Proceedings of the Royal Society of Edinburgh, **151**, 377–390, *2021*
DOI:10.1017/prm.2020.19

Nonsingular bilinear maps revisited

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(Received 2 February 2020; Accepted 18 February 2020)

A bilinear map $\Phi : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n$ is nonsingular if $\Phi(\overrightarrow{a}, \overrightarrow{b}) = \overrightarrow{0}$ implies $\overrightarrow{a} = \overrightarrow{0}$ or $\overrightarrow{b} = \overrightarrow{0}$. These maps are of interest to topologists, and are instrumental for the study of vector bundles over real projective spaces. The main purpose of this paper is to produce examples of such maps in the range $24 \leqslant r \leqslant 32,\ 24 \leqslant s \leqslant 32$, using the arithmetic of octonions (otherwise known as Cayley numbers) as an effective tool. While previous constructions in lower dimensional cases use ad hoc techniques, our construction follows a systematic procedure and subsumes those techniques into a uniform perspective.

Keywords: Bilinear maps; Polynomial product; Immersion; Vector bundles; Projective spaces

2010 Mathematics Subject Classification: 57N35, 11E39

1. Introduction

In this paper bilinearity always means $\mathbb{R}$-bilinearity. A bilinear map $\Phi : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n$ is nonsingular if $\Phi(\overrightarrow{a}, \overrightarrow{b}) = \overrightarrow{0}$ implies $\overrightarrow{a} = \overrightarrow{0}$ or $\overrightarrow{b} = \overrightarrow{0}$. These maps generalize the multiplication of the classical division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{K}$ of real, complex, quaternion or octonion numbers, which correspond to the cases $r = s = n = 1, 2, 4$ and 8, respectively. The study of such maps by topological methods began with Hopf [**5**] and Stiefel [**15**], leading eventually to their applications to embedding and immersion of real projective spaces $\mathbb{R}P^{r-1}$ into Euclidean Space [**4**, **12**, **14**]. Applications to homotopy groups of spheres can be found in [**10**].

The problem, for what triples (r, s, n) can there exist a nonsingular bilinear map Φ, remains unsolved up to the present day. See [**9**, **11**] for an overall discussion. From past experience nonexistence results will have to involve increasingly sophisticated tools in algebraic topology [**6**], while existence results can be obtained through skillful constructions via algebra. In particular, use of the octonions $\mathbb{K}$, with its ‘restricted associativity property’ listed in section 2, has led to constructions by Lam [7] and Adem [1–3].

More than 47 years have passed since [3, 7, 8, 12] without any new example of nonsingular bilinear maps appearing in the literature. In particular, an open question about commutators in $\mathbb{K}$, posed in the introduction of [3], remains unanswered. The purpose of this paper is to construct a new family of nonsingular bilinear maps, in section 6 below, and to comment on their topological implications. In particular, we answer Adem's open question in the affirmative.

The second author would like to dedicate this work to the memory of professor Elmer Rees (1941–2019), a highly esteemed colleague and long time friend. Elmer's paper [13], in particular, has been a source of inspiration for the present article. Both authors are indebted to the late professor José Adem, whose papers [1–3] set the stage for theorem 6.1 bellow.

2. Restricted associativity of the octonions $\mathbb{K}$

For elements $a, b, c, \dots$ in $\mathbb{K}$ we shall use $\mathbb{R}(a, b, c, \dots)$ to denote the subalgebra they generate. The following properties are well known, and shall be frequently used in the sequel.

1. $\mathbb{R}(a)$ always contains the conjugate $\overline{a}$.
2. If $ab = ba$, then $\mathbb{R}(a, b)$ is commutative as well as associative. There exists then $d \in \mathbb{K}$ such that $\mathbb{R}(a, b) = \mathbb{R}(d) =$ a field, isomorphic either to $\mathbb{R}$ or to $\mathbb{C}$.
3. (Restricted associativity). If $ab = ba$, then for any $c \in \mathbb{K}, \mathbb{R}(a, b, c) = \mathbb{R}(d, c)$ is associative. Restricted associative laws hold:

$$a(bc) = (ab)c; \quad (ca)b = c(ab).$$

4. In particular, since a always commutes with $\overline{a}$, one has

$$a(\overline{a}c) = (a\overline{a})c = |a|^2 c = (ca)\overline{a}.$$

3. Twisted polynomial multiplication and bilinear maps

A primary source of nonsingular bilinear maps is polynomial multiplication. In this paper we shall mainly deal with the polynomial ring $\Lambda[X]$ with coefficient ring $\Lambda = \mathbb{K}$, de-emphasizing the cases $\Lambda = \mathbb{H}, \mathbb{C}$ or $\mathbb{R}$. If

$$p(X) = a_0 + a_1 X + \cdots + a_r X^r$$
$$q(X) = b_0 + b_1 X + \cdots + b_s X^s,$$

then $p(X)q(X)$ is traditionally defined to be

$$p(X)q(X) = c_0 + c_1 X + c_2 X^2 + \cdots + c_{r+s} X^{r+s}$$

where $c_k = a_0 b_k + a_1 b_{k-1} + \cdots a_k b_0$ for $0 \leqslant k \leqslant r + s$.

