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A DETERMINANT IDENTITY IMPLYING THE LAGRANGE–GOOD INVERSION FORMULA

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Abstract In this paper a determinant identity is established, from which a simple proof of the multivariate Lagrange–Good inversion formula follows directly. Further discussion on a discrete analogue of the Lagrange–Good inversion formula is also presented.

Keywords: Lagrange–Bürmann; Lagrange–Good; inversion formula; matrix inversion; formal derivative; formal power series

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1. Introduction

Recall that the classical (one-variable) Lagrange–Bürmann inversion formula, a landmark discovery in the history of analysis with regard to the expansion of functions into series, often reads as follows.

Theorem 1.1 (the Lagrange–Bürmann inversion formula; see Whittaker and Watson [28, § 7.32]). *Assume that $F(x)$ and $\phi(x)$ are analytic around $x = 0$, $\phi(0) \neq 0$. Then*

(1)
$$F(x) = F(0) + \sum_{n=1}^{\infty} a_n \left(\frac{x}{\phi(x)}\right)^n, \tag{1.1}$$

where
$$a_n = \frac{1}{n!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}[F'(x)\phi^n(x)]_{|x=0};$$

(2)
$$\frac{F(x)}{1 - x\phi'(x)/\phi(x)} = \sum_{n=0}^{\infty} b_n \left(\frac{x}{\phi(x)}\right)^n \tag{1.2}$$

where
$$b_n = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}[F(x)\phi^n(x)]_{|x=0}.$$

Hereafter, $\mathrm{d}/\mathrm{d}x$ *denotes the usual derivative operator.*

The problem of finding 'good' (q)-extensions of the Lagrange–Bürmann inversion formula, including generalizing it to multivariate cases, has received a lot of attention from many mathematicians. For the former, we refer the reader to [**2**, **10**, **11**, **15**, **17**, **18**], and in particular to the good survey of Stanton [**26**] for a more comprehensive treatment. Regarding the problem of multivariate generalizations, early contributions in this direction can be found in [**3**,**4**,**8**,**10**,**12**,**14**,**22**,**29**], but the discovery of the general multivariate formula is certainly attributed to Good [**12**, Theorem 8].

In this paper we concern ourselves with Tutte's generalization in the context of the theory of formal power series, which can be found in [**27**] or the book by Goulden and Jackson [**13**].

Theorem 1.2 (the multivariate Lagrange–Good inversion formula). *Let $x_1, x_2, \ldots, x_m$ be indeterminate, let $\boldsymbol{x} = (x_1, x_2, \ldots, x_m)$, and let $\phi_i(\boldsymbol{x})$ $(1 \leqslant i \leqslant m)$ be m formal power series over $\mathbb{C}$ such that $\phi_i(\boldsymbol{0}) \neq 0$. Suppose that*

$$z_i = x_i/\phi_i(\boldsymbol{x}), \quad i = 1, 2, \ldots, m,$$

and write for short

$$\boldsymbol{z}^{\boldsymbol{n}} = \prod_{i=1}^{m} z_i^{n_i}, \qquad \phi^{\boldsymbol{n}}(\boldsymbol{x}) = \prod_{i=1}^{m} \phi_i^{n_i}(\boldsymbol{x}).$$

Then, for an arbitrary formal power series $H(\boldsymbol{x})$ there holds

(1)

$$H(\boldsymbol{x}) = \sum_{\forall \boldsymbol{n} \in \mathbb{N}^m} c_{\boldsymbol{n}} \boldsymbol{z}^{\boldsymbol{n}}, \tag{1.3 a}$$

where

$$c_{\boldsymbol{n}} = [\boldsymbol{x}^{\boldsymbol{n}}] H(\boldsymbol{x}) \phi^{\boldsymbol{n}}(\boldsymbol{x}) \det_{1 \leqslant i,j \leqslant m} (\delta_{i,j} - z_i [\partial \phi]_{i,j}); \tag{1.3 b}$$

(2)

$$\frac{H(\boldsymbol{x})}{\det_{1 \leqslant i,j \leqslant m} (\delta_{i,j} - z_i [\partial \phi]_{i,j})} = \sum_{\forall \boldsymbol{n} \in \mathbb{N}^m} d_{\boldsymbol{n}} \boldsymbol{z}^{\boldsymbol{n}}, \tag{1.4 a}$$

where

$$d_{\boldsymbol{n}} = [\boldsymbol{x}^{\boldsymbol{n}}] H(\boldsymbol{x}) \phi^{\boldsymbol{n}}(\boldsymbol{x}). \tag{1.4 b}$$

Hereafter, $\det_{1 \leqslant i,j \leqslant m}(a_{i,j})$ denotes the usual $m \times m$ determinant with the (i, j)-entry $a_{i,j}$.

In what follows, since we are interested only in presenting a simple proof of Theorem 1.2, we will not bother with applications of these two formulae in various mathematical contexts. For more details, we refer the reader to the books by Henrici [**15**], Goulden and Jackson [**13**, Chapter 5] and Flajolet and Sedgewick [**7**], as well as the expository papers [**17**, **23**].

Up to now, there have been many different proofs of Theorem 1.2. We refer the reader to [**1**, **12**, **16**] for analytic approaches and to [**8**–**10**, **13**, **27**, **29**] for some combinatorial

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proofs. However, to the best of our knowledge, the appearance of determinants in these results lacks natural and transparent interpretations, except for the Jacobian of the transformation in the theory of differential calculus in several variables. In this paper we will show that the core of Theorem 1.2 lies in the following theorem.

