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THE LATTICE OF EQUATIONAL CLASSES OF SEMIGROUPS WITH ZERO

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In contrast to the very complicated structure of the lattice of equational classes of commutative semigroups (see [5]), the lattice of equational classes of commutative monoids (semigroups with unit) is isomorphic with $N \times N^*$ with a unit adjoined, where N is the lattice of natural numbers with the usual order and N^* is the lattice of natural numbers ordered by division. (See [4].) However, the lattice of equational classes of commutative semigroups-with-zero is not so simple to describe. The present paper shows that the lattice of equational classes of semigroups-with-zero is isomorphic to a particular sublattice of the lattice of equational classes of semigroups; as a corollary we obtain a characterization of the lattice of equational classes of commutative semigroups-with-zero in terms of the lattice of equational classes of commutative semigroups. Moreover, in the light of Gerhard's description [3] of the lattice of equational classes of idempotent semigroups, we get a description of the lattice of equational classes of idempotent semigroups-with-zero.

1. **The embedding.** For a class $\mathfrak{H}$ of semigroups, let $H(\mathfrak{H})$, $S(\mathfrak{H})$, $P(\mathfrak{H})$ be, respectively, the classes of homomorphic images, subsemigroups, and products of members of $\mathfrak{H}$. For a class $\mathfrak{K}$ of semigroups-with-zero, let $H^0(\mathfrak{K})$, $S^0(\mathfrak{K})$, and $P^0(\mathfrak{K})$ be, respectively, the classes of 0-homomorphic images, 0-subsemigroups, and products of members of $\mathfrak{K}$. For a semigroup-with-zero, A, let $|A|$ be the semigroup obtained from A by forgetting about the extra operation; for a class $\mathfrak{K}$ of semigroups-with-zero, define $|\mathfrak{K}|$ accordingly. Note that $|S^0(\mathfrak{K})| \subseteq S(|\mathfrak{K}|)$, $|H^0(\mathfrak{K})| \subseteq H(|\mathfrak{K}|)$, and $|P^0(\mathfrak{K})| = P(|\mathfrak{K}|)$.

Moreover, if α is a semigroup homomorphism from a semigroup A onto a semigroup B, and if A has a zero element, 0_A, then, for $b \in B$, $b=\alpha(a)$, say, for $a \in A$, $b\alpha(0_A)=\alpha(a)\alpha(0_A)=\alpha(a0_A)=\alpha(0_A)=\alpha(0_Aa)=\alpha(0_A)\alpha(a)=\alpha(0_A)b$. Thus B has a zero, namely $\alpha(0_A)$, and α is a 0-homomorphism. It follows that for a class $\mathfrak{K}$ of semigroups-with-zero, $|H^0(\mathfrak{K})|=H(|\mathfrak{K}|)$.

Let $\mathscr{L}$ be the lattice of equational classes of semigroups, $\mathscr{L}^0$ the lattice of equational classes of semigroups-with-zero. For $\mathfrak{H} \in \mathscr{L}$, let $\mathfrak{H}^0$ be the class of all semigroups-with-zero, A, such that $|A| \in \mathfrak{H}$. Since a semigroup can have at most one zero, there is a one-to-one correspondence between the semigroups in $\mathfrak{H}$ that have a

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zero, and the elements of $\mathfrak{H}^0$. Since $\mathfrak{H}$ is an equational class of semigroups, $H^0S^0P^0(\mathfrak{H}^0)\subseteq\mathfrak{H}^0$; thus $\mathfrak{H}^0\in\mathscr{L}^0$. The map $\mathfrak{H}\rightsquigarrow\mathfrak{H}^0$ is clearly order preserving.

For $\mathfrak{K}\in\mathscr{L}^0$, let $\mathfrak{K}^s=HS(|\mathfrak{K}|)$. Since $\mathfrak{K}$ is closed under the formation of products, $\mathfrak{K}^s$ is an equational class of semigroups, i.e., $\mathfrak{K}^s\in\mathscr{L}$. Also, the mapping $\mathfrak{K}\rightsquigarrow\mathfrak{K}^s$ is order preserving.

