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Gradient theory of phase transitions in composite media

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We study the behaviour of non-convex functionals singularly perturbed by a possibly oscillating inhomogeneous gradient term, in the spirit of the gradient theory of phase transitions. We show that a limit problem giving a sharp interface, as the perturbation vanishes, always exists, but may be inhomogeneous or anisotropic. We specialize this study when the perturbation oscillates periodically, highlighting three types of regimes, depending on the frequency of the oscillations. In the two extreme cases, a separation of scales effect is described.

1. Introduction

In the classical theory of phase transitions for mixtures of two immiscible fluids (or for two phases of the same fluid). it is assumed that, at equilibrium, the two fluids arrange themselves in such a way that the area of the interface that separates the regions occupied by the two phases is minimal. This 'minimal-interface criterion' can be interpreted in mathematical terms as an energy-minimization process. We can describe every configuration of the system by a function u defined on Ω (the 'container' of the fluids), taking the value 0 on the first phase and 1 on the second one. In addition, u satisfies a 'volume constraint' $\int_\Omega u \,\mathrm{d}x = V$, where V is the assigned total volume of the second fluid. The set of discontinuity points of u parametrizes the interface between the two fluids in the corresponding configuration and is denoted by $S(u)$. We then postulate that the energy of such a u is proportional to the area of the interfaces, i.e. it is given by

$$F(u) = \sigma_0 \mathcal{H}^2(S(u)),$$

where $\mathcal{H}^2$ denotes the two-dimensional (Hausdorff) surface measure and σ_0 (the '*surface tension*') is a strictly positive constant, characteristic of the fluids. In such

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a way, the optimal configurations are obtained by minimizing this surface energy among all admissible configurations.

The 'gradient theory' of phase transitions is an alternative way to study these systems of fluids, by assuming that the transition between the phases is not concentrated on a interfacial surface, but takes place on a thin 'transition layer'. In this way, we allow fine mixtures of the two fluids, and an admissible configuration u will be a function taking its values in $[0,1]$, so that $u(x)$ will be interpreted as a local average density or concentration of the second fluid. Following this model proposed by Cahn and Hilliard [11], to such a u, we associate the energy

$$E_\varepsilon(u) = \int_\Omega (W(u) + \varepsilon^2|Du|^2)\,\mathrm{d}x,$$

where W is a 'double-well energy' with wells at 0 and 1 (i.e. a non-negative function vanishing only at 0 and 1), and ε is a small parameter linked to the width of the transition layer. In addition, the admissible configurations will always satisfy the same volume constraint as above. The competing effects of the two integrals in E_ε are to favour the configurations that take values close to 0 and 1 by the first term and at the same time to penalize spatial inhomogeneities of u (and hence the introduction of too many transition regions) by the second term.

The connection between these two standpoints had been conjectured by Gurtin [17], and was proved by Modica [18] (after an earlier work by Modica and Mortola [19]) by showing that minimum problems for the functional E_ε tend to minimum problems for F if the constant σ_0 is chosen as $\sigma_0 = 2\int_0^1 \sqrt{W(s)}\,\mathrm{d}s$. In [20], one can also find the proof of the Γ-convergence of the scaled functionals

$$\frac{1}{\varepsilon}E_\varepsilon(u) = \int_\Omega \left(\frac{W(u)}{\varepsilon} + \varepsilon|Du|^2\right)\mathrm{d}x$$

to F. Loosely speaking, this convergence means that minimal configurations u_ε for E_ε will tend to have transition layers that 'concentrate' as $\varepsilon \to 0$ on the interface $S(u)$ of a minimizer u of F. Moreover, the scaled minimal values $(1/\varepsilon)E_\varepsilon(u_\varepsilon)$ will converge to the value $F(u)$. It is interesting to note that the proof of the Modica–Mortola result is essentially one dimensional. The key point is to show that, for minimizers of E_ε, the profile of the transition layer approximately depends only on the direction orthogonal to $S(u)$ and is a scaling of an 'optimal profile'. After noticing this, the convergence result can be proved first, with the due changes in the statement, if Ω is one dimensional (in which case, interfaces are points), and then the three-dimensional case can be recovered by a 'slicing' argument (see, for example, [1,6]).

