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DELAY DIFFERENTIAL EQUATIONS OF ODD ORDER SATISFYING PROPERTY P_k

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Abstract

The property P_k $(k = 0, 1, \cdots, n)$ is formulated. For $k = 0$, n this property reduces to conditions A and B defined by Kiguradze (1962) for a class of ordinary differential equations. Sufficient conditions are then given which guarantee that a class of delay differential equations of odd order possesses property P_k. The property P_k is also seen to be useful in reducing the number of types of positive solutions of a related nonhomogeneous delay differential equation.

The equation

$$D^m y(t) + F[t, y(t)] = 0, \quad m \geqq 2 \tag{1}$$

has been considered by various authors subject to additional sign and monotone properties on $F(t, u)$. Briefly, a solution of (1) or of (2) below is called oscillatory on $[a, \infty)$ if for each $\alpha > a$ there is a $\beta > \alpha$ such that $y(\beta) = 0$. It is called nonoscillatory otherwise. Paralleling the development by Kiguradze (1962) we adopt the following terminology.

DEFINITION 1. *For $m = 2n + 1$, a positive solution y of (1) is of type A_k $(k = 0, \cdots, n)$ if for t sufficiently large $D^j y(t) > 0$ for $j = 0, \cdots, 2k$ and $(-1)^j D^j y(t) > 0$ for $j = 2k + 1, \cdots, 2n$.*

DEFINITION 2. *For $m = 2n$, a positive solution y of (1) is of type A_k $(k = 0, \cdots, n - 1)$ if for t sufficiently large $D^j y(t) > 0$ for $j = 0, \cdots, 2k + 1$ and $(-1)^{j+1} D^j y(t) > 0$ for $j = 2k + 2, \cdots, 2n - 1$.*

DEFINITION 3. *Equation (1) is said to satisfy condition A if (1) has an oscillatory solution and every nonoscillatory solution tends to zero monotonically as $t \to \infty$.*

DEFINITION 4. *Equation (1) satisfies condition B if a solution y is either oscillatory or $\lim_{t \to \infty} D^{m-1} y(t) = 0$.*

It has been shown in Kiguradze (1962) that a positive solution of (1) is necessarily of type A_k for some admissible k. For $m = 2n$, (1) fulfills condition A if, and only if, all solutions are oscillatory. For $m = 2n + 1$, (1) satisfies condition A if, and only if, there are no solutions of type A_r $(r = 1, \cdots, n)$ and every solution of type A_0 tends to zero monotonically as $t \to \infty$.

In section one we consider a homogeneous delay differential equation of odd order and formulate a property P_k which includes both conditions A and B as special cases. Section two is devoted to providing sufficient conditions for the equation to possess property P_k. In section three an *a priori* classification according to types C_k^R is introduced for the positive solutions of a nonhomogeneous delay differential equation of odd order. The property P_k is seen to be useful in reducing the kinds of positive solutions admissible.

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In this section and the next we shall consider the homogeneous delay differential equation

$$D^{2n+1-i}[r(t)D^i y(t)] + y_\tau(t) f[t, y_\tau(t)] = 0, \tag{2}$$

where $0 < m \leqq r(t) \leqq M < \infty$, $0 \leqq \tau(t) \leqq T < \infty$, $y_\tau(t) = y[t - \tau(t)]$ and $f(t, u)$ satisfies the following properties:

(F1) $f(t, u)$ is a continuous real-valued function on $[0, \infty) \times R$;

(F2) for each fixed $t \in [0, \infty)$, $f(t, u) < f(t, v)$ for $0 < u < v$; and

(F3) for each fixed $t \in [0, \infty)$, $f(t, u) > 0$ and $f(t, -u) = f(t, u)$ for $u \neq 0$.

We first let

$$y_j(t) = \begin{cases} D^j y(t), & j = 0, \cdots, i-1 \\ D^{j-i}[r(t) D^i y(t)], & j = i, \cdots, 2n. \end{cases}$$

Analogous to Definition 1 we shall classify the positive solutions of (2).

DEFINITION 5. *A positive solution y of* (2) *is of type C_k on $[T_0, \infty)$ if for $t \geqq T_0$ $y_j(t) > 0$ $(j = 0, \cdots, 2k)$ and $(-1)^j y_j(t) > 0$ $(j = 2k + 1, \cdots, 2n)$.*

As in Terry (1973, 1974), it is evident that a positive solution of (2) is necessarily of type C_k for some $k = 0, \cdots, n$. Moreover, the following two lemmas may be established.

LEMMA 1. *Let y be a solution of* (2) *of type C_k, $k \geqq 1$. Then there exist numbers $N_j^k > 0$ $(j = 0, \cdots, 2k)$ such that*

$$(t - T_1) y_j(t) \leqq N_j^k y_{j-1}(t), \; t \geqq T_1 = T_0 + T \text{ and}$$

$$t y_j(t) \leqq 2N_j^k y_{j-1}(t), \qquad t \geqq 2T_1.$$

LEMMA 2. *Let y be a solution of* (2) *of type C_k, $k \geqq 1$. Then there exist numbers $k_j > 0$ and $t_j \geqq T_1$ ($j = 0, \cdots, 2k-1$) such that*

$$y_{j\tau}(t) = y_j(t - \tau(t)) \geqq k_j y_j(t), \qquad t \geqq t_j.$$

While it is of interest to obtain specific estimates for the numbers N_j^k, this is unnecessary for the subsequent development of this paper. As in Terry (1973), these two lemmas and the later results may be extended to the case where $\tau(t)$ satisfies either of the two conditions

(T1) $\quad 0 \leqq \tau(t) \leqq \mu t, \qquad 0 \leqq \mu < m/(m+M)$, or

(T2) $\quad 0 \leqq \tau(t) \leqq \mu t^{\beta}, \qquad 0 \leqq \mu < \infty$ and $0 \leqq \beta < 1$,

provided T_1 is reinterpreted as $\min\{t > T_0 \colon t - \tau(t) \geqq T_0 \text{ for } t \geqq T_1\}$.