If $\mathfrak{K}\in\mathscr{L}^0$ and $A\in\mathfrak{K}$, then $|A|\in\mathfrak{K}^s$ and $|A|$ has a zero, thus $A\in\mathfrak{K}^{s0}$. It follows that $\mathfrak{K}\subseteq\mathfrak{K}^{s0}$.

Moreover, if $\mathfrak{H}\in\mathscr{L}$, since $|\mathfrak{H}^0|\subseteq\mathfrak{H}$, and $\mathfrak{H}$ is closed under H and S, it follows that $\mathfrak{H}^{0s}\subseteq\mathfrak{H}$.

Thus the mappings $\mathfrak{K}\rightsquigarrow\mathfrak{K}^s$ for $\mathfrak{K}\in\mathscr{L}^0$ and $\mathfrak{H}\rightsquigarrow\mathfrak{H}^0$ for $\mathfrak{H}\in\mathscr{L}$ give a Galois correspondence of mixed type between $\mathscr{L}$ and $\mathscr{L}^0$. It follows that the mapping of $\mathscr{L}^0$ into $\mathscr{L}$ given by $\mathfrak{K}\rightsquigarrow\mathfrak{K}^s$ is join-preserving.

LEMMA 1. *For all* $\mathfrak{K}\in\mathscr{L}^0$, $\mathfrak{K}=\mathfrak{K}^{s0}$.

Proof. If $A\in\mathfrak{K}^{s0}$, then $|A|\in HS(|\mathfrak{K}|)$ and $|A|$ has a zero, 0_A. Thus there exists a semigroup homomorphism α from B onto $|A|$, and $C\in\mathfrak{K}$ such that B is a subsemigroup of $|C|$. If $0_C\in B$, then $B\in|S^0(\mathfrak{K})|$ and thus $A\in H^0S^0(\mathfrak{K})\subseteq\mathfrak{K}$. If $0_C\notin B$, then $B\cup\{0_C\}\in|S^0(\mathfrak{K})|$ and the mapping $\bar{\alpha}$ from $B\cup\{0_C\}$ to $|A|$ defined by $\bar{\alpha}\mid B=\alpha$, $\bar{\alpha}(0_C)=0_A$ is a homomorphism that preserves 0. Thus $A\in H^0S^0(\mathfrak{K})\subseteq\mathfrak{K}$. It follows that $\mathfrak{K}^{s0}\subseteq\mathfrak{K}$. We already know that $\mathfrak{K}\subseteq\mathfrak{K}^{s0}$, so this completes the proof.

LEMMA 2. *If* $\mathfrak{K}_1,\mathfrak{K}_2\in\mathscr{L}^0$ *then* $HS(|\mathfrak{K}_1|)\cap HS(|\mathfrak{K}_2|)=HS(|\mathfrak{K}_1\cap\mathfrak{K}_2|)$.

Proof. It is clear that $HS(|\mathfrak{K}_1\cap\mathfrak{K}_2|)\subseteq HS(|\mathfrak{K}_1|)\cap HS(|\mathfrak{K}_2|)$. Moreover,

$$(HS(|\mathfrak{K}_1|)\cap HS(|\mathfrak{K}_2|))^0=HS(|\mathfrak{K}_1|)^0\cap HS(|\mathfrak{K}_2|)^0=\mathfrak{K}_1^{s0}\cap\mathfrak{K}_2^{s0}=\mathfrak{K}_1\cap\mathfrak{K}_2$$