Theorem 1.3. *Assume the same conditions as in Theorem 1.2. Then, for* $\boldsymbol{n} \geqslant \boldsymbol{0} \in \mathbb{N}^m$,

$$[\boldsymbol{x}^{\boldsymbol{n}}]\phi^{\boldsymbol{n}}(\boldsymbol{x}) \det_{1\leqslant i,j\leqslant m}(\delta_{i,j} - z_i[\partial\phi]_{i,j}) = \delta_{\boldsymbol{n},\boldsymbol{0}}, \tag{1.5}$$

where $\delta_{\boldsymbol{n},\boldsymbol{k}}$ *denotes the usual Kronecker symbol. In general, for* $\boldsymbol{m}, \boldsymbol{n} > \boldsymbol{0} \in \mathbb{N}^m$,

$$\begin{aligned} &\boldsymbol{n}[\boldsymbol{x}^{\boldsymbol{m}}]\phi^{\boldsymbol{n}}(\boldsymbol{x}) \det_{1\leqslant i,j\leqslant m}(\delta_{i,j} - z_i[\partial\phi]_{i,j}) \\ &\qquad = \sum_{\substack{\forall \boldsymbol{k}(i)=(k_{i,1},k_{i,2},\dots,k_{i,m})\geqslant \boldsymbol{0},\\ \sum_{i=1}^m \boldsymbol{k}(i)=\boldsymbol{m}}} \det_{1\leqslant i,j\leqslant m}(n_i\delta_{i,j} - k_{i,j}) \prod_{i=1}^m A(i;\boldsymbol{k}(i)), \end{aligned} \tag{1.6}$$

where the coefficients $A(i;\boldsymbol{k})$ *are given by (2.1) below.*

As we will see later, the argument of Theorem 1.3 uses nothing more than row and column operations of determinants in conjunction with the algebra of formal power series. Nevertheless, it sheds light on the core of the Lagrange–Good inversion formula.

Our paper is organized as follows. The full and elementary proofs of Theorems 1.2 and 1.3 are given in the next section. In § 3 we show a multidimensional matrix inversion due to Krattenthaler and Schlosser in exactly the same way. In this sense, we claim that Krattenthaler and Schlosser's result may to some extent be regarded as a discrete analogue of the Lagrange–Good inversion formula.

Throughout our discussions we will use the standard multi-index notation. More precisely, we abbreviate vectors from $\mathbb{N}^m$ ($\mathbb{N}$ is the set of non-negative integers) by boldface symbols. For any $\boldsymbol{n} = (n_1, n_2, \dots, n_m)$, $\boldsymbol{k} = (k_1, k_2, \dots, k_m) \in \mathbb{N}^m$, we employ the following compact notation:

$$\begin{aligned} \boldsymbol{n}! &= n_1!n_2!\cdots n_m!, \\ \boldsymbol{n} > \boldsymbol{k} &\iff n_i > k_i \quad (i = 1, 2, \dots, m), \\ \boldsymbol{n} \geqslant \boldsymbol{k} &\iff n_i \geqslant k_i \quad (i = 1, 2, \dots, m), \\ \boldsymbol{n} - \boldsymbol{k} &= (n_1 - k_1, n_2 - k_2, \dots, n_m - k_m), \\ |\boldsymbol{n}| &= n_1 + n_2 + \cdots + n_m, \\ \boldsymbol{x}^{\boldsymbol{n}} &= x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m} \end{aligned}$$

for $\boldsymbol{x} = (x_1, x_2 \dots, x_m)$, $\boldsymbol{n} = n_1 n_2 \cdots n_m$ if $\boldsymbol{n}$ appears in the context as coefficients. Also, we will use $\mathbb{C}[\![\boldsymbol{x}]\!]$ to denote the algebra of a formal power series in m indeterminates x_i over $\mathbb{C}$ and, for all $f(\boldsymbol{x}) = \sum_{\boldsymbol{n}\geqslant \boldsymbol{0}} a_{\boldsymbol{n}} \boldsymbol{x}^{\boldsymbol{n}} \in \mathbb{C}[\![\boldsymbol{x}]\!]$, the coefficient functional (extraction of the coefficient)

$$[\boldsymbol{x}^{\boldsymbol{n}}]f(\boldsymbol{x}) = a_{\boldsymbol{n}}, \quad a_{\boldsymbol{0}} = f(\boldsymbol{0}).$$

In the case in which $f(\boldsymbol{x})$ is analytic at $\boldsymbol{x}=\boldsymbol{0}$, it is understood that

$$[\boldsymbol{x}^{\boldsymbol{n}}]f(\boldsymbol{x}) = \frac{1}{\boldsymbol{n}!}\frac{\partial^{|\boldsymbol{n}|}f(\boldsymbol{x})}{\partial_{x_1}^{n_1}\partial_{x_2}^{n_2}\cdots\partial_{x_m}^{n_m}}\bigg|_{\boldsymbol{x}=\boldsymbol{0}}.$$

In particular, we will employ the notation

$$[\partial f]_{i,j} = \frac{\partial f_i(\boldsymbol{x})}{\partial x_j}$$

when a system of formal power series $f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \dots, f_m(\boldsymbol{x})$ is concerned.