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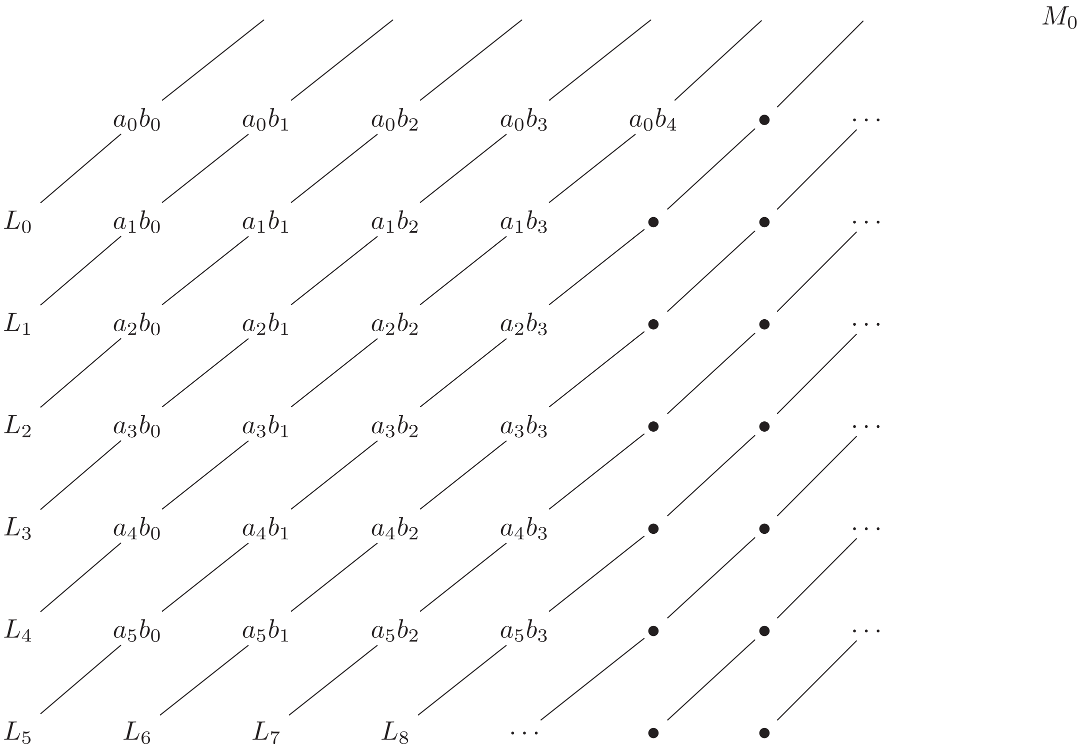
This multiplication produces immediate examples of nonsingular bilinear maps

$$\Phi_0 : \mathbb{R}^{8r+8} \times \mathbb{R}^{8s+8} \to \mathbb{R}^{8r+8s+8}$$

One simply identifies $\mathbb{R}^{8r+8}$ first with $\mathbb{K}^{r+1}$, and then visualizes a typical vector $\overrightarrow{a} = (a_0, a_1, \ldots, a_r)$ of $\mathbb{K}^{r+1}$ to be the polynomial $p(X)$, similarly with $\mathbb{R}^{8s+8}$ and $q(X)$. Then $\Phi_0(p(X), q(X))$ is none other than $p(X)q(X)$. The nonsingularity of Φ_0 is tantamount to the claim that $\mathbb{K}[X]$, like $\mathbb{K}$, has no zero divisors. This can be easily proved by induction on the total degree $r + s$.

It would be convenient to encode polynomial multiplication using a matrix scheme M_0 as follows



(3.1)

Here we put $a_{i-1}b_{j-1}$ at the (i, j)th entry of the matrix M_0. The segmented lines L_0, L_1, L_2, etc., are successive lines of slope 1, with unital decreases in y-intercept, passing through various lattice points at which the $a_{i-1}b_{j-1}$ terms are situated. For instance, L_2 passes through a_2b_0, a_1b_1 and a_0b_2. We shall use $\sum L_2$ to mean $a_2b_0 + a_1b_1 + a_0b_2$ for convenience. Indeed, the purpose of this encoding is to allow one to use $\sum L_k$ to symbolically stand for the coefficient c_k of X^k in the product $p(X)q(X)$ given above. When degree $p(X) = r$, degree $q(X) = s$, M_0 is of size $(r + 1) \times (s + 1)$, and can be used simultaneously to encode the bilinear map Φ_0 above as

$$\Phi_0 = \Phi_{M_0} : \mathbb{K}^{r+1} \times \mathbb{K}^{s+1} \to \mathbb{K}^{r+s+1}$$

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in the format

$$\Phi_{M_0}((a_0, a_1, \dots, a_r), (b_0, b_1, \dots, b_s)) = \left(\sum L_0, \sum L_1, \dots, \sum L_{r+s}\right).$$

Remark 3.1. As a matter of fact, given any matrix M of size $(r+1) \times (s+1)$, possibly $r = \infty$ or $s = \infty$, of which each entry is an arbitrary bilinear form in $\overrightarrow{a}, \overrightarrow{b}$, a bilinear map Φ_M will be automatically defined, with components $\sum L_0, \sum L_1, \sum L_2, \dots$, etc. The main effort of this paper is to seek out some $M's$ for which Φ_M would be nonsingular.

