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POSITIVE SOLUTIONS FOR A DEGENERATE KIRCHHOFF PROBLEM

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Abstract By assuming that the Kirchhoff term has K degeneracy points and other appropriated conditions, we have proved the existence of at least K positive solutions other than those obtained in Santos Júnior and Siciliano [Positive solutions for a Kirchhoff problem with vanishing nonlocal term, J. Differ. Equ. **265** (2018), 2034–2043], which also have ordered $H_0^1(\Omega)$-norms. A concentration phenomena of these solutions is verified as a parameter related to the area of a region under the graph of the reaction term goes to zero.

Keywords: degenerate Kirchhoff problem; variational methods; concentration phenomena

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Secondary: 35Q74

1. Introduction

In the last decade, non-local Kirchhoff problems have turned into an extremely active field of research in nonlinear partial differential equations. In particular, from different perspectives and approaches, many authors have investigated problems like (or its variations)

$$\begin{cases} -m(\|u\|^2)\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a smooth bounded domain, $\|u\| := |\nabla u|_2$ is the usual norm in the Sobolev space $H_0^1(\Omega)$, $m : [0, \infty) \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are continuous functions with m positive and away from zero. Without any intention to provide a complete overview about the matter, we just refer the reader to the following references [1, 7, 9–12] which can give a first glimpse about mathematical, physical and historical aspects involving stationary Kirchhoff models.

Nevertheless, most recently, it has grown the interest of some authors, attracted by inherent mathematical difficulties, in investigating the case in which m is a degenerate function, i.e., when m is not required to be bounded from below by a positive constant. In fact, Ambrosetti and Arcoya [2, 3] took a first step towards the understanding of this class of problems. In [3], the function m is allowed to verify $m(0) = 0$ and/or $\lim_{t\to+\infty} m(t) = 0$. It is worth to point out that in such a paper remains the assumption $m(t) > 0$ for $t > 0$. On the other hand, Theorem 2.1 (see also Remark 2.2) in [2] assures the existence of a non-trivial solution even if $m(t) = 0$ for some $t > 0$.

Inspired by [5, 8], in [13], the authors considered the case in which m has many different positive zeroes and they proved that the number of positive solutions of (P) is related to the number of points which vanish m. In order to state the result, we denote by t_k, $k = 1, 2, \dots K$, the zeroes of m, $F(t) := \int_0^t f$, for every $t \geq 0$, λ_1 the first eigenvalue of Laplacian operator with homogeneous Dirichlet boundary condition and

$$\alpha_k := \max_{u\in H_0^1(\Omega), \|u\|\leq t_k^{1/2}} \int_\Omega F^*(u)\,\mathrm{d}x, \quad k \in \{1, \dots, K\}. \tag{1.1}$$

where

$$F^*(t) = \int_0^t f^*(s)\,\mathrm{d}s, \text{ with } f^*(t) = \begin{cases} f(0) & \text{if } t < 0, \\ f(t) & \text{if } 0 \leq t < s_*, \\ 0 & \text{if } s_* \leq t. \end{cases} \tag{1.2}$$

and $s_* > 0$ such that $f(t) > 0$ in $(0, s_*)$ and $f(s_*) = 0$.

It is proved the following theorem.

Theorem 1.1 (Santos Júnior and Siciliano [13]). *If $m : [0, \infty) \to \mathbb{R}$ is a continuous function and $f : [0, \infty) \to \mathbb{R}$ is a locally Lipschitz continuous function satisfying*:

(m) there exist positive numbers $0 =: t_0 < t_1 < t_2 < \cdots < t_K$ such that

$$m(t_k) = 0 \text{ and} m > 0 \text{in} (t_{k-1}, t_k), \text{for all} k = 1, \dots, K;$$

(f) there exists $s_* > 0$ such that $f(t) > 0$ in $(0, s_*)$ and $f(s_*) = 0$;

(A) $\alpha_k < \frac{1}{2}\int_{t_{k-1}}^{t_k} m(s)\,ds < |\Omega| \int_0^{s_*} f$, for all $k = 1, \dots, K$,

then, problem (P) possesses at least K non-trivial positive solutions. Furthermore, these solutions are ordered in the $H_0^1(\Omega)$-norm, i.e.,

$$0 < \|u_1\|^2 < t_1 < \|u_2\|^2 < t_2 < \ldots < t_{K-1} < \|u_K\|^2 < t_K.$$

In this paper, we improve Theorem 1.1 by proving under some appropriated conditions on functions m and f, the existence of K further positive solutions for (1.1). Moreover, we show concentration phenomena of these solutions at the zeroes of m, when the area $F(s_*) = \int_0^{s_*} f$ tends to zero. Specifically, we have the following result.