2. The proofs

Let us start with the proof of Theorem 1.3.

Proof of Theorem 1.3. To show (1.5), without loss of generality we assume that $\boldsymbol{n} = (n_1, n_2, \dots, n_m) > \boldsymbol{0}$, namely, each $n_i \geqslant 1$ for $i \in \{1, 2, \dots, m\}$. Because if there are some $n_i = 0$, then all terms in the expansion of the determinant (namely, (2.3)) corresponding to $z_i[\partial\phi]_{i,j}$ will be divisible by x_i, and hence will not contribute to the coefficient of $\boldsymbol{x}^{\boldsymbol{n}}$. Now, let us consider

$$\phi_i^{n_i}(\boldsymbol{x}) = \sum_{\forall \boldsymbol{k}\in\mathbb{N}^m} A(i;\boldsymbol{k})\boldsymbol{x}^{\boldsymbol{k}}. \tag{2.1}$$

Taking the formal derivative of (2.1) with respect to x_j and multiplying by x_i on both sides yields

$$\begin{aligned} x_i n_i \phi_i^{n_i-1}(\boldsymbol{x})[\partial\phi]_{i,j} &= x_i \sum_{\forall \boldsymbol{k}=(k_1,k_2,\dots,k_m)\geqslant\boldsymbol{0}} A(i;\boldsymbol{k}) x_1^{k_1} x_2^{k_2}\cdots(k_j x_j^{k_j-1})\cdots x_m^{k_m} \\ &= \frac{x_i}{x_j} \sum_{\forall \boldsymbol{k}=(k_1,k_2,\dots,k_m)\geqslant\boldsymbol{0}} k_j A(i;\boldsymbol{k}) x_1^{k_1} x_2^{k_2}\cdots x_j^{k_j}\cdots x_m^{k_m}. \end{aligned} \tag{2.2}$$

Start with (2.1) and simplify

$$\boldsymbol{n}\phi^{\boldsymbol{n}}(\boldsymbol{x}) \det_{1\leqslant i,j\leqslant m}(\delta_{i,j} - z_i[\partial\phi]_{i,j}) \tag{2.3}$$

by writing for clarity

$$\boldsymbol{n}\phi^{\boldsymbol{n}}(\boldsymbol{x}) = \prod_{i=1}^{m} n_i\phi_i^{n_i}(\boldsymbol{x}).$$

To this end, on multiplying all elements in the ith row in the determinant by $n_i\phi_i^{n_i}(\boldsymbol{x})$, we arrive at

$$\boldsymbol{n}\phi^{\boldsymbol{n}}(\boldsymbol{x}) \det_{1\leqslant i,j\leqslant m}(\delta_{i,j} - z_i[\partial\phi]_{i,j}) = \det_{1\leqslant i,j\leqslant m}(n_i\phi_i^{n_i}(\boldsymbol{x})\delta_{i,j} - x_i n_i \phi_i^{n_i-1}(\boldsymbol{x})[\partial\phi]_{i,j}).$$

Inserting (2.1) and (2.2) into the last identity and then performing further simplification on the resulting equality by removing the factors x_i and x_j from each row and column, we thereby have

$$\boldsymbol{n}\phi^{\boldsymbol{n}}(\boldsymbol{x}) \det_{1\leqslant i,j\leqslant m}(\delta_{i,j} - z_i[\partial\phi]_{i,j}) = \det_{1\leqslant i,j\leqslant m}(\chi(i,j)), \tag{2.4}$$

where the (i,j)-entry

$$\chi(i,j) = \sum_{\forall \boldsymbol{k}=(k_1,k_2,\dots,k_m)} (n_i\delta_{i,j} - k_j)A(i;\boldsymbol{k})\boldsymbol{x}^{\boldsymbol{k}}. \tag{2.5}$$

Next, expand the right-hand side of (2.4) using the definition of the determinant and with a bit of rearranging of terms. We can easily find that

$$\begin{aligned}\boldsymbol{n}\phi^{\boldsymbol{n}}(\boldsymbol{x}) &\det_{1\leqslant i,j\leqslant m}(\delta_{i,j} - z_i[\partial\phi]_{i,j}) \\ &= \sum_{\forall \boldsymbol{k}(1),\boldsymbol{k}(2),\dots,\boldsymbol{k}(m)\geqslant \mathbf{0}} \det_{1\leqslant i,j\leqslant m}(n_i\delta_{i,j} - k_{i,j}) \prod_{i=1}^{m} A(i;\boldsymbol{k}(i))\boldsymbol{x}^{\sum_{i=1}^{m}\boldsymbol{k}(i)},\end{aligned} \tag{2.6}$$

where $\boldsymbol{k}(i) = (k_{i,1}, k_{i,2}, \dots, k_{i,m})$. Recalling that

$$\boldsymbol{x}^{\sum_{i=1}^{m}\boldsymbol{k}(i)} = x_1^{\sum_{i=1}^{m}k_{i,1}} x_2^{\sum_{i=1}^{m}k_{i,2}} \cdots x_m^{\sum_{i=1}^{m}k_{i,m}}$$

and then applying $[\boldsymbol{x}^{\boldsymbol{n}}]$ to both sides of (2.6), we thereby obtain

$$\begin{aligned}[\boldsymbol{x}^{\boldsymbol{n}}]\boldsymbol{n}\phi^{\boldsymbol{n}}(\boldsymbol{x}) &\det_{1\leqslant i,j\leqslant m}(\delta_{i,j} - z_i[\partial\phi]_{i,j}) \\ &= \sum_{\substack{\forall \boldsymbol{k}(i)=(k_{i,1},k_{i,2},\dots,k_{i,m})\geqslant \mathbf{0},\\ \sum_{i=1}^{m}k_{i,j}=n_j}} \det_{1\leqslant i,j\leqslant m}(n_i\delta_{i,j} - k_{i,j}) \prod_{i=1}^{m} A(i;\boldsymbol{k}(i)).\end{aligned}$$