In this paper we investigate the effect of the presence of small-scale heterogeneities on the passage to the limit described above. More precisely, we assume that the gradient term in the definition of E_ε may depend on the space variable x, so that we are led to the study of the asymptotic behaviour of functionals of the form

$$F_\varepsilon(u) = \int_\Omega \left(\frac{W(u)}{\varepsilon} + \varepsilon f_\varepsilon(x, Du)\right)\mathrm{d}x,$$

where f_ε are Borel functions with quadratic growth in the second variable. In this case, by a simple comparison argument with the case studied by Modica, we may

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see that the domain of the Γ-limit will be the same as that of the energy F above. However, the determination of the actual form of the limit is much more complex. By following the 'direct methods of Γ-convergence' (see [10, 13]), we have first given a general compactness result for Γ-limits of functionals F_ε as above, and then explicitly characterized the limit functional when f_ε is rapidly oscillating in the first variable. In our general framework (as it was already done by Modica and Mortola), we do not restrict to the case of space dimension $n = 3$.

Our compactness result (theorems 3.3 and 3.5) shows that, from every sequence (F_{ε_j}) of functionals as above, it is possible to extract a subsequence that converges to a functional F_0 of the form

$$F_0(u) = \int_{S(u)} \sigma(x, \nu_u)\, \mathrm{d}\mathcal{H}^{n-1}$$

defined on functions $u : \Omega \to \{0, 1\}$ of bounded variation. In this case, ν_u represents the measure-theoretical normal to $S(u)$. Note that, in this case, the limit may be *anisotropic and inhomogeneous*, but it is always in the same 'class' of the functional F above, which we recover when σ is a constant. To prove this result, we follow a procedure that is by now customary in Γ-convergence, consisting of combining localization and integral representation arguments. First, we extend the definition of F_ε to every open set of $\mathbb{R}^n$ by

$$F_\varepsilon(u, A) = \int_A \left(\frac{W(u)}{\varepsilon} + \varepsilon f_\varepsilon(x, Du) \right) \mathrm{d}x.$$

We then prove the existence of converging subsequences to an (abstract) functional $F_0(u, A)$, which is, among other things, (the restriction of) a measure in the second variable and, by comparison, we get $F_0 \leqslant cF$ for some $c > 0$ We conclude then that

$$F_0(u, A) = \int_{S(u)\cap A} \sigma(x, \nu_u)\, \mathrm{d}\mathcal{H}^{n-1}$$

for some Borel function σ by suitable representation results (see [5,9]). This method is well established in the case of functionals defined on Sobolev spaces (see [10, 13]) and had been previously used within the framework of Caccioppoli partitions [2] or, in a way similar to the present paper, to characterize limits of non-local functionals [12].

It is interesting to note that the key point in the complex procedure above is proving that the set function $F_0(u, \cdot)$ is subadditive, and this was the object of an early lemma by Dal Maso and Modica [14]. Their result was inspired by De Giorgi, clearly aiming to illustrate how the direct methods of Γ-convergence could also be applied outside the framework of Sobolev spaces. Only now do we have at our disposal powerful integral representation techniques for functionals defined on functions with bounded variation that allow us to conclude this argument.

The main part of paper is § 4, where we specialize the convergence result in the case of rapidly oscillating perturbations. We fix a function $\delta = \delta(\varepsilon)$ such that $\delta \to 0$ as $\varepsilon \to 0$, a function f periodic in the first variable and positively homogeneous of degree two in the second variable, and take

$$f_\varepsilon(x, z) = f\left(\frac{x}{\delta}, z\right),$$

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so that

$$F_\varepsilon(u) = \int_\Omega \left(\frac{W(u)}{\varepsilon} + \varepsilon f\left(\frac{x}{\delta}, Du\right)\right) \mathrm{d}x.$$

We may interpret this situation as modelling the presence of *heterogeneities* at a scale δ, which locally favour or disfavour the onset of a transition layer. This may be viewed as a dishomogeneity of the fluid, or interpreted, more appropriately, as a microscopic property of a medium subject to solid–solid phase transitions. However, in order to treat this second case in depth, we would need to extend the theory to u vector valued and subject to differential constraints, which goes beyond the scope of this paper.