DEFINITION 6. *Equation* (2) *fulfills property P_k if, and only if,* (2) *has no solutions of types C_r ($r = k+1, \cdots, n$) and for any solution $y(t)$ of type C_k the intermediate function $y_{2k}(t)$ tends to zero monotonically as $t \to \infty$.*

When $r \equiv 1$ and $\tau \equiv 0$, the classification of solutions of (2) according to types C_k coincides with that of Kiguradze (1962). Moreover, property P_0 is the natural analogue of condition A; property P_n corresponds to condition B.

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We now seek to prescribe conditions which ensure that equation (2) fulfills the property P_k.

THEOREM 1. *Let y be a positive solution of* (2) *of type C_k. Then $y_{2k}(t)$ tends to zero monotonically as $t \to \infty$ if for all positive constants C*

$$\int^{\infty} t^{2n} f(t, Ct^{2k})\,dt = +\infty. \tag{3}$$

PROOF. Let y be a solution of (2) of type C_k on $[T_0, \infty)$. Then for $t \geqq T_1$, $y_\tau(t) > 0$, $y_j(t) > 0$ ($j = 0, \cdots, 2k$) and $(-1)^j y_j(t) > 0$ ($j = 2k+1, \cdots, 2n$). Multiplying (2) by t^{2n-2k} and integrating from T_1 to $t \geqq T_1$

$$\int_{T_1}^{t} s^{2n-2k} Dy_{2n}(s)\,ds + \int_{T_1}^{t} s^{2n-2k} y_\tau(s) f[s, y_\tau(s)]\,ds = 0.$$

Integrating the first term by parts

$$I \equiv \int_{T_1}^{t} s^{2n-2k} Dy_{2n}(s)\,ds = [s^{2n-2k} y_{2n}(s)]_{T_1}^{t}$$
$$- (2n-2k) \int_{T_1}^{t} s^{2n-2k-1} y_{2n}(s)\,ds.$$

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An easy induction yields

$$I = \left[\sum_{j=0}^{l} (-1)^j (2n-2k)_j s^{2n-2k-j} y_{2n-j}(s)\right]_{T_1}^{t} \tag{4}$$
$$+ (-1)^{l+1}(2n-2k)_{l+1} \int_{T_1}^{t} s^{2n-2k-l-1} y_{2n-l}(s)\,ds,$$

where $0 \leqq j \leqq l \leqq 2n-i$, $(n)_0 = 1$ and $(n)_k = n \cdots (n-k+1)$ for $k \geqq 1$. If $2k \geqq i$, $2n-2k \leqq 2n-i$ and we may let $l = 2n-2k-1$ in (4) to obtain

$$I = \left[\sum_{j=0}^{2n-2k-1} (-1)^j (2n-2k)_j s^{2n-2k-j} y_{2n-j}(s)\right]_{T_1}^{t} \tag{5}$$
$$+ (-1)^{2n-2k}(2n-2k)_{2n-2k} \int_{T_1}^{t} y_{2k+1}(s)\,ds.$$

On the other hand, if $1 < 2k < i$, then $2n-2k > 2n-i$. We let $l = 2n-i$ in (4) and observe that

$$\begin{aligned}(-1)^{l+1} y_{2n-l}(s) &= (-1)^{2n-i+1} y_i(s) = (-1)^{i+1} y_i(s)\\ &= (-1)^{i+1} r(s) D^i y(s)\\ &\geqq (-1)^{i+1} M D^i y(s).\end{aligned}$$

Then (4) becomes

$$I \geqq \left[\sum_{j=0}^{2n-i} (-1)^j (2n-2k)_j s^{2n-2k-j} y_{2n-j}(s)\right]_{T_1}^{t} \tag{6}$$
$$+ (-1)^{2n-i+1} M (2n-2k)_{2n-i+1} \int_{T1}^{t} s^{i-2k-1} D y_{i-1}(s)\,ds.$$

We now examine the latter integral. An integration by parts results in

$$J \equiv \int_{T_1}^{t} s^{i-2k-1} D y_{i-1}(s)\,ds = [s^{i-2k-1} y_{i-1}(s)]_{T_1}^{t}$$
$$- (i-2k-1) \int_{T_1}^{t} s^{i-2k-2} y_{i-1}(s)\,ds.$$

This serves as the anchor for another inductive argument based on further integration by parts. It follows that

$$J = \left[\sum_{j=1}^{L} (-1)^{j+1} (i-2k-1)_{j-1} s^{i-2k-j} y_{i-j}(s)\right]_{T_1}^{t}$$
$$+ (-1)^L (i-2k-1)_L \int_{T_1}^{t} s^{i-2k-j-1} y_{i-L}(s)\,ds,$$

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where $1 \leqq j \leqq L \leqq i-2k-1$. Combining (6) and (7), we obtain

$$I \geqq \Bigg[\sum_{j=0}^{2n-i}(-1)^j(2n-2k)_j s^{2n-2k-j} y_{2n-j}(s) + N_1 \sum_{j=1}^{L}(-1)^{j+1}(i-2k-1)_j s^{i-2k-j} y_{i-j}(s)\Bigg]_{T_1}^{t} + N_1 N_2 \int_{T_1}^{t} y_{i-L}(s)\,ds,$$

where $N_1 = (-1)^{i+1} M(2n-2k)_{2n-i+1}$ and $N_2 = (-1)^L (i-2k-1)_L$. We may let $L = i-2k-1$ in this and observe that

$$\operatorname{sgn} N_1 N_2 = (-1)^{i+1}(-1)^L = (-1)^{i+1}(-1)^{i-2k-1} = +1.$$