It is certainly clear now that the determinant on the right-hand side

$$\det_{1\leqslant i,j\leqslant m}(n_i\delta_{i,j} - k_{i,j}) \equiv 0 \quad \text{for } \sum_{i=1}^{m} k_{i,j} = n_j \ (1 \leqslant j \leqslant m),$$

giving rise to identity (1.5) immediately.

From the above derivation, it is easy to see that (1.6) follows immediately upon applying $[\boldsymbol{x}^{\boldsymbol{m}}]$, instead of $[\boldsymbol{x}^{\boldsymbol{n}}]$, to both sides of (2.6). The theorem is thus proved. □

With Theorem 1.3 on hand, we are now able to show the Lagrange–Good inversion formula in a few lines.

Proof of Theorem 1.2. Since the set

$$\left\{ \boldsymbol{z}^{\boldsymbol{n}} = \prod_{i=1}^{m}(x_i/\phi_i(\boldsymbol{x}))^{n_i} \;\middle|\; \forall \boldsymbol{n} = (n_1, n_2, \dots, n_m) \geqslant \mathbf{0} \right\}$$

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consists of a base of $\mathbb{C}[\![\boldsymbol{x}]\!]$, given $H(x)$ there exist some proper coefficients $c_{\boldsymbol{n}}$ such that (1.3 a) holds. All that remains is to show that these $c_{\boldsymbol{n}}$ satisfy (1.3 b). For this, by multiplying both sides of (1.3 a) with

$$\phi^{\boldsymbol{k}}(\boldsymbol{x}) \det_{1\leqslant i,j\leqslant m}(\delta_{i,j} - z_i[\partial\phi]_{i,j})$$

we immediately get that for $\boldsymbol{k} \in \mathbb{N}^m$,

$$H(\boldsymbol{x})\phi^{\boldsymbol{k}}(\boldsymbol{x}) \det_{1\leqslant i,j\leqslant m}(\delta_{i,j} - z_i[\partial\phi]_{i,j}) = \sum_{\forall \boldsymbol{n}\in\mathbb{N}^m} c_{\boldsymbol{n}} \det_{1\leqslant i,j\leqslant m}(\delta_{i,j} - z_i[\partial\phi]_{i,j})\boldsymbol{x}^{\boldsymbol{n}}\phi^{\boldsymbol{k}-\boldsymbol{n}}(\boldsymbol{x}).$$

Comparing the coefficients of $\boldsymbol{x}^{\boldsymbol{k}} = \prod_{i=1}^m x_i^{k_i}$ on both sides, we arrive at

$$\begin{aligned}
[\boldsymbol{x}^{\boldsymbol{k}}]H(\boldsymbol{x})\phi^{\boldsymbol{k}}(\boldsymbol{x}) & \det_{1\leqslant i,j\leqslant m}(\delta_{i,j} - z_i[\partial\phi]_{i,j}) \\
&= \sum_{\forall \boldsymbol{n}\in\mathbb{N}^m} c_{\boldsymbol{n}}[\boldsymbol{x}^{\boldsymbol{k}-\boldsymbol{n}}]\phi^{\boldsymbol{k}-\boldsymbol{n}}(\boldsymbol{x}) \det_{1\leqslant i,j\leqslant m}(\delta_{i,j} - z_i[\partial\phi]_{i,j}) \\
&= \sum_{\forall \boldsymbol{n}\in\mathbb{N}^m} c_{\boldsymbol{n}}\delta_{\boldsymbol{k}-\boldsymbol{n},\boldsymbol{0}} \\
&= c_{\boldsymbol{k}}.
\end{aligned}$$

Note that the penultimate equality is built on Theorem 1.3. Hence, (1.3 b) is confirmed. The equalities (1.4 a) and (1.4 b) in Theorem 1.2 follow immediately from (1.3 a) and (1.3 b), respectively, by replacing $H(\boldsymbol{x})$ with

$$\frac{H(\boldsymbol{x})}{\det_{1\leqslant i,j\leqslant m}(\delta_{i,j} - z_i[\partial\phi]_{i,j})}.$$