(by Lemma 1)$=(HS(|\mathfrak{K}_1\cap\mathfrak{K}_2|))^0$. Thus, if $A\in HS(|\mathfrak{K}_1|)\cap HS(|\mathfrak{K}_2|)$ and A has a zero, then $A\in HS(|\mathfrak{K}_1\cap\mathfrak{K}_2|)$. If $A\in HS(|\mathfrak{K}_1|)\cap HS(|\mathfrak{K}_2|)$ and A has no zero, then for $i=1,2$, there exists $C_i\in\mathfrak{K}_i$, a subsemigroup B_i of $|C_i|$ and a homomorphism φ_i of B_i onto A. Since A does not have a zero, it follows that neither B_1 nor B_2 have zeroes. $|C_1|$ and $|C_2|$ have zeroes, thus $B_i\cup\{0\}$ is a subsemigroup of $|C_i|$ for $i=1,2$. Define $\bar{\varphi}_i$ from $B_i\cup\{0\}$ to $A\cup\{0\}$ by: $\bar{\varphi}_i\mid B_i=\varphi_i$, $\bar{\varphi}_i(0)=0$. Then $\bar{\varphi}_i$ is a homomorphism. It follows that $A\cup\{0\}\in|(HS(|\mathfrak{K}_1|)\cap HS(|\mathfrak{K}_2|))^0|$ $\subseteq HS(|\mathfrak{K}_1\cap\mathfrak{K}_2|)$ and thus $A\in HS(|\mathfrak{K}_1\cap\mathfrak{K}_2|)$. This yields $HS(|\mathfrak{K}_1|)\cap HS(|\mathfrak{K}_2|)$ $\subseteq HS(|\mathfrak{K}_1\cap\mathfrak{K}_2|)$, completing the proof.

A direct consequence of the above lemmas is that the mapping from $\mathscr{L}^0$ into $\mathscr{L}$ given by $\mathfrak{K}\rightsquigarrow\mathfrak{K}^s$ is a lattice monomorphism. $\mathfrak{H}\in\mathscr{L}$ is in the image of this monomorphism iff $\mathfrak{H}=\mathfrak{H}^{0s}$. We will now determine which equational classes $\mathfrak{H}$ of semigroups have the property that $\mathfrak{H}=\mathfrak{H}^{0s}$.

2. **The image.** A semigroup equation e is called regular if every variable that appears on one side of e also appears on the other side. If e is regular, and a semigroup A satisfies e, then so does $A\cup\{0\}$.

An equational class of semigroups is called regular if it satisfies only regular equations. The set of all regular equational classes of semigroups is exactly the principal filter $\mathscr{B}$ of all equational classes containing the class of idempotent commutative semigroups.

If $\mathfrak{H} \in \mathscr{L}$ is regular then for all $A \in \mathfrak{H}$, $A \cup \{0\} \in \mathfrak{H}$, thus $A \in \mathfrak{H}^{0s}$. It follows that $\mathfrak{H} \subseteq \mathfrak{H}^{0s}$; and thus $\mathfrak{H} = \mathfrak{H}^{0s}$. Thus every regular equational class of semigroups is in the image of the above monomorphism.

Now assume that $\mathfrak{H} \in \mathscr{L}$ is not regular. Then $\mathfrak{H}$ satisfies an equation e with a variable x, say, appearing only on the right-hand side of e. If $\mathfrak{H} = \mathfrak{H}^{0s}$, then **F**, the $\mathfrak{H}$-free semigroup on countably many generators, is a homomorphic image of a subsemigroup of $|A|$ for some $A \in \mathfrak{H}^0$. For $a \in A$, substituting 0_A for x and a for all the other variables in e yields $a^n = 0$, where n is the length of the left-hand side of e. Thus for all $a, b \in A$, $a^n b = b a^n = a^n$. Thus $|A|$ satisfies $x^n y = y x^n = x^n$. Since **F** is a homomorphic image of a subsemigroup of $|A|$, it follows that **F** satisfies $x^n y = y x^n = x^n$. Thus $\mathfrak{H}$ satisfies $x^n y = y x^n = x^n$. On the other hand, if $\mathfrak{H}$ satisfies $x^n y = y x^n = x^n$ for some n, then every nonempty semigroup A in $\mathfrak{H}$ has a zero, and $a^n = 0$ for all $a \in A$. Thus $\mathfrak{H} = |\mathfrak{H}^0| \cup \{\varnothing\}$ and hence $\mathfrak{H} \subseteq \mathfrak{H}^{0s}$, thus $\mathfrak{H} = \mathfrak{H}^{0s}$. Thus if $\mathfrak{H} \in \mathscr{L}$ is *not* regular, then $\mathfrak{H} = \mathfrak{H}^{0s}$ iff $\mathfrak{H}$ satisfies $x^n y = y x^n = x^n$ for some n.