Theorem 1.2. *Suppose that hypotheses (m) and (f) hold.*

(i) If α_k given by (1.1) satisfies that
(A') $\alpha_k \leq \frac{1}{2}\int_{t_{k-1}}^{t_k} m(s)\, ds$, for all $k = 1, \dots, K$,
then, problem (P) possesses at least $K-1$ positive solutions which are ordered in $H_0^1(\Omega)$-norm, i.e.,

$$t_1 < \|v_2\|^2 < t_2 < \cdots < t_{K-1} < \|v_K\|^2 < t_K.$$

(ii) Furthermore, if, in addition, the following condition holds
(mf) there exists $\gamma := \lim_{t\searrow 0} \frac{f(t)}{t} \in (0,\infty)$ and $m(0) < \gamma/\lambda_1$,
then there is one more positive solution $v_1 \in H_0^1(\Omega)$ with

$$0 < \|v_1\|^2 < t_1.$$

(iii) If we assume the stronger condition (A) instead of (A'), then for every $k = 1, \dots, K$, the solution u_k provided in Theorem 1.1 verifies $u_k \neq v_k$ and hence problem (P) has at least $2\,K$ positive solutions.

(iv) Moreover, if for each $\lambda > 0$, we denote by (P_λ) the problem (P) with $F(s_*) = \lambda$ and by $v_{k,\lambda}$, $k = 1, \dots, K$, the solutions of (P_λ) obtained in (i)–(ii), then,

$$\lim_{\lambda\searrow 0} \|v_{k,\lambda}\|^2 = t_{k-1}, \quad \forall\, k = 1, \dots, K.$$

Observe that hypothesis (A') is weaker than (A).

Let f_λ be a function f satisfying (f), (mf), (A') and $F(s_*) = \lambda$, where λ is a positive parameter. In Theorem 1.2-iv), we are interested in proving that if $f_\lambda^* := (f_\lambda)^*$ and λ goes to zero, then the square of the H_0^1-norm of the solutions of

$$\begin{cases} -m(\|u\|^2)\Delta u = f_\lambda^*(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P$_\lambda$}$$

are concentrating at the points which vanish function m. This concentration result is somewhat more general than those commonly found in the literature, mainly because the nonlinearity is not required tending to zero in a uniform way. In fact, in [6], for example, in order to study concentration phenomena of the solutions, when λ tends to infinity, of the problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

the authors require λf goes uniformly to infinity. Instead, we are just assuming $F(s_*) \to 0$. This assumption allows f_λ^* going to zero in different ways for each $t \in (0, s_*)$.

The paper is organized as follows. In §2, we provide a technical lemma which plays an important role in the existence result. In §3, we prove items (i), (ii) and (iii) of Theorem 1.2 by looking for critical points with norm in the interval $(t_{k-1}^{1/2}, t_k^{1/2})$ of a truncated functional I_k defined in $H_0^1(\Omega)$. In §4, we study the part (iv) of the cited theorem, that is, the concentration phenomena of the solutions obtained in Theorem 1.2 as $\lambda = F(s_*)$ tends to zero.

2. A technical lemma

Taking into account that f^* is continuous and bounded, we deduce by the compact embedding Rellich–Kondrachov theorem that the function $J : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$J(v) = \int_\Omega F^*(v)\,\mathrm{d}x, \quad \forall\, v \in H_0^1(\Omega)$$

is weakly continuous and then it attains its maximum in any closed ball in $H_0^1(\Omega)$. In particular, the map $\alpha : [0,\infty) \to \mathbb{R}$ given by

$$\alpha(t) = \max_{u\in H_0^1(\Omega), \|u\|\le t^{1/2}} \int_\Omega F^*(u)\,\mathrm{d}x, \quad \forall\, t \geq 0, \tag{2.1}$$

is well defined; i.e., the set

$$S_t := \{u \in H_0^1(\Omega) : \|u\|^2 \leq t \text{ and } \int_\Omega F^*(u)\,\mathrm{d}x = \alpha(t)\} \tag{2.2}$$

is non-empty for every $t \geq 0$.

The following technical lemma collects the main properties of α and S_t. In order to state it, we define the function $M_k : \mathbb{R} \to \mathbb{R}$ by

$$M_k(t) = \int_0^t m_k, \text{ with } m_k(t) = \begin{cases} m(t), & \text{if } t_{k-1} \leq t \leq t_k, \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

Lemma 2.1. *Assume that* (f) *holds and consider the map* $\alpha : [0,\infty) \to \mathbb{R}$ *defined by* (2.1) *and the set* S_t *given by* (2.2) *for* $t \geq 0$. *We have:*

(i) for every $t > 0$, $\alpha(t) > 0$ *and each maximizer* $u \in S_t$, *satisfies*

- $\|u\|^2 = t$, $0 < u \leq s_*$,
- $u \in C^{2,\beta}(\overline{\Omega})$ *for some* $\beta \in (0,1)$ *and*
- $\partial u/\partial\tau < 0$ *on* $\partial\Omega$, *where* τ *stands for the outward unit normal vector;*

(ii) the map α *is differentiable in* $(0,\infty)$ *with* $\alpha'(t) = (1/2t)\max_{u\in S_t}\int_\Omega f^*(u)u\,dx$ *and* α' *is upper semicontinuous. In particular,* α *is increasing. Moreover, if there exists* $\gamma = \lim_{t\to 0} f(t)/t$, *then*

$$\lim_{t\searrow 0} \alpha'(t) = \frac{\gamma}{2\lambda_1} \quad \textit{and} \quad \liminf_{t\to t_*} \alpha'(t) > 0, \quad \forall\, t_* > 0;$$

(iii) if, in addition, condition (m) holds, then for every $k \in \{2,\ldots,K\}$, *there exists* $\delta_{k-1} > 0$ *such that the map* $g_k(t) = (1/2)M_k(t) - \alpha(t)$ *is decreasing in* $(0, t_{k-1} + \delta_{k-1})$. *Moreover, if (mf) holds, the same is true in the case* $k = 1$.