There is no short supply of such $M's$. We note that in the previous paragraph the guts of the induction argument for Φ_0's nonsingularity is that, if $p(X)q(X)$ had vanishing leading coefficient $a_r b_s$, then either $a_r = 0$ or $b_s = 0$. The same conclusion, of course, also follows from $a_r \overline{b_s} = 0$, or from $-\overline{b_s}\,\overline{a_r} = 0$, and so on. This motivates us to bring in the following

Definition 3.2. By a modification of the scheme M_0, we mean a matrix M of the same size, obtained from M_0 by replacing each $a_i b_j$ with any of the following choices

$$\pm a_i b_j, \pm a_i \overline{b_j}, \pm \overline{a_i} b_j, \pm \overline{a_i}\,\overline{b_j},$$
$$\pm b_j a_i, \pm \overline{b_j} a_i, \pm b_j \overline{a_i}, \pm \overline{b_j}\,\overline{a_i}.$$

There is a total of 16 possibilities at each entry. Any modification M of M_0 leads to a twisted polynomial multiplication $\odot_M$ for $\mathbb{K}[X]$ different from the traditional one, namely

Definition 3.3. The M-twisted product, or simply M-product, of $p(X)$ and $q(X)$ is defined to be

$$p(X) \odot_M q(X) = \hat{c}_0 + \hat{c}_1 X + \hat{c}_2 X^2 + \cdots$$

where $\hat{c}_k = \sum L_k$ is the sum of all entries of M falling on the line L_k depicted as in the scheme (3.1).

The space of polynomials with coefficients in $\mathbb{K}$ under twisted product $\odot_M$ becomes a ring $\mathbb{K}_M[X]$ with not many nice properties. For example the constant polynomial 1 may not be a two-sided multiplicative identity. However $\mathbb{K}_M[X]$ is still free of zero divisors, just like $\mathbb{K}[X](= \mathbb{K}_{M_0}[X])$. An induction proof for the former works, almost verbatim, as it does for the latter. Thus we have

Theorem 3.4. *Let M be any $(r+1) \times (s+1)$ matrix obtained from the matrix M_0 of equal size through modification. Then the bilinear map*

$$\Phi_M : \mathbb{K}^{r+1} \times \mathbb{K}^{s+1} \to \mathbb{K}^{r+s+1}$$

defined by $\Phi_M(p(X), q(X)) = p(X) \odot_M q(X)$, or equivalently by

$$\Phi_M(\overrightarrow{a}, \overrightarrow{b}) = \left(\sum L_0, \sum L_1, \dots, \sum L_{r+s}\right)$$

is again nonsingular.

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Remark 3.5. At first sight this theorem is not useful, as Φ_M is of real type $(8r+8, 8s+8, 8r+8s+8)$, exactly the same as the type of Φ_0 (or Φ_{M_0}). What, then, is the point of modifying?

But some modifications do indeed lead to novelty. The bilinearity of Φ_{M_0} induces, in the most obvious way, an adjoint map

$$adjM_0 : \mathbb{K}^{r+1} - \{\overrightarrow{0}\} \to Mono_{\mathbb{R}}(\mathbb{K}^{s+1}, \mathbb{K}^{r+s+1}),$$

where $Mono_{\mathbb{R}}$ means the space of monomorphisms from one real vector space to another. When M_0 is modified into M, there is no reason why $adjM$ should be homotopic to $adjM_0$. It is this potentially new homotopy feature of Φ_M that could be perhaps exploited to produce new families of nonsingular bilinear maps. This strategy will be carried out in a case of 4×4 matrices in section 6.

4. Neat factorization of polynomials and quasi irreducibility

Let M_0 be the $(r+1) \times (s+1)$ matrix of § 3 that encodes standard multiplication in $\mathbb{K}[X]$ of one polynomial $p(X)$ of degree $\leqslant r$ with another polynomial $q(X)$ of degree $\leqslant s$. Let M be a modification of M_0 encoding a twisted multiplication $\odot_M$ of the same two polynomials.

Definition 4.1. A polynomial $g(X)$ of degree $\leqslant r+s$ is said to be neatly factorized into an M-product of $p(X)$ and $q(X)$ if

1. $g(X) = p(X) \odot_M q(X)$,
2. $p(X)$ and $q(X)$ are of positive degree and
3. The constant terms of $p(X)$ and $q(X)$ commute, i.e., $p(0)q(0)$ equals $q(0)p(0)$ as octonions.

Definition 4.2. A polynomial $g(X)$ is said to be quasi M-irreducible, if it does not admit any neat factorization in $\mathbb{K}_M[X]$.

As a matter of generality, this definition could be understood in the following way. One allows the M_0 in § 3 to have countable number of rows and columns, so that M is also allowed to be such; but one imposes the requirement that all modification of entries occur within the upper $(r+1) \times (s+1)$ block of M_0, for some finite r and finite s. Throughout this paper, whether such generality ought to be in effect shall be clear from the context.

Two mini examples will serve to illustrate definitions 4.1 and 4.2.