Thus, the theorem is proved. □

3. Discrete Lagrange–Good inversion formula

Before proceeding further, we would like to address here that it was Henrici who first pointed out [**15**, Chapter 1] that the Lagrange–Bürmann inversion formula is essentially equivalent to matrix inversions in the context of combinatorial analysis. Recall that a matrix inversion, sometimes called an inversion formula or an inverse (reciprocal) relation in the literature (see [**24**, Chapters 2 and 3] or [**6**, Definition 3.1.1]), is commonly defined as a pair of multidimensional infinite lower-triangular (ILT) matrices over the complex field $\mathbb{C}$, say, $F = (f_{\boldsymbol{n},\boldsymbol{k}})_{\boldsymbol{n}\geqslant\boldsymbol{k}\geqslant 0}$ and $G = (g_{\boldsymbol{n},\boldsymbol{k}})_{\boldsymbol{n}\geqslant\boldsymbol{k}\geqslant 0}$ satisfying

$$\sum_{\boldsymbol{n}\geqslant\boldsymbol{i}\geqslant\boldsymbol{k}} f_{\boldsymbol{n},\boldsymbol{i}}g_{\boldsymbol{i},\boldsymbol{k}} = \sum_{\boldsymbol{n}\geqslant\boldsymbol{i}\geqslant\boldsymbol{k}} g_{\boldsymbol{n},\boldsymbol{i}}f_{\boldsymbol{i},\boldsymbol{k}} = \delta_{\boldsymbol{n},\boldsymbol{k}} \quad \text{for all } \boldsymbol{n},\boldsymbol{k} \in \mathbb{N}^m. \tag{3.1}$$

Here, the adjective ILT means that each entry $f_{\boldsymbol{n},\boldsymbol{k}} = 0$ unless $\boldsymbol{n} \geqslant \boldsymbol{k}$.

Matrix inversions, found by a process now known as the *inverse technique* [**5**], have proved very fruitful in the study of summations and transformations of hypergeometric series. Once viewed from Henrici's viewpoint in reverse, it is easy to see that the study of matrix inversions amounts theoretically to the study of discrete analogues of the Lagrange–Good inversion formula. As cogent evidence in support of such a view, we would like to reconsider the following multidimensional matrix inversion found by Krattenthaler and Schlosser [**21**]. It contains the famous Krattenthaler inversion formula [**20**] as the special case in which $m = 1$.

Theorem 3.1 (discrete Lagrange–Good inversion formula; see Krattenthaler and Schlosser [21, Theorem 3.3]). *Let* $\{a_i(t)\}_{t\in\mathbb{N}}$, $\{b_{ij}(t)\}_{t\in\mathbb{N}}$, $\{c_i(t)\}_{t\in\mathbb{N}}$, $1 \leqslant i, j \leqslant m$, *be arbitrary sequences such that* $c_i(t_r) \neq c_i(t_s)$ *for* $r \neq s$. *Then two multidimensional ILT matrices* $(f_{\boldsymbol{n},\boldsymbol{k}})_{\boldsymbol{n},\boldsymbol{k}\in\mathbb{N}^m}$ *and* $(g_{\boldsymbol{n},\boldsymbol{k}})_{\boldsymbol{n},\boldsymbol{k}\in\mathbb{N}^m}$ *with entries given by*

$$f_{\boldsymbol{n},\boldsymbol{k}} = C(\boldsymbol{n},\boldsymbol{k}) \prod_{i=1}^{m} \frac{\prod_{t_i=k_i}^{n_i-1} \Gamma_i(t_i;\boldsymbol{k})}{\prod_{t_i=k_i+1}^{n_i}\{c_i(t_i) - c_i(k_i)\}}, \tag{3.2 a}$$

$$g_{\boldsymbol{n},\boldsymbol{k}} = \prod_{i=1}^{m} \frac{\prod_{t_i=k_i+1}^{n_i} \Gamma_i(t_i;\boldsymbol{n})}{\prod_{t_i=k_i}^{n_i-1}\{c_i(t_i) - c_i(n_i)\}} \tag{3.2 b}$$

for arbitrary $\boldsymbol{n} = (n_1, n_2, \ldots, n_m) \geqslant \boldsymbol{k} = (k_1, k_2, \ldots, k_m) \in \mathbb{N}^m$ *are inverse to each other, where*

$$C(\boldsymbol{n},\boldsymbol{k}) = \frac{\det_{1\leqslant i,j\leqslant m}(\Gamma_i(n_i;\boldsymbol{k})\delta_{i,j} + b_{ij}(n_i)\{c_i(n_i) - c_i(k_i)\})}{\prod_{i=1}^{m} \Gamma_i(k_i;\boldsymbol{k})}$$

and, hereafter,

$$\Gamma_i(t;\boldsymbol{k}) := a_i(t) + \sum_{j=1}^{m} b_{ij}(t)c_j(k_j). \tag{3.3}$$

One of the main reasons we call Theorem 3.1 the *discrete Lagrange–Good inversion formula* is because this result can also be proved using exactly the same argument as described in the foregoing section. As we will see later, such a proof seems more elementary than the original proof of Krattenthaler and Schlosser [**21**] via Krattenthaler's operator method [**19**].

Our argument for Theorem 3.1 depends on a key fact from the theory of finite differences.

Lemma 3.2 (cf. Stanley [25, 1.9.2 Proposition]). *Let* $H(x)$ *be a polynomial in* x *of degree no more than* n *and let* $x_1, x_2, \ldots, x_{n+1}$ *be* $n + 1$ *distinct complex numbers. Then*

$$\sum_{i=1}^{n+1} \frac{H(x_i)}{\prod_{j=1,j\neq i}^{n+1}(x_i - x_j)} = 0. \tag{3.4}$$

As one may expect, the validity of Theorem 3.1 is closely related to the following property of the determinant.