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Proof. (i) Fix $t > 0$. For any positive function $\varphi \in H_0^1(\Omega)$ with $\|\varphi\|^2 \le t$, we have

$$0 < \int_\Omega F^*(\varphi)\,\mathrm{d}x \le \max_{v\in H_0^1(\Omega),\|v\|\le t^{1/2}} \int_\Omega F^*(v)\,\mathrm{d}x = \alpha(t),$$

and then $\alpha(t) > 0$. In particular, if $u \in S_t$, then $u \ne 0$. Since

$$\int_\Omega F^*(v)\,\mathrm{d}x \le \int_\Omega F^*(|v|)\,\mathrm{d}x \quad \text{and} \quad \|v\| = \||v|\|,$$

with strict inequality if $v \not\equiv |v|$, we derive that a maximizer $u \in S_t$ is also non-negative.

In addition, we have $\|u\| = t^{1/2}$. Indeed, if, by contradiction, it would be satisfied that $\|u\| < t^{1/2}$, then u would be a local maximum point in the interior of the ball $B_{t^{1/2}}(0)$ of the functional $J \in C^1(H_0^1(\Omega))$. We would get

$$0 = J'(u)v = \int_\Omega f^*(u)v\,\mathrm{d}x, \quad \forall\, v \in H_0^1(\Omega),$$

and then $f^*(u) = 0$, contradicting that u is non-negative and non-trivial and proving the claim.

In particular, by Lagrange multipliers theorem, there exists $\lambda \in \mathbb{R}$ verifying

$$\int_\Omega f^*(u)v\,\mathrm{d}x = 2\lambda \int_\Omega \nabla u \nabla v\,\mathrm{d}x, \quad \forall\, v \in H_0^1(\Omega).$$

Choosing $v = u$ in the previous equality, we obtain

$$\lambda = \frac{1}{2t}\int_\Omega f^*(u)u\,\mathrm{d}x > 0.$$

Thus, u is a non-trivial and non-negative weak solution of

$$\begin{cases} -\Delta u = \dfrac{1}{2\lambda} f^*(u) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

To see that $u \le s_*$, it is sufficient to choose $v = (u - s_*)^+$ as test function in the above boundary value problem. Since $f^*(u) = f(u) \in L^\infty(\Omega)$, it also follows from [4, Theorem 0.5(ii)] that $u \in C^{1,\beta}(\overline{\Omega})$ for some $0 < \beta < 1$. Now, since f is locally Lipschitz continuous, we get $f(u) \in C^{0,\beta}(\overline{\Omega})$. Hence, by [4, Theorem 0.5(iii)], we have $u \in C^{2,\beta}(\overline{\Omega})$. Applying then the strong maximum principle and Hopf lemma, we obtain $u > 0$ in Ω and $\partial u/\partial\tau < 0$ on $\partial\Omega$ and case (i) is proved.

(ii) We begin by proving the continuity of α in $(0, \infty)$. Let $\{t_n\}$ be a sequence of positive numbers with $t_n \to t_*$, $t_* > 0$, and $u_n \in S_{t_n}$. Since $\{u_n\}$ is bounded in $H_0^1(\Omega)$, there exists $u_* \in H_0^1(\Omega)$ such that, up to a subsequence, $\{u_n\}$ is weakly convergent to u_*. Using that the norm is weakly lower semicontinuous and the weak continuity of J,

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we get

$$\|u_*\|^2 \le \liminf_{n\to\infty} \|u_n\|^2 = t_* \text{ and } \lim_{n\to\infty} \alpha(t_n) = \lim_{n\to\infty} \int_\Omega F^*(u_n)\,\mathrm{d}x = \int_\Omega F^*(u_*)\,\mathrm{d}x.$$

Hence, by definition of α, it is clear that

$$\int_\Omega F^*(u_*) \le \alpha(t_*).$$

On the other hand, for the converse inequality, choosing $u_0 \in S_{t_*}$ we have

$$\int_\Omega F^*\left(\frac{t_n^{1/2}}{t_*^{1/2}} u_0\right) \mathrm{d}x \le \int_\Omega F^*(u_n)\,\mathrm{d}x, \quad \forall\, n \in \mathbb{N},$$

and passing to the limit as n goes to infinity, we get

$$\int_\Omega F^*(u_0)\,\mathrm{d}x = \alpha(t_*) \le \int_\Omega F^*(u_*)\,\mathrm{d}x,$$

concluding that

$$\lim_{n\to\infty} \alpha(t_n) = \lim_{n\to\infty} \int_\Omega F^*(u_n)\,\mathrm{d}x \to \int_\Omega F^*(u_*)\,\mathrm{d}x = \alpha(t_*),$$

and the continuity of α has been proved.