Let $\mathscr{I} = \{\mathfrak{H} \in \mathscr{L} \mid \mathfrak{H}$ satisfies $x^n y = y x^n = x^n$ for some $n \in N\}$. Then $\mathscr{I}$ is an ideal in the lattice $\mathscr{L}$.

It follows from the above results that $\mathscr{B} \cup \mathscr{I}$ is a sublattice of $\mathscr{L}$, and the mapping $\mathfrak{K} \rightsquigarrow \mathfrak{K}^s$ of $\mathscr{L}^0$ into $\mathscr{L}$ is a lattice monomorphism mapping onto $\mathscr{B} \cup \mathscr{I}$.

The restriction of this monomorphism to the sublattice of equational classes of commutative semigroups-with-zero gives an embedding of the lattice of equational classes of commutative semigroups-with-zero into $\mathscr{L}_c$, the lattice of equational classes of commutative semigroups. It follows from the above, and from the results in [5] that the image of this embedding is $\{\mathfrak{K} \in \mathscr{L}_c \mid V(\mathfrak{K}) \geq 1\} \cup \{\mathfrak{K} \in \mathscr{L}_c \mid V(\mathfrak{K}) = 1, D(\mathfrak{K}) = 0\}$. (For definitions of V, D see [5].)

In the same vein, we have an embedding of the lattice of equational classes of idempotent semigroups-with-zero into the lattice $\mathscr{L}_i$ of equational classes of idempotent semigroups. It follows from the above results that the image under this embedding is the sublattice of $\mathscr{L}_i$ consisting of all equational classes of idempotent semigroups containing the class of idempotent commutative semigroups, plus the class of semigroups satisfying $x = y$. In [3], Gerhard gives a complete description of $\mathscr{L}_i$; it now follows from his results that the lattice of equational classes of idempotent semigroups-with-zero is isomorphic to $\mathscr{L}_i - \{R, L, R \vee L\}$, where R is the class of idempotent semigroups satisfying $yx = x$, L is the class of idempotent semigroups satisfying $xy = x$ and $R \vee L$ is the join of these two classes, i.e., the class of idempotent semigroups satisfying $xyz = xz$. In Gerhard's notation these three classes are characterized by $\not{E}$, $\not{H}$, H^*; $\not{E}$, H, $\not{H}^*$; and $\not{E}$, H, H^* respectively, in the diagram of [3, p. 222].

Finally, since the embedding of the dual of Π_∞ into $\mathscr{L}$ given in [2] uses only

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regular equations, it follows that the dual of Π_∞ can be embedded in $\mathscr{L}^0$. Also it follows from the results in [1] and [5] that for each natural number n, the dual of Π_n can be embedded in the lattice of equational classes of commutative semigroups-with-zero, and thus (by [6]), this lattice satisfies no special lattice laws.

Added in proof. W. H. Carlisle (Doctoral Dissertation, Emory University) has also shown that $\mathscr{L}^0$ can be embedded in $\mathscr{L}$; his embedding is described in terms of equations but can be seen to be the same embedding as the one described here.

References

1. S. Burris and E. Nelson, *Embedding the dual of* Π_m *in the lattice of equational classes of commutative semigroups*, Proc. Amer. Math. Soc. (to appear).

2. ———, *Embedding the dual of* Π_∞ *in the lattice of equational classes of semigroups*, Algebra Universalis (to appear).

3. J. A. Gerhard, *The lattice of equational classes of idempotent semigroups*, J. Algebra **15** (1970), 195–224.

4. T. J. Head, *The varieties of commutative monoids*, Nieuw Archief voor Wiskunde (3) **XVI**, (1968), 203–206.

5. E. Nelson, *The lattice of equational classes of commutative semigroups*, Canad. J. Math. (to appear).

6. D. Sachs, *Identities in finite partition lattices*, Proc. Amer. Math. Soc. **12** (1969), 944–945.

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