Example 4.3. In $\mathbb{R}[X]$ usual multiplication of two linear polynomials into a quadratic is encoded by

$$M_0 = \begin{bmatrix} a_0b_0 & a_0b_1 \\ a_1b_0 & a_1b_1 \end{bmatrix}$$

With respect to M_0, $1 + X^2$ is irreducible. If M_0 is modified into

$$M^- = \begin{bmatrix} a_0b_0 & a_0b_1 \\ a_1b_0 & -a_1b_1 \end{bmatrix}$$

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then $1+X^2$, as a quadratic in $\mathbb{R}_{M^-}[X]$, is factorizable, and of course any M^- factorization is neat.

EXAMPLE 4.4. Replace $\mathbb{R}$ by $\mathbb{K}$ in example 4.3, so that now all matrix entries are octonions. Let the M_0 in 4.3 be modified into M where

$$M = \begin{bmatrix} a_0 b_0 & a_0 \overline{b_1} \\ \overline{b_0} a_1 & a_1 b_1 \end{bmatrix}$$

Let $i, j, k \in \mathbb{H} \subset \mathbb{K}$ be the usual imaginary quaternion units, considered here as octonions. Then one has an $\odot_M$ factorization of $i + iX^2$ into two linear factors, namely

$$i + iX^2 = (j + iX) \odot_M (k + X).$$

Here $a_0 = j, a_1 = i, b_0 = k, b_1 = 1; \sum L_0 = i, \sum L_1 = 0$ and $\sum L_2 = i$. Accordingly $i + iX^2$ is M-reducible.

But the above is not a neat factorization at all because the constant terms of the two factors do not commute: $jk \neq kj$. In fact it is not hard to check here that $i + iX^2$ admits no neat factorization and is M-quasi irreducible. In hindsight, such quasi-irreducibility is the key feature that facilitates the construction of nonsingular bilinear maps in [7]. As we shall see, it also facilitates the new construction in theorem 6.1 below.

5. Twisted multiplication of two cubics

Traditional multiplication of two polynomials of at most cubic degree is encoded, as in § 3, by the 4×4 matrix $M_0 = [a_i b_j], 0 \leqslant i, j \leqslant 3$, with octonion entries. In this section we consider a twisted multiplication $\odot_M$ given by a specific modification M of M_0, where

$$M = \begin{bmatrix} a_0 b_0 & a_0 \overline{b_1} & a_0 \overline{b_2} & \overline{a_0} b_3 \\ \overline{b_0} a_1 & a_1 b_1 & -\overline{b_2} a_1 & -b_3 \overline{a_1} \\ \overline{b_0} a_2 & a_2 \overline{b_1} & a_2 b_2 & a_2 b_3 \\ b_0 a_3 & a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}$$

THEOREM 5.1. *For any nonzero $c \in \mathbb{K}$ and any strictly positive real number λ, the polynomial $g(X) = c + \lambda c X^4$ in $\mathbb{K}_M[X]$ is quasi-irreducible. In other words $g(X)$ does not admit any neat $\odot_M$ factorization.*

Proof. Since $g(X)$ has degree 4, its possible factorization must occur as either

Case 1: a linear times a cubic, or

Case 2: a cubic times a linear, or

Case 3: a quadratic times another quadratic.

Corresponding to these cases are the submatrices M_1, M_2 and M_3 of M, of sizes $2 \times 4, 4 \times 2$ and 3×3 respectively. M_1 is formed by $M's$ first two rows, M_2 by its first two columns, and M_3 is $M's$ principal 3×3 sub-block. We display each one explicitly below, for easy tracking later.

$$M_1 = \begin{bmatrix} a_0b_0 & a_0\overline{b_1} & a_0\overline{b_2} & \overline{a_0}b_3 \\ \overline{b_0}a_1 & a_1b_1 & -\overline{b_2}a_1 & -b_3\overline{a_1} \end{bmatrix}$$

$$M_2 = \begin{bmatrix} a_0b_0 & a_0\overline{b_1} \\ \overline{b_0}a_1 & a_1b_1 \\ \overline{b_0}a_2 & a_2\overline{b_1} \\ b_0a_3 & a_3b_1 \end{bmatrix} \qquad M_3 = \begin{bmatrix} a_0b_0 & a_0\overline{b_1} & a_0\overline{b_2} \\ \overline{b_0}a_1 & a_1b_1 & -\overline{b_2}a_1 \\ \overline{b_0}a_2 & a_2\overline{b_1} & a_2b_2 \end{bmatrix}$$

It suffices to establish that $g(X)$ is quasi M_i-irreducible for $i = 1, 2$ and 3. All cases are done by reduction to absurdity; that is, supposing $g(X) = p(X) \odot_{M_i} q(X)$ neatly and deriving therefrom a contradiction, $i = 1, 2, 3$.