Fix now $t_0 > 0$. For each $t \in (-t_0, \infty)$, we take a function u_{t_0} (respectively, u_{t_0+t}) in S_{t_0} (respectively, in S_{t+t_0}). Thus, by definition of S_{t+t_0},

$$\begin{aligned}
\frac{\alpha(t_0+t)-\alpha(t_0)}{t} &= \frac{1}{t}\int_\Omega [F^*(u_{t_0+t}) - F^*(u_{t_0})]\,\mathrm{d}x \\
&\ge \frac{1}{t}\int_\Omega \left[F^*\left(\left(1+\frac{t}{t_0}\right)^{1/2} u_{t_0}\right) - F^*(u_{t_0})\right]\mathrm{d}x \\
&= \frac{1}{2t_0(1+tc_t/t_0)^{1/2}} \int_\Omega f^*\left(\left(1+\frac{tc_t}{t_0}\right)^{1/2} u_{t_0}\right) u_{t_0}\,\mathrm{d}x,
\end{aligned}$$

where $0 < |c_t| < 1$. Consequently,

$$\liminf_{t\to 0}\left[\frac{\alpha(t_0+t)-\alpha(t_0)}{t}\right] \ge \frac{1}{2t_0}\int_\Omega f^*(u_{t_0})u_{t_0}\,\mathrm{d}x, \quad \forall\, u_{t_0} \in S_{t_0}.$$

Since f^* is bounded, it is a consequence of the definition of S_t that $\max_{u\in S_{t_0}} \int_\Omega f^*(u)u\,\mathrm{d}x$ is attained. Therefore,

$$\liminf_{t\to 0}\left[\frac{\alpha(t_0+t)-\alpha(t_0)}{t}\right] \ge \frac{1}{2t_0}\max_{u\in S_{t_0}} \int_\Omega f^*(u)u\,\mathrm{d}x. \tag{2.4}$$

On the other hand, by definition of S_{t_0}, we get

$$\frac{\alpha(t_0+t)-\alpha(t_0)}{t} \leq \frac{1}{t}\int_\Omega \left[F^*\left(u_{t_0+t}\right) - F^*\left(\left(1-\frac{t}{t_0+t}\right)^{1/2} u_{t_0+t}\right)\right] \mathrm{d}x$$
$$= \frac{1}{2(t_0+t)(1-td_t/(t_0+t))} \int_\Omega f^*\left(\left(1-\frac{td_t}{t_0+t}\right)^{1/2} u_{t_0+t}\right)\left(1-\frac{td_t}{t_0+t}\right)^{1/2} u_{t_0+t}\,\mathrm{d}x,$$

where $0<|d_t|<1$. Therefore,

$$\frac{\alpha(t_0+t)-\alpha(t_0)}{t} \leq \frac{1}{2(t_0+t)(1-td_t/(t_0+t))} \max_{u\in S_{t_0+t(1-d_t)}} \int_\Omega f^*(u)u\,\mathrm{d}x. \tag{2.5}$$

□

Claim. $\limsup_{t\to 0}[\max_{u\in S_{t_0+t(1-d_t)}} \int_\Omega f^*(u)u\,\mathrm{d}x] \leq \max_{u\in S_{t_0}} \int_\Omega f^*(u)u\,\mathrm{d}x$.

Let $\{t_n\}$ be a sequence of real numbers converging to zero. For each n, choose $u_n \in S_{t_0+t_n(1-d_{t_n})}$ such that

$$\int_\Omega f^*(u_n)u_n\,\mathrm{d}x = \max_{u\in S_{t_0+t_n(1-d_{t_n})}} \int_\Omega f^*(u)u\,\mathrm{d}x.$$

Since $\|u_n\|^2 \leq t_0 + t_n(1-d_{t_n})$, we have that $\{u_n\}$ is bounded in $H_0^1(\Omega)$ and, as before, there exists $u_0\in H_0^1(\Omega)$ such that, up to a subsequence,

$$u_n \rightharpoonup u_0 \text{ in} H_0^1(\Omega),$$

and

$$\|u_0\|^2 \leq \liminf_{n\to\infty}\|u_n\|^2 \leq t_0.$$

Moreover, by the continuity of α,

$$\alpha(t_0) = \lim_{n\to\infty}\alpha(t_0+t_n(1-d_{t_n})) = \lim_{n\to\infty}\int_\Omega F^*(u_n)\,\mathrm{d}x = \int_\Omega F^*(u_0)\,\mathrm{d}x.$$

Consequently, $u_0\in S_{t_0}$ and, by (i), $\|u_0\|^2 = t_0$. The claim follows now by observing that the weak convergence of u_n to u_0 implies that

$$\limsup_{n\to\infty} \max_{u\in S_{t_0+t_n(1-d_{t_n})}} \int_\Omega f^*(u)u\,\mathrm{d}x = \limsup_{n\to\infty}\int_\Omega f^*(u_n)u_n\,\mathrm{d}x$$
$$= \int_\Omega f^*(u_0)u_0\,\mathrm{d}x$$
$$\leq \max_{u\in S_{t_0}}\int_\Omega f^*(u)u\,\mathrm{d}x.$$