Lemma 3.3. *With the same notation as in Theorem 3.1, for any* $\boldsymbol{n} \geqslant \boldsymbol{0}$ *define*

$$G(y_1, y_2, \ldots, y_m) := \det_{1\leqslant i,j\leqslant m}\left(\left\{\Gamma_i(n_i;\boldsymbol{n}) - \sum_{s=1}^{m} b_{is}(n_i)y_s\right\}\delta_{i,j} + b_{ij}(n_i)y_j\right).$$

Then there exist some coefficients $\eta_{\boldsymbol{p}}$ *and* $\lambda_{\boldsymbol{p}}$ *that are polynomial in* $a_i(n_i)$, $b_{ij}(n_i)$ *and* $c_i(n_i)$ *such that*

(1)

$$G(y_1, y_2, \ldots, y_m) = \sum_{\substack{\forall p_i \geqslant 0, \\ |\boldsymbol{p}=(p_1,p_2,\ldots,p_m)| \leqslant m-1}} \eta_{\boldsymbol{p}} y_1^{p_1} y_2^{p_2} \cdots y_m^{p_m}; \tag{3.5}$$

(2) *for any* $\boldsymbol{n}, \boldsymbol{k} \geqslant \mathbf{0}$,

$$\begin{aligned} &\det_{1 \leqslant i,j \leqslant m} (\Gamma_i(n_i; \boldsymbol{k})\delta_{i,j} + b_{ij}(n_i)\{c_i(n_i) - c_i(k_i)\}) \\ &\qquad = \sum_{\substack{\forall p_i \geqslant 0, \\ |\boldsymbol{p}=(p_1,p_2,\ldots,p_m)| \leqslant m-1}} \lambda_{\boldsymbol{p}} c_1^{p_1}(k_1) c_2^{p_2}(k_2) \cdots c_m^{p_m}(k_m). \end{aligned} \tag{3.6}$$

Proof of Lemma 3.3. To achieve (3.5), by the definition of a determinant, $G(y_1, y_2, \ldots, y_m)$ is exactly polynomial in $y_1, y_2, \ldots, y_m$ with degree no more than m. Thus, we only need to show that all terms $y_1^{l_1} y_2^{l_2} \cdots y_m^{l_m}$ with $l_1 + l_2 + \cdots + l_m = m$ do not appear in $G(y_1, y_2, \ldots, y_m)$. Considering that all $\Gamma_i(n_i; \boldsymbol{n})$ are independent of y_j, it is quite clear that all such terms $y_1^{l_1} y_2^{l_2} \cdots y_m^{l_m}$ with degree exactly m are uniquely determined by

$$\det_{1 \leqslant i,j \leqslant m} \left(-\left\{ \sum_{s=1}^{m} b_{is}(n_i) y_s \right\} \delta_{i,j} + b_{ij}(n_i) y_j \right),$$

which turns out to be zero since the sum of the elements in each row is zero, namely,

$$\sum_{j=1}^{m} \left\{ -\sum_{s=1}^{m} b_{is}(n_i) y_s \delta_{i,j} + b_{ij}(n_i) y_j \right\} = -\sum_{s=1}^{m} b_{is}(n_i) y_s \sum_{j=1}^{m} \delta_{i,j} + \sum_{j=1}^{m} b_{ij}(n_i) y_j = 0.$$

Thus, (3.5) is proved.

We now proceed to show (3.6). To do this, we remove the factor $c_i(n_i) - c_i(k_i)$ from the ith row, then multiply by it all elements in the ith column of the determinants in (3.6), and subsequently deduce

$$\begin{aligned} &\det_{1 \leqslant i,j \leqslant m} (\Gamma_i(n_i; \boldsymbol{k})\delta_{i,j} + b_{ij}(n_i)\{c_i(n_i) - c_i(k_i)\}) \\ &\qquad = \det_{1 \leqslant i,j \leqslant m} \left(\frac{\Gamma_i(n_i; \boldsymbol{k})}{c_i(n_i) - c_i(k_i)} \delta_{i,j} + b_{ij}(n_i) \right) \times \prod_{i=1}^{m} \{c_i(n_i) - c_i(k_i)\} \\ &\qquad = \det_{1 \leqslant i,j \leqslant m} \left(\Gamma_i(n_i; \boldsymbol{k}) \frac{c_j(n_j) - c_j(k_j)}{c_i(n_i) - c_i(k_i)} \delta_{i,j} + b_{ij}(n_i)\{c_j(n_j) - c_j(k_j)\} \right) \\ &\qquad = \det_{1 \leqslant i,j \leqslant m} (\Gamma_i(n_i; \boldsymbol{k})\delta_{i,j} + b_{ij}(n_i)\{c_j(n_j) - c_j(k_j)\}). \end{aligned} \tag{3.7}$$

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A comparison of the rightmost summation of (3.7) with (3.5) leads us to

$$\begin{aligned}\det_{1\leqslant i,j\leqslant m}(\Gamma_i(n_i;\boldsymbol{k})\delta_{i,j}+b_{ij}(n_i)\{c_i(n_i)-c_i(k_i)\}) \\ &= G(y_1,y_2,\ldots,y_m)|_{y_j=c_j(n_j)-c_j(k_j)} \\ &= \sum_{\substack{\forall p_i\geqslant 0,\\ |\boldsymbol{p}=(p_1,p_2,\ldots,p_m)|\leqslant m-1}} \eta_{\boldsymbol{p}}\prod_{j=1}^{m}(c_j(n_j)-c_j(k_j))^{p_j},\end{aligned}$$

being further reformulated in the form

$$\sum_{\substack{\forall p_i\geqslant 0,\\ |\boldsymbol{p}=(p_1,p_2,\ldots,p_m)|\leqslant m-1}} \lambda_{\boldsymbol{p}}c_1^{p_1}(k_1)c_2^{p_2}(k_2)\cdots c_m^{p_m}(k_m).$$

This completes the proof of (3.6). □

With the aforementioned lemmas on hand, we are now in a good position to give a full proof of the so-called discrete (multivariate) Lagrange–Good inversion formula.