It follows that

$$[F_{2k}(s)]_{T_1}^{t} + (2n-2k)! \int_{T_1}^{t} y_{2k+1}(s)\,ds + \int_{T_1}^{t} s^{2n-2k} y_\tau(s) f[s, y_\tau(s)]\,ds = 0 \tag{8a}$$

for $i \leqq 2k$ and

$$[\bar{F}_{2k}(s)]_{T_1}^{t} + N_1 N_2 \int_{T_1}^{t} y_{2k+1}(s)\,ds + \int_{T_1}^{t} s^{2n-2k} y_\tau(s) f[s, y_\tau(s)]\,ds \leqq 0 \tag{8b}$$

for $i > 2k$, where

$$F_{2k}(s) = \sum_{j=0}^{2n-2k-1}(-1)^j(2n-2k)_j s^{2n-2k-j} y_{2n-j}(s) \quad \text{and}$$

$$\bar{F}_{2k}(s) = \sum_{j=0}^{2n-i}(-1)^j(2n-2k)_j s^{2n-2k-j} y_{2n-j}(s) + N_1 \sum_{j=1}^{i-2k-1}(-1)^j(i-2k-1)_j s^{i-2k-j} y_{i-j}(s).$$

We note that each term of $F_{2k}(s)$ is positive on $[T_1, \infty)$ since

$$(-1)^j y_{2n-j}(s) = (-1)^{2n-j}(s) = (-1)^p y_p(s)$$

and $p = 2n-j \geqq 2k+1$ since $0 \leqq j \leqq 2n-2k-1$. Similarly, each term of $\bar{F}_{2k}(s)$ is positive since

$$(-1)^{i+1}(-1)^{j+1} y_{i-j}(s) = (-1)^{i-j} y_{i-j}(s) = (-1)^q y_q(s)$$

and $q = i-j \geqq 2k+1$ since $j \leqq i-2k-1$. Moreover,

$$\int_{T_1}^{t} y_{2k+1}(s)ds = y_{2k}(t) - y_{2k}(T_1) \quad \text{if} \quad 2k+1 \neq i$$

$$= \int_{T_1}^{t} r(s)D^i y(s) \quad \text{if} \quad 2k+1 = i$$

$$\geqq M\int_{T_1}^{t} D^i y(s) = M[y_{2k}(t) - y_{2k}(T_1)].$$

Let us assume that $\lim_{t\to\infty} y_{2k}(t) = \gamma > 0$. Since $y_{2k+1}(t) < 0$ on $[T_1, \infty)$, $y_{2k}(t)$ is a decreasing function of t on $[T_1, \infty)$ and $y_{2k}(t) \geqq \gamma$, $t \geqq T_1$. By Lemma 1

$$y(s) \geqq Ns^{2k} y_{2k}(s) \geqq N\gamma s^{2k}, \text{ where } N^{-1} = N_1^k N_2^k \cdots N_{2k}^k.$$

Hence,

$$y_\tau(s) \geqq N\gamma[s - \tau(s)]^{2k} \geqq N\gamma(1-\mu)^{2k} s^{2k}$$

if τ satisfies (T1). On the other hand, if τ satisfies (T2), there is a $T_2 \geqq T_1$ such that $s - \tau(s) \geqq s/2$ for $s \geqq T_2$, which implies that $y_\tau(s) \geqq N\gamma 2^{-2k} s^{2k}$ for $s \geqq T_2$. As we may replace T_1 by T_2 in the above considerations, we may assume, *without loss of generality, that* $T_2 = T_1$. *Thus, in either case* $y_\tau(s) \geqq Cs^{2k}$ *on the* appropriate interval so that

$$s^{2n-2k} y_\tau(s) f[s, y_\tau(s)] \geqq s^{2n-2k} C s^{2k} f[s, Cs^{2k}]$$
$$= Cs^{2n} f(s, Cs^{2k}).$$

For $i \leqq 2k$ we substitute in (8a)

$$[F_{2k}(s)]_{T_1}^t + (2n-2k)!\,[y_{2k}(s)]_{T_1}^t s^{2n} f(s, Cs^{2k})ds \leqq 0.$$

Transposing,

$$\int_{T_1}^{t} s^{2n} f(s, Cs^{2k})ds \leqq C^{-1}[F_{2k}(T_1) + (2n-2k)!\, y_{2k}(t)].$$

This contradicts (3) in the case τ satisfies $(T1)$; if τ satisfies $(T2)$, we replace T_1 by T_2 and obtain the same contradiction. For $i > 2k$, we substitute in (8b) instead.

THEOREM 2. *Let* ϕ *be a function satisfying* $\phi(y) > 0$, $\phi'(y) \geqq 0$ *and*

(9a) $$\int^{\infty} \frac{dy}{y\phi(y)} < \infty;$$

then (2) *fulfills property* P_k *if for all positive constants* C

(9b) $$\int^{\infty} t^{2n} f(t, Ct^{2k}) \phi^{-1}(t)dt = +\infty.$$

PROOF. Suppose that y is a solution of (2) of type C_k on $[T_0, \infty)$, where $k \geqq 1$. Then multiplying equation (2) by $t^{2n}[\phi(t)y(t)]^{-1}$ and integrating from T_1 to $t > T_1$

$$\int_{T_1}^{t} s^{2n} D y_{2n}(s)[\phi(s)y(s)]^{-1} ds + \int_{T_1}^{t} s^{2n} \frac{f[s, y_\tau(s)]}{\phi(s)y(s)} ds = 0. \tag{10}$$

We denote the first integral by I_1. An integration by parts yields

$$I_1 = [\gamma(s)(\phi(s)y(s))^{-1}]_{T_1}^{t} - \int_{T_1}^{t} \gamma(s) D((\phi(s)y(s))^{-1}) ds, \tag{11}$$

where $\gamma(s) = D^{-1}(s^{2n} D y_{2n}(s))$. Specifically, one integration by parts gives

$$\gamma(s) = s^{2n} y_{2n}(s) - 2n D^{-1}(s^{2n-1} y_{2n}(s)).$$

We may establish by induction that

$$\begin{aligned} \gamma(s) &= \sum_{j=0}^{p} (-1)^j (2n)_j s^{2n-j} y_{2n-j}(s) \\ &\quad + (-1^{p+1}(2n)_{p+1} D^{-1}[s^{2n-p-1} y_{2n-p}(s)] \end{aligned} \tag{12}$$

for $0 \leqq j \leqq p \leqq 2n - i$. If $2k \geqq i$, $2n - 2k \leqq 2n - i$ and we may let $p = 2n - 2k$ in (12) to obtain