We show that the behaviour of the whole family (F_ε) can be completely described and depends on the mutual speed of convergence to 0 of δ and ε. The limit functional F_0 is homogeneous, but may be anisotropic,

$$F_0(u) = \int_{S(u)} \sigma(\nu_u)\, \mathrm{d}\mathcal{H}^{n-1}.$$

In the first case, $\varepsilon \ll \delta$, the final result is that we have a 'separation of scales' effect. We may first regard δ as fixed and let $\varepsilon \to 0$, and subsequently let $\delta \to 0$. In this way, we first obtain an inhomogeneous functional by applying the Modica–Mortola procedure, which can be explicitly computed as

$$F^\delta(u) = \sigma_0 \int_{S(u)} \sqrt{f\left(\frac{x}{\delta}, \nu_u\right)}\, \mathrm{d}\mathcal{H}^{n-1}$$

(for this anisotropic version, see also [6, ch. 4.3]). The limit as $\delta \to 0$ of these types of functionals falls within the framework of Γ-convergence of functionals defined on Caccioppoli partitions [2] and can also be seen as a particular case of homogenization on BV spaces [4]. By applying either of these two procedures, we obtain a formula for σ (see also [8]). A second case is when ε and δ are comparable (for simplicity, $\varepsilon = \delta$). In this case, the two effects cannot be separated, and $\sigma(\nu)$ is described through an asymptotic formula that describes the optimal profile, which, in this case, is not depending only on the direction ν. Finally, when $\delta \ll \varepsilon$, we again find a separation of scales phenomenon. The total effect is as if first we freeze ε. In this case, letting $\delta \to 0$, we obtain a functional of the form

$$F^\varepsilon(u) = \int_\Omega \left(\frac{W(u)}{\varepsilon} + \varepsilon f_{\mathrm{hom}}(Du)\right) \mathrm{d}x,$$

where f_{hom} is the *homogenized integrand* of f (see, for example, [10]). We eventually let $\varepsilon \to 0$, so that, by applying the Modica–Mortola procedure, we have $\sigma(\nu) = \sigma_0 \sqrt{f_{\mathrm{hom}}(\nu)}$. Note that, by the inequality $w^2 + z^2 \geqslant 2wz$, we always have the estimate

$$F_\varepsilon(u) \geqslant \int_\Omega 2\sqrt{W(u) f\left(\frac{x}{\delta}, Du\right)}\, \mathrm{d}x,$$

which turns out to be optimal if $\varepsilon \ll \delta$, but is not sharp in all other cases.

To briefly illustrate the difference in the separation of scales effect, as an example, we may consider the case of a simple inhomogeneous isotropic f_ε,

$$F_\varepsilon(u) = \int_\Omega \left(\frac{W(u)}{\varepsilon} + \varepsilon a\left(\frac{x}{\delta}\right)|Du|^2 \right) \mathrm{d}x,$$

where $n = 2$ and a, for example, is a 'chessboard coefficient' (taking the values α on 'white squares' and $\beta > \sqrt{2}\alpha$ on 'black squares'). If $\varepsilon \ll \delta$, then

$$F^\delta(u) = \sigma_0 \int_{S(u)} \sqrt{a\left(\frac{x}{\delta}\right)}\, \mathrm{d}\mathcal{H}^1,$$

and

$$\sigma(\nu) = \sigma_0 \alpha((\sqrt{2}-1)|\nu_1| \wedge |\nu_2| + |\nu_1| \vee |\nu_2|)$$

(see [8, example 5.3]). If $\delta \ll \varepsilon$, then we have, by the classical Dychne formula (see [22]),

$$F^\varepsilon(u) = \int_\Omega \left(\frac{W(u)}{\varepsilon} + \varepsilon\sqrt{\alpha\beta}|Du|^2 \right) \mathrm{d}x,$$

and, eventually,

$$\sigma(\nu) = \sigma_0(\alpha\beta)^{1/4}.$$

We finally point out that throughout the paper we have chosen to make some hypotheses on f in order to simplify formulae. First, we have made the technical assumption that f is positively homogeneous of degree two in the second variable, so that the Modica–Mortola procedure may be applied to the corresponding homogeneous case. In addition, we make some continuity hypothesis, without which, formulae should take into account complex relaxation results in BV spaces. The reader interested in the problems connected to general Borel integrands is referred to [5, 8, 9].

