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NEW INSIGHTS FROM THE CANONICAL RAMSEY–CASS–KOOPMANS GROWTH MODEL

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The present article presents novel results on the Ramsey–Cass–Koopmans growth model. It is shown that the shadow price of capital goes to infinity as the capital stock goes to zero even if all functions are bounded with finite derivatives and that imposing the Inada condition of infinite derivative of the per capita production function at zero stock is irrelevant. It is also shown that unless marginal utility at zero consumption is infinity, there will be a non-empty interval where the Keynes–Ramsey rule does not hold. The paper also shows that the stable saddle path in a phase diagram with the state variable and the shadow price has an unrecognized economic interpretation that enables us to illustrate the value function as the integral of the stable saddle path.

Keywords: Economic Growth, Ramsey–Cass–Koopmans Model, Optimal Control, Stable Saddle Path.

1. INTRODUCTION

The Ramsey–Cass–Koopmans (RCK) growth model is one of the corner stones of economic growth theory and as such studied by the majority of economics graduate students.[1] It still forms the basis of much theoretical and empirical work (Becker and Mitra (2012), Guo and Jiang (2017), Biljanovska (2019)). Given its ubiquity in text books and journal articles, one could be forgiven for thinking that economists have gleaned all the insights that may be acquired from the standard RCK growth model. However, one issue is usually overlooked. A number of textbooks illustrate the phase diagram for the standard RCK model and they all fail to draw complete stable saddle paths or even discuss the problem of the behaviour of the stable saddle path close to the axes[2] (Romer (2006, p. 60), Acemoglu (2009, p. 303), Bénassy (2011, p. 149)).[3,4] Blanchard and Fischer (1989) also draw a phase diagram where the stable saddle path is left dangling in the interior of the phase diagram, but they do discuss the issue and claim that where the stable saddle path intersects the k-axis depends on the elasticity of substitution. Below it is

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shown that this is only correct as far as the elasticity of substitution depends on marginal utility.

One may think that this is a trifling issue, but in examining the behaviour of the stable saddle path a number of other results become apparent. One important insight that has been overlooked in the literature is that if the solution to an optimal control problem is illustrated in a phase diagram with the state variable along the x-axis and the shadow price along the y-axis, the stable saddle path has an interesting economic interpretation. One may think of the stable saddle path as a function that takes the state variable as an argument and the shadow price along the optimal path as the output. It follows that the stable saddle path is in fact the derivative of the value function. It follows further that in the RCK model if zero capital stock implies that if the value function is a finite number, typically zero, one can illustrate the value function as the integral under the stable saddle path.[5] The paper also proves the following results: (1) The shadow price of capital will always go to infinity as the capital stock goes to zero even if instantaneous marginal utility and/or productivity have finite derivatives at the origin and the production and utility functions are bounded. (2) Imposing the familiar Inada condition that the derivative of the per capita production function at zero stock is infinity is in fact an irrelevant assumption. As long as an interior steady state exists, this particular condition can be integrated out of the optimal solution. (3) Marginal instantaneous utility equal to infinity at zero consumption is a necessary and sufficient condition for consumption to be positive for all positive capital levels and thus conversely that marginal instantaneous utility less than infinity implies that for sufficiently low levels of the capital stock, consumption per capita is zero.

These results will hopefully make the exposition of the RCK model in textbooks more complete and benefit researchers who are working on extending the RCK model in their research.

2. THE MODEL

Assume that instantaneous utility depends on consumption, c, and is given by $U(c)$, where $0 < U'(c) \leq \infty$ and $U''(c) < 0$. The cases $U'(0) < \infty$ and $U'(0) = \infty$ will be discussed separately. Assume further that production per worker is given by $f(k)$, where k is the capital intensity and $0 \leq f'(k) \leq \infty$ and $f''(k) < 0$. We shall assume that $f(k) - (n + \delta)k$ is bounded and that $f(0) = 0$. Note that Inada conditions for $f(0)$ are not explicitly imposed. It is well known that one can derive the following differential equation for k:

$$\dot{k} = f(k) - (n + \delta)k - c, \quad k(0) = k_0 > 0 \tag{1}$$

Here, n is the labour force growth rate, and δ is the rate of capital depreciation. (B&F set δ to zero. This does not matter conceptually for the arguments below, and δ is included for generality.) It is assumed $f(k) - (n + \delta)k$ has the property that

