*Proceedings of the Edinburgh Mathematical Society* (2020) **63**, 579–607
doi:10.1017/S0013091520000012

# ON STEENROD $\mathbb{L}$-HOMOLOGY, GENERALIZED MANIFOLDS, AND SURGERY

FRIEDRICH HEGENBARTH[1] AND DUŠAN REPOVŠ[2]

[1] *Dipartimento di Matematica 'Federigo Enriques', Università degli studi di Milano, 20133 Milano, Italy* (friedrich.hegenbarth@unimi.it)

[2] *Faculty of Education and Faculty of Mathematics and Physics, University of Ljubljana & Institute of Mathematics, Physics and Mechanics, 1000 Ljubljana, Slovenia* (dusan.repovs@guest.arnes.si)

(Received 19 December 2019; first published online 20 March 2020)

*Dedicated to the memory of Professor Andrew Ranicki (1948–2018)*

*Abstract* The aim of this paper is to show the importance of the Steenrod construction of homology theories for the disassembly process in surgery on a *generalized* $n$-manifold $X^n$, in order to produce an element of generalized homology theory, which is basic for calculations. In particular, we show how to construct an element of the $n$th Steenrod homology group $H_n^{st}(X^n, \mathbb{L}^+)$, where $\mathbb{L}^+$ is the connected covering spectrum of the periodic surgery spectrum $\mathbb{L}$, avoiding the use of the *geometric* splitting procedure, the use of which is standard in surgery on *topological* manifolds.

*Keywords:* Poincaré duality complex; generalized manifold; Steenrod $\mathbb{L}$-homology; periodic surgery spectrum $\mathbb{L}$; fundamental complex; $\mathbb{L}$-homology class; Quinn index

2010 *Mathematics subject classification:* Primary 55N07; 55R20 57P10; 57R67
Secondary 18F15; 55M05 55N20; 57P05; 57P99 57R65

## 1. Introduction

In order to study global objects it can be useful to decompose them into similar smaller pieces. This process of *disassembly* also applies to surgery theory. If one does it in an appropriate way, it produces an element of a generalized homology theory which is basic for calculations. Here, 'appropriate' means 'semisimplicially' defined spectra (this holds for all spectra considered in the paper).

Geometrically, one uses transversality to attain the goal. This works well for piecewise-linear (PL) *topological* manifolds, but it does not work for *generalized* manifolds. The aim of this paper is to show that for generalized manifolds an appropriate tool to overcome this problem is the Steenrod construction of homology theory.

Steenrod homology is a homology theory which is highly appropriate for compact metric spaces which have certain bad local properties. Generalized Steenrod homology

© 2020 The Edinburgh Mathematical Society 

https://doi.org/10.1017/S0013091520000012 Published online by Cambridge University Press

theory has been well presented by Ferry [5] (for more, see also Milnor [17]). A rigorous development of this theory was given earlier by Kahn *et al.* [13].

The underlying spectra of homology theory which we shall consider are $\Omega^N$, $\Omega^{PD}$, $\Omega^{NPD}$, and $\mathbb{L}$. They are defined simplicially, in terms of adic objects (see Nicas [18], Quinn [20] and Ranicki [25]). Objects in $\Omega^N$ (respectively, $\Omega^{PD}$) are adic normal spaces (respectively, adic Poincaré duality complexes), and objects in $\Omega^{NPD}$ are adic normal spaces with boundaries being adic Poincaré duality complexes (see Quinn [21]).

Our main interest will be the periodic surgery spectrum $\mathbb{L}$ with

$$\mathbb{L}_0 = \mathbb{Z} \times {}^{G}/_{TOP}$$

and its connected covering spectrum $\mathbb{L}\langle 1\rangle$, which we shall denote by $\mathbb{L}^+$. Elements of $\mathbb{L}^+$ are adic surgery problems (see Nicas [18]), and there is a fibre sequence of spectra

$$\mathbb{L}^+ \to \mathbb{L} \to \mathbb{K}(\mathbb{Z}, 0),$$

where $\mathbb{K}(\mathbb{Z}, 0)$ is the Eilenberg–MacLane spectrum.

Steenrod homology is defined on compact metric spaces $X$, and we write $H_*^{st}(X, \mathcal{S})$, where $\mathcal{S}$ is any one of the above spectra. If $X$ is a PL topological manifold, then $H_*^{st}(X, \mathcal{S})$ coincides with the ordinarily defined generalized homology $H_*(X, \mathcal{S})$.

It is important to note that $\mathbb{L}^+$ (respectively, $\mathbb{L}$) can be defined algebraically and that the following theorem holds.

**Theorem 1.1 (Ranicki [24, 25]).** *There is a map of spectra*

$$\Omega^{NPD} \to \Sigma\mathbb{L}^+,$$

*where $\Sigma\mathbb{L}^+$ is the suspension spectrum of $\mathbb{L}^+$ (see Ranicki [24, p. 287]). Moreover, the induced morphism*

$$H_n(K, \Omega^{NPD}) \to H_{n-1}(K, \mathbb{L}^+)$$

*is an isomorphism for $n \geq 4$, where $K$ is a finite polyhedron (see Hausmann and Vogel [8], Jones [12], Levitt [15] and Quinn [21]).*

Steenrod homology is related to locally finite homology.

**Theorem 1.2 (Ferry [5], Milnor [17]).** *For every compact metric pair $(X, X')$, the natural homomorphism*

$$H_*^{st}(X, X', \mathcal{S}) \to H_*^{lf}(X \setminus X', \mathcal{S})$$

*is an isomorphism.*

We shall apply this property only for $\mathcal{S} = \mathbb{L}^+$. The definition of $H_*^{lf}(\cdot, \mathbb{L}^+)$ can be found in Ranicki [25, Appendix C].

In order to verify the axioms of Steenrod homology theory, one has to use the following result.

https://doi.org/10.1017/S0013091520000012 Published online by Cambridge University Press

**Theorem 1.3 (Ferry [5], Milnor [17]).** *Any compact metric pair $(X, X')$ can be embedded into a compact metric pair $(T, T')$ so that*

(1) *$T$ and $T'$ are contractible,*

(2) *$T \setminus X$ is a CW-complex and $T' \setminus X' \subset T \setminus X$ is a subcomplex.*

*Moreover, the construction of $(T, T')$ is natural with respect to maps between compact pairs $(X, X') \to (Y, Y')$.*

We shall adopt the notation from Ferry [5] and write $T \setminus X = \mathrm{OFC}(X)$ for the open fundamental complex of $X$, and $T = \mathrm{CFC}(X)$ for the closed fundamental complex of $X$. Our construction of $\mathrm{OFC}(X)$ comes with a basepoint $b_0 \in X$. We shall describe these fundamental complexes below, because we shall construct an element in

$$H^{lf}_{n+2}(\mathrm{OFC}(X^n) \setminus \{b_0\}, \Omega^{NPD}),$$

associated to a degree-one normal map $f : M^n \to X^n$, where $X^n$ is a generalized $n$-manifold, and $b : \nu_{M^n} \to \xi$ is an appropriate bundle map.

More precisely, we have to fix a degree-one normal map

$$\{f_0, b_0\} : M_0 \to X$$

and associate to $\{f, b\}$ an element in

$$H^{lf}_{n+2}(\mathrm{OFC}(X^n) \setminus \{b_0\}, \Omega^{NPD}),$$

which we shall denote $\{f, b\} - \{f_0, b_0\}$.

By the above theorems we have the following chain of morphisms:

$$\begin{aligned} H^{lf}_{n+2}(\mathrm{OFC}(X^n) \setminus \{b_0\}, \Omega^{NPD}) &\xrightarrow{\cong} H^{st}_{n+2}(\mathrm{CFC}(X^n), X^n \amalg \{b_0\}, \Omega^{NPD}) \\ &\xrightarrow{\cong} H^{st}_{n+1}(\mathrm{CFC}(X^n), X^n \amalg \{b_0\}, \mathbb{L}^+) \\ &\to H^{st}_{n}(X^n \amalg \{b_0\}, \mathbb{L}^+), \end{aligned}$$

that is, $\{f, b\} - \{f_0, b_0\}$ determines an element

$$[f, b] - [f_0, b_0] \in H^{st}_n(X^n \amalg \{b_0\}, \mathbb{L}^+).$$

If $X^n$ is a topological $n$-manifold which carries a simplicial structure, then the construction of the element

$$[f, b] - [f_0, b_0] \in H^{st}_n(X^n, \mathbb{L}^+) \cong H_n(X^n, \mathbb{L}^+)$$

follows from the splitting procedure: the surgery problem

$$(f, b) = \left(M^n \xrightarrow{f} X^n,\ \nu_{M^n} \xrightarrow{b} \xi\right)$$

can be split into adic surgery problems which define $[f, b]$ (see Hegenbarth and Repovš [9] and Ranicki [25]). This is due to transversality with respect to a dual cell structure on $X^n$.

https://doi.org/10.1017/S0013091520000012 Published online by Cambridge University Press

It is the purpose of this paper to present a construction, based on Theorems 1.1–1.3, to obtain an element of $H_n^{st}(X^n, \mathbb{L}^+)$ which avoids this geometric splitting. We point out that algebraic splitting is also possible (see Pedersen *et al.* [**19**]) and leads to an identification of $\mathbb{L}$-homology groups with controlled Wall groups.

We conclude this introduction by describing the structure of our paper. In § 2 we shall recall preliminary material about nerves $N(\mathcal{U})$ and canonical maps

$$\varphi : X^n \to N(\mathcal{U})$$

between the underlying space $X^n$ and the nerve $N(\mathcal{U})$.