2. Notation and preliminary results

Let Ω be an open subset of $\mathbb{R}^n$. We denote by $\mathcal{A}$ and $\mathcal{B}$ the families of all bounded open and Borel subsets of $\mathbb{R}^n$, respectively. We denote by χ_E the *characteristic function* of E. We introduce the notation

$$Q(x,\rho) = x + \rho(-\tfrac{1}{2}, \tfrac{1}{2})^n,$$

in particular, $Q = Q(0,1)$; $Q^\nu_\rho(x)$ denotes an open cube of $\mathbb{R}^n$ centred at x, having side length ρ and one face orthogonal to ν; $Q^\nu_\rho = Q^\nu_\rho(0)$ and $Q^\nu = Q^\nu_1(0)$. By $[t]$ we denote the integer part of $t \in \mathbb{R}$.

Let U and U' be open sets with $U' \subset\subset U$. We say that $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a *cut-off function* related to U and U' if $\varphi \in \mathcal{C}^\infty_0(U')$ and $0 \leqslant \varphi \leqslant 1$ with $\varphi \equiv 1$ in a neighbourhood of $\bar{U}$.

Given a vector-valued measure μ on Ω, we adopt the notation $|\mu|$ for its total variation (see [15]) and $\mathcal{M}(\Omega)$ is the set of all signed measures on Ω with bounded total variation. The Lebesgue measure of a set E is denoted by $|E|$. The Hausdorff $(n-1)$-dimensional measure in $\mathbb{R}^n$ is denoted by $\mathcal{H}^{n-1}$.

We say that $u \in L^1(\Omega)$ is a *function of bounded variation*, and we write $u \in BV(\Omega)$, if its distributional first derivatives $D_i u$ belong to $\mathcal{M}(\Omega)$. We denote by Du the $\mathbb{R}^n$-valued measure whose components are $D_1 u, \dots, D_n u$.

We will say that a set E is of *finite perimeter* in Ω, or a *Caccioppoli set*, if $\chi_E \in BV(\Omega)$, and, for every open subset Ω of $\mathbb{R}^n$, we let

$$\mathcal{P}_\Omega(E) = |D\chi_E|(\Omega),$$

the *perimeter* of E in Ω. The family of Caccioppoli sets can be identified with the functions $u \in BV(\Omega; \{0,1\})$, the set of $BV(\Omega)$ functions that take almost everywhere the values 0 or 1.

In this case (if $u \in BV(\Omega; \{0,1\})$), the vector-valued measure Du can be represented as

$$Du(B) = \int_{B \cap S(u)} \nu_u \,\mathrm{d}\mathcal{H}^{n-1}$$

for every Borel set $B \subseteq \Omega$, where $S(u)$ denotes the complement of the Lebesgue set of u and $\nu_u \in \mathbb{R}^n$ is a unit vector that is $\mathcal{H}^{n-1}$-a.e. defined in $S(u)$, interpreted as the *normal* to $S(u)$. Moreover, one can prove that, if $E = \{x : u(x) = 1\}$,

$$\mathcal{P}_\Omega(E) = |Du|(\Omega) = \mathcal{H}^{n-1}(S(u) \cap \Omega).$$

For the general exposition of the theory of functions of bounded variation, we refer to [3, 15, 16, 21, 23].

Since we will consider either functions in Sobolev spaces or characteristic functions of sets of finite perimeter, with a slight abuse, we will use the notation $Du = (D_1 u, \dots, D_n u)$ both for the gradient of a Sobolev function and for the distributional derivative of u, as no confusion may arise.