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if k is large enough we have that $\dot{k} < 0$ for all positive values of c. The discount rate is assumed to be fixed and given by ρ. The problem to be solved is:[6]

$$V(k_0) = \max_{c(t)\geq 0 \forall t} \int_0^\infty U(c) e^{-\rho t} dt \qquad (2)$$

subject to (1). $V(\cdot)$ is of course the value function. The maximum principle yields the following conditions for the optimality of a triple (c, k, μ):

$$U'(c) - \mu \leq 0 \; \forall t \quad (= 0 \text{ if } c > 0) \qquad (3)$$

$$\dot{\mu} = \rho\mu - \left[f'(k) - (n + \delta)\right] \mu \qquad (4)$$

From (3) it is immediately clear that if c goes to zero and $U'(0) = \infty$, then μ must go to infinity as well. However, if $U'(0) < \infty$, then it is not possible to see from (3) whether μ goes to infinity or not as c goes to zero. Equation (3) may be rewritten as:

$$\mu(t) \geq U'(0) \Leftrightarrow c(t) = 0, \quad \mu(t) < U'(0) \Rightarrow U'\left[c(t)\right] = \mu(t) \qquad (5)$$

Note that the last inequality and the implication is a statement about the relationship between $U'(0)$ and $\mu(t)$. If $\mu(t)$ is lower than $U'(0)$, the inequality in (3) holds with equality and has an interior solution with strictly positive $c(t)$. As $U'(\cdot)$ is strictly decreasing, it has an inverse so we may write (5) as:

$$c = \max\left[U'^{-1}(\mu), 0\right] \qquad (6)$$

Additionally, we must impose transversality conditions and k must be determined by (1). The optimality conditions can be used to construct phase diagrams. These are typically drawn in (k, c) coordinates, but the phase diagram drawn here will use (k, μ) coordinates as this helps illustrate results. By combining the condition (6) in with (1), we get the following expression for the isocline where $\dot{k} = 0$:

$$\mu = U'\left[f(k) - (n + \delta)k\right] \qquad (7)$$

In the illustrations below, it is assumed there exists a $\bar{k}$ such that $f(\bar{k}) - (n + \delta)\bar{k} = 0$. This does not matter for the results but helps in drawing the phase diagram as it then holds that:

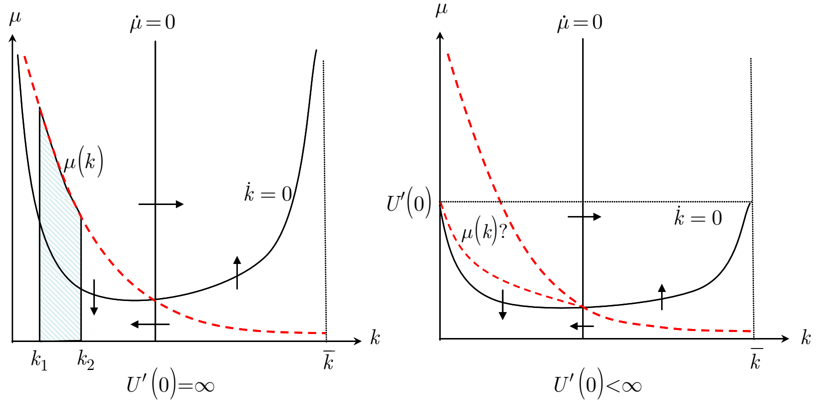
$$U'\left[f(0) - (n + \delta) \times 0\right] = U'\left[f(\bar{k}) - (n + \delta)\bar{k}\right] = U'(0) \qquad (8)$$