$$\gamma(s) = \sum_{j=0}^{2n-2k} (-1)^j (2n)_j s^{2n-j} y_{2n-j}(s) + N_0 D^{-1}[s^{2k-1} y_{2k}(s)],$$

where $N_0 = (-1)^{2n-2k+1}(2n)_{2n-2k+1}$. We define $\gamma_0(s)$ by

$$\gamma(s) = \gamma_0(s) + N_0 D^{-1}[s^{2k-1} y_{2k}(s)].$$

Then, a substitution in (11) produces

$$\begin{aligned} I_1 &= [(\gamma_0(s) + N_0 D^{-1}(s^{2k-1} y_{2k}(s)))(\phi(s)y(s))^{-1}]_{T_1}^{t} \\ &\qquad - \int_{T_1}^{t} (\gamma_0(s) + N_0 D^{-1}(s^{2k-1} y_{2k}(s)) D((\phi(s)y(s))^{-1}) ds \\ &= [\gamma_0(s)(\phi(s)y(s))^{-1}]_{T_1}^{t} - \int_{T_1}^{t} \gamma_0(s) D((\phi(s)y(s))^{-1}) ds \\ &\qquad + N_0 [D^{-1}(s^{2k-1} y_{2k}(s))/\phi(s)y(s)]_{T_1}^{t} \\ &\qquad - N_0 \int_{T_1}^{t} D^{-1}(s^{2k-1} y_{2k}(s)) D((\phi(s)y(s))^{-1}) ds. \end{aligned}$$

Applying the integration-by-parts formula in reverse, we recombine the last two terms to obtain

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$$I_1 = [\gamma_0(s)(\phi(s)y(s))^{-1}]_{T_1}^{t} - \int_{T_1}^{t} \gamma_0(s) D((\phi(s)y(s))^{-1}) ds$$
$$+ N_0 \int_{T_1}^{t} \frac{s^{2k-1} y_{2k}(s)}{\phi(s)y(s)} ds.$$

By Lemma 1 there is a number $N_{2k} = N_1^k \cdots N_{2k}^k > 0$ such that

$$t^{2k} y_{2k}(t) \leqq N_{2k} y(t) \text{ for } t \geqq 2T_1.$$

Thus

$$\frac{s^{2k-1} y_{2k}(s)}{\phi(s)y(s)} = \frac{s^{2k} y_{2k}(s)}{s\phi(s)y(s)} \leqq \frac{N_{2k}}{s\phi(s)}.$$

Since $N_0 = -|N_0| < 0$,

$$I_1 \geqq [\gamma_0(s)(\phi(s)y(s))^{-1}]_{T_1}^{t} - \int_{T_1}^{t} \gamma_0(s) D((\phi(s)y(s))^{-1}) ds$$
(13)
$$- |N_0| N_{2k} \int_{2T_1}^{t} \frac{ds}{s\phi(s)}.$$

Otherwise, if $2k < i$, we continue the inductive procedure defined by (12) until $p = 2n - i$. Then (12) becomes

$$\gamma(s) = \sum_{j=0}^{2n-i} (-1)^j (2n)_j s^{2n-j} y_{2n-j}(s) + N_1 D^{-1}[s^{i-1} y_i(s)],$$

where $N_1 = (-1)^{2n-i+1}(2n)_{2n-i+1}$. Letting

$$\gamma(s) = \gamma_1(s) + N_1 D^{-1}[s^{i-1} y_i(s)];$$

it follows as before upon substitution in (11) that

$$I_1 = [\gamma_1(s)(\phi(s)y(s))^{-1}]_{T_1}^{t} - \int_{T_1}^{t} \gamma_1(s) D((\phi(s)y(s))^{-1}) ds$$
$$+ N_1 \int_{T_1}^{t} \frac{s^{i-1} y_i(s)}{\phi(s)y(s)} ds.$$

We observe that

$$N_1 y_i(s) = |N_1| (-1)^{2n-i+1} y_i(s) = |N_1| (-1)^{i+1} y_i(s)$$
$$= |N_1| (-1)^{i+1} r(s) D y_{i-1}(s)$$
$$\geqq |N_1| (-1)^{i+1} M D y_{i-1}(s) = M N_1 D y_{i-1}(s).$$

Thus,

$$I_1 \geqq [\gamma_1(s)(\phi(s)y(s))^{-1}]_{T_1}^{t} - \int_{T_1}^{t} \gamma_1(s) D((\phi(s)y(s))^{-1}) ds$$
(14)
$$+ N_1 M \int_{T_1}^{t} \frac{s^{i-1} D y_{i-1}(s)}{\phi(s)y(s)} ds.$$

Analogous to (11) we have

$$I_2 = \int_{T_1}^{t} \frac{s^{i-1}Dy_{i-1}(s)}{\phi(s)y(s)}\,ds = [\gamma_2(s)\ (\phi(s)y(s))^{-1}]_{T_1}^{t} - \int_{T_1}^{t} \gamma_2(s)D((\phi(s)y(s))^{-1})ds,$$

where $\gamma_2(s) = D^{-1}[s^{i-1}Dy_{i-1}(s)]$. We find upon one integration that

$$\gamma_2(s) = s^{i-1}y_{i-1}(s) - (i-1)D^{-1}[s^{i-2}y_{i-1}(s)].$$

An inductive argument yields

$$\gamma_2(s) = \sum_{j=1}^{p} (-1)^{j+1}(i-1)_{j-1}s^{i-j}y_{i-j}(s) + (-1)^p(i-1)_p D^{-1}[s^{i-p-1}y_{i-p}(s)],$$

for $i \leqq j \leqq p \leqq i-2k$. Letting $p = i-2k$ results in

$$\begin{aligned}\gamma_2(s) &= \sum_{j=1}^{i-2k} (-1)^{j+1}(i-1)_{j-1}s^{i-j}y_{i-j}(s) + (-1)^{i-2k}(i-1)_{i-2k}D^{-1}[s^{2k-1}y_{2k}(s)]\\ &= \gamma_3(s) + N_2D^{-1}[s^{2k-1}y_{2k}(s)],\end{aligned}$$