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Now, by previous claim, passing to the upper limit as t tends to zero in (2.5), we get

$$\limsup_{t\to 0}\left[\frac{\alpha(t_0+t)-\alpha(t_0)}{t}\right] \le \frac{1}{2t_0}\max_{u\in S_{t_0}}\int_\Omega f^*(u)u\,\mathrm{d}x.$$

By comparing this inequality with (2.4), we conclude that α is differentiable in t_0 and

$$\alpha'(t_0) = \frac{1}{2t_0}\max_{u\in S_{t_0}}\int_\Omega f^*(u)u\,\mathrm{d}x.$$

To prove the upper semicontinuity of the derivative, let $\{t_n\}$ be a sequence of real numbers converging to a positive number t_0. For each n, choose $u_n \in S_{t_n}$ such that

$$\int_\Omega f^*(u_n)u_n\,\mathrm{d}x = \max_{u\in S_{t_n}}\int_\Omega f^*(u)u\,\mathrm{d}x.$$

Since $u_n \in S_{t_n}$, as previously, it follows that there exists $u_0 \in H_0^1(\Omega)$ such that, up to a subsequence,

$$\begin{aligned}
&u_n \rightharpoonup u_0 \ \mathrm{in} H_0^1(\Omega),\\
&\|u_0\|^2 \le \liminf_{n\to\infty}\|u_n\|^2 \le t_0,\\
&\lim_{n\to\infty}\alpha(t_n) = \lim_{n\to\infty}\int_\Omega F^*(u_n)\,\mathrm{d}x = \int_\Omega F^*(u_0)\,\mathrm{d}x,\\
&\lim_{n\to\infty}\int_\Omega f^*(u_n)u_n\,\mathrm{d}x = \int_\Omega f^*(u_0)u_0\,\mathrm{d}x.
\end{aligned}$$

Using the continuity of α,

$$\alpha(t_0) = \int_\Omega F^*(u_0)\,\mathrm{d}x,$$

and consequently

$$u_0 \in S_{t_0},$$

and by the computation of the derivative $\alpha'(t_0)$, we obtain

$$\limsup_{n\to\infty}\alpha'(t_n) = \limsup_{n\to\infty}\frac{1}{2t_n}\int_\Omega f^*(u_n)u_n\,\mathrm{d}x = \frac{1}{2t_0}\int_\Omega f^*(u_0)u_0\,\mathrm{d}x \le \alpha'(t_0).$$

To prove the last part of (ii), let $t_n \searrow 0$ and $u_n \in S_{t_n}$ be such that

$$\alpha'(t_n) = \frac{1}{2t_n}\int_\Omega f^*(u_n)u_n\,\mathrm{d}x.$$

Define $v_n := u_n/t_n^{1/2}$ and observe that, by (i), $\|u_n\| = t_n^{1/2}$ and thus $\{v_n\}$ is bounded in $H_0^1(\Omega)$. Consequently, there exists v_* such that, up to a subsequence,

$$v_n \rightharpoonup v_* \quad \text{in } H_0^1(\Omega). \tag{2.6}$$

Thus, by Lebesgue Dominated Convergence Theorem and the definition of γ, we get

$$\alpha'(t_n) = \frac{1}{2t_n}\int_\Omega f^*(u_n)u_n\,\mathrm{d}x = \frac{1}{2}\int_\Omega \left[\frac{f^*(t_n^{1/2}v_n)}{t_n^{1/2}v_n}\right] v_n^2\,\mathrm{d}x \to \frac{\gamma}{2}\int_\Omega v_*^2\,\mathrm{d}x. \tag{2.7}$$

Moreover, by the definition of γ again and the L'Hospital rule, we have $\lim_{t\searrow 0} F^*(t)/t^2 = \gamma/2$. Thence, from (2.6), up to a subsequence, we obtain

$$\lim_{n\to\infty}\frac{F^*(u_n)}{t_n} = \lim_{n\to\infty}\left[\frac{F^*(t_n^{1/2}v_n)}{(t_n^{1/2}v_n)^2}\right] v_n^2 = \frac{\gamma}{2}v_*^2 \text{ a.e. in } \Omega.$$

and

$$\left|\frac{F^*(u_n)}{t_n}\right| = \left|\frac{F^*(t_n^{1/2}v_n)}{(t_n^{1/2}v_n)^2}\right| |v_n^2| \le \gamma h(x) \text{ a.e. in } \Omega,$$

for some $h \in L^1(\Omega)$. (Observe that the convergence (2.6) implies, passing to a subsequence if necessary, the existence of such a function $h(x)$). Therefore, by Lebesgue Dominated Convergence Theorem

$$\int_\Omega \frac{F^*(u_n)}{t_n}\,\mathrm{d}x \to \frac{\gamma}{2}\int_\Omega v_*^2\,\mathrm{d}x. \tag{2.8}$$