Section 3 will be devoted to the construction of appropriate fundamental complexes of $X^n$. Section 4 is the core of the paper: for any Euclidean neighbourhood retract Poincaré duality space $X^n$ we shall apply Theorems 1.1–1.3 to construct the $\mathbb{L}$-homology class

$$[f, b] \in H_n^{st}(X^n, \mathbb{L}^+)$$

for an arbitrary surgery problem

$$(f, b) = \big(M^n \xrightarrow{f} X^n,\ \nu_{M^n} \xrightarrow{b} \xi\big).$$

However, we shall see that this class depends on the canonical surgery problem (see Lemma 4.4).

In § 5 we shall present some improvements and give an outlook. Finally, in § 6 we shall discuss selected remaining related problems.

For more background information on Poincaré complexes, surgery theory and generalized manifolds we refer the reader to, for example, [**1**, **2**, **6**, **8**, **23**, **26**–**29**].

## 2. Coverings, nerves, and canonical maps

Throughout the paper we shall consider compact metric spaces $X$. Our main interest will be in *closed generalized $n$-manifolds* $X^n$, that is, $X^n$ is a *Euclidean neighbourhood retract* (ENR) and a *$\mathbb{Z}$-homology $n$-manifold*:

$$H_*(X^n, X^n \setminus \{x\}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}), \quad \text{for every } x \in X^n$$

(see, for example, Cavicchioli *et al.* [**2**]).

Open coverings $\mathcal{U} = \{U\}_{j\in J}$ will always be assumed to be locally finite. We shall denote the simplicial complex of $\mathcal{U}$ by $N(\mathcal{U})$. The vertex corresponding to $U_j \in \mathcal{U}$ will be denoted by $\langle U_j \rangle$, and if

$$\bigcap_{0\le i\le k} U_{j_i} \neq \emptyset$$

then the $k$-simplex determined by $U_{j_0}, \ldots, U_{j_k} \in \mathcal{U}$ will be denoted by

$$\sigma = \langle U_{j_0}, \ldots, U_{j_k} \rangle \in N(\mathcal{U}).$$

We shall abbreviate and write

$$\bigcap_{0\le i\le k} U_{j_i} = \bigcap \sigma.$$

https://doi.org/10.1017/S0013091520000012 Published online by Cambridge University Press

We shall also write $N(\mathcal{U})$ for its topological realization. The space $N(\mathcal{U})$ can be given the Whitehead or the metric topology. However, since we shall only consider locally finite coverings, these two topologies are identical (see, for example, Dugundji [**3**, p. 99]).

### 2.1. The map $\varphi : X \to N(\mathcal{U})$

Let

$$\mathrm{mesh}(\mathcal{U}) = \sup\{\mathrm{diam}(U) | U \in \mathcal{U}\},$$

where $\mathrm{diam}(U)$ denotes the diameter of $U \subset X$. We shall now describe the first of our canonical maps.

A partition of unity $\{\varphi_j\}_{j \in J}$ subordinate to $\mathcal{U}$ gives rise to the map $\varphi : X \to N(\mathcal{U})$ defined by

$$\varphi(x) = \sum_j \varphi_j(x) \langle U_j \rangle .$$

If $\{\overline{\varphi}_j\}_{j \in J}$ is another partition subordinate to $\mathcal{U}$, it defines the map $\overline{\varphi} : X \to N(\mathcal{U})$. The homotopies

$$\{t\overline{\varphi}_j + (1-t)\varphi_j \mid 0 \le t \le 1\}_{j \in J}$$

then define a homotopy between $\overline{\varphi}$ and $\varphi$; that is, up to homotopy, the map $\varphi$ is unique.

### 2.2. Maps induced by refinements

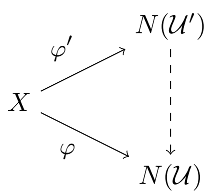
Next, we shall consider refinements of coverings and induced maps. Let

$$\mathcal{U}' = \{U'_{j'}\}_{j' \in J'}$$

be a refinement of $\mathcal{U}$, that is, there is a map $s : J' \to J$ such that

$$U'_{j'} \subset U_{s(j')}, \quad \text{for every } j' \in J'.$$

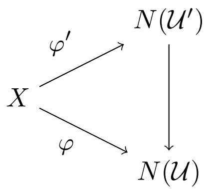
Let $\varphi' : X \to N(\mathcal{U}')$ be a map as defined in § 2.1 by the partition of unity $\{\varphi'_{j'}\}_{j' \in J'}$. We want to complete the diagram




by a map indicated by the dashed line, so that it is homotopy commutative even in the controlled way.

https://doi.org/10.1017/S0013091520000012 Published online by Cambridge University Press

We can get such maps from, for example, Hu [11, Theorem 8.1, p. 146]. There exists a refinement $\mathcal{V}$ of $\mathcal{U}$, such that for every refinement $\mathcal{U}'$ of $\mathcal{V}$ there is a simplicial map $N(\mathcal{U}') \to N(\mathcal{U})$ such that




commutes up to a homotopy $h_t$ with

$$\{h_t(x) | 0 \leq t \leq 1\} \subset \overset{\circ}{st} \langle U \rangle, \quad \text{for some } U \in \mathcal{U}.$$

Such maps are called *bridge maps* in Hu [11] and *projections* in Milnor [17].

### 2.3. Maps from nerves to the space (dominations)

We now describe the construction of maps $N(\mathcal{U}) \to X$, using the construction presented by Ferry [4, Theorems 29.7 and 29.9]. Given

$$\sigma = \langle U_{j_0} \dots U_{j_k} \rangle \in N(\mathcal{U}),$$

we pick a point $x_\sigma \in \bigcap \sigma$ and define a non-continuous map

$$\rho : N(\mathcal{U}) \to X \qquad \text{by } \rho(\sigma) = x_\sigma.$$

Let $W \subset \mathbb{R}^m$ be an appropriate regular neighbourhood of some embedding $X \subset \mathbb{R}^m$. Then the map

$$N(\mathcal{U}) \xrightarrow{\rho} X \subset W$$

can be approximated by a continuous map

$$\psi' : N(\mathcal{U}) \to W.$$

The composition with the retraction

$$\pi : W \to X$$

then gives the map

$$\psi = \pi \circ \psi' : N(\mathcal{U}) \to X.$$

By sufficiently subdividing $N(\mathcal{U})$, one can achieve that

$$\text{dist}(\psi, \rho) < \delta, \quad \text{for arbitrary small } \delta > 0.$$

For a given $\varepsilon > 0$, one then finds coverings $\mathcal{U}$ with $\text{mesh}(\mathcal{U})$ sufficiently small, so that

$$\text{dist}(Id_X, \rho \circ \varphi) < \varepsilon,$$

https://doi.org/10.1017/S0013091520000012 Published online by Cambridge University Press

and therefore

$$\mathrm{dist}(Id_X, \psi \circ \varphi) < \varepsilon + \delta.$$

By invoking Ferry [4, Corollary 29.9], we can then conclude that

$$Id_X \text{ and } \psi \circ \varphi \text{ are } \varepsilon'\text{-homotopic.}$$

Beginning with an $\varepsilon' > 0$, one then finds coverings $\mathcal{U}$ of $X$ such that

$$\psi \circ \varphi \text{ is } \varepsilon'\text{-homotopic to } Id_X.$$

This is well known (see, for example, Hu [11, Theorem 6.1, p. 138]), but we shall need some of the details from above in the sequel.

**Theorem 2.1 (Hu [11]).** *Let $X$ be an absolute neighbourhood retract (ANR). Then the following properties hold.*

(i) *Given an open covering $\mathcal{U}$ of $X$, there exist maps*

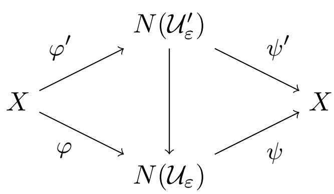
$$\varphi : X \to N(\mathcal{U}) \quad \text{and} \quad \psi : N(\mathcal{U}) \to X.$$

(ii) *Given $\varepsilon > 0$, there exists an open covering $\mathcal{U}_\varepsilon$ of $X$ such that the composite map*

$$X \xrightarrow{\varphi} N(\mathcal{U}_\varepsilon) \xrightarrow{\psi} X$$

*is $\varepsilon$-homotopic to $Id_X$.*

(iii) *Given $\mathcal{U}_\varepsilon$ as in* (ii), *there exist a refinement $\mathcal{U}'_\varepsilon$ and a map $N(\mathcal{U}'_\varepsilon) \to N(\mathcal{U}_\varepsilon)$ such that the diagram*




*and its subdiagrams are commutative up to $\varepsilon$-homotopy.*

## 3. Fundamental complexes

Let $X$ be a compact metric space. As explained in § 2, we can choose a covering $\mathcal{U}$ of $X$ such that the composite map

$$X \xrightarrow{\varphi} N(\mathcal{U}) \xrightarrow{\psi} X$$

is an $\varepsilon$-equivalence for a given $\varepsilon > 0$. Then we can choose a refinement $\mathcal{U}'$ of $\mathcal{U}$ such that the composite map

$$X \xrightarrow{\varphi'} N(\mathcal{U}') \xrightarrow{\psi'} X$$

is an $\varepsilon'$-equivalence for a given $\varepsilon' < \varepsilon$, etc.

https://doi.org/10.1017/S0013091520000012 Published online by Cambridge University Press

In this way we can get a sequence of coverings $\{\mathcal{U}_1, \mathcal{U}_2, \ldots\}$ such that $\mathcal{U}_{j+1}$ refines $\mathcal{U}_j$ for every $j \in \mathbb{N}$, and the composite map

$$X \xrightarrow{\varphi_j} N(\mathcal{U}_j) \xrightarrow{\psi_j} X$$

is an $\varepsilon_j$-equivalence with $\varepsilon_j \to 0$ for $j \to \infty$. Moreover, we have simplicial maps

$$s_j : N(\mathcal{U}_{j+1}) \to N(\mathcal{U}_j)$$

so that the diagram

$$\begin{array}{ccccc} & & N(\mathcal{U}_{j+1}) & & \\ & \overset{\varphi_{j+1}}{\nearrow} & & \overset{\psi_{j+1}}{\searrow} & \\ X & & \Big\downarrow s_j & & X \\ & \underset{\varphi_j}{\searrow} & & \underset{\psi_j}{\nearrow} & \\ & & N(\mathcal{U}_j) & & \end{array}$$

and its subdiagrams commute up to homotopy.