where

$$\gamma_3(s) = \sum_{j=1}^{i-2k} (-1)^{j+1}(i-1)_{j-1}s^{i-j}y_{i-j}(s)$$

and
$$N_2 = (-1)^{i-2k}(i-1)_{i-2k}.$$

To simplify the expressions involved, let

$$\begin{aligned}\Gamma(s) &= \gamma_1(s) + MN_1\gamma_2(s)\\ &= \gamma_1(s) + MN_1[\gamma_3(s) + N_2D^{-1}(s^{2k-1}y_{2k}(s))]\\ &= \gamma_1(s) + MN_1\gamma_3(s) + MN_1N_2D^{-1}[s^{2k-1}y_{2k}(s)]\\ &= \Gamma_0(s) + MN_1N_2D^{-1}[s^{2k-1}y_{2k}(s)]\\ &= \Gamma_0(s) - M\,|N_1N_2|D^{-1}[s^{2k-1}y_{2k}(s)].\end{aligned}$$

We note here that $\operatorname{sgn} N_1N_2 = (-1)^{2n-i+1}(-1)^{i-2k} = (-1)^1 = -1$. Substituting in (14),

$$\begin{aligned}I_1 \geqq{}& [\gamma_1(s)(\phi(s)y(s))^{-1}]_{T_1}^{t} - \int_{T_1}^{t} \gamma_1(s)D((\phi(s)y(s))^{-1})ds\\ &+ MN_1[(\gamma_3(s) + N_2D^{-1}(s^{2k-1}y_{2k}(s)))/\phi(s)y(s)]_{T_1}^{t}\\ &- MN_1\int_{T_1}^{t} (\gamma_3(s) + N_2D^{-1}(s^{2k-1}y_{2k}(s)))D((\phi(s)y(s))^{-1})ds\end{aligned}$$

$$= [\Gamma_0(s)(\phi(s)y(s))^{-1}]_{T_1}^t - \int_{T_1}^t \Gamma_0(s)D((\phi(s)y(s))^{-1})ds$$

$$+ MN_1N_2\int_{T_1}^t \frac{s^{2k-1}y_{2k}(s)}{\phi(s)y(s)}ds.$$

As in the discussion preceding (13) we conclude that

$$I_1\tau[\Gamma_0(s)(\phi(s)y(s))^{-1}]_{T_1}^t - \int_{T_1}^t \Gamma_0(s)D((\phi(s)y(s))^{-1})ds \tag{15}$$
$$- M|N_1N_2|\int_{2T_1}^t \frac{ds}{s\phi(s)}.$$

We next consider the second integral in (10). By Lemma 2 there is a $k_0 > 0$ and a $t_0 > T_1$ such that $y_\tau(s) \geqq k_0 y(s)$ for $s \geqq t_0$. Since $y_{2k}(s) > 0$, $y_{2k-1}(s)$ is increasing on $[T_0, \infty)$ and there is a $C_0 > 0$ such that $y_{2k-1}(s) \geqq C_0$ for $s \geqq T_1$. Moreover, by Lemma 1 there is an $N_{2k-1} = N_1^k \cdots N_{2k-1}^k$ such that $s^{2k-1}y_{2k-1}(s) \leqq N_{2k-1}y(s)$ for $s \geqq 2T_1$. Thus,

$$y_\tau(s) \geqq k_0 y(s) \geqq k_0 N_{2k-1}^{-1} s^{2k-1} y_{2k-1}(s) \geqq k_0 N_{2k-1}^{-1} C_0 s^{2k-1}.$$

By (iii)

$$s^{2n}f[s, y_\tau(s)]y_\tau(s)y^{-1}(s)\phi^{-1}(s) \geqq k_0 s^{2n} f(s, C_1 s^{2k-1})\phi^{-1}(s),$$

where $C_1 = k_0 N_{2k-1}^{-1} C_0$ and $s \geqq T_* = \max\{t_0, 2T_1\}$. As a result,

$$\int_{T_1}^t \frac{s^{2n}f[s, y_\tau(s)]y_\tau(s)ds}{\phi(s)y(s)} \geqq \int_{T_*}^t \frac{s^{2n}f[s, y_\tau(s)]y_\tau(s)ds}{\phi(s)y(s)} \tag{16}$$
$$\geqq k_0\int_{T_*}^t s^{2n}f(s, C_1 s^{2k-1})ds/\phi(s).$$

Substituting (16) together with (13) or (15) in (10), we obtain

$$[\gamma_0(s)(\phi(s)y(s))^{-1}]_{T_1}^t - \int_{T_1}^t \gamma_0(s)D((\phi(s)y(s))^{-1})ds \tag{17a}$$
$$-|N_0|N_{2k}\int_{2T_1}^t \frac{ds}{s\phi(s)} + k_0\int_{T_*}^t s^{2n}f(s, C_1 s^{2k-1})ds/\phi(s) \leqq 0 \qquad \text{for } 2k \geqq i$$

or

$$[\Gamma_0(s)(\phi(s)y(s))^{-1}]_{T_1}^t - \int_{T_1}^t \Gamma_0(s)D((\phi(s)y(s))^{-1})ds \tag{17b}$$
$$-M|N_1N_2|N_{2k}\int_{2T_1}^t \frac{ds}{s\phi(s)} + k_0\int_{T_*}^t s^{2n}f(s, C_1 s^{2k-1})ds/\phi(s) \leqq 0 \qquad \text{for } 2k < i.$$

We note that each term of $\gamma_0(s)$ or of $\Gamma_0(s)$ is positive on $[T_1, \infty)$. Since $k \geqq 1$, $y'(s) \geqq 0$ and

$$-D((\phi(s)y(s))^{-1}) = \frac{D(\phi(s)y(s))}{[\phi(s)y(s)]^2} = \frac{\phi(s)y'(s) + \phi'(s)y(s)}{[\phi(s)y(s)]^2} > 0.$$