Equation (8) implies that when constructing phase diagrams we only need to draw it for $0 \leq k \leq \bar{k}$. The isocline for $\dot{\mu} = 0$ is given by:

$$\mu = 0 \text{ and } k = f'^{-1}(\rho + n + \delta) \qquad (9)$$

Here, $f'^{-1}(\cdot)$ is the inverse function of $f'(\cdot)$. Note that we must assume that there actually exist a value k^*, such that $f'(k^*) = \rho + n + \delta$. From (6) and (7), it follows that μ^* is determined by:

$$\mu^* = U'\left[f(k^*) - (n + \delta)k^*\right] \qquad (10)$$



Data Source: The diagram on the left is straight forward and consistent with, for example, a constant elasticity of substitution utility function. As $U'(0)=\infty$, the isocline for $\dot{k}=0$ goes to infinity for $k=0$ and $k=\bar{k}$. The stable saddle path, illustrated by the dashed line, must lie above the isocline for $\dot{k}=0$ for all k lower than the steady state level and the saddle path therefore goes to infinity as k goes to zero. Note that the shape of the stable saddle path confirms that $\mu(k)$ is a decreasing function of k so the value function is concave. As $\mu(k)$ is the derivative of the value function, $V(k)$, we can illustrate the value function as the sum of $V(0)$ and the integral of $\mu(.)$ over $[0, k]$. This is only possible if $V(0)\neq-\infty$. But even if $V(0)=-\infty$, we can illustrate the difference $V(k_2)-V(k_1)$ as the integral of $\mu(k)$ over $[k_1, k_2]$ as illustrated by the shaded area. When $U'(0)<\infty$ as in the phase diagram on the right, things are less clear. There are two possible stable saddle paths that satisfy directional derivatives in the interior of the (k, μ)-space. One lies everywhere below the line $\mu=U'(0)$ except at $k=0$ and one where the saddle path lies above the line $\mu=U'(0)$ for k sufficiently small. Proposition 3 below proves that also in this case will the stable saddle path in fact go to infinity as k goes to zero.

FIGURE 1. Phase diagrams for the RCK model.

We can draw these isoclines in a phase diagram in the standard manner. Depending on whether $U'(0)=\infty$ or $U'(0)<\infty$, this diagram takes one of two possible forms.

The two cases are illustrated in Figure 1 and discussed in the caption. We also need a procedure to compute the stable saddle path. The stable saddle path is a function that gives the shadow price as a function of k along the optimal path and is here denoted $\mu(k)$ as opposed to the function $\mu(t\,|\,k_0)$ usually analysed in optimal control models which gives the optimal shadow price as a function of time. $\mu(k)$ can be calculated by eliminating time as a variable and reducing the differential equations in (1) and (4) to a single differential equation with the steady state $(k,\mu)=(k^*,\mu^*)$ as the initial condition (Judd (1998, Ch. 10.7)).

$$\frac{\dot{\mu}}{\dot{k}}=\frac{d\mu}{dk}=\frac{\rho\mu-\left[f'(k)-(n+\delta)\right]\mu}{f(k)-(n+\delta)k-c},\quad \mu(k^*)=\mu^* \tag{11}$$

The function $\mu(k)$ that solves (11) is the stable saddle path and it should be clear that this function is the derivative of the value function.

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As mentioned above, there is a ubiquitous failure to identify the intersection of the stable saddle path. The diagrams simply fail to identify what values the control should take if the state variable is close to zero. If $U'(0)=\infty$, the issue may be settled by examining the phase diagram as explained in Figure 1. The stable saddle path drawn in (k, μ)-space will go to infinity as consumption, which cannot be positive if there is no capital, goes to zero. We shall here show that the stable saddle path always goes to infinity when k goes to zero. We shall prove this in a few steps, one of which is to prove that if $U'(0)<\infty$, then c is always zero for sufficiently low values of k.

PROPOSITION 1. *Assume that $U'(0)<\infty$ and $f'(0)<\infty$, then the stable saddle path will not intersect the μ-axis at $\mu \leq U'(0)$.*

Proof. It is not possible for the stable saddle path to intersect the μ-axis at μ-values below $U'(0)$ as this is inconsistent with (3). If the proposition is false, therefore $\mu(0)=U'(0)$. If $\mu(0)=U'(0)$, the slope of the stable saddle path at $(k, \mu)=\big(0, U'(0)\big)$ is given by:

$$\frac{\mathrm{d}\mu}{\mathrm{d}k}_{k=0,\mu=U'(0)} = \frac{\dot{\mu}}{\dot{k}}_{k=0,\mu=U'(0)} = \frac{\left[\rho+\delta+n-f'(0)\right]U'(0)}{f(0)-(n+\delta)\times 0} = -\infty \tag{12}$$