We recall the definition of *Γ-convergence* of a sequence of functionals F_j defined on $L^1(\Omega)$ (with respect to the $L^1(\Omega)$-convergence). We say that (F_j) Γ-converges to F_0 on $L^1(\Omega)$ if, for all $u \in L^1(\Omega)$,

(i) (*Γ-liminf inequality*) for all (u_j), sequences of functions in $L^1(\Omega)$ converging to u in $L^1(\Omega)$, we have

$$F_0(u) \leqslant \liminf_j F_j(u_j);$$

(ii) (*Γ-limsup inequality*) there exists a sequence (u_j) of functions in $L^1(\Omega)$ converging to u in $L^1(\Omega)$ such that

$$F_0(u) \geqslant \limsup_j F_j(u_j).$$

We will say that a family (F_ε) Γ-converges to F_0 if, for all sequences (ε_j) of positive numbers converging to 0, conditions (i) and (ii) above are satisfied, with F_{ε_j} in place of F_j. For a comprehensive study of Γ-convergence, we refer to [13] (for a simplified introduction, see [7]), while a detailed analysis of some of its applications to homogenization theory can be found in [10].

The model example of Γ-convergence we have in mind is the following result (see [1, 6, 18, 19]).

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THEOREM 2.1. *Let $W : \mathbb{R} \to [0, +\infty)$ be a continuous function such that*

$$\{z \in \mathbb{R} : W(z) = 0\} = \{0, 1\}, \tag{2.1}$$
$$c_1(|z|^\gamma - 1) \leqslant W(z) \leqslant c_2(|z|^\gamma + 1) \quad \text{for every } z \in \mathbb{R}, \tag{2.2}$$

with $\gamma \geqslant 2$.

Then the functionals

$$E_\varepsilon(u, A) = \begin{cases} \displaystyle\int_A \left(\frac{W(u)}{\varepsilon} + \varepsilon|Du|^2\right) \mathrm{d}x & \text{if } u \in W^{1,\gamma}(A), \\ +\infty & \text{otherwise} \end{cases}$$

Γ-converge as $\varepsilon \to 0$ to the functional

$$E(u, A) = c_0 \Phi(u, A)$$

for every Lipschitz set $A \in \mathcal{A}$ and every function $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, where

$$\Phi(u, A) = \begin{cases} \mathcal{H}^{n-1}(S(u) \cap A) = |Du|(A) = \mathcal{P}_A(\{u = 1\}) \\ \qquad\qquad\qquad\qquad \text{if } u \in BV(A; \{0, 1\}), \\ +\infty \qquad\qquad\qquad \text{otherwise} \end{cases} \tag{2.3}$$

and

$$c_0 = 2 \int_0^1 \sqrt{W(z)}\, \mathrm{d}z. \tag{2.4}$$

From this theorem and the properties of convergence of minima of Γ-limits, the following corollary, which describes the limit behaviour of the gradient theory of phase transitions, holds (see [18, proposition 3]).

COROLLARY 2.2. *Let $0 \leqslant V \leqslant |\Omega|$. Let $\gamma > 2$ and let $u_\varepsilon \in W^{1,\gamma}(\Omega)$ be a solution of problem*

$$m_\varepsilon = \min\left\{ \int_\Omega (W(u) + \varepsilon^2 |Du|^2)\, \mathrm{d}x : \int_\Omega u\, \mathrm{d}x = V \right\}.$$

Then, upon extracting a subsequence, $u_\varepsilon \to u \in BV(\Omega; \{0, 1\})$ in $L^1(\Omega)$, where u is a solution of the problem

$$m = \min\left\{ |Du|(\Omega) : u \in BV(\Omega; \{0, 1\}), \int_\Omega u\, \mathrm{d}x = V \right\} = \min\{\mathcal{P}_\Omega(E) : |E| = V\}$$

and $m_\varepsilon/\varepsilon \to c_0 m$.

3. A compactness result

For all $\varepsilon > 0$, let $f_\varepsilon : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ be a Borel function satisfying the growth condition

$$c_1|\xi|^2 - c_3 \leqslant f_\varepsilon(y, \xi) \leqslant c_2(1 + |\xi|^2) \quad \text{for a.e. } y \in \mathbb{R}^n \quad \text{for every } \xi \in \mathbb{R}^n, \tag{3.1}$$

with $0 < c_1 \leqslant c_2$ and $c_3 > 0$, independent of ε.