We add to this sequence the trivial covering $\mathcal{U}_0 = \{X\}$ with

$$s_0 : N(\mathcal{U}_1) \to N(\mathcal{U}_0) = \langle X \rangle$$

the constant map. The union of the mapping cylinders of the simplicial maps $\{s_j\}_{j\in\{0,1,\ldots\}}$, denoted by

$$F = \bigcup_{j \geq 0} N(\mathcal{U}_{j+1}) \times I \underset{s_j}{\cup} N(\mathcal{U}_j),$$

is an open fundamental complex of $X$.

Let

$$F_l = \bigcup_{l \geq j \geq 0} N(\mathcal{U}_{j+1}) \times I \underset{s_j}{\cup} N(\mathcal{U}_j),$$

that is to say, $F_l \subset F_{l+1}$ is a deformation retract and

$$r_l : F_{l+1} \to F_l$$

is the obvious retraction. Then

$$CF = \varprojlim_{l} F_l$$

is a closed fundamental complex.

https://doi.org/10.1017/S0013091520000012 Published online by Cambridge University Press

Both complexes are contractible, $F \subset CF$, and

$$CF \setminus F = \varprojlim_{s_j} N(\mathcal{U}_{j+1}).$$

Identifying $N(\mathcal{U}_j)$ with the mapping cylinder

$$N(\mathcal{U}_{j+1}) \times I \underset{s_j}{\cup} N(\mathcal{U}_j),$$

we can form $\bigcap_j N(\mathcal{U}_j)$ and complete $F$ by it, that is,

$$\bigcap_j N(\mathcal{U}_j) = \varprojlim_{s_j} N(\mathcal{U}_{j+1}).$$

**Theorem 3.1.** *The maps*

$$\psi_j : N(\mathcal{U}_j) \to X$$

*fit together to form the map*

$$\psi : \varprojlim_j N(\mathcal{U}_j) \to X.$$

**Proof.** Let

$$h : N(\mathcal{U}_{j+1}) \times I \to X$$

be a homotopy between $\psi_{j+1}$ and $\psi_j \circ s_j$. It induces a map

$$\Lambda_j : N(\mathcal{U}_{j+1}) \times I \underset{s_j}{\cup} N(\mathcal{U}_j) \to X$$

which restricts to

$$\psi_{j+1} \text{ on } N(\mathcal{U}_{j+1}) \times \{0\} \quad \text{and} \quad \psi_j \text{ on } N(\mathcal{U}_j),$$

hence it can be glued to give maps

$$F_l^\circ = \bigcup_{l \geq j \geq 1} N(\mathcal{U}_{j+1}) \times I \to X.$$

Since the diagram

$$\begin{array}{ccc} N(\mathcal{U}_{j+1}) \times I \underset{s_j}{\cup} N(\mathcal{U}_j) & \xrightarrow{r_j} & N(\mathcal{U}_j) = N(\mathcal{U}_j) \times \{0\} \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

commutes, it induces a map

$$\varprojlim_{l \geq 1} F_l^\circ \to X$$

whose restriction to $\varprojlim_{l \geq 1} N(\mathcal{U}_l)$ then gives the map $\psi$. □

https://doi.org/10.1017/S0013091520000012 Published online by Cambridge University Press

## 4. Construction of $\mathbb{L}$-homology classes

In this section $X^n$ will denote an oriented generalized $n$-manifold, $n \geq 5$, with a fundamental class

$$[X] \in H_n(X^n, \mathbb{Z}).$$

Then $X^n$ has a Spivak normal fibration $\nu_{X^n}$ (see Quinn [**22**, Example 2.3]). Moreover, $\nu_{X^n}$ has topological reductions (see Ferry and Pedersen [**7**, Theorem 16.6]). We shall consider a sequence of coverings $\{\mathcal{U}_j\}_{j\in\{0,1,\dots\}}$ as described in § 3.

**Theorem 4.1.** *There is a map*

$$\Gamma_j : X^n \times I \to N(\mathcal{U}_{j+1}) \times I \underset{s_j}{\cup} N(\mathcal{U}_j)$$

*such that* $\Gamma_j$ *restricts to*

$$\varphi_{j+1} : X^n \times \{0\} \to N(\mathcal{U}_{j+1}) \times \{0\}$$

*and*

$$\varphi_j : X^n \times \{1\} \to N(\mathcal{U}_j).$$

**Proof.** We consider the composite map

$$\overline{\varphi}_{j+1} : X^n \times I \xrightarrow{\varphi_{j+1}\times Id} N(\mathcal{U}_{j+1}) \times I \to N(\mathcal{U}_{j+1}) \times I \underset{s_j}{\cup} N(\mathcal{U}_j).$$

It restricts to

$$\varphi_{j+1} : X^n \times \{0\} \to N(\mathcal{U}_{j+1}) \times I \underset{s_j}{\cup} N(\mathcal{U}_j)$$

and

$$s_j \circ \varphi_{j+1} : X^n \times \{1\} \to N(\mathcal{U}_{j+1}) \times I \underset{s_j}{\cup} N(\mathcal{U}_j).$$

However,

$$s_j \circ \varphi_{j+1} \simeq \varphi_j \quad \text{via the homotopy } g : X^n \times I \to N(\mathcal{U}_j).$$

Composing $\overline{\varphi}_{j+1}$ and $g$ in the obvious way,

$$X^n \times I \cup X^n \times [1,2] \xrightarrow{\overline{\varphi}_{j+1}\cup g} N(\mathcal{U}_{j+1}) \times I \underset{s_j}{\cup} N(\mathcal{U}_j),$$

one gets the required map

$$\Gamma_j : X^n \times I \simeq X^n \times I \cup X^n \times [1,2] \to N(\mathcal{U}_{j+1}) \times I \underset{s_j}{\cup} N(\mathcal{U}_j). \qquad \square$$

Let us denote

$$F_0 = \bigcup_{j\geq 1} N(\mathcal{U}_{j+1}) \times I \underset{s_j}{\cup} N(\mathcal{U}_j),$$

that is, $F_0 \sim F \setminus \{b_0\}$, where

$$b_0 \in N(\mathcal{U}_1) \times I \underset{s_0}{\cup} N(\mathcal{U}_0)$$

is the basepoint of $N(\mathcal{U}_0)$. Then we get the following corollary.

https://doi.org/10.1017/S0013091520000012 Published online by Cambridge University Press

**Corollary 4.2.** *The maps $\Lambda_j$ and $\Gamma_j$ in Theorems 3.1 and 4.1 fit together to give maps*

$$X^n \times \mathbb{R}_+ \xrightarrow{\Gamma} F_0 \xrightarrow{\Lambda} X^n$$

*such that $\Lambda \circ \Gamma$ restricts to*

$$\psi_1 \circ \varphi_1 : X^n \times \{0\} \to X^n.$$

We can now construct a normal space with underlying space $F_0$ as follows. Let $\xi$ be a topological reduction of $\nu_{X^n}$ and set $\eta = \Lambda^*(\xi)$. Since

$$\Lambda \circ \Gamma \sim \psi_1 \circ \varphi_1 \sim Id,$$

we get

$$\Gamma^*(\eta) \cong \xi \times \mathbb{R}_+.$$

Then

$$\beta : S^m \times \mathbb{R}_+ \xrightarrow{\alpha \times Id} T(\xi) \times \mathbb{R}_+ \cong T(\Gamma^*(\eta)) \to T(\eta)$$

defines the structure map of the bundle $\eta$ over $F_0$. Here, $T(\cdot)$ denotes the Thom space and

$$\alpha : S^m \to T(\xi) \sim T(\nu_{X^n})$$

is the structure map of $(X^n, \nu_{X^n})$, where we assume that $X^n \subset S^m$. Therefore $\xi$ is an $\mathbb{R}^{m-n}$-bundle over $X^n$.

Let

$$(f, b) = (f : M^n \to X^n,\ b : \nu_{M^n} \to \xi)$$

be a surgery problem. It defines a normal map

$$(F, B) = \big(M^n \times \mathbb{R}_+ \xrightarrow{f \times Id} X^n \times \mathbb{R}_+ \xrightarrow{\Gamma} F_0,\ \nu_{M^n} \times \mathbb{R}_+ \xrightarrow{b \times Id} \xi \times \mathbb{R}_+ \xrightarrow{\widetilde{\Gamma}} \eta\big),$$

where $\widetilde{\Gamma}$ is the obvious bundle map covering $\Gamma$.