Case 1. Suppose that, neatly,

$$c + \lambda c X^4 = (a_0 + a_1X) \odot_{M_1} (b_0 + b_1X + b_2X^2 + b_3X^3)$$

The neatness requirement is

$$a_0b_0 (= c) = b_0a_0 \quad \text{(neatness)} \tag{5.1}$$

Quick comparison of coefficients, using the scheme of L-lines of § 3 for the displayed M_1, gives

$$\overline{b_0}a_1 + a_0\overline{b_1} = 0 \tag{5.2}$$

$$a_1b_1 + a_0\overline{b_2} = 0 \tag{5.3}$$

$$-\overline{b_2}a_1 + \overline{a_0}b_3 = 0 \tag{5.4}$$

$$-b_3\overline{a_1} = \lambda c = \lambda a_0b_0, \lambda > 0 \tag{5.5}$$

where $c \neq 0$ entails $a_0 \neq 0 \neq b_0$. It further entails $b_3 \neq 0 \neq a_1$ via (5.5) and also $b_2 \neq 0$ via (5.4). That, in turn, forces $b_1 \neq 0 \neq a_1$ via (5.3). With obvious meaning for the notation evaluate $[b_0(5.2)]b_1$ to obtain

$$[b_0(\overline{b_0}a_1)]b_1 + [b_0(a_0\overline{b_1})]b_1 = 0 \tag{5.6}$$

Because a_0 and b_0 commute, restricted associativity shows that the second term on left equals $[(b_0a_0)\overline{b_1}]b_1$ which in turn equals $(b_0a_0)|b_1|^2$, because $\overline{b_1}$ and b_1 commute. Applying similar arguments to the first term we reduce (5.6) to

$$|b_0|^2a_1b_1 + |b_1|^2a_0b_0 = 0 \tag{5.7}$$

Substituting (5.3) into (5.7) gives

$$-|b_0|^2a_0\overline{b_2} + |b_1|^2a_0b_0 = 0 \tag{5.8}$$

which confirms $b_1 \neq 0$ again.

Left cancelling the nonzero a_0 factor from (5.8) one confirms $\overline{b_2}$ to be a real multiple of b_0, and obtains the crucial fact that $\overline{b_2}$ commutes with a_0 since b_0 does.

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Next, evaluate $[a_0(5.4)]\overline{a_1}$ and simplify the result using restricted associativity to obtain

$$-|a_1|^2 a_0\overline{b_2} + |a_0|^2 b_3\overline{a_1} = 0 \tag{5.9}$$

Equations (5.5), (5.8) and (5.9) can be reorganized into three homogeneous linear relationships, with real coefficients, amongst the three octonions a_0b_0, $a_0\overline{b_2}$ and $b_3\overline{a_1}$. Since $a_0b_0(=c) \neq 0$, the 3×3 matrix that collectively summarizes all such relation must have zero determinant. This matrix is

$$\begin{bmatrix} -\lambda & 0 & -1 \\ |b_1|^2 & -|b_0|^2 & 0 \\ 0 & -|a_1|^2 & |a_0|^2 \end{bmatrix}$$

Its determinant is $\lambda|a_0|^2|b_0|^2 + |b_1|^2|a_1|^2$, which is strictly positive. We have thus arrived at a contradiction.

Case 2: This is very similar to case 1 and we can afford to be brief. Suppose that, neatly,

$$c + \lambda cX^4 = (a_0 + a_1X + a_2X^2 + a_3X^3) \odot_{M_2} (b_0 + b_1X).$$

Then

$$a_0b_0(=c) = b_0a_0 \quad \text{(neatness)} \tag{5.10}$$

$$\overline{b_0}a_1 + a_0\overline{b_1} = 0 \tag{5.11}$$

$$\overline{b_0}a_2 + a_1b_1 = 0 \tag{5.12}$$

$$b_0a_3 + a_2\overline{b_1} = 0 \tag{5.13}$$

$$a_3b_1(=\lambda c) = \lambda a_0b_0, \lambda > 0 \tag{5.14}$$

We now get, using $[b_0(5.11)]b_1$ followed by substitution

$$|b_0|^2a_1b_1 + |b_1|^2a_0b_0 = 0 \tag{5.15}$$

$$-|b_0|^2\overline{b_0}a_2 + |b_1|^2a_0b_0 = 0 \tag{5.16}$$

Use $b_0(5.16)$ to recognize a_2 as a real multiple of $b_0(a_0b_0)$, commuting, therefore, with a_0b_0 and $\overline{b_0}$. This crucial commutativity helps produce, via $[\overline{b_0}(5.13)]b_1$, the linear relationship

$$|b_0|^2a_3b_1 + |b_1|^2\overline{b_0}a_2 = 0 \tag{5.17}$$

Together with (5.14) and (5.16) there are three such relationships amongst $a_0b_0, \overline{b_0}a_2$ and a_3b_1. Again $a_0b_0 \neq 0$ forces the coefficient matrix to have zero determinant. This matrix is

$$\begin{bmatrix} -\lambda & 0 & 1 \\ |b_1|^2 & -|b_0|^2 & 0 \\ 0 & |b_1|^2 & |b_0|^2 \end{bmatrix}$$

with determinant $\lambda|b_0|^4 + |b_1|^4$ strictly positive. Contradiction !