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Let $W:\mathbb{R}\to[0,+\infty)$ be a continuous function satisfying (2.1), (2.2). We will consider the functionals $G_\varepsilon : L^1_{\rm loc}(\mathbb{R}^n)\times\mathcal{A}\to[0,+\infty]$ defined by

$$G_\varepsilon(u,A)=\begin{cases}\displaystyle\int_A\Big(\frac{W(u)}{\varepsilon}+\varepsilon f_\varepsilon(x,Du)\Big)\,\mathrm{d}x & \text{if } u\in W^{1,\gamma}(A),\\ +\infty & \text{otherwise.}\end{cases}\tag{3.2}$$

Remark 3.1. By (3.1), it follows immediately that

$$\int_A\Big(\frac{W(u)}{\varepsilon}+\varepsilon c_1|Du|^2\Big)\,\mathrm{d}x-c_3\varepsilon|A|\leqslant G_\varepsilon(u,A)$$
$$\leqslant\int_A\Big(\frac{W(u)}{\varepsilon}+\varepsilon c_2|Du|^2\Big)\,\mathrm{d}x+c_2\varepsilon|A|$$

for each $u\in W^{1,\gamma}(A)$, and hence, if we set

$$G'(u,A)=\Gamma\text{-}\liminf_{\varepsilon\to0}G_\varepsilon(u,A),$$
$$G''(u,A)=\Gamma\text{-}\limsup_{\varepsilon\to0}G_\varepsilon(u,A),$$

then, by theorem 2.1, $G'(u,A)=G''(u,A)=+\infty$ whenever $u\notin BV(A;\{0,1\})$. Moreover, if $A\in\mathcal{A}$ is a Lipschitz set, we have the estimate

$$c_0\sqrt{c_1}\Phi(u,A)\leqslant G'(u,A)\leqslant G''(u,A)\leqslant c_0\sqrt{c_2}\Phi(u,A)\tag{3.3}$$

where Φ is defined in (2.3).

The following lemma is crucial in the description of the behaviour of the Γ-limits with respect to the set variable.

Lemma 3.2 (the fundamental estimate). *Let G_ε be defined by (3.2). Then, for every $\varepsilon>0$, for every bounded open set U, U', V, with $U\subset\subset U'$, and for every $u,v\in L^1_{\rm loc}(\mathbb{R}^n)$, there exists a cut-off function φ related to U and U', which may depend on ε, U, U', V, u, v, such that*

$$G_\varepsilon(\varphi u+(1-\varphi)v,U\cup V)\leqslant G_\varepsilon(u,U')+G_\varepsilon(v,V)+\delta_\varepsilon(u,v,U,U',V),$$

where $\delta_\varepsilon : L^1_{\rm loc}(\mathbb{R}^n)^2\times\mathcal{A}^3\to[0,+\infty[$ are functions depending only on ε and G_ε such that

$$\lim_{\varepsilon\to0}\delta_\varepsilon(u_\varepsilon,v_\varepsilon,U,U',V)=0$$

whenever $U,U',V\in\mathcal{A}$, $U\subset\subset U'$ and $u_\varepsilon,v_\varepsilon\in L^1_{\rm loc}(\mathbb{R}^n)$ have the same limit as $\varepsilon\to0$ in $L^1((U'\setminus\bar U)\cap V)$ and satisfy

$$\sup_{\varepsilon>0}(G_\varepsilon(u_\varepsilon,U')+G_\varepsilon(v_\varepsilon,V))<+\infty.$$

Proof. The proof follows the lines of that contained in the appendix of [14], with slight modifications. However, we include it, since the changes in the notation are heavy.

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We fix ε, $U, U', V \in \mathcal{A}$ with $U \subset\subset U'$. Let k_ε denote the integer part of $1/\varepsilon$, let $d = \operatorname{dist}(U, \mathbb{R}^n \setminus U')$ and choose $k_\varepsilon + 1$ open sets $U_1, \dots, U_{k_\varepsilon+1} \in \mathcal{A}$ such that

$$U \subset\subset U_1 \subset\subset \cdots \subset\subset U_{k_\varepsilon+1} \subset\subset U'$$

and

$$\operatorname{dist}(U_i, \mathbb{R}^n \setminus U_{i+1}) \geqslant \frac{d}{k_\varepsilon + 2}, \quad i = 1, 2, \dots, k_\varepsilon.$$

