d}x+\int\frac{p'(w_\epsilon^2+\sigma(w_\epsilon^2-w^{\epsilon 2}))(w_\epsilon+w^\epsilon)}{2\epsilon^2 w_\epsilon}\psi_x^2\,\mathrm{d}x$$
$$\leqslant\frac{C(1+D^8)}{\epsilon^2}(\|\psi\|_1^2+\|\eta\|_1^2+\epsilon^2\|\psi_t\|^2+\epsilon^2\|\psi_{xx}\|^2)+C\epsilon^2.$$

In order to obtain a higher-order estimate, we differentiate with respect to $x$; therefore, by repeating the previous steps, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\frac{1}{2\epsilon^2}(\partial_x^\alpha\psi)^2+\partial_x^\alpha\psi\partial_x^\alpha\psi_t+(\partial_x^\alpha\psi_t)^2+\tfrac{1}{4}(\partial_x^\alpha\psi_{xx})^2\right)\mathrm{d}x+\frac{1}{4}\int(\partial_x^\alpha\psi_{xx})^2\,\mathrm{d}x$$
$$+\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{p'(w_\epsilon^2+\sigma(w_\epsilon^2-w^{\epsilon 2}))(w_\epsilon+w^\epsilon)}{2\epsilon^2 w_\epsilon}\partial_x^\alpha\psi_x^2\,\mathrm{d}x$$
$$+\left(\frac{27}{16\epsilon^2}-\frac{17}{16}\right)\int(\partial_x^\alpha\psi_t)^2\,\mathrm{d}x+\int\frac{p'(w_\epsilon^2+\sigma(w_\epsilon^2-w^{\epsilon 2}))(w_\epsilon+w^\epsilon)}{2\epsilon^2 w_\epsilon}\partial_x^\alpha\psi_x^2\,\mathrm{d}x$$
$$\leqslant\frac{C(1+\epsilon^2 D^8)}{\epsilon^2}(\|(\psi,\eta)\|_{1+\alpha}^2+\epsilon^2\|\partial_x^\alpha\psi_{xx}\|^2+\epsilon^2\|\partial_x^\alpha\psi_t\|^2)+C\epsilon^2.$$

This completes the proof. □

Combining lemmas 3.2–3.4 yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(\partial_x^\alpha\psi,\partial_x^\alpha\psi_x,\epsilon\partial_x^\alpha\psi_t,\partial_x^\alpha\eta,\epsilon\partial_x^\alpha\psi_{xx})\|^2+C\|(\epsilon\partial_x^\alpha\psi_{xx},\partial_x^\alpha\psi_t,\partial_x^\alpha\psi_x,\partial_x^\alpha\eta)\|^2$$
$$\leqslant C\epsilon^4+C(1+D^8)(\|(\psi,\eta)\|_{\alpha+1}^2+\epsilon^2\|\partial_x^\alpha\psi_t\|^2+\epsilon^2\|\partial_x^\alpha\psi_{xx}\|^2). \tag{3.18}$$

Then we integrate (3.18) from 0 to $T$ with $[0,T]\subset[0,\min\{T_\epsilon,T_*\})$ to obtain

$$\|(\partial_x^\alpha\psi,\partial_x^\alpha\psi_x,\partial_x^\alpha\eta,\epsilon\partial_x^\alpha\psi_{xx},\epsilon\partial_x^\alpha\psi_t)\|^2+C\int_0^T\|(\epsilon\partial_x^\alpha\psi_{xx},\partial_x^\alpha\psi_x,\partial_x^\alpha\psi_t,\partial_x^\alpha\eta)\|^2\,\mathrm{d}t$$
$$\leqslant CT\epsilon^4+C\int_0^T(1+D^8)(\|(\psi,\eta)\|_{\alpha+1}^2+\epsilon^2\|\partial_x^\alpha\psi_{xx}\|^2+\epsilon^2\|\partial_x^\alpha\psi_t\|^2)\,\mathrm{d}t.$$

Here we have used the fact the initial data are in equilibrium. Summing up the last inequality over all $\alpha$ satisfying $\alpha\leqslant 2$ and noting that (3.13), we get

$$\|(\psi,\epsilon\psi_t)\|_2^2+\|\eta\|_3^2+\epsilon^2\|\psi_{xxx}\|_1^2+C\int_0^T(\|\psi\|_2^2+\epsilon^2\|\psi_{xxx}\|_1^2+\epsilon^2\|\psi_t\|_2^2+\|\eta\|_3^2)\,\mathrm{d}t$$
$$\leqslant CT_*\epsilon^4+C\int_0^T(1+D^8)(\|\psi\|_2^2+\|\eta\|_3^2+\epsilon^2\|\psi_t\|_2^2+\epsilon^2\|\psi_{xxx}\|_1^2)\,\mathrm{d}t. \tag{3.19}$$

We apply Gronwall's lemma to (3.19) to get

$$\|\psi\|_2^2+\|\eta\|_3^2+\epsilon^2\|\psi_t\|_2^2+\epsilon^2\|\psi_{xxx}\|_1^2\leqslant CT_*\epsilon^4\exp\left[C\int_0^T(1+D^8)\,\mathrm{d}t\right]. \tag{3.20}$$

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Since $\|\psi\|_2^2 + \|\eta\|_3^2 + \epsilon^2\|\psi_t\|_2^2 + \epsilon^2\|\psi_{xxx}\|_1^2 = \epsilon^2 D^2$, it follows from (3.20) that

$$D(T)^2 \leqslant CT_*\epsilon^4 \exp\left[C\int_0^T (1 + D^8)\,\mathrm{d}t\right] \equiv \Phi(T). \tag{3.21}$$

Thus, it holds that

$$\Phi'(t) = C(1 + D^8)\Phi(t) \leqslant C\Phi(t) + C\Phi^5(t). \tag{3.22}$$

Applying the nonlinear Gronwall-type inequality in [24, 25] to the last inequality yields

$$\Phi(t) \leqslant \mathrm{e}^{CT_*} \tag{3.23}$$

for $t \in [0, \min\{T_*, T_\epsilon\})$ if we choose $\epsilon$ so small that

$$\Phi(0) = CT_*\epsilon^2 \leqslant \mathrm{e}^{-CT_*}.$$

Due to (3.21), there exists a constant $c$, independent of $\epsilon$, such that

$$D(T) \leqslant c \tag{3.24}$$

for $T \in [0, \min\{T_*, T_\epsilon\})$. Finally, the theorem is concluded from (3.20) and (3.24). This completes the proof.

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