However, the slope of the isocline $\dot{k}=0$ at $(k, \mu)=\left[0, U'(0)\right]$ can be calculated from (7) to be $U''(0)\left[f'(0)-(\delta+n)\right]$ which is finite. This implies that if the stable saddle path originates at $(k, \mu)=\left[0, U'(0)\right]$, it enters the region, where $\dot{k}<0$ which is a contradiction as the stable saddle path cannot be located in that region. ■

Proposition 1 assumed that $f'(0)<\infty$. The case $f'(0)=\infty$ can also be proved. As long as $U'(0)<\infty$, the stable saddle path has a slope given by (12) if it originates in $(k, \mu)=\left[0, U'(0)\right]$. Inspection of (12) and the slope of the isocline $\dot{k}=0$ reveals that, roughly speaking, when $k\to 0$, the slope of the stable saddle path is proportional to the slope of the isocline $\dot{k}=0$ divided by zero which implies that the stable saddle path has a steeper slope very close to $k=0$ which again implies it would enter the region where $\dot{k}<0$. Below is a numerical model illustrating the particular case where $U'(0)<\infty$ and $f'(0)=\infty$.

Using Proposition 1 we can prove the following trivial but important result.

PROPOSITION 2. *Assume that $U'(0)<\infty$. Then there exists an interval $[0, \tilde{k}]$, where optimal c is given by* 0.

Proof. An immediate consequence of Proposition 1 and (3) is that stable saddle path must intersect the $\mu=U'(0)$ line at some positive value of k from here on denoted as $\tilde{k}$ and from (3) it follows that $c=0$ for $k\leq\tilde{k}$. ■

We can now examine the stable saddle path in more detail. As noted above, the stable saddle path is calculated by solving (11). In general this equation has no closed form solution but must be solved numerically. But from Proposition 2 we have the existence of an interval $[0, \tilde{k}]$ where we can write (11) as separable differential equation:

$$\frac{1}{\mu}\frac{d\mu}{dk} = \frac{(\rho - n) - \left[f'(k) - (n+\delta)\right]}{f(k) - (\delta + n)\,k}, \quad \mu(\tilde{k}) = U'(0) \tag{13}$$

Equation (13) may be solved in the following steps:

$$\begin{aligned} -\int_{k}^{\tilde{k}} \frac{1}{\mu}\frac{d\mu}{dy}dy &= \int_{\tilde{k}}^{k} \frac{\rho}{f(y) - (\delta+n)\,y}dy - \int_{\tilde{k}}^{k} \frac{f'(y) - (n+\delta)}{f(y) - (\delta+n)\,y}dy \\ -\ln\frac{\mu(k)}{U'(0)} &= \int_{\tilde{k}}^{k} \frac{\rho}{f(y) - (\delta+n)\,y}dy - \int_{f(\tilde{k})-(\delta+n)\tilde{k}}^{f(k)-(\delta+n)k} \frac{1}{z}dz \end{aligned} \tag{14}$$

Note, from (14) that when the last integral is solved through integration by substitution $f'(k)$ disappears which implies that $f'(0) = \infty$ is an irrelevant assumption as long as $f'(k^*) - (n+\delta) > 0$. A few further calculations yield:

$$\mu(k) = U'(0)\frac{f(\tilde{k}) - (\delta+n)\,\tilde{k}}{f(k) - (\delta+n)\,k} \exp\left[-\int_{k}^{\tilde{k}} \frac{\rho}{f(y) - (\delta+n)\,y}dy\right] \tag{15}$$

This expression is valid for any functional form of $f(k)$ and clearly is independent of the shape of $U(c)$ except for the dependence on $U'(0)$. We can now prove that $\mu(k)$ goes to infinity when k goes to zero.