For each $i = 1, \dots, k_\varepsilon$, let φ_i be a cut-off function between U_i and U_{i+1} such that

$$\max |D\varphi_i| \leqslant \frac{2(k_\varepsilon + 2)}{d}. \tag{3.4}$$

We have, for every $i = 1, \dots, k_\varepsilon$, that

$$\begin{aligned} G_\varepsilon(\varphi_i u + (1-\varphi_i)v, U \cup V) &= G_\varepsilon(\varphi_i u + (1-\varphi_i)v, (U \cup V) \cap \bar{U}_i) \\ &\quad + G_\varepsilon(\varphi_i u + (1-\varphi_i)v, (U \cup V) \cap (\mathbb{R}^n \setminus U_{i+1})) \\ &\quad + G_\varepsilon(\varphi_i u + (1-\varphi_i)v, (U \cup V) \cap (U_{i+1} \setminus \bar{U}_i)) \\ &= G_\varepsilon(u, (U \cup V) \cap \bar{U}_i) + G_\varepsilon(v, V \cap (\mathbb{R}^n \setminus U_{i+1})) \\ &\quad + G_\varepsilon(\varphi_i u + (1-\varphi_i)v, (U_{i+1} \setminus \bar{U}_i) \cap V) \\ &\leqslant G_\varepsilon(u, U') + G_\varepsilon(v, V) + G_\varepsilon(\varphi_i u + (1-\varphi_i)v, (U_{i+1} \setminus \bar{U}_i) \cap V). \end{aligned} \tag{3.5}$$

We now estimate the last term in (3.5). We write $S_i = (U_{i+1} \setminus \bar{U}_i) \cap V$. By the growth conditions (3.1) and (3.4), we have that

$$\begin{aligned} G_\varepsilon(\varphi_i u + (1-\varphi_i)v, S_i) &= \int_{S_i} \frac{1}{\varepsilon} W(\varphi_i u + (1-\varphi_i)v) + \varepsilon f\left(\frac{x}{\delta}, D(\varphi_i u + (1-\varphi_i)v)\right) \mathrm{d}x \\ &\leqslant \int_{S_i} \frac{1}{\varepsilon} W(\varphi_i u + (1-\varphi_i)v)\, \mathrm{d}x + \int_{S_i} \varepsilon c_2 |D(\varphi_i u + (1-\varphi_i)v)|^2\, \mathrm{d}x \\ &\leqslant \int_{S_i} \frac{1}{\varepsilon} W(\varphi_i u + (1-\varphi_i)v)\, \mathrm{d}x + \int_{S_i} \varepsilon c(|Du|^2 + |Dv|^2 + |D\varphi_i|^2 |u-v|^2)\, \mathrm{d}x \\ &\leqslant \int_{S_i} \frac{1}{\varepsilon} W(\varphi_i u + (1-\varphi_i)v)\, \mathrm{d}x \\ &\quad + \varepsilon c \left(\frac{2(k_\varepsilon + 2)}{d}\right)^2 \int_{S_i} |u-v|^2\, \mathrm{d}x + c(G_\varepsilon(u, S_i) + G_\varepsilon(v, S_i)). \end{aligned}$$

Summing on i, we get

$$\begin{aligned} \sum_{i=1}^{k_\varepsilon} G_\varepsilon(\varphi_i u + (1-\varphi_i)v, S_i) &\leqslant \sum_{i=1}^{k_\varepsilon} \int_{S_i} \frac{1}{\varepsilon} W(\varphi_i u + (1-\varphi_i)v)\, \mathrm{d}x \\ &\quad + \varepsilon c \left(\frac{2(k_\varepsilon + 2)}{d}\right)^2 \int_S |u-v|^2\, \mathrm{d}x + c(G_\varepsilon(u, S) + G_\varepsilon(v, S)), \end{aligned}$$