The mapping cylinder $M(F, B)$ is well known to be a normal space with boundary equal to

$$\partial M(F, B) = M^n \times \mathbb{R}_+ \amalg F_0.$$

We shall only consider the restriction of $(F, B)$ to

$$M^n \times (0, \infty) \to F \setminus N(\mathcal{U}_1) \times I \underset{s_0}{\cup} N(\mathcal{U}_0),$$

and also denote it by $(F, B)$. Since

$$F \setminus N(\mathcal{U}_1) \times I \underset{s_0}{\cup} N(\mathcal{U}_0)$$

is a locally finite complex, normal transversality is used to decompose $M(F, B)$ into adic normal complexes.

https://doi.org/10.1017/S0013091520000012 Published online by Cambridge University Press

If $\xi'$ is another topological reduction of $\nu_{X^n}$ then the same construction gives $M(F', B')$. One now glues

$$M(F, B) \cup -M(F', B') \text{ along } F_0$$

to obtain an element

$$\{f', b'\} - \{f, b\} \in H^{lf}_{n+2}(F_0, \Omega^{NPD}).$$

Here, $-M(F, B)$ indicates that the orientation on $M^n$ is reversed.

By Theorem 1.1, this is isomorphic to

$$H^{lf}_{n+1}(F_0, \mathbb{L}^+)$$

which in turn, by Theorem 1.2, is isomorphic to

$$H^{st}_{n+1}(CF, \varprojlim_j N(\mathcal{U}_j) \amalg N(\mathcal{U}_1) \times I \cup_{s_0} N(\mathcal{U}_0), \mathbb{L}^+).$$

Under the homology boundary morphism it maps to an element in

$$H^{st}_n(\varprojlim_j N(\mathcal{U}_j), \mathbb{L}^+).$$

Finally,

$$\psi_* : H^{st}_n(\varprojlim_j N(\mathcal{U}_j), \mathbb{L}^+) \to H^{st}_n(X, \mathbb{L}^+)$$

gives the desired element $[f', b'] - [f, b]$.

**Remark 4.3.** We thank the referee for pointing out an error here in our previous version (we have claimed that $M(F, B)$ defines an element already in $H^{lf}_{n+2}(F_0, \Omega^{NPD})$).

The element $\{f', b'\} - \{f, b\}$ is represented by a compatible family of adic objects

$$(\{f', b'\} - \{f, b\})_\sigma$$

belonging to the semisimplicially defined spectrum $\Omega^{NPD}$, where $\sigma$ is a simplex in $F_0$. Since $\sigma$ belongs to some

$$N(\mathcal{U}_{l+1}) \times I \cup_{s_l} N(\mathcal{U}_l),$$

one can break $\{f', b'\} - \{f, b\}$ into pieces

$$\begin{aligned} \{f', b'\}_l - \{f, b\}_l &\in H^{lf}_{n+2}\big(N(\mathcal{U}_{l+1}) \times I \cup_{s_l} N(\mathcal{U}_l), \Omega^{NPD}\big) \\ &= H_{n+2}\big(N(\mathcal{U}_{l+1}) \times I \cup_{s_l} N(\mathcal{U}_l), \Omega^{NPD}\big). \end{aligned}$$

We shall return to this splitting later on. A detailed construction of the adic elements

$$(\{f', b'\} - \{f, b\})_\sigma$$

which works also in our case is given in Kúhl *et al.* [**14**, Construction 11.3, p. 236].

https://doi.org/10.1017/S0013091520000012 Published online by Cambridge University Press

Before stating the main result of this section, recall the following well-known fact (see Ferry and Pedersen [7, Theorem 16.6]).

**Lemma 4.4.** *The canonical topological reduction $\xi_0$ of the generalized manifold $X$ defines, up to a normal cobordism, a unique surgery problem*

$$(f_0, b_0) : M_0 \to X,$$

*called the canonical surgery problem.*

**Proof.** Since $X$ is a compact metric space, it is homotopy equivalent to a finite complex $K$, hence $K$ is a $PD_n$-complex (see West [30]). There is a fibre homotopy equivalence $\nu_K \sim \nu_X$ covering the homotopy equivalence $K \sim X$. The latter induces a reduction $\xi_0$ on $K$ and a structure map

$$S^m \to T(\xi_0).$$

Transversality applies here to define a surgery problem (the Pontryagin–Thom construction):

$$(f_0, b_0) : M_0 \to K \sim X.$$

If $K'$ is another finite complex homotopy equivalent to $X$, it can be easily proved that the resulting surgery problem is normally cobordant to $(f_0, b_0)$. □

In summary, we have obtained the following result.

**Theorem 4.5.** *Let $X$ be an oriented generalized $n$-manifold, $n \geq 5$, with the canonical reduction $\xi_0$ of $\nu_X$ whose associated canonical surgery problem is*

$$(f_0, b_0) : M_0 \to X.$$

*Then the procedure explained after Corollary 4.2 yields, for any degree-one normal map*

$$(f, b) : M \to X,$$

*a well-defined element*

$$[f, b] - [f_0, b_0] \in H_n^{st}(X, \mathbb{L}^+).$$

## 5. Improvements and outlook

As in the previous chapter, let $X$ be a generalized manifold and $\mathcal{N}(X)$ the set of all normal bordism classes of degree-one normal maps $(f, b) : M \to X$. Theorem 4.5 can be improved to give the following result.

**Theorem 5.1.** *The association*

$$(f, b) \rightsquigarrow [f, b] - [f_0, b_0]$$

*in Theorem 4.5 defines a map*

$$t : \mathcal{N}(X) \to H_n^{st}(X, \mathbb{L}^+).$$

https://doi.org/10.1017/S0013091520000012 Published online by Cambridge University Press

**Proof.** We have to show, first, that the construction does not depend on the choice of the normal bordism class of $(f, b)$ and, second, that it does not depend on the choice of the sequence $\{\mathcal{U}_j\}$ described in § 3 either.

**Lemma 5.2.** *Fix the sequence of coverings $\{\mathcal{U}_j\}$ of $X$ described in § 3. Suppose that*

$$(f, b) : M \to X$$

*is normally bordant to*

$$(f', b') : M' \to X.$$

*Then*

$$\{f', b'\} - \{f_0, b_0\} = \{f, b\} - \{f_0, b_0\} \in H^{lf}_{n+2}(F_0, \Omega^{NPD}).$$

**Proof.** Let

$$(g, c) : W \to X \times I$$

be a normal cobordism between $(f, b)$ and $(f', b')$. Consider also the product normal cobordism

$$(g_0, c_0) : M_0 \times I \to X \times I.$$

The mapping cylinders of the obvious normal maps

$$(G, C) : W \times \mathbb{R}_+ \to X \times I \times \mathbb{R}_+ \to F_0 \times I$$

and

$$(G_0, C_0) : M_0 \times I \times \mathbb{R}_+ \to X \times I \times \mathbb{R}_+ \to F_0 \times I$$

can be glued along $F_0 \times I$ to give a normal cobordism between

$$M(F, B) \cup -M(F_0, B_0)$$

and

$$M(F', B') \cup -M(F_0, B_0),$$

implying the claim. For definitions of $M(F, B), -M(F_0, B_0)$, and $M(F', B')$ see the previous section. □

For the second step of the proof of Theorem 5.1, we let $\{\mathcal{U}_j\}, \{\mathcal{U}'_j\}$ be two sequences defining

$$F_0 = \bigcup_{j \geq 1} N(\mathcal{U}_{j+1}) \times I \underset{s_j}{\cup} N(\mathcal{U}_j)$$

and

$$F'_0 = \bigcup_{j \geq 1} N(\mathcal{U}'_{j+1}) \times I \underset{s'_j}{\cup} N(\mathcal{U}'_j).$$

Let

$$\mathcal{U}''_j = \{U \cap U' \mid U \in \mathcal{U}_j, U' \in \mathcal{U}'_j\}.$$

Observe that

$$\text{mesh}(\mathcal{U}''_j) \leq \min\{\text{mesh}(\mathcal{U}_j),\ \text{mesh}(\mathcal{U}'_j)\},$$

where $\{\mathcal{U}''_j\}$ is a sequence as described in § 3, defining $F''_0$ and the open (respectively, closed) fundamental complex $F''$ (respectively, $CF''$).

https://doi.org/10.1017/S0013091520000012 Published online by Cambridge University Press

Let $(f, b) : M \to X$ be given. Our strategy will be to compare the elements

$$\{f, b\} - \{f_0, b_0\} \in H^{lf}_{n+2}(F_0, \Omega^{NPD})$$

and

$$\{f, b\}' - \{f_0, b_0\}' \in H^{lf}_{n+2}(F'_0, \Omega^{NPD}),$$

with

$$\{f, b\}'' - \{f_0, b_0\}'' \in H^{lf}_{n+2}(F''_0, \Omega^{NPD}).$$

Since $\mathcal{U}''_j$ refines $\mathcal{U}_j$, there are maps

$$p_j : N(\mathcal{U}''_j) \to N(\mathcal{U}_j)$$

such that the diagram

$$\begin{array}{ccc} N(\mathcal{U}''_{j+1}) & \xrightarrow{\;s''_j\;} & N(\mathcal{U}''_j) \\ \Big\downarrow{\scriptstyle p_{j+1}} & & \Big\downarrow{\scriptstyle p_j} \\ N(\mathcal{U}_{j+1}) & \xrightarrow[\;s_j\;]{} & N(\mathcal{U}_j) \end{array}$$

commutes up to homotopy.