If it were $v_* = 0$, then by (2.7) and (2.8), we would conclude that

$$0 = \lim_{n\to\infty}\alpha'(t_n) = \frac{1}{2}\lim_{n\to\infty}\frac{1}{t_n}\int_\Omega f^*(u_n)u_n\,\mathrm{d}x = \lim_{n\to\infty}\int_\Omega \frac{F^*(u_n)}{t_n}\,\mathrm{d}x. \tag{2.9}$$

On the other hand, since $u_n \in S_{t_n}$, it follows again from Lebesgue Dominated Convergence Theorem and L'Hospital rule, that

$$\lim_{n\to\infty}\int_\Omega \frac{F^*(u_n)}{t_n}\,\mathrm{d}x \ge \lim_{n\to\infty}\int_\Omega \frac{F^*(t_n^{1/2}\varphi_1)}{(t_n^{1/2}\varphi_1)^2}\varphi_1^2\,\mathrm{d}x = \frac{\gamma}{2\lambda_1}, \tag{2.10}$$

where φ_1 is the eigenfunction of Laplacian operator with homogeneous Dirichlet boundary condition, associated with λ_1 with $\|\varphi_1\| = 1$. By comparing (2.9) and (2.10), we get a contradiction proving that $v_* \neq 0$ ($v_* \ge 0$).

Finally, since $u_n \in S_{t_n}$ and $\|u_n\| = t_n^{1/2}$, we can again use the Lagrange multipliers theorem (see proof of item (i)) to conclude that

$$\begin{cases} -\Delta u_n = \dfrac{1}{2\alpha'(t_n)} f^*(u_n) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Consequently,

$$\int_\Omega \nabla v_n \nabla w\,\mathrm{d}x = \frac{1}{2\alpha'(t_n)}\int_\Omega \left[\frac{f^*(t_n^{1/2}v_n)}{t_n^{1/2}v_n}\right] v_n w\,\mathrm{d}x, \quad \forall\, w \in H_0^1(\Omega). \tag{2.11}$$

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By passing to the limit as n goes to infinity in (2.11), it follows from (2.6) and (2.7) that

$$\int_\Omega \nabla v_* \nabla w \, \mathrm{d}x = \frac{1}{\int_\Omega v_*^2 \, \mathrm{d}x} \int_\Omega v_* w \, \mathrm{d}x, \quad \forall\, w \in H_0^1(\Omega).$$

Since $v^* \geq 0$, $v^* \not\equiv 0$, we have $\int_\Omega v_*^2 \, \mathrm{d}x = 1/\lambda_1$ and (2.7) implies that $\lim_{n\to\infty} \alpha'(t_n) = \gamma/(2\lambda_1)$.

Finally, suppose by contradiction that there exists a sequence of positive numbers $\{t_n\}$ and $\{u_n\} \subset S_{t_n}$ such that $t_n \to t_*$ (for some $t_* > 0$), $\|u_n\| = t_n^{1/2}$, $\int_\Omega F^*(u_n) \, \mathrm{d}x = \alpha(t_n)$ and

$$\alpha'(t_n) = \frac{1}{2t_n} \int_\Omega f^*(u_n) u_n \, \mathrm{d}x \to 0. \tag{2.12}$$

In particular, $\{u_n\}$ is bounded and hence there exists $u_* \in H_0^1(\Omega)$ with $u_* \geq 0$ (because $u_n > 0$) and such that, up to a subsequence,

$$u_n \rightharpoonup u_* \ \mathrm{in} H_0^1(\Omega).$$

The weak continuity of J and of the map $\int_\Omega f^*(v) v \, \mathrm{d}x$, (2.12), (i) and the continuity of α imply by taking limits when n tends to infinity that

$$\int_\Omega F^*(u_*) \, \mathrm{d}x = \alpha(t_*) > 0 \tag{2.13}$$

and

$$\frac{1}{2t_*} \int_\Omega f^*(u_*) u_* \, \mathrm{d}x = 0 \tag{2.14}$$

Identity (2.14) implies that $u_* = 0$, but that leads us to a contradiction with (2.13). The result is now proved.

(iii) Let us consider first $k = 2, \ldots, K$. By (ii), $g_k : (0, \infty) \to \mathbb{R}$ is differentiable with $g_k'(t) = m_k(t) - \alpha'(t)$ for every $t > 0$. If $t \in (0, t_{k-1}]$, by the definition of m_k we have $g_k(t) = -\alpha(t)$ and thus

$$g_k'(t) = -\alpha'(t) = -\frac{1}{2t} \max_{u \in S_t} \int_\Omega f^*(u) u \, \mathrm{d}x < 0, \quad \forall\, t \in (0, t_{k-1}].$$

On the other hand, by (m), $\lim_{t \searrow t_{k-1}} m_k(t) = m_k(t_{k-1}) = 0$ and (ii)

$$\limsup_{t \searrow t_{k-1}} g_k'(t) = m_k(t_{k-1}) - \liminf_{t \searrow t_{k-1}} \alpha'(t) = -\liminf_{t \searrow t_{k-1}} \alpha'(t) < 0.$$

Finally, assuming hypothesis (mf), if $k = 1$ then, by $g_1(0) = 0$ and (ii), we have

$$\lim_{t \searrow 0} g_1'(t) = \frac{1}{2}\left[m(0) - \frac{\gamma}{\lambda_1}\right] < 0.$$

The result follows.