Consequently,

$$\int_T^t s^{2n} f(s, C_1 s^{2k-1}) \phi^{-1}(s) ds \leqq \begin{cases} k_0^{-1}\left[\gamma_0(\phi y)(T_1) + |N_0| N_{2k} \int_{2T_1}^t \frac{ds}{s\phi(s)}\right] \\ k_0^{-1}\left[\Gamma_0(\phi y)(T_1) + M\,|N_1 N_2| N_{2k} \int_{2T_1}^t \frac{ds}{s\phi(s)}\right]. \end{cases}$$

Thus the condition

$$\int^\infty t^{2n} f(t, Ct^{2k-1}) \phi^{-1}(t) dt = \infty, \qquad k \geqq 1$$

will imply that (2) has no C_r-solutions $(r = k, \cdots, n)$, that is the condition

$$\int^\infty t^{2n} f(t, Ct^{2k+1}) \phi^{-1}(t) dt = \infty, \qquad k \geqq 0$$

will imply that (2) has no C_r-solution $r = k+1, \cdots, n)$. *A fortiori*, (9b) implies that (2) has no C_r-solutions $(r = k+1, \cdots, n)$, where $k \geqq 0$. In addition, the conditions $\phi > 0$ and $\phi' \geqq 0$ show that there is a $k > 0$ such that $\phi(t) \geqq k$ so that

$$\int^\infty t^{2n} f(t, Ct^{2k}) dt = k\left[\frac{1}{k}\int^\infty t^{2n} f(t, Ct^{2k}) dt\right]$$
$$\geqq k \int^\infty t^{2n} f(t, Ct^{2k}) \phi^{-1}(t) dt.$$

Thus, the integral condition of (9b) implies that of (3). By Theorem 1, any C_k-solution y of (2) will satisfy $\lim_{t\to\infty} y_{2k}(t) = 0$. It follows that (2) possesses property P_k.

By modifying the conditions on f and ϕ, we may obtain a simpler criterion for the presence of property P_k.

DEFINITION 7. *The function f is nonlinear with strength coefficient $2n+1-j$ $(j = 0, \cdots, 2n+1)$ if, and only if, there is a function ϕ satisfying $\phi(u) > 0$, $\phi'(u) \geqq 0$,*

$$\int^\infty \frac{du}{u\phi(u)} < \infty$$

and $f(t, u) \geqq \phi(u) f(t, Ct^j)$.

When $j = 0$, the strength coefficient is maximal and $f(t, u)$ is called strongly nonlinear.

In the presence of some degree of nonlinearity, the hypotheses of Theorem 2 may be proportionately weakened. Suppose that f is nonlinear with strength coefficient $2n+1-2j$ and $\phi''(u)<0$. Then, multiplying (2) by $t^{2n}[\phi(y)y]^{-1}$, we obtain as in the proof of Theorem 2

$$[\gamma_0(s)(y(s)\phi(y(s)))^{-1}]_{T_1}^{t} - \int_{T_1}^{t} \gamma_0(s)D((y(s)\phi(y(s)))^{-1})ds$$
$$-\int_{T_1}^{t} \frac{s^{2k-1}y_{2k}(s)}{y(s)\phi(y(s))}ds + \int_{T_1}^{t} s^{2n}y_\tau(s)f[s, y_\tau(s)]ds/\phi(y(s))y(s) = 0$$

or

$$0 \geqq [\Gamma_0(s)(y(s)\phi(y(s)))^{-1}]_{T_1}^{t} - \int_{T_1}^{t} \Gamma_0(s)D((y(s)\phi(y(s)))^{-1})ds$$
$$-M|N_1N_2| \int_{T_1}^{t} \frac{s^{2k-1}y_{2k}(s)}{y(s)\phi(y(s))}ds$$
$$+\int_{T_1}^{t} s^{2n}y_\tau(s)f[s, y_\tau(s)]ds/y(s)\phi(y(s)).$$

Since $\phi>0$, $\phi'\geqq 0$ and $\phi''\leqq 0$, ϕ' is a positive decreasing function on $[T_0,\infty)$ so that

$$\phi(t)-\phi(T_1) = \int_{T_1}^{t} \phi'(s)ds \geqq (t-T_1)\phi'(t),$$

that is.

$$(t-T_1)\phi'(t) \leqq \phi(t)-\phi(T_1) < \phi(t)$$

for $t \geqq T_1$. Thus, $t\phi'(t) \leqq \phi(t)$ for $t \geqq 2T_1$. Either of these inequalities implies that $\lim_{t\to\infty}\phi'(t)/\phi(t) = 0$. We consider

$$\left|\frac{\phi(y_\tau(t))}{\phi(y(t))} - 1\right| = \frac{\phi(y(t))-\phi(y_\tau(t))}{\phi(y(t))} = \frac{\tau(t)\phi'(\mu)}{\phi(y)}$$

for some μ, where $y_\tau(t)<\mu<y(t)$. Since ϕ is an increasing function, $\phi(y_\tau(t))<\phi(\mu)<\phi(y(t))$, which implies that $1/\phi(y(t))<1/\phi(y_\tau(t))$. Similarly, ϕ' is a decreasing function so that $\phi'(\mu)<\phi'(y_\tau(t))$. Thus, $\phi'(\mu)/\phi(y(t))<\phi'(y_\tau(t))/\phi(y_\tau(t))$. Moreover, because $k\geqq 1$, $\lim_{t\to\infty} y_\tau(t)=\infty$, which shows that

$$\lim_{t\to\infty}\frac{\phi(y_\tau(t))}{\phi(y(t))} = 1.$$

Thus, for any ε with $0<\varepsilon<1$, there is a $t_\varepsilon \geqq T_1$ such that

$$\phi(y_\tau(t)) \geqq (1-\varepsilon)\phi(y(t)), \qquad t \geqq t_\varepsilon.$$

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By Lemma 2, there is a $k_0 > 0$ and a $t_0 \geqq T_1$ such that $y_\tau(t) \geqq k_0 y(t)$ for $t \geqq t_0$. Thus, for $s \geqq T = \max\{t_0, t_\varepsilon, 2T_1\}$

$$s^{2n}\frac{y_T(s)}{y(s)}\frac{f[s, y_\tau(s)]}{\phi(y(s))} \geqq k_0(1-\varepsilon)s^{2n}\frac{f[s, y_\tau(s)]}{\phi(y(s))}$$

$$\geqq k_0(1-\varepsilon)s^{2n}f(s, Cs^j).$$

By Lemma 1 and a change of variables

$$\int_{T_1}^{t}\frac{s^{2k-1}y_{2k}(s)}{y(s)\phi(y(s))}ds \geqq N_{2k}\int_{T_1}^{t}\frac{y'(s)ds}{y(s)\phi(y(s))}$$

$$N_{2k}\int_{y(T_1)}^{y(t)}\frac{du}{u\phi(u)}.$$

We may now duplicate the rest of the arguments of the proof of Theorem 2 to obtain the following result.