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Case 3. With the 3×3 matrix M_3 defining a twisted multiplication of two quadratics, the argument proceeds similarly, only that the route towards contradiction is a bit more devious. Supposing a neat factorization

$$c + \lambda c X^4 = (a_0 + a_1 X + a_2 X^2) \odot_{M_3} (b_0 + b_1 X + b_2 X^2)$$

leads to

$$a_0 b_0 (= c) = b_0 a_0 \quad \text{(neatness)} \tag{5.18}$$

$$\overline{b_0} a_1 + a_0 \overline{b_1} = 0 \tag{5.19}$$

$$\overline{b_0} a_2 + a_1 b_1 + a_0 \overline{b_2} = 0 \tag{5.20}$$

$$a_2 \overline{b_1} - \overline{b_2} a_1 = 0 \tag{5.21}$$

$$a_2 b_2 (= \lambda c) = \lambda a_0 b_0, \lambda > 0 \tag{5.22}$$

where $c \neq 0$ by hypothesis, thereby entailing $a_2 \neq 0 \neq b_2$.

Exactly as before one can get

$$|b_0|^2 a_1 b_1 + |b_1|^2 a_0 b_0 = 0 \tag{5.23}$$

which shows $a_1 b_1$ to be a real multiple of $a_0 b_0$, commuting, therefore, with a_0 and with b_0. Now evaluate $[b_0(5.20)]b_2$ to get

$$|b_0|^2 a_2 b_2 + [b_0(a_1 b_1)] b_2 + |b_2|^2 a_0 b_0 = 0 \tag{5.24}$$

In (5.24) the middle term equals $[(a_1 b_1) b_0] b_2$ which in turn equals $[a_1 b_1] b_0 b_2$ by restricted associativity.

Using (5.23) this term becomes

$$-[|b_0|^{-2} |b_1|^2 a_0 b_0](b_0 b_2)$$

which has $a_0 b_0$ as a left factor. Observe that the first term of (5.24) has $a_0 b_0$ as left factor too, on account of (5.22). Left cancelling this common factor reduces (5.24) to

$$\lambda |b_0|^2 - |b_0|^{-2} |b_1|^2 b_0 b_2 + |b_2|^2 = 0 \tag{5.25}$$

Since $b_0 \neq 0$ and $\lambda > 0$ by hypothesis, (5.25) forces $b_1 \neq 0$ and shows $b_0 b_2$ to be a positive real number. Thus b_2 is just $\overline{b_0}$ up to a real multiple, in resemblance with the conclusion from the earlier (5.8). By properties of $\mathbb{K}$ listed in § 2, $b_2 \in \mathbb{R}(b_0) \subset \mathbb{R}(a_0, b_0)$. Equation (5.22) then implies $a_2 \in \mathbb{R}(a_0, b_0)$ so that a_2 and b_2 commute. This crucial commutativity allows one to simplify $[b_2(5.21)]b_1$ into

$$|b_1|^2 a_2 b_2 - |b_2|^2 a_1 b_1 = 0 \tag{5.26}$$

The octonions $a_0 b_0, a_1 b_1$ and $a_2 b_2$, with $a_0 b_0 \neq 0$, are now subject to homogeneous linear relations (5.22),(5.23) and (5.26) with real coefficients. Again the

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relevant coefficient matrix must have zero determinant. When that matrix is written down, like what was done in cases 1 and 2, its determinant is evaluated to be

$$|b_1|^2(\lambda|b_0|^2 + |b_2|^2).$$

Recalling $b_1 \neq 0$ from (5.25), we have reached our final contradiction. This completely establishes theorem 5.1 □

REMARK 5.2. The polynomial $g(X) = c + \lambda c X^4$ may or may not have an M-factorization that is not neat. When $c = i, \lambda = 1$, one has

$$i + iX^4 = (j + iX^2) \odot_M (k + X^2),$$

an analogue of mini example 4.4 in § 4.

When $c = 1, \lambda = 1, 1 + X^4$ has no factorization in the twisted polynomial ring $\mathbb{K}_M[X]$ whatsoever, because any M-factorization has to be neat, and thus contrary to theorem 5.1.

REMARK 5.3. The effect of 'twisting', i.e., conjugating, negating and factor transposing, becomes apparent if we recall that in the untwisted $\mathbb{K}[X]$ the following factorization is well known:

$$1 + X^4 = (1 + \sqrt{2}X + X^2)(1 - \sqrt{2}X + X^2).$$

Theorem 5.1 thus brings out the subtlety of octonionic arithmetic.

6. The nonsingular bilinear map $\Phi_{\widetilde{M}} : \mathbb{K}^4 \times \mathbb{K}^4 \to \mathbb{K}^7$ and octonion commutators

We continue to study the 4×4 matrix M of § 5. By theorem 3.4 it already defines a nonsingular map

$$\Phi_M : \mathbb{R}^{32} \times \mathbb{R}^{32} \to \mathbb{R}^{56}.$$

As it turns out M can actually be adjusted slightly to become an $\widetilde{M}$ that defines better maps. To do so introduce the 4×4 matrix,

$$N = \begin{bmatrix} b_0a_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_0a_0 \\ 0 & 0 & b_0a_0 & 0 \\ 0 & b_0a_0 & 0 & 0 \end{bmatrix}$$

We take the liberty to think of N as encoding a very esoteric multiplication $\odot_N$ of cubic polynomials, namely

$$(a_0 + a_1X + a_2X^2 + a_3X^3) \odot_N (b_0 + b_1X + b_2X^2 + b_3X^3) = b_0a_0 + 3b_0a_0X^4.$$