PROPOSITION 3. $\lim_{k\downarrow 0} \mu(k) = \infty$

Proof. If $U'(0) < \infty$ and the integral in (15) converges, the proposition is a trivial consequence of the denominator $f(k) - (\delta + n)k$ going to zero as k goes to zero. If the integral does not converge, we can use L'Hôpital's rule and calculate that:

$$\begin{aligned} \lim_{k\to 0} \mu(k) &= U'(0)\left[f(\tilde{k}) - (\delta+n)\,\tilde{k}\right]\frac{\lim_{k\to 0} \frac{d}{dk} \exp\left[-\int_{k}^{\tilde{k}} \frac{\rho}{f(y)-(\delta+n)y}dy\right]}{\lim_{k\to 0} \frac{d}{dk}\left[f(k)-(\delta+n)k\right]} \\ &= \frac{\rho}{f'(0)-(\delta+n)} \lim_{k\to 0}\left\{\left[U'(0)\frac{f(\tilde{k})-(\delta+n)\tilde{k}}{f(k)-(\delta+n)k}\right] \exp\left[-\int_{k}^{\tilde{k}} \frac{\rho}{f(y)-(\delta+n)y}dy\right]\right\} \end{aligned} \tag{16}$$

The last equality implies that:

$$\lim_{k\to 0} \mu(k) = \frac{\rho}{f'(0) - (\delta+n)} \lim_{k\to 0} \mu(k) \tag{17}$$

As $f'(0) - (\delta+n) > f'(k^*) - (\delta+n) = \rho$, it follows that $\rho/\left[f'(0) - (\delta+n)\right] \neq 1$, so $\lim_{k\to 0}\mu(k)$ is either 0 or infinity. As $\mu(k)$ is a decreasing function of k, which can be affirmed by inspecting (13), $\lim_{k\to 0}\mu(k) = \infty$ follows. ■

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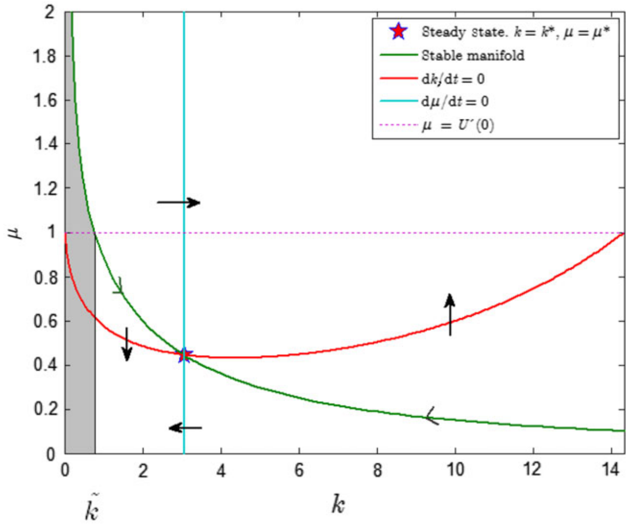

Data Source: The parameters used are $a = 1, b = 1, \alpha = 0.7$, and $n + \delta = 0.45$. This implies a maximum value of k at 14.32. The stable saddle path seems to go to infinity as k goes to zero. For $k = 10^{-50}$, we have that $\mu(10^{-50}) = 3.7327 \times 10^{34}$. For $k < \tilde{k}$, we have that $\mu(k) > U'(0)$ so $c = 0$ for all $k \in [0, \tilde{k}]$ where $\tilde{k} \approx 0.75$. As the value function at $k = 0$ is zero and the stable saddle path is the derivative of the value function, the gray area gives the numerical value of the value function at $k = \tilde{k}$. The figure is generated by a Matlab program available from the author upon request.

FIGURE 2. Phase diagram for the model in (18).

3. NUMERICAL EXAMPLE

Armed with these propositions, we can draw a complete phase diagram. The exact formulation is:

$$V[k(0)] = \max_{s \in [0,1]} \int_0^\infty \left(a - ae^{-bc}\right) e^{-\rho t} dt \qquad (18)$$

$$\text{subject to: } \dot{k} = k^\alpha - (\delta + n) k - c, \quad k(0) \text{ given}$$

Here $a > 0, b > 0, 0 < \alpha < 1, \delta > 0$, and $n > 0$. Marginal utility at zero consumption is finite and equal to ab. Marginal productivity at $k = 0$ is infinity. The optimality conditions are straightforward to derive and are omitted. The solution to (18) is illustrated in Figure 2 and discussed in the caption.

It is customary to draw phase diagrams with k along the x-axis and c along the y-axis. The choice to here measure μ along the y-axis was motivated by making the results easier to explain. It is easy to translate the results from this paper into a phase diagram with c along the y-axis if so desired. However, by using (6), one may draw level curves for combinations of k and μ in (k, μ) – space and show how the control depends on the state variable and shadow price. Obviously, in the RCK model such curves would be horizontal lines and make this process quite straightforward.