Corollary 1. *Let f be nonlinear with strength coefficient $2n+1-j$. Let the associated function ϕ satisfy $\phi''(u) < 0$. Then* (2) *fulfills property P_0 if for all positive constants C*

$$\int^{\infty} t^{2n}f(t, Ct^j)dt = \infty.$$

Remark 1. If $\tau \equiv 0$, the ratio $y_\tau(t)/y(t)$ does not occur and the condition $\phi''(u) < 0$ may be omitted.

Remark 2. When $r \equiv 1$, $\tau \equiv 0$ and $k \equiv 0$, Theorem 1 reduces to the sufficiency of Theorem 1 of Kiguradze (1962). Under the same conditions the conclusion of Theorem 2 is that (2) satisfies property P_0, that is, condition A; Theorem 2 then coincides with Theorem 3 of Kiguradze (1962). In view of Remark 1, when $r \equiv 1$, $\tau \equiv 0$ and $j = 0$, Theorem 3 reduces to the sufficiency of Theorem 5 of Kiguradze (1962).

Remark 3. When $r \equiv 1$, $\tau \equiv 0$, $k = n$, the conclusion of Theorem 2 is that any C_n-solution $y(t)$ of (2) must satisfy $\lim_{t\to\infty} y_{2n}(t) = 0$. This is not quite condition B since we have not shown that any positive solution of (2) has this property. If $r(t) \equiv 1$, property P_n reduces to condition B. For suppose that $y(t)$ is any solution of (2) of type C_k, where $k < n$, then $D^{2n-1}y(t) = y_{2n-1}(t) < 0$. We now invoke a lemma most recently stated in generalized form by Ladas (1971) which we adapt here as follows.

Lemma 3. *Let y be a positive solution of* (2) *on $[t_0, \infty)$ with $r \equiv 1$. Then*

$$\lim_{t\to\infty} D^{2n}y(t) = \lim_{t\to\infty}(j-1)!\,(t-t_0)^{1-j}D^{2n+1-j}y(t)$$

where $j = 1, \cdots, 2n+1$.

Letting $j = 2$, we note that $D^{2n}y(t) > 0$ so that $\lim_{t\to\infty} D^{2n}y(t) \geqq 0$. On the other hand, $D^{2n-1}y(t) < 0$, which implies that the limit on the right-hand side is nonpositive and $\lim_{t\to\infty} D^{2n}y(t) = 0$.

When $r(t) \not\equiv 1$, the statement of this lemma is more complicated. We may, however, state a weak analogue.

LEMMA 3′. *Let $y(t)$ be a positive solution of (2) on $[t_0, \infty)$ with $j \geqq 2n + 1 - i$. Then*

$$\lim_{t\to\infty} y_{2n}(t) = \lim_{t\to\infty} (j-1)!\,(t-t_0)^{1-j} y_{2n+1-j}(t),$$

where $j = 1, \cdots, 2n + 1 - i$.

Now let y be a C_k-solution of (2) on $[T_0, \infty)$ for $k = 0, \cdots, n-1$. Then $y_{2n-1}(t) < 0$ for $t \geqq T_0$. We see then that if $i \leqq 2n - 1$, $2n - i \geqq 1$ and $2n - i + 1 \geqq 2$. Thus, we may let $j = 2$ in the statement of Lemma 3′ to obtain as before that $\lim_{t\to\infty} y_{2n}(t) = 0$. It follows that if $i \leqq 2n - 1$, the conclusion of Theorem 2 may be restated as: (2) fulfills condition B. Specifically, property P_n and condition B coincide for the equation

$$D^{n+1}[r(t)D^n y(t)] + y_\tau(t) f[t, y_\tau(t)] = 0, \qquad n \geqq 1$$

since $i = n \leqq 2n - 1$ if $n \geqq 1$. The same remark holds for

$$D^{n}[r(t)D^{n+1} y(t)] + y_\tau(t) f[t, y_\tau(t)] = 0, \qquad n \geqq 2$$

since $i = n + 1 \leqq 2n - 1$ if $n \geqq 2$.

REMARK 4. Use of the preliminary transformation $Y(t) = -y(t)$ will enable us to formulate criteria for the nonexistence of negative solutions y of (2) for which $-y$ is of type C_k ($k = 0, \cdots, n$). Moreover, if (2) has property P_k, then there are no negative solutions y such that $-y$ is of type C_r ($r = k+1, \cdots, n$) and any negative solution y for which $-y$ is of type C_k will satisfy $\lim_{t\to\infty} y_{2k}(t) = 0$.