The mapping cylinder construction now applies to obtain maps

$$q_j : N(\mathcal{U}''_{j+1}) \times I \underset{s''_j}{\cup} N(\mathcal{U}''_j) \to N(\mathcal{U}_{j+1}) \times I \underset{s_j}{\cup} N(\mathcal{U}_j)$$

which restrict to $p_{j+1}$ (respectively, $p_j$) on the boundary. Therefore they can be pieced together to yield a map

$$q = \cup q_j : F''_0 \to F_0.$$

The completion of this process then gives the map which we shall also denote by $q$,

$$q : \big(CF'', F''_0, \varprojlim_j N(\mathcal{U}''_j)\big) \to \big(CF, F_0, \varprojlim_j N(\mathcal{U}_j)\big).$$

We shall also need the following lemma.

**Lemma 5.3.** *Under the map*

$$q_* : H^{lf}_{n+2}(F''_0, \Omega^{NPD}) \to H^{lf}_{n+2}(F_0, \Omega^{NPD}),$$

*the element*

$$\{f, b\}'' - \{f_0, b_0\}''$$

*maps to the element*

$$\{f, b\} - \{f_0, b_0\}.$$

https://doi.org/10.1017/S0013091520000012 Published online by Cambridge University Press

**Proof.** To prove the lemma, we 'break up'

$$\{f,b\}'' - \{f_0,b_0\}'' \quad (\text{respectively, } \{f,b\} - \{f_0,b_0\})$$

into pieces

$$\{f,b\}''_j - \{f_0,b_0\}''_j \quad (\text{respectively, } \{f,b\}_j - \{f_0,b_0\}_j)$$

and we show that they correspond under $q_*$.

To this end, consider the normal map

$$(F_j,B_j) : M \times [j,j+1] \xrightarrow{(f,b)\times Id} X \times [j,j+1] \xrightarrow{\Gamma_j} N(\mathcal{U}_{j+1}) \times I \underset{s_j}{\cup} N(\mathcal{U}_j),$$

where $B_j$ is the obvious bundle map with the target in $\Lambda^*_j(\xi)$. As above, here $b : \nu_M \to \xi$ is the bundle map of $(f,b) : M \to X$. Observe that

$$\Lambda^*(\xi)|_{N(\mathcal{U}_{j+1})\times I \underset{s_j}{\cup} N(\mathcal{U}_j)} \cong \Lambda^*_j(\xi).$$

The mapping cylinder $M(F_j,B_j)$ is then a normal complex with boundary. We do the same for

$$(f_0,b_0) : M_0 \to X$$

and obtain $(F^\circ_j, B^\circ_j)$.

Then

$$M(F_j,B_j) \cup -M(F^\circ_j,B^\circ_j)$$

defines an element

$$\{f,b\}_j - \{f_0,b_0\}_j \in H_{n+2}\big(N(\mathcal{U}_{j+1}) \times I \underset{s_j}{\cup} N(\mathcal{U}_j), \Omega^{NPD}\big).$$

The inclusions

$$N(\mathcal{U}_{j+1}) \times I \underset{s_j}{\cup} N(\mathcal{U}_j) \to F_0$$

represent $\{f,b\} - \{f_0,b_0\}$ as an infinite (locally finite) sum

$$\sum_j (\{f,b\}_j - \{f_0,b_0\}_j).$$

The same process yields a representation for $\{f,b\}'' - \{f_0,b_0\}''$ as an infinite (locally finite) sum

$$\sum_j (\{f,b\}''_j - \{f_0,b_0\}''_j).$$

https://doi.org/10.1017/S0013091520000012 Published online by Cambridge University Press

We now consider the following (homotopy) commutative diagram:

$$\begin{array}{ccccccc}
M \times I & \xrightarrow{\ f \times Id\ } & X \times I & \xrightarrow{\ \Gamma_j''\ } & N(\mathcal{U}_{j+1}'') \times I \cup_{s_j''} N(\mathcal{U}_j'') & \xrightarrow{\ \Lambda_j''\ } & X \\
\downarrow{=} & & \downarrow{=} & & \downarrow{q_j} & & \downarrow{=} \\
M \times I & \xrightarrow{\ f \times Id\ } & X \times I & \xrightarrow{\ \Gamma_j\ } & N(\mathcal{U}_{j+1}) \times I \cup_{s_j} N(\mathcal{U}_j) & \xrightarrow{\ \Lambda_j\ } & X
\end{array}$$

First, we deduce from this diagram that

$$q_j^* \Lambda_j^*(\xi) \cong \Lambda_j''^*(\xi),$$

hence $q_j$ can be covered by a bundle map

$$\overline{q}_j : \Lambda_j''^*(\xi) \to \Lambda_j^*(\xi).$$

Next, we have

$$\Gamma_j''^*(\Lambda_j''^*(\xi)) \cong \xi \times I \quad \text{and} \quad \Gamma_j^*(\Lambda_j^*(\xi)) \cong \xi \times I.$$

From this we obtain the bundle maps

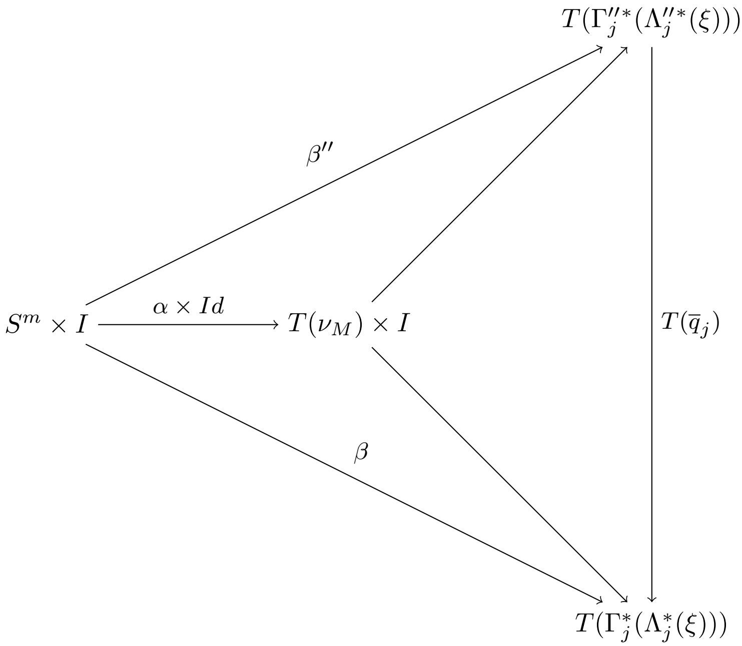
$$b'' : \nu_M \times I \xrightarrow{b \times Id} \xi \times I \cong \Gamma_j''^*(\Lambda_j''^*(\xi))$$

and

$$b : \nu_M \times I \xrightarrow{b \times Id} \xi \times I \cong \Gamma_j^*(\Lambda_j^*(\xi))$$

together with the (homotopy) commutative diagram

https://doi.org/10.1017/S0013091520000012 Published online by Cambridge University Press




Now

$$(N(\mathcal{U}''_{j+1}) \times I \underset{s''_j}{\cup} N(\mathcal{U}''_j), \Gamma_j''^*(\Lambda_j''^*(\xi)), \beta'')$$

and

$$(N(\mathcal{U}_{j+1}) \times I \underset{s_j}{\cup} N(\mathcal{U}_j), \Gamma_j^*(\Lambda_j^*(\xi)), \beta)$$

determine $M(F''_j, B''_j)$ and $M(F_j, B_j)$, respectively. Therefore we can conclude that

$$(*) \quad q_{j^*} : \Omega_{n+2}^{NPD}(N(\mathcal{U}''_{j+1}) \times I \underset{s''_j}{\cup} N(\mathcal{U}''_j)) \to \Omega_{n+2}^{NPD}(N(\mathcal{U}_{j+1}) \times I \underset{s_j}{\cup} N(\mathcal{U}_j))$$

maps $M(F''_j, B''_j)$ to $M(F_j, B_j)$.

The same holds for $M(F_j''^{\circ}, B_j''^{\circ})$ and $M(F_j^{\circ}, B_j^{\circ})$, if we take $(f_0, b_0) : M_0 \to X$ instead of $(f, b) : M \to X$.

Since the differences have manifold boundaries, we get

$$\begin{aligned} M(F''_j, B''_j) - M(F_j''^{\circ}, B_j''^{\circ}) &\in \Omega_{n+2}^{NTOP}\big(N(\mathcal{U}''_{j+1}) \times I \underset{s''_j}{\cup} N(\mathcal{U}''_j)\big) \\ &\cong H_{n+2}\big(N(\mathcal{U}''_{j+1}) \times I \underset{s''_j}{\cup} N(\mathcal{U}''_j), \Omega^{NTOP}\big), \end{aligned}$$

https://doi.org/10.1017/S0013091520000012 Published online by Cambridge University Press

and similarly,

$$M(F_j, B_j) - M(F_j^\circ, B_j^\circ) \in \Omega_{n+2}^{NTOP}\big(N(\mathcal{U}_{j+1}) \times I \underset{s_j}{\cup} N(\mathcal{U}_j)\big)$$
$$\cong H_{n+2}\big(N(\mathcal{U}_{j+1}) \times I \underset{s_j}{\cup} N(\mathcal{U}_j), \Omega^{NTOP}\big)$$

(see Remark 5.4 below).

The canonical map of spectra $\Omega^{NTOP} \to \Omega^{NPD}$ maps these elements to

$$\{f, b\}_j'' - \{f_0, b_0\}_j'' \quad \text{and} \quad \{f, b\}_j - \{f_0, b_0\}_j,$$

respectively.

The property $(*)$ above now implies that

$$q_{j^*} : H_{n+2}\big(N(\mathcal{U}_{j+1}'') \times I \underset{s_j''}{\cup} N(\mathcal{U}_j''), \Omega^{NPD}\big)$$
$$\to H_{n+2}\big(N(\mathcal{U}_{j+1}) \times I \underset{s_j}{\cup} N(\mathcal{U}_j), \Omega^{NPD}\big)$$

maps $\{f, b\}_j'' - \{f_0, b_0\}_j''$ to $\{f, b\}_j - \{f_0, b_0\}_j$.