Alter M to $\widetilde{M} = M - N$. Each entry of $\widetilde{M}$ is bilinear in $\overrightarrow{a}$, $\overrightarrow{b}$, so by remark 3.1 it defines a bilinear

$$\Phi_{\widetilde{M}} : \mathbb{K}^4 \times \mathbb{K}^4 \to \mathbb{K}^7$$

where the first component of $\Phi_{\widetilde{M}}(\overrightarrow{a}, \overrightarrow{b}) \in \mathbb{K}^7$ is an octonion commutator $a_0b_0 - b_0a_0$. This is the map of the section title.

https://doi.org/10.1017/prm.2020.19 Published online by Cambridge University Press

THEOREM 6.1. *The bilinear map $\Phi_{\widetilde{M}}$ is nonsingular. Also, because an octonion commutator has no real part, the type of $\Phi_{\widetilde{M}}$ should be more accurately exhibited as*

$$\Phi_{\widetilde{M}} : \mathbb{R}^{32} \times \mathbb{R}^{32} \to \mathbb{R}^{55}.$$

lowering the range of $\Phi_{\widetilde{M}}$ from $\mathbb{R}^{56}$ to $\mathbb{R}^{55}$.

Proof. The proof of nonsingularity is not hard thanks to the preparation in § 5. The strategy is to suppose

$$\Phi_{\widetilde{M}}(\overrightarrow{a}, \overrightarrow{b}) = \overrightarrow{0} \tag{6.1}$$

and deduce that

$$\text{either } \overrightarrow{a} = \overrightarrow{0} \text{ or } \overrightarrow{b} = \overrightarrow{0} \tag{6.2}$$

As is easily seen via the scheme of L-lines in § 3, $\Phi_{\widetilde{M}} = \Phi_M - \Phi_N$, so that (6.1) means

$$\Phi_M(\overrightarrow{a}, \overrightarrow{b}) = \Phi_N(\overrightarrow{a}, \overrightarrow{b}) \tag{6.3}$$

In terms of twisted polynomial product this is

$$(a_0 + a_1X + a_2X^2 + a_3X^3) \odot_M (b_0 + b_1X + b_2X^2 + b_3X^3) = b_0a_0 + 3b_0a_0X^4 \tag{6.4}$$

Comparing constant terms yields $a_0b_0 = b_0a_0$. We denote this common value by c so that the right-hand side of (6.4) reads $c + 3cX^4$.

Comparing coefficients in degrees 6 and 5 yields

$$a_3b_3 = 0,\ a_3b_2 + a_2b_3 = 0.$$

This can happen only in one of the three circumstances below (cf. § 5).

Case $(1)^\sim$ $a_2 = 0 = a_3$ so that the left factor in (6.4) is at most linear, or

Case $(2)^\sim$ $b_2 = b_3 = 0$ so that the right factor in (6.4) is at most linear, or

Case $(3)^\sim$ $a_3 = 0,\ b_3 = 0$ so that both factors are at most quadratic.

We first show how to reach the desired conclusion (6.2) for case $(3)^\sim$. If both $\odot_M$ factors in (6.4) have positive degree, then (6.4) shows $c + 3cX^4$ to be neatly $\odot_M$ factorizable into two quadratics, contrary to case 3 of theorem 5.1. Therefore either left or right factor must be a constant polynomial. Say left factor$= a_0$, with $a_1 = a_2 = 0$ (in addition to the case specification $a_3 = 0$). Directly from the encoding scheme of § 3, the left-hand side of (6.4) now reduces to

$$a_0b_0 + a_0\overline{b_1}X + a_0\overline{b_2}X^2 + \overline{a_0}b_3X^3.$$

Coefficient comparison with right-hand side gives

$$a_0b_0 = b_0a_0,\ a_0\overline{b_1} = 0,\ a_0\overline{b_2} = 0,\ \overline{a_0}b_3 = 0,\ 0 = 3b_0a_0$$

This implies either $\overrightarrow{b} = \overrightarrow{0}$ or $a_0 = 0$ (and thus $\overrightarrow{a} = \overrightarrow{0}$), which is the desired (6.2).

For the possibility that right factor $= b_0$, with $b_1 = b_2 = 0$ (in addition to the case specification $b_3 = 0$), the argument to reach (6.2) is entirely parallel.

Finally, handling cases $(1)^\sim$ and $(2)^\sim$ through appeals to cases 1 and 2 of theorem 5.1 proceeds in exact analogy with the $(3)^\sim$ case, and needs no further comment. The nonsingularity of $\Phi_{\widetilde{M}}$ is now fully established. □

It may be relevant to point out that the proof of theorem 6.1 does follow the strategy suggested in remark 3.5. The 1-parameter family of matrices $M - tN$, $0 \leqslant t \leqslant 1$, reveals that, once M_0 is modified into M, $adj M$ would acquire the noteworthy property of homotopically factoring through a subspace $Mono_{\mathbb{R}}(\mathbb{K}^4, \mathbb{R}^{55})$ of $Mono_{\mathbb{R}}(\mathbb{K}^4, \mathbb{K}^7)$, whereas it is by no means clear that the original $adj M_0$ had any analogous property as such. In this sense, M is a preferred choice for construction purposes.