3

In this section we consider the nonhomogeneous delay differential equation

$$D^{2n+1-i}[r(t)D^i y(t)] + y_\tau(t) f[t, y_\tau(t)] = Q(t). \tag{18}$$

Following the procedure introduced and used most effectively by Kartsatos and Manougian (to appear), we shall assume that R is a solution of the ordinary differential equation

$$D^{2n+1-i}[r(t)D^i R(t)] = Q(t). \tag{19}$$

This permits the transformation of (18) to a homogeneous delay equation of order $2n+1$ for which the methods of the previous sections may be applied. Let us assume that y is a positive solution of (18) and let $u(t) = y(t) - R(t)$. Then

$$D^{2n+1-i}[r(t)D^i u(t)] = D^{2n+1-i}[r(t)D^i y(t)] - D^{2n+1-i}[r(t)D^i R(t)]$$
$$= -y_\tau(t) f[t, y_\tau(t)] = -(u+R)_\tau(t) f[t,(u+R)_\tau(t)],$$

so that u is a solution of the homogeneous equation

$$D^{2n+1-i}[r(t)D^i u(t)] + (u+R)_\tau(t) f[t,(u+R)_\tau(t)] = 0. \tag{20}$$

Since $y(t) > 0$ for $t \geqq T_0$, $(u+R)_\tau(t) > 0$ for $t \geqq T_1$ and $D^{2n+1-i}[r(t)D^i u(t)] < 0$ for $t \geqq T_1$, which implies that $u(t)$ is a nonoscillatory solution of (20). If $u(t) < 0$, then we further transform the equation by letting $v(t) = -u(t)$. It follows that v is a positive solution of

$$D^{2n+1-i}[r(t)D^i v(t)] - (R-v)_\tau(t) f[t,(R-v)_\tau(t)] = 0. \tag{21}$$

DEFINITION 8. *A positive solution y of (18) is of type C_k^R on $[T_0,\infty)$ for $k = 0,\cdots,n$ if $u = y - R$ is a positive solution of (20) of type C_k on $[T_0,\infty)$.*

DEFINITION 9. *A positive solution y of (18) is of type $\hat{C}_k^R$ on $[T_0,\infty)$ for $k = 0,\cdots,n-1$ if $v = R - y$ is a positive solution of (21) which for $t \geqq T_0$ satisfies*

$$v_i(t) > 0,\ i = 0,\cdots,2k+1 \text{ and } (-1)^{i+1} v_i(t) > 0,\ i = 2k+2,\cdots,2n.$$

It is of type $\hat{C}_n^R$ if $v_i(t) > 0$ for $i = 0,\cdots,2n$.

DEFINITION 10. *Equation (18) has property P_k^R if, and only if, (20) has property P_k.*

A positive solution of (18) is evidently of type C_k^R $(k = 0,\cdots,n)$ or of type $\hat{C}_k^R$ $(k = 0,\cdots,n)$ for some k. We now formulate criteria under which (18) possesses property P_k^R.

THEOREM 3. *Let R be a solution of (19). Any solution y of type C_k^R will satisfy*

$$\lim_{t\to\infty} [y(t) - R(t)]_{2k} = 0 \tag{22}$$

if for all positive constants C

$$\int^\infty t^{2n-2k}(R_\tau(t) + Ct^{2k}) f[t, R_\tau(t) + Ct^{2k}]\,dt = \infty. \tag{23}$$

PROOF. Let y be a solution of (18) of type C_k^R on $[T_0,\infty)$. As in Theorem 1, we obtain

$$\int^t s^{2n-2k} Dy_{2n}(s)ds + \int^t s^{2n-2k}(u+R)_\tau(s) f[s,(u+R)_\tau(s)]ds = 0.$$

The first integral is handled as in Theorem 1. It remains to estimate the second integral. Since u is of type C_k on $[T_0,\infty)$, there are numbers $k_0 > 0$, $N_{2k} > 0$, $C > 0$ such that

$$u_\tau(t) \geqq k_0 u(t) \geqq k_0 N_{2k}^{-1} t^{2k} u_{2k}(t) \geqq k_0 N_{2k}^{-1} C t^{2k}$$

provided we assume that $\lim_{t\to\infty} u_{2k}(t) = C > 0$. The above inequalities will lead to the same contradiction as in the proof of Theorem 1.

THEOREM 4. *Suppose that R is as in the hypothesis of Theorem* 3 *and that ϕ is a function satisfying $\phi(y) > 0$, $\phi'(y) \geqq 0$ and* (9a). *Equation* (18) *possesses property P_k^R if in addition to* (23) *for all positive constants C*

$$\int^\infty t^{2n} f(t, R_\tau(t) + Ct^{2k})st/\phi(t) = \infty. \tag{24}$$

PROOF. As in the proof of Theorem 2, we first show that

$$\int^\infty t^{2n} f(t, R_\tau(t) + Ct^{2k-1})dt/\phi(t) = \infty, \qquad k \geqq 1$$

is sufficient to exclude solutions of type C_s^R $(s = k, \cdots, n)$. Then

$$\int^\infty t^{2n} f(t, R_\tau(t) + Ct^{2k+1})dt/\phi(t) = \infty, \qquad k \geqq 0$$

and hence (24) will exclude solutions of (18) of type C_s^R $(s = k+1, \cdots, n)$. The details are left to the reader.

REMARK 5. If $R(t) > 0$, the condition of Theorem 3 may be replaced by

$$\int^\infty t^{2n} f(t, Ct^{2k})dt = \infty. \tag{25}$$

The same replacement in Theorem 4 may be made. Thus (25) will ensure that (18) has property P_k^R.

REMARK 6. If R is oscillatory or negative, there can be no solutions of (18) of type $\hat{C}_k^R$. The conclusion of Theorem 4 is thereby strengthened.

REMARK 7. Use of the transformation $w_k(t) = y_{2n}(t) y_{2k-1}^{-1}(t)$ results in a stronger criterion for the nonexistence of C_k^R-solutions of (18) independent of the existence of an auxiliary function $\phi(t)$ satisfying the hypotheses of Theorem 4. Specifically, we may obtain:

THEOREM 5. *Let R be a solution of* (19) *with $R(t) = 0(t^{2k-1-\varepsilon})$ for some ε such that $0 < \varepsilon < 2k-1$;* (18) *has no positive solutions of type C_s^R $(s = k, \cdots, n)$ if for all positive constants C*

$$\int^{\infty} t^{2k-1} f(t, R_\tau(t) + Ct^{2k-1}) dt = \infty.$$

REMARK 8. As in Remark 4, the preliminary transformation $Y(t) = -y(t)$ will enable us to formulate analogous criteria for the nonexistence of certain negative solutions y of (18) for which $-y$ is a positive C_k^{-R}-solution of the transformed non-homogeneous delay differential equation.

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