Proof of Theorem 3.1. Obviously, it suffices to verify that

$$\sum_{\boldsymbol{k}=\boldsymbol{l}}^{\boldsymbol{n}} f_{\boldsymbol{n},\boldsymbol{k}}g_{\boldsymbol{k},\boldsymbol{l}}=\delta_{\boldsymbol{n},\boldsymbol{l}}, \tag{3.8}$$

which is valid for $\boldsymbol{n}=(n_1,n_2,\ldots,n_m)=(l_1,l_2,\ldots,l_m)=\boldsymbol{l}\in\mathbb{N}^m$. Next, we only consider the case in which $\boldsymbol{n}>\boldsymbol{l}\in\mathbb{N}^m$, since if there are r pairs of integers $n_i=l_i$, then (3.8) will reduce, after some simplification, to such a case for $\mathbb{N}^{m-r}$. To confirm (3.8), we compute in a straightforward way

$$\begin{aligned}\sum_{\boldsymbol{k}=\boldsymbol{l}}^{\boldsymbol{n}} f_{\boldsymbol{n},\boldsymbol{k}}g_{\boldsymbol{k},\boldsymbol{l}} &= \sum_{\boldsymbol{k}=\boldsymbol{l}}^{\boldsymbol{n}}\frac{\det_{1\leqslant i,j\leqslant m}(\Gamma_i(n_i;\boldsymbol{k})\delta_{i,j}+b_{ij}(n_i)\{c_i(n_i)-c_i(k_i)\})}{\prod_{i=1}^{m}\Gamma_i(k_i;\boldsymbol{k})} \\ &\quad\times\prod_{i=1}^{m}\frac{\prod_{t_i=k_i}^{n_i-1}\Gamma_i(t_i;\boldsymbol{k})}{\prod_{t_i=k_i+1}^{n_i}(c_i(t_i)-c_i(k_i))}\prod_{i=1}^{m}\frac{\prod_{t_i=l_i+1}^{k_i}\Gamma_i(t_i;\boldsymbol{k})}{\prod_{t_i=l_i}^{k_i-1}(c_i(t_i)-c_i(k_i))} \\ &= \sum_{\boldsymbol{k}=\boldsymbol{l}}^{\boldsymbol{n}}\det_{1\leqslant i,j\leqslant m}(\Gamma_i(n_i;\boldsymbol{k})\delta_{i,j}+b_{ij}(n_i)\{c_i(n_i)-c_i(k_i)\}) \\ &\quad\times\prod_{i=1}^{m}\frac{\prod_{t_i=l_i+1}^{n_i-1}\Gamma_i(t_i;\boldsymbol{k})}{\prod_{t_i=l_i,t_i\neq k_i}^{n_i}(c_i(t_i)-c_i(k_i))}.\end{aligned}$$

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Now, by using of Lemma 3.3, we replace the determinant with the expression on the right-hand side of (3.6) and then change the order of summation, finally arriving at

$$\sum_{\boldsymbol{k}=\boldsymbol{l}}^{\boldsymbol{n}} f_{\boldsymbol{n},\boldsymbol{k}} g_{\boldsymbol{k},\boldsymbol{l}} = \sum_{\substack{\forall p_i \geqslant 0,\\ |\boldsymbol{p}|\leqslant m-1}} \lambda_{\boldsymbol{p}} \sum_{\boldsymbol{k}=\boldsymbol{l}}^{\boldsymbol{n}} c_1^{p_1}(k_1) c_2^{p_2}(k_2)\cdots c_m^{p_m}(k_m) \frac{F(\boldsymbol{k};\boldsymbol{n},\boldsymbol{l})}{\prod_{i=1}^{m}\prod_{t_i=l_i,\, t_i\neq k_i}^{n_i}(c_i(t_i)-c_i(k_i))}, \tag{3.9}$$

where

$$F(\boldsymbol{k};\boldsymbol{n},\boldsymbol{l}) := \prod_{i=1}^{m}\prod_{t_i=l_i+1}^{n_i-1} \Gamma_i(t_i;\boldsymbol{k}).$$

Recalling the definition of $\Gamma_i(t_i;\boldsymbol{k})$, we may furthermore assume that

$$F(\boldsymbol{k};\boldsymbol{n},\boldsymbol{l}) = \sum_{\boldsymbol{q}=(q_1,q_2,\dots,q_m)\geqslant\boldsymbol{0}} \mu_{\boldsymbol{q}} c_1^{q_1}(k_1) c_2^{q_2}(k_2)\cdots c_m^{q_m}(k_m). \tag{3.10}$$