This completes the proof of the lemma. □

**Remark 5.4.** Transversality implies that the assembly construction defines an isomorphism between $\Omega^{NTOP}$ homology groups and $\Omega^{NTOP}$ bordism groups. This is not true for $\Omega^{NPD}$.

We now continue with the proof of Theorem 5.1. Denote by

$$\langle f, b\rangle - \langle f_0, b_0\rangle \in H_n^{st}\big(\varprojlim_j N(\mathcal{U}_j), \mathbb{L}^+\big)$$

the image of $\{f, b\} - \{f_0, b_0\}$ under the composition

$$H_{n+2}^{lf}\big(F_0, \Omega^{NPD}\big) \cong H_{n+1}^{st}\big(CF, \varprojlim_j N(\mathcal{U}_j) \cup \{*\}, \mathbb{L}^+\big) \xrightarrow{\partial_*'} H_n^{st}\big(\varprojlim_j N(\mathcal{U}_j), \mathbb{L}^+\big),$$

where $\partial_*'$ is the composition of the boundary homomorphism with the projection

$$H_n^{st}\big(\varprojlim_j N(\mathcal{U}_j), \mathbb{L}^+\big) \oplus H_n^{st}(\{*\}, \mathbb{L}^+) \to H_n^{st}\big(\varprojlim_j N(\mathcal{U}_j), \mathbb{L}^+\big).$$

Lemma 5.3 now implies that

$$\big(\, q|_{\varprojlim_j N(\mathcal{U}_j)}\big)_*\big(\langle f, b\rangle'' - \langle f_0, b_0\rangle''\big) = \langle f, b\rangle - \langle f_0, b_0\rangle,$$

where $\langle f, b\rangle''$ and $\langle f_0, b_0\rangle''$ denote the corresponding images of $\{f, b\}''$ and $\{f_0, b_0\}''$, respectively.

https://doi.org/10.1017/S0013091520000012 Published online by Cambridge University Press

In order to complete the proof of Theorem 5.1, we have to pass to

$$H_n^{st}(X, \mathbb{L}^+)$$

via the homomorphism

$$\psi_* : H_n^{st}\big(\varprojlim_j N(\mathcal{U}_j), \mathbb{L}^+\big) \to H_n^{st}(X, \mathbb{L}^+),$$

induced by the map

$$\psi : \varprojlim_j N(\mathcal{U}_j) \to X$$

which was defined in Theorem 3.1. Similarly for the map

$$\psi'' : \varprojlim_j N(\mathcal{U}_j'') \to X.$$

Now observe that

$$q|_{N(\mathcal{U}_j'')} = p_j$$

and that $\psi_j''$ is homotopic to $\psi_j \circ p_j$. Hence the following diagram commutes:

$$\begin{array}{ccc}
H_*^{st}(N(\mathcal{U}_j''), \mathbb{L}^+) & & \\
\Big\downarrow (q|_{N(\mathcal{U}_j'')})_* & \searrow^{(\psi_j'')_*} & \\
 & & H_*^{st}(X, \mathbb{L}^+) \\
 & \nearrow_{(\psi_j)_*} & \\
H_*^{st}(N(\mathcal{U}_j), \mathbb{L}^+) & &
\end{array}$$

It follows that the diagram

$$\begin{array}{ccc}
H_*^{st}\big(\varprojlim_j N(\mathcal{U}_j''), \mathbb{L}^+\big) & & \\
\Big\downarrow (q|_{N(\mathcal{U}_j'')})_* & \searrow^{(\psi_j'')_*} & \\
 & & H_*^{st}(X, \mathbb{L}^+) \\
 & \nearrow_{(\psi_j)_*} & \\
H_*^{st}\big(\varprojlim_j N(\mathcal{U}_j), \mathbb{L}^+\big) & &
\end{array}$$

https://doi.org/10.1017/S0013091520000012 Published online by Cambridge University Press

also commutes, thus we can see that

$$\psi''_*(\langle f,b\rangle' - \langle f_0,b_0\rangle'') = \psi_*(\langle f,b\rangle - \langle f_0,b_0\rangle).$$

Analogously, one obtains

$$q' : \left(CF'', F''_0, \varprojlim_j N(\mathcal{U}''_j)\right) \to \left(CF', F'_0, \varprojlim_j N(\mathcal{U}'_j)\right)$$

such that

$$\psi''_*\left(\langle f,b\rangle'' - \langle f_0,b_0\rangle''\right) = \psi'_*\left(\langle f,b\rangle' - \langle f_0,b_0\rangle'\right).$$

This proves

$$[f,b]' - [f_0,b_0]' = [f,b] - [f_0,b_0],$$

and hence finally, completes the proof of Theorem 5.1. □

We shall now apply our construction to the case where $X$ is a manifold with simplicial structure. The given degree-one normal map $(f,b) : M \to X$ then decomposes into adic pieces to define an element

$$\sigma^c_*(f,b) \in H_n(X, \mathbb{L}^+).$$

This element is the controlled surgery obstruction of $(f,b)$ over $Id : X \to X$ (see Pedersen *et al.* [**19**]). We take $(f_0,b_0) = Id : X \to X$.

**Supplement.** In § 4 we associated to given normal degree-one maps

$$(f_0,b_0) : M_0 \to X \quad \text{and} \quad (f,b) : M \to X,$$

where $X$ is a generalized manifold, the element

$$[f,b] - [f_0,b_0] \in H^{st}_n(X, \mathbb{L}^+).$$

The normal maps $(f_0,b_0)$ and $(f,b)$ give rise to a normal space with boundary

$$M_0 \times (0,\infty) \quad \text{and} \quad M \times (0,\infty).$$

At this point, transversality for normal spaces (with TOP-manifold boundaries) is used to split (disassemble) the normal space, in order to obtain an element in the $\Omega^{NPD}$-homology group.

Actually, it belongs to the $\Omega^{NTOP}$-homology, but we pass to $\Omega^{NPD}$ via

$$\Omega^{NTOP} \to \Omega^{NPD}.$$

A detailed splitting construction can be found in Kúhl *et al.* [**14**, Construction 11.3, p. 236].

If $X$ is a manifold with simplicial structure, transversality directly applies to split $(f,b)$ and $(f_0,b_0)$ into pieces in order to obtain an element in $H_n(X, \mathbb{L})$. It is now natural to take

$$(f_0,b_0) = Id : X \to X.$$

Since $Id : X \to X$ does not contribute to $\mathbb{L}$-homology, one gets an element depending on $(f,b)$ which we shall denote by

$$\sigma(f,b) \in H_n(X, \mathbb{L})$$

https://doi.org/10.1017/S0013091520000012 Published online by Cambridge University Press

(this corresponds to $\mathrm{sig}_X^{\mathbb{L}}(f,b)$ in Kúhl *et al.* [**14**, Definition 8.14]). Moreover,

$$\sigma(f,b) \in H_n(X,\mathbb{L}^+).$$

Since $H_n(X,\mathbb{L})$ is the controlled surgery obstruction group, the element

$$\sigma(f,b) \in H_n(X,\mathbb{L}^+) \subset H_n(X,\mathbb{L})$$

is sometimes denoted by $\sigma^c(f,b)$.

The reason is that the zero-dimensional components come from

$$f^{-1}(D(\sigma,X)) \to D(\sigma,X),$$

where $\sigma \prec X$ runs through the $n$-simplices of $X$ and $D(\sigma,X)$ is its dual with respect to a subdivision $X'$ of $X$. Hence $D(\sigma,X)$ is a point $x \in X$ and, by transversality,

$$f^{-1}(D(\sigma,X)) = \{\pm y_1,\dots,\pm y_k\} \subset M.$$

Since $f$ has degree one, it is equivalent to $y \to x$, which is the trivial object. (We have also addressed such questions in Hegenbarth and Repovš [**10**, Lemma 2.1].)

If $X$ is only a generalized manifold, this leads to the so-called zero-dimensional signature of $f$. This is misleading, since it is the signature obstruction of a $4k$-dimensional surgery problem, which is 'moved' to $\pi_0(\mathbb{L}) = L_0$ by periodicity of $\mathbb{L}$ (see Hegenbarth and Repovš [**10**, p. 79]).

The aim of the next theorem is to show that for a given degree-one normal map $(f,b): M \to X$, where $X$ is a manifold with simplicial structure, the construction via normal spaces from § 4 gives an element which coincides with the element $\sigma(f,b)$.