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4. A COMMENT ON THE EXPOSITION OF THE RCK MODEL

The usual approach when presenting the RCK model is to assume that (3) holds with equality and that one can therefore differentiate (3) and obtain $U''(c)\dot{c} = \dot{\mu}$ and by combining this expression with (3) and (4), one can derive the familiar Keynes–Ramsey rule:

$$\dot{c} = \frac{U'(c)}{U''(c)} \left(\rho + \delta + n - f'(k)\right) \tag{19}$$

An immediate consequence of Proposition 2 is that if $U'(0) < \infty$, then for some values of k we have that $c = 0$ and therefore $\dot{c} = 0$. In this case there exists an interval $[0, \tilde{k})$ where the Keynes–Ramsey rule does not hold. This highlights the danger of simply assuming interior solutions in dynamic economic models and the importance of drawing complete saddle paths when illustrating dynamic models where the steady state is a saddle point. Of course, if $U(c)$ exhibits constant intertemporal elasticity of substitution, then $U'(0) = \infty$ and the solution to (19) goes through the origin.

5. SUMMARY

The present paper has tidied up a few loose ends in the Ramsey–Cass–Koopmans growth model. Some hitherto unnoticed results have been shown. We show that the assumption that zero capital stock implies zero growth has some important consequences for the shape of the stable saddle path. Also, the standard phase diagram has been shown to contain more information than typically acknowledged as the stable saddle path is in fact the derivative of the value function, thus enabling us to illustrate the value function as the area under the stable saddle path.

NOTES

1. Spear and Young (2014) provide a good historical treatment of the evolution of the RCK model and argue convincingly that the Ramsey–Cass–Koopmans model or Cass–Koopmans model should be renamed Cass–Malinvaud–Koopmans. I keep the original name to avoid confusion.
2. I cannot claim to have exhaustively examined the large number of books and articles discussing the RCK model, but have checked quite a few. If I have missed some exposition that covers this topic I apologize. My only excuse is that if this is the case I am not alone in being guilty of this omission.
3. Wikipedia's entry on the Ramsey–Cass–Koopmans model has an easily accessible and typical illustration of dangling stable saddle paths, Wikipedia contributors (2019).
4. This failure is not contained to the economic growth literature.
5. Indeed, any growth model where zero stock implies that the value function and the growth function are zero has this property.
6. There are a number of formulations of the RCK model. The objective function is often specified as integrating $U(c)e^{-(\rho-n)t}$ implying Benthamite utilitarianism. Here, we follow the formulation in Blanchard and Fischer (1989), pp. 39–41, where the objective represents the preferences of a representative family. All results below hold for the instantaneous utility function $U(c)e^{-(\rho-n)t}$, if ρ is replaced by $\rho - n$ and $\rho - n$ is assumed positive.

REFERENCES

Acemoglu, D. (2009) *Introduction to Modern Economic Growth*. Princeton: Princeton University Press.

Becker, R. A. and T. Mitra (2012) Efficient ramsey equilibria. *Macroeconomic Dynamics* 16(S1), 18–32.

Bénassy, J.-P. (2011) *Macroeconomic Theory*. New York: Oxford University Press.

Biljanovska, N. (2019) Optimal policy in collateral constrained economies. *Macroeconomic Dynamics* 23(2), 798–836.

Blanchard, O. J. and S. Fischer (1989). *Lectures on Macroeconomics*. Cambridge: MIT Press.

Guo, S. and Z. Jiang (2017) Optimal fiscal policy with land financing in China. *Macroeconomic Dynamics* 21(8), 1–26.

Judd, K. L. (1998) *Numerical Methods in Economics*. Cambridge MA: MIT Press.

Romer, D. (2006) *Advanced Macroeconomics (Third edition)*. New York: McGrawHill/Irwin.

Spear, S. E. and W. Young (2014) Optimum savings and optimal growth: The Cass–Malinvaud–Koopmans nexus. *Macroeconomic Dynamics* 18(1), 215–243.

Wikipedia contributors (2019). Ramsey–Cass–Koopmans model. Wikipedia, The Free Encyclopedia, 17 June 2019. Web. 14 August 2019.