The question of existence of a bilinear map having the same type as $\Phi_{\widetilde{M}}$, with an octonion commutator in one component, was posed by Adem in [**3**]. Theorem 6.1 answers this question in the affirmative.

A good number of nonsingular bilinear maps now follow, some new, some previously recorded. All are obtained from $\Phi_{\widetilde{M}}$ by restrictions of domain and range. The choice of domain/range to restrict to is guided by the properties of octonionic commutators $a_0 b_0 - b_0 a_0$. We refer to [**1**] or [**2**] for a full account of possible choices. Just for example, one can restrict $\Phi_{\widetilde{M}}$ to

$$(\mathbb{C} \oplus \mathbb{K}^3) \times (\mathbb{C} \oplus \mathbb{K}^3) \to \{\overrightarrow{0}\} \oplus \mathbb{K}^6,$$

to obtain a type $\mathbb{R}^{26} \times \mathbb{R}^{26} \to \mathbb{R}^{48}$ originally envisaged by the first author as an extension of Adem's $\mathbb{R}^{18} \times \mathbb{R}^{18} \to \mathbb{R}^{32}$ in [**1**].

For yet another example, take $V \subset \mathbb{K}$ to be the 3-dimensional real subspace spanned by the imaginary quaternionic units $i.j, k$, with 5-dimensional orthogonal complement $V^\perp \subset \mathbb{K}$. Then because the commutator map for $\mathbb{K}$ restricts to $V^\perp \times V^\perp \xrightarrow{[,]} V$, one obtains

$$(V^\perp \oplus \mathbb{K}^3) \times (V^\perp \oplus \mathbb{K}^3) \to V \oplus \mathbb{K}^6$$

to be another legitimate restriction of $\Phi_{\widetilde{M}}$, leading to a new type $\mathbb{R}^{29} \times \mathbb{R}^{29} \to \mathbb{R}^{51}$ which is the most interesting among all possible restrictions. It generalizes the map $\mathbb{R}^{13} \times \mathbb{R}^{13} \to \mathbb{R}^{19}$ in the second author's Ph.D. thesis. It also gives an immersion of $\mathbb{R}P^{28}$ into $\mathbb{R}^{50}$ without any need to use Postnikov obstruction theory.

We use the following table and propositions to summarize the many nonsingular bilinear maps Φ that can result from restricting $\Phi_{\widetilde{M}}$. Many of these are new, superseding, for example, the maps constructed by Adem in [**3**, proposition 4.3].

h	32	32	31	29	27	26	26	25
k	32	26	27	29	27	30	26	32
m	55	54	53	51	49	52	48	48

(6.5)

Proposition 6.2. *For each triple (h, k, m) tabulated, there exits a nonsingular bilinear $\Phi : \mathbb{R}^h \times \mathbb{R}^k \to \mathbb{R}^m$ obtained through restricting the domain and range of $\Phi_{\widetilde{M}}$ in a suitable way.*

https://doi.org/10.1017/prm.2020.19 Published online by Cambridge University Press

Proposition 6.3. *Additionally, by further restricting Φ's domain and range, one can obtain a second level of nonsingular bilinear maps Φ^-, of types $\mathbb{R}^{h-8} \times \mathbb{R}^k \to \mathbb{R}^{m-8}$ as well as $\mathbb{R}^h \times \mathbb{R}^{k-8} \to \mathbb{R}^{m-8}$. These types either match, or supersede, all nearby types documented so far in the literature.*

Of the above, the maps Φ and Φ^- corresponding to columns 3, 4 and 5 in the table are new. The Φ^- maps of type $(26, 24, 46)$, $(26, 22, 44)$ and $(26, 18, 40)$ are also new.

Proposition 6.4. *Further restriction of domain and range of Φ^- yields a third level of nonsingular bilinear maps*

$$\Phi^{--} : \mathbb{R}^{h-8} \times \mathbb{R}^{k-8} \to \mathbb{R}^{m-16}$$

*We again leave out the details for selecting such restrictions, but point out that these Φ^{--} coincide precisely with Adem's eight maps constructed in [**1**, Theorem 3.6].*

In this sense propositions 6.2 and 6.3 become direct expansion of Adem's Theorem. Ultimately, to return the subject to its debut, one could further restrict the domain and range of Φ^{--}, to produce a lowest level of nonsingular bilinear maps. These are essentially the ones in [**7**].

Even though quite a number of maps constructed in this paper are new, in the range $h \leqslant 32, k \leqslant 32$ there are maps in existing literature which supersede ours. One notable example is Milgram's $\mathbb{R}^{32} \times \mathbb{R}^{32} \to \mathbb{R}^{54}$ in [**12**], reformulated by Adem in [**3**]. Milgram [**12**] has no explicit use of octonion commutators.

One naturally wonders whether theorem 6.1 can have higher dimensional analogue. For example, is there a nonsingular bilinear

$$\mathbb{K}^8 \times \mathbb{K}^8 \to \mathbb{K}^{15}$$

with a commutator component? An examination of the pattern of proof in § 6 and 5 shows that, to get an answer one needs to struggle through a jungle of octonion arithmetic, or to have new ideas. We leave this as an invitation to interested readers.

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