In such a case, all coefficients $\mu_{\boldsymbol{q}}$, just like $\lambda_{\boldsymbol{p}}$ in Lemma 3.3, are merely polynomials in $a_i(n_i)$, $b_{ij}(n_i)$ and $c_i(n_i)$, being independent of $c_i(k_i)$ and k_i. Meanwhile, it is easily seen that the right-hand sum of (3.10) is just polynomial of degree no more than $|\boldsymbol{n}|-|\boldsymbol{l}|-m$, that is, $|\boldsymbol{q}| \leqslant |\boldsymbol{n}|-|\boldsymbol{l}|-m$. Thus, substituting expression (3.10) for $F(\boldsymbol{k};\boldsymbol{n},\boldsymbol{l})$ in (3.9) and changing the order of summation immediately yields

$$\sum_{\boldsymbol{k}=\boldsymbol{l}}^{\boldsymbol{n}} f_{\boldsymbol{n},\boldsymbol{k}} g_{\boldsymbol{k},\boldsymbol{l}} = \sum_{\substack{\forall p_i,q_i \geqslant 0,\\ |\boldsymbol{p}|\leqslant m-1,\\ |\boldsymbol{q}|\leqslant|\boldsymbol{n}|-|\boldsymbol{l}|-m}} \lambda_{\boldsymbol{p}}\mu_{\boldsymbol{q}} \sum_{\boldsymbol{k}=\boldsymbol{l}}^{\boldsymbol{n}} \frac{c_1^{p_1+q_1}(k_1) c_2^{p_2+q_2}(k_2)\cdots c_m^{p_m+q_m}(k_m)}{\prod_{i=1}^{m}\prod_{t_i=l_i,t_i\neq k_i}^{n_i}(c_i(t_i)-c_i(k_i))}.$$

It is crucial to realise that, under the restrictions

$$\forall p_i, q_i \geqslant 0, \quad |\boldsymbol{p}| \leqslant m-1 \quad \text{and} \quad |\boldsymbol{q}| \leqslant |\boldsymbol{n}|-|\boldsymbol{l}|-m,$$

we must have

$$|\boldsymbol{p}|+|\boldsymbol{q}| \leqslant |\boldsymbol{n}|-|\boldsymbol{l}|-1.$$

From this relation, we conclude that there exists at least one $i=\iota\in\{1,2,\dots,m\}$ such that $p_\iota+q_\iota \leqslant n_\iota - l_\iota - 1$. Keeping this in mind, we now reformulate

$$\sum_{\boldsymbol{k}=\boldsymbol{l}}^{\boldsymbol{n}} f_{\boldsymbol{n},\boldsymbol{k}} g_{\boldsymbol{k},\boldsymbol{l}} = \sum_{\substack{\forall p_i,q_i \geqslant 0,\\ |\boldsymbol{p}|\leqslant m-1,\\ |\boldsymbol{q}|\leqslant|\boldsymbol{n}|-|\boldsymbol{l}|-m}} \lambda_{\boldsymbol{p}}\mu_{\boldsymbol{q}} \prod_{i=1}^{m} \Delta^{(i)}(\boldsymbol{n},\boldsymbol{l}),$$

where, for brevity, we write for each integer i that

$$\Delta^{(i)}(\boldsymbol{n},\boldsymbol{l}) := \sum_{k_i=l_i}^{n_i} \frac{c_i^{p_i+q_i}(k_i)}{\prod_{t_i=l_i,t_i\neq k_i}^{n_i}(c_i(t_i)-c_i(k_i))},$$

$$\Delta^{(i)}(\boldsymbol{n},\boldsymbol{l})\Delta^{(j)}(\boldsymbol{n},\boldsymbol{l}) := \sum_{k_i=l_i}^{n_i} \frac{c_1^{p_i+q_i}(k_i)}{\prod_{t_i=l_i,t_i\neq k_i}^{n_i}(c_i(t_i)-c_i(k_i))} \times \sum_{k_j=l_j}^{n_j} \frac{c_j^{p_j+q_j}(k_j)}{\prod_{t_j=l_j,t_j\neq k_j}^{n_j}(c_j(t_j)-c_j(k_j))}$$

and so on. Consequently, we can easily find that the summation corresponding to $i=\iota$,

$$\Delta^{(\iota)}(\boldsymbol{n},\boldsymbol{l}) = \sum_{k_\iota=l_\iota}^{n_\iota} \frac{c_\iota^{p_\iota+q_\iota}(k_\iota)}{\prod_{t_\iota=l_\iota,t_\iota\neq k_\iota}^{n_\iota}(c_\iota(t_\iota)-c_\iota(k_\iota))},$$

is nothing but the $(n_\iota - l_\iota)$th difference with respect to $c_\iota(k_\iota)$. By a direct application of Lemma 3.2, we obtain $\Delta^{(\iota)}(\boldsymbol{n},\boldsymbol{l}) = 0$, i.e. the left-hand side of (3.8) is zero, which in turn confirms that

$$\sum_{\boldsymbol{k}=\boldsymbol{l}}^{\boldsymbol{n}} f_{\boldsymbol{n},\boldsymbol{k}} g_{\boldsymbol{k},\boldsymbol{l}} = 0 \quad \text{for } \boldsymbol{n} > \boldsymbol{l}.$$

Hence, the theorem is confirmed. □

We end this paper by remarking that in [**3**, **4**] Chu found some multidimensional analogues of the Gould–Hsu inverse relations. His results may also be considered as special cases of the discrete Lagrange–Good inversion formula, Theorem 3.1.

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