**Theorem 5.5.** *The controlled surgery obstruction of* $(f,b): M \to X$ *coincides with* $[f,b] - [f_0,b_0]$.

**Proof.** Choose a sequence $\{\mathcal{U}_j\}$ of coverings of $X$ as above. Since $X$ is a manifold with simplicial structure, we can define

$$\overline{\{f,b\}} - \overline{\{f_0,b_0\}} \in H^{lf}_{n+2}(X\times(0,\infty),\Omega^{NPD}).$$

Here, $\overline{\{f,b\}}$ denotes the normal space, defined by the mapping cylinder of the map

$$(f\times Id, b\times Id): M\times(0,\infty) \to X\times(0,\infty),$$

and similarly for $\overline{\{f_0,b_0\}}$.

https://doi.org/10.1017/S0013091520000012 Published online by Cambridge University Press

Now, $\overline{\{f,b\}} - \overline{\{f_0,b_0\}}$ maps under the induced map

$$\Gamma : X \times (0,\infty) \to F_0$$

to

$$\{f,b\} - \{f_0,b_0\} \in H^{lf}_{n+2}(F_0, \Omega^{NPD}).$$

Under the composition

$$\begin{aligned} H^{lf}_{n+2}(X \times (0,\infty), \Omega^{NPD}) &\cong H^{lf}_{n+1}(X \times (0,\infty), \mathbb{L}^+) \\ &\cong H^{st}_{n+1}(X \times [0,\infty]/(X \times \{\infty\}), X \cup \{*\}, \mathbb{L}^+) \\ &\xrightarrow{\partial'_*} H^{st}_n(X, \mathbb{L}^+), \end{aligned}$$

$\overline{\{f,b\}} - \overline{\{f_0,b_0\}}$ maps to $\sigma^c_*(f,b)$. This is because $\overline{\{f,b\}} - \overline{\{f_0,b_0\}}$ is represented by the mapping cylinders of

$$(f \times Id, b \times Id) \quad \text{and} \quad (f_0 \times Id, b_0 \times Id).$$

The latter does not contribute to $\mathbb{L}$-homology because we have chosen $(f_0,b_0) = (Id, Id)$. Under the composition it therefore goes to the element defined by splitting $(f,b) : M \to X$, that is, to $\sigma(f,b)$.

Here,

$$X \times [0,\infty]/(X \times \{\infty\})$$

is the completion of $X \times (0,\infty)$ obtained as the inverse limit, similarly to

$$CF = \varprojlim_{l} F_l$$

(see § 3). However, under

$$\Gamma_* : H^{lf}_{n+2}(X \times (0,\infty), \Omega^{NPD}) \to H^{lf}_{n+2}(F_0, \Omega^{NPD})$$

the difference $\overline{\{f,b\}} - \overline{\{f_0,b_0\}}$ maps to $\{f,b\} - \{f_0,b_0\}$.

Consider now (using previous notation)

$$X \times I \xrightarrow{\Gamma_j} N(\mathcal{U}_{j+1}) \times I \underset{s_j}{\cup} N(\mathcal{U}_j) \xrightarrow{\overline{\Lambda}_j} X \times I,$$

where

$$\overline{\Lambda}_j(u,t) = (\Lambda_j(u), t)$$

so

$$\overline{\Lambda}_j \circ \Gamma_j(x,0) = ((\psi_{j+1} \circ \varphi_{j+1})(x), 0)$$

and

$$\overline{\Lambda}_j \circ \Gamma_j(x,1) = ((\psi_j \circ \varphi_j)(x), 1).$$

https://doi.org/10.1017/S0013091520000012 Published online by Cambridge University Press

Since $\psi_k \circ \varphi_k \sim Id_X$ we can use these homotopies to glue the maps and obtain

$$X \times \mathbb{R}_+ \xrightarrow{\Gamma} F_0 \xrightarrow{\overline{\Lambda}} X \times \mathbb{R}_+,$$

restricting to

$$X \times \{0\} \xrightarrow{\varphi} \varprojlim_j N(\mathcal{U}_j) \xrightarrow{\psi} X \times \{0\},$$

that is, we get maps

$$X \times [0,\infty]/(X \times \{\infty\}) \to CF \to X \times [0,\infty]/(X \times \{\infty\}).$$

Therefore $\Gamma$ induces a morphism between the sequences

$$\begin{aligned} H_{n+2}^{lf}(X \times [0,\infty), \Omega^{NPD}) &\cong H_{n+1}^{lf}(X \times [0,\infty), \mathbb{L}^+) \\ &\cong H_{n+1}^{st}(X \times [0,\infty]/(X \times \{\infty\}), X \cup \{*\}, \mathbb{L}^+) \\ &\xrightarrow{\partial'_*} H_n^{st}(X, \mathbb{L}^+) \end{aligned}$$

and

$$\begin{aligned} H_{n+2}^{lf}(F_0, \Omega^{NPD}) &\cong H_{n+1}^{lf}(F_0, \mathbb{L}^+) \\ &\cong H_{n+1}^{st}\big(CF, \varprojlim_j N(\mathcal{U}_j) \cup \{*\}, \mathbb{L}^+\big) \xrightarrow{\partial'_*} H_n^{st}\big(\varprojlim_j N(\mathcal{U}_j), \mathbb{L}^+\big). \end{aligned}$$

It follows that

$$\varphi_*(\sigma_*^c(f,b)) = \langle f, b\rangle - \langle f_0, b_0\rangle \in H_n^{st}(\varprojlim_j N(\mathcal{U}_j), \mathbb{L}^+).$$

Since $\psi \circ \varphi \sim Id$, we can conclude that

$$[f,b] - [f_0, b_0] = \psi_*(\langle f, b\rangle - \langle f_0, b_0\rangle) = \psi_*(\varphi_*(\sigma_*^c(f,b))) = \sigma_*^c(f,b). \qquad \square$$

We conclude this section with some remarks on the map $t$. In the PL manifold case there is an $\mathbb{L}^\bullet$-orientation

$$\mathcal{U}_{\mathbb{L}^\bullet} \in \overline{H}^{m-n}(T(\nu_X), \mathbb{L}^\bullet),$$

where $\mathbb{L}^\bullet$ is the symmetric $\mathbb{L}$-spectrum. Furthermore, $\mathbb{L}^\bullet$ is a ring spectrum and $\mathbb{L}^+$ is an $\mathbb{L}^\bullet$-module spectrum, and the cup product

$$.\bigcup \mathcal{U}_{\mathbb{L}^\bullet} : [X, G/TOP] = H^\circ(X, \mathbb{L}^+) \to \overline{H}^{m-n}(T(\nu_X), \mathbb{L}^+)$$

is an isomorphism. Here we are assuming that $X \subset \mathbb{R}^m$.

https://doi.org/10.1017/S0013091520000012 Published online by Cambridge University Press

The difference between $(f,b): M \to X$ and $(f_0,b_0): X \to X$ defines a map

$$\mathcal{N}(X) \to [X, G/TOP].$$

Combining with the Alexander–Spanier duality

$$\overline{H}^{m-n}(T(\nu_X),\mathbb{L}^+) \cong H_n(X,\mathbb{L}^+),$$

we obtain a bijective map

$$\mathcal{N}(X) \to H_n(X,\mathbb{L}^+).$$

This is the map $t$ (see Ranicki [**25**, Chapter 17, pp. 191–193]).

In the case of a generalized manifold we can embed $X$ into $\mathbb{R}^m$ with a cylindrical neighbourhood, also obtaining an isomorphism

$$\overline{H}^{m-n}(T(\nu_X),\mathbb{L}^+) \cong H_n^{st}(X,\mathbb{L}^+).$$

Let

$$N = \partial N \times I \cup_p X$$

be a mapping cylinder neighbourhood of $X \subset S^{m+1}$. It can be used to prove the following fact.

**Theorem 5.6.** *There exist an* $\mathbb{L}^\bullet$*-orientation*

$$\mathcal{U} \in H^{m+1-n}(N,\partial N,\mathbb{L}^\bullet)$$

*and an isomorphism*

$$.\bigcup \mathcal{U} : H^0(X,\mathbb{L}^+) \xrightarrow{\cong} H^{m+1-n}(N,\partial N,\mathbb{L}^+).$$

With this theorem one obtains the following isomorphisms:

$$\begin{aligned} H^0(X,\mathbb{L}^+) &\cong H^{m+1-n}(N,\partial N,\mathbb{L}^+) \\ &\cong H^{m+1-n}(S^{m+1}, S^{m+1}\setminus N,\mathbb{L}^+) \\ &\cong H^{m+1-n}(S^{m+1}, S^{m+1}\setminus X,\mathbb{L}^+) \\ &\cong \overline{H}^{m-n}(S^{m+1}\setminus X,\mathbb{L}^+) \\ &\cong H_n^{st}(X,\mathbb{L}^+). \end{aligned}$$

The last isomorphism is the Steenrod duality (see Kahn *et al.* [**13**, Theorem B], where one must take the reduced $\mathbb{L}^+$-homology).

We shall omit the proof of Theorem 5.6 because it is not obvious that the composition

$$\mathcal{N}(X) \to H^0(X,\mathbb{L}^+) \to H_n^{st}(X,\mathbb{L}^+)$$

coincides with the association

$$(f,b) \to [f,b] - [f_0,b_0].$$

This will be included in a future paper.

https://doi.org/10.1017/S0013091520000012 Published online by Cambridge University Press

## 6. Discussion

**(I)**. The homotopy groups of the spectrum $\mathbb{L}^+$ are the Wall groups of the trivial group, that is,

$$\pi_n(\mathbb{L}^+) \cong L_n(1) \cong L_n \quad \text{for every } n \geq 1.$$

Since the simplicial complex

$$N(\mathcal{U}_1) \times I \underset{s_0}{\cup} N(\mathcal{U}_0)$$

is contractible, we have

$$H_n^{st}(N(\mathcal{U}_1) \times I \underset{s}{\cup} N(\mathcal{U}_0), \mathbb{L}^+) \cong H_n(N(\mathcal{U}_1) \times I \underset{s_0}{\cup} N(\mathcal{U}_0), \mathbb{L}^+)$$
$$\cong H_n(\{b_0\}, \mathbb{L}^+) \cong L_n.$$

Therefore the above mentioned homology boundary homomorphism is

$$H_{n+1}^{st}\big(CF, \varprojlim_j N(\mathcal{U}_j) \amalg \{b_0\}, \mathbb{L}^+\big) \to H_n^{st}\big(\varprojlim_j N(\mathcal{U}_j), \mathbb{L}^+\big) \oplus L_n.$$

The component in $L_n$ is the surgery obstruction of

$$(f, b) = (M^n \to X^n,\ \nu_{M^n} \to \xi)$$

mapped to $L_n$ under

$$L_n(\pi_1(X^n)) \to L_n(1) \cong L_n,$$

where the morphism is induced by $X^n \to \{*\}$.
**(II).** We have used the map

$$\psi_* : H_n^{st}\big(\varprojlim_j N(\mathcal{U}_j), \mathbb{L}^+\big) \to H_n^{st}(X^n, \mathbb{L}^+)$$

to obtain our element

$$[f, b] \in H_n^{st}(X^n, \mathbb{L}^+).$$

We did not need the fact that it is an isomorphism.