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3. Existence of K further solutions for (p)

As in [13], our approach to prove the existence of a solution v_k for each $k \in \{1, \ldots, K\}$ is based on looking for critical points with norm in the interval $(t_{k-1}^{1/2}, t_k^{1/2})$ of the truncated functional $I_k : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$I_k(u) = \frac{1}{2} M_k(\|u\|^2) - \int_\Omega F^*(u)\,\mathrm{d}x, \quad \forall\, u \in H_0^1(\Omega), \tag{3.1}$$

where F^* is given by (1.2) and M_k by (2.3).

Indeed, every critical point $v_k \in H_0^1(\Omega)$ of I_k with $\|v_k\|^2 \in (t_{k-1}, t_k)$ satisfies

$$\int_\Omega \nabla v_k \nabla \varphi\,\mathrm{d}x = \frac{1}{m(\|v_k\|^2)} \int_\Omega f^*(v_k)\varphi\,\mathrm{d}x \geq 0, \quad \forall\, \varphi \in H_0^1(\Omega), \varphi \geq 0,$$

(because $m_k(\|v_k\|^2) = m(\|v_k\|^2)$). Hence, by the maximum principle, we conclude that $v_k \geq 0$ in Ω. In addition, by choosing $\varphi = (v_k - s_*)^+$ as a test function, we get

$$0 \leq m(\|v_k\|^2)\|(v_k - s_*)^+\|^2 = \int_\Omega f^*(v_k)(v_k - s_*)^+\,\mathrm{d}x = 0.$$

Since $m(\|v_k\|^2) > 0$ (by (m)), we conclude that $0 \leq v_k \leq s_*$ is a weak solution of (P). By elliptic regularity and the maximum principle, $v_k > 0$ is a classical solution.

Proof of case (i) of Theorem 1.2. Fix $k = 2, \ldots, K$. It follows from (m) that $M_k \geq 0$ and thus by the definition of α_k that

$$I_k(u) \geq -\int_\Omega F^*(u)\,\mathrm{d}x \geq -\alpha_k,$$

for all $u \in H_0^1(\Omega)$ with $\|u\| \leq t_k^{1/2}$. Consequently, it makes sense to define the real number

$$b_k := \inf_{u \in H_0^1(\Omega), \|u\| \leq t_k^{1/2}} I_k(u) = \inf_{0 < t \leq t_k} \left\{ \inf_{u \in H_0^1(\Omega), \|u\|^2 = t} I_k(u) \right\} = \inf_{0 < t \leq t_k} g_k(t).$$

We deduce from (iii) of Lemma 2.1 that

$$b_k < g_k(t_{k-1}) = -\alpha_{k-1}. \tag{3.2}$$

To conclude the proof, it suffices to show that b_k is a critical value of I_k with an associated critical point $v_k \in H_0^1(\Omega)$ of norm $t_{k-1}^{1/2} < \|v_k\| < t_k^{1/2}$. To prove it, let $\{u_n\} \subset H_0^1(\Omega)$ be

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a minimizing sequence: i.e., such that

$$\lim_{n\to\infty} I_k(u_n) = b_k \quad \text{and} \quad \|u_n\| \le t_k^{1/2}.$$

Hence, up to a subsequence, there exists $v_k \in H_0^1(\Omega)$ and $\tau_k \in [0, t_k]$ such that

$$u_n \rightharpoonup v_k \text{ in} H_0^1(\Omega) \quad \text{and} \quad \|u_n\| \to \tau_k^{1/2},$$
$$\|v_k\| \le \liminf_{n\to\infty} \|u_n\| = \tau_k^{1/2} \le t_k^{1/2},$$

and, by the weak continuity of J,

$$\lim_{n\to\infty} \int_\Omega F^*(u_n)\,\mathrm{d}x = \int_\Omega F^*(v_k)\,\mathrm{d}x.$$

Observe that

$$b_k = \lim_{n\to\infty} I_k(u_n) = \frac{1}{2}\lim_{n\to\infty} M_k(\|u_n\|^2) - \lim_{n\to\infty}\int_\Omega F^*(u_n)\,\mathrm{d}x = \frac{1}{2}M_k(\tau_k) - \int_\Omega F^*(v_k)\,\mathrm{d}x. \tag{3.3}$$