In fact, the relation between $\varprojlim_j N(\mathcal{U}_j)$ and $X^n$ seems to be insufficiently documented. It was claimed in Milnor [**17**, Lemma 2] that they are identical. It was also asserted in Ferry [**5**, Footnote, p. 156] that they are strongly shape equivalent.

To this end, we state the following theorem. First, recall that, given $\varepsilon > 0$, a map $f : X \to Y$ of metric spaces $X$ and $Y$ is called an $\varepsilon$-map if for every $y \in Y$, $\operatorname{diam}(f^{-1}(y)) < \varepsilon$.

**Theorem 6.1.** *The maps*

$$\varphi_j : X^n \to N(\mathcal{U}_{j+1}) \times I \underset{s_j}{\cup} N(\mathcal{U}_j)$$

*fit together to produce the map*

$$\varphi : X^n \to \varprojlim_j N(\mathcal{U}_j),$$

*which is an $\varepsilon$-map onto the image of $\varphi$ for all $\varepsilon > 0$.*

https://doi.org/10.1017/S0013091520000012 Published online by Cambridge University Press

**Proof.** The maps $\Gamma_j$ can be glued to get maps

$$X_l^n = X^n \times [0, l+1] \to F_l^\circ,$$

such that the diagram

$$\begin{array}{ccc} X_l^n & \longrightarrow & F_l^\circ \\ \Big\downarrow {\scriptstyle pr} & & \Big\downarrow {\scriptstyle r_l} \\ X_{l-1}^n & \longrightarrow & F_{l-1}^\circ \end{array}$$

commutes.

Hence we get a map

$$X^n \times [0, \infty] \to \varprojlim_l F_l^\circ$$

which restricts to

$$\varphi : X^n \times \{\infty\} \to \varprojlim_j N(\mathcal{U}_j).$$

If now

$$p_l : \varprojlim_j N(\mathcal{U}_j) \to N(\mathcal{U}_l)$$

is the projection, then $p_l \circ \varphi = \varphi_l$.

Let $x \in \mathrm{Im}\varphi$. Then

$$x_l = p_l(x) \in N(\mathcal{U}_l)$$

belongs to some $\mathrm{st}(\langle U \rangle)$, for some vertex $\langle U \rangle \in N(\mathcal{U}_l)$, where $U \in \mathcal{U}_l$.

Therefore

$$\varphi^{-1}(x) \subset \varphi_l^{-1}(x_l) \subset U$$

(see Dugundji [**3**, Theorem 5.4, Chapter VIII]). Hence

$$\mathrm{diam}(\varphi^{-1}(x)) \leq \mathrm{mesh}(\mathcal{U}_l)$$

and since $\mathrm{mesh}(\mathcal{U}_j) \to 0$ for $j \to \infty$, the assertion follows. □

**Remark 6.2.** It would be interesting to know if the Bing shrinking criterion (see Marin and Visetti [**16**]) can be applied to improve Theorem 6.1.

**Acknowledgements.** This research was supported by the Slovenian Research Agency grants P1-0292, J1-8131, N1-0064, N1-0083, and N1-0114. We very gratefully acknowledge the referee for several important comments and suggestions which have considerably improved the presentation.

https://doi.org/10.1017/S0013091520000012 Published online by Cambridge University Press

## References


1. J. L. Bryant, S. Ferry, W. Mio and S. Weinberger, Topology of homology manifolds, *Ann. Math.* **143**(2) (1996), 435–467.
2. A. Cavicchioli, F. Hegenbarth and D. Repovš, *Higher-dimensional generalized manifolds: surgery and constructions*, EMS Series of Lectures in Mathematics, Volume 23 (European Mathematical Society, Zurich, 2016).
3. J. Dugundji, *Topology* (Allyn & Bacon, Boston, 1973).
4. S. C. Ferry, *Geometric topology notes*, Rutgers University, Piscataway, NJ, 2008. sites.math.rutgers.edu/∼sferry/ps/geotop.pdf.
5. S. C. Ferry, Remarks on Steenrod homology, in *Novikov conjectures, index theorems, and rigidity* (eds. S. Ferry, A. Ranicki and J. Rosenberg), pp. 148–166, London Mathematical Society Lecture Note Series 227, Volume 2 (Cambridge University Press, Cambridge, 1995).
6. S. C. Ferry, Epsilon-delta surgery over $\mathbb{Z}$, *Geom. Dedicata* **148** (2010), 71–101.
7. S. C. Ferry and E. K. Pedersen, Epsilon surgery theory, in *Novikov conjecture, index theorem and rigidity* (eds. S. Ferry, A. Ranicki and J. Rosenberg), pp. 167–226, London Mathematical Society, Lecture Notes Series 227, Volume 2 (Cambridge University Press, Cambridge, 1995).
8. J. C. Hausmann and P. Vogel, *Geometry on Poincaré spaces*, Mathematical Notes, 41 (Princeton University Press, Princeton, NJ, 1993).
9. F. Hegenbarth and D. Repovš, Controlled homotopy equivalences on structure sets of manifolds, *Proc. Amer. Math. Soc.* **142** (2014), 3987–3999.
10. F. Hegenbarth and D. Repovš, Controlled surgery and $\mathbb{L}$-homology, *Mediterr. J. Math.* **16** (2019), no. 3, Art. 79.
11. S. T. Hu, *Theory of retracts* (Wayne State University Press, Detroit, 1965).
12. L. Jones, Patch spaces: a geometric representation for Poincaré spaces, *Ann. Math. (2)* **97** (1973), 306–343. Corrections, *Ann. of Math. (2)* **102** (1975), 183–185.
13. D. S. Kahn, J. Kaminker and C. Schochet, Generalized homology theories on compact metric spaces, *Michigan Math. J.* **24**(2) (1977), 203–224.
14. P. Kühl, T. Macko and A. Mole, The total surgery obstruction revisited, *Münster J. Math.* **6** (2013), 181–269.
15. N. Levitt, Poincaré duality cobordism, *Ann. Math. (2)* **96** (1972), 211–244.
16. A. Marin and Y. M. Visetti, A general proof of Bing's shrinkability criterion, *Proc. Amer. Math. Soc.* **53**(2) (1975), 501–507.
17. J. Milnor, On the Steenrod homology theory, in *Novikov conjectures, index theorems, and rigidity* (eds. S. Ferry, A. Ranicki and J. Rosenberg), pp. 79–96, London Mathematical Society Lecture Note Series 226, Volume 1 (Cambridge University Press, Cambridge, 1995).
18. A. Nicas, Induction theorems for groups of homotopy manifold structures, *Mem. Amer. Math. Soc.* **39**(267) (1982).
19. E. K. Pedersen, F. Quinn and A. Ranicki, Controlled surgery with trivial local fundamental groups, in *High-dimensional manifold topology* (eds. F.T. Farrell and W. Lück), pp. 421–426 (World Scientific Publishing, River Edge, NJ, 2003).
20. F. S. Quinn, *A geometric formulation of surgery*, Doctoral Dissertation (Princeton University, Princeton, NJ, 1969).
21. F. S. Quinn, Surgery on Poincaré and normal spaces, *Bull. Amer. Math. Soc.* **78** (1972), 262–267.
22. F. S. Quinn, Resolutions of homology manifolds and the topological characterization of manifolds, *Invent. Math.* **72**(2) (1983), 267–284; Corrigendum, *Invent. Math.* 85 (3) (1986), 653.


https://doi.org/10.1017/S0013091520000012 Published online by Cambridge University Press

23. F. S. Quinn, An obstruction to the resolution of homology manifolds, *Michigan Math. J.* **34**(2) (1987), 285–291.
24. A. A. Ranicki, The total surgery obstruction, in *Proc. Alg. Topol. Conf. Aarhus 1978*, pp. 275–316, Lecture Notes Mathematics, Volume 763 (Springer-Verlag, Berlin, 1979).
25. A. A. Ranicki, *Algebraic L-theory and topological manifolds*, Cambridge Tracts in Mathematics, Volume 102 (Cambridge University Press, Cambridge, 1992).
26. M. Yamasaki, $L$-groups of crystallographic groups, *Invent. Math.* **88**(3) (1987), 571–602.
27. C. T. C. Wall, Poincaré complexes, I., *Ann. Math.* **86**(2) (1967), 213–245.
28. C. T. C. Wall, *Surgery on compact manifolds*, 2nd edn, edited and with a foreword by A.A. Ranicki, Mathematical Surveys and Monographs, Volume 69 (American Mathematical Society, Providence, RI, 1999).
29. S. Weinberger, Homology manifolds, in *Handbook of geometric topology*, pp. 1085–1102 (North-Holland, Amsterdam, 2002).
30. J. E. West, Mapping Hilbert cube manifolds to ANR's: a solution of a conjecture of Borsuk, *Ann. Math. (2)* **106** (1977), 1–18.

https://doi.org/10.1017/S0013091520000012 Published online by Cambridge University Press