Moreover, it is worth to point out that

$$\tau_k > t_{k-1}, \tag{3.4}$$

because in the contrary case we will have $M_k(\tau_k) = 0$, $\|v_k\|^2 \le t_{k-1}$ and we will deduce by (3.3) and the definition of α_{k-1} that

$$b_k = -\int_\Omega F^*(v_k)\,\mathrm{d}x \ge -\alpha_{k-1},$$

contradicting (3.2). Therefore, (3.4) holds. Now, we can prove that $\|v_k\|^2 = \tau_k$. Indeed, we get a contradiction if we assume that $\|v_k\|^2 < \tau_k$ because this implies by (m) and (3.4) that $M_k(\tau_k) > M_k(\|v_k\|^2)$ and hence (3.3) gives

$$b_k > \frac{1}{2}M_k(\|v_k\|^2) - \int_\Omega F^*(v_k)\,\mathrm{d}x = I_k(v_k),$$

a contradiction with the definition of b_k. Therefore, $\{u_n\}$ is weakly convergent to v_k with the norms $\{\|u_n\|\}$ converging also to $\|v_k\|$, which is equivalent to the strong convergence of $\{u_n\}$ to v_k. As a consequence,

$$b_k = I_k(v_k)$$

and v_k is a minimum point of I_k on the ball with radio $\tau_k^{1/2}$ and centered at the origin. To prove that v_k is a critical point of I_k with level $b_k < 0$ (by (3.2) and (i) of Lemma 2.1), it remains us to show that $\|v_k\| < t_k^{1/2}$. Suppose by contradiction that $\|v_k\| = t_k^{1/2}$. It follows from (A') that

$$b_k = I_k(v_k) \ge \frac{1}{2}\int_{t_{k-1}}^{t_k} m(s)\,\mathrm{d}s - \alpha_k \ge 0,$$

which contradicts that $b_k < 0$ and therefore we have proved that v_k is a critical point of I_k with $t_{k-1} < \|v_k\|^2 < t_k$. □

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Proof of case (ii) of Theorem 1.2. In this case, we can give exactly the same proof as in the previous case. The hypothesis (mf) is only required due to item (iii) of the Lemma 2.1. □

Proof of case (iii) of Theorem 1.2. For every $k = 1, \ldots, K$, the solution v_k, provided by items (i) and (ii) in Theorem 1.2, corresponds with a critical point of the truncated functional $I_k : H_0^1(\Omega) \to \mathbb{R}$ given by (3.1) with $I_k(v_k) < 0$. On the other hand, the solution u_k provided in [13, Theorem 1.1] corresponds with a critical point of the truncated functional I_k with $0 < I_k(u_k)$. Since $I_k(v_k) < 0 < I_k(u_k)$, we deduce that $v_k \neq u_k$ obtaining then 2 K positive solutions of (P). □

4. Concentration phenomena of solutions

Throughout this section, we are going to denote by f_λ a function f which satisfies (f), (mf), (A′) and $F(s_*) = \lambda$, where λ is a positive parameter. We also denote the dependence on this parameter λ of the functional $I_k = I_{k,\lambda}$ given by (3.1). It follows from previous section that problem (P_λ) has K positive solutions $v_{k,\lambda}$, with $I_{k,\lambda}(v_k) < 0$, $k = 1, \ldots, K$. Before proving the main result of this section, we need the following lemma

Lemma 4.1. *Let $b_{k,\lambda} = \inf_{u \in H_0^1(\Omega), \|u\| \leq t_k^{1/2}} I_{k,\lambda}(u)$, where $I_{k,\lambda}$ is the truncated functional associated with the problem (P_λ) (and given in (3.1)). Then,*

$$\lim_{\lambda \searrow 0} b_{k,\lambda} = \lim_{\lambda \searrow 0} \int_\Omega F_\lambda^*(v_{k,\lambda})\, dx = 0, \quad \forall\, k = 1, \ldots, K.$$

Proof. It is sufficient to note that

$$0 > b_{k,\lambda} = I_{k,\lambda}(v_{k,\lambda}) > -\int_\Omega F_\lambda^*(v_{k,\lambda}) \geq -\lambda|\Omega|,$$

where F_λ^* is the primitive of f_λ^*. The proof will be finished by passing to the limit as $\lambda \searrow 0$ in the previous inequality. □

Proof of case (iv) of Theorem 1.2. Assume by contradiction that for some fixed $k = 1, \ldots, K$ there exists a sequence of positive numbers such that $\lambda_n \searrow 0$ and

$$\|v_{k,\lambda_n}\| \to \tau_{k-1}^{1/2} \text{ as} n \to \infty,$$

for some $t_{k-1} < \tau_{k-1} \leq t_k$. Since

$$t_{k-1} < \|v_{k,\lambda_n}\|^2 < t_k, \quad \forall\, n \in \mathbb{N},$$

there exists $v_{k,0} \in H_0^1(\Omega)$ such that, up to a subsequence, as n tends to infinity, we obtain

$$v_{k,\lambda_n} \rightharpoonup v_{k,0} \text{ in } H_0^1(\Omega).$$

By Lemma 4.1, we deduce that

$$0 = \lim_{n\to\infty} b_{k,\lambda_n} = \lim_{n\to\infty} I_{k,\lambda}(v_{k,\lambda_n}) = \lim_{n\to\infty} \left[\frac{1}{2} M_k(\|v_{k,\lambda_n}\|^2) - \int_\Omega F_{\lambda_n}^*(v_{k,\lambda_n})\, \mathrm{d}x\right]$$
$$= \frac{1}{2} M_k(\tau_{k-1})$$

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and taking into account that by hypothesis (m), we have $M_k(\tau_{k-1}) > M_k(t_{k-1}) = 0$, contradicting the previous inequality. □

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