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where $S = (U' \setminus \bar{U}) \cap V$. Then there exists φ_h among $\varphi_1, \dots, \varphi_{k_\varepsilon}$ such that

$$\begin{aligned} G_\varepsilon(\varphi_h u + (1-\varphi_h)v, S_h) &\leqslant \frac{1}{\varepsilon k_\varepsilon}\biggl(\sum_{i=1}^{k_\varepsilon}\int_{S_i} W(\varphi_i u + (1-\varphi_i)v)\,\mathrm{d}x\biggr) \\ &\quad + c\frac{\varepsilon}{k_\varepsilon}\biggl(\frac{2(k_\varepsilon+2)}{d}\biggr)^2\int_S |u-v|^2\,\mathrm{d}x + \frac{c}{k_\varepsilon}(G_\varepsilon(u,S) + G_\varepsilon(v,S)). \end{aligned}$$

If we define

$$\begin{aligned} \delta_\varepsilon(u,v,U,U',V) &= \frac{1}{\varepsilon k_\varepsilon}\biggl(\sum_{i=1}^{k_\varepsilon}\int_{S_i} W(\varphi_i u + (1-\varphi_i)v)\,\mathrm{d}x\biggr) \\ &\quad + c\frac{\varepsilon}{k_\varepsilon}\biggl(\frac{2(k_\varepsilon+2)}{d}\biggr)^2\int_S |u-v|^2\,\mathrm{d}x + \frac{c}{k_\varepsilon}(G_\varepsilon(u,S) + G_\varepsilon(v,S)) \end{aligned} \tag{3.6}$$

and choose $\varphi = \varphi_h$ cut-off function between U_h and U_{h+1}, by (3.5), we have that

$$G_\varepsilon(\varphi u + (1-\varphi)v, U \cup V) \leqslant G_\varepsilon(u, U') + G_\varepsilon(v, V) + \delta_\varepsilon(u, v, U, U', V).$$

Let u_ε and v_ε be two sequences in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ with the same limit in $L^1(S)$ and with $\sup_{\varepsilon>0}(G_\varepsilon(u_\varepsilon, U') + G_\varepsilon(v_\varepsilon, V)) \leqslant M$. Under these conditions, we can prove that the sequences u_ε and v_ε converge to the same limit also in $L^\gamma(S)$. In fact, let w be the common limit of u_ε and v_ε in $L^1(S)$ and let $r \in \mathbb{R}$ be such that

$$W(z) \geqslant \tfrac{1}{2}c_1|z|^\gamma \quad \text{if } |z| > r.$$

We define

$$w^r(x) = -r \vee (r \wedge w(x)), \quad x \in \mathbb{R}^n,$$

and, analogously, u^r_ε and v^r_ε. It can be easily seen that u^r_ε and v^r_ε converge to w^r in $L^\gamma(S)$. Moreover,

$$\begin{aligned} \int_S |u_\varepsilon(x) - u^r_\varepsilon(x)|^\gamma\,\mathrm{d}x &\leqslant \int_{\{x\in S:|u_\varepsilon|>r\}} |u_\varepsilon(x)|^\gamma\,\mathrm{d}x \\ &\leqslant \frac{2}{c_1}\int_S W(u_\varepsilon(x))\,\mathrm{d}x \\ &\leqslant \frac{2}{c_1}\varepsilon G_\varepsilon(u_\varepsilon, S) \\ &\leqslant \frac{2M}{c_1}\varepsilon. \end{aligned}$$

Hence we can conclude that u_ε and v_ε converge to w^r in $L^\gamma(S)$. As they converge to w in $L^1(S)$, we have $w^r \equiv w$. To prove that

$$\lim_{\varepsilon\to 0}\delta_\varepsilon(u_\varepsilon, v_\varepsilon, U, U', V) = 0,$$

it remains to study the convergence to zero of the first term in (3.6), since, for the other ones, it is obvious.

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Note that $1/\varepsilon k_\varepsilon$ is bounded. Hence it is sufficient to prove that

$$\lim_{\varepsilon\to 0}\sum_{i=1}^{k_\varepsilon}\int_{S_i} W(\varphi_i u_\varepsilon + (1-\varphi_i)v_\varepsilon)\,\mathrm{d}x = 0.$$

We define, for $x \in \mathbb{R}^n$,

$$w_\varepsilon = \begin{cases} \varphi_i u_\varepsilon + (1-\varphi_i)v_\varepsilon & \text{if } x \in S_i \text{ for some } i = 1,\dots,k_\varepsilon,\\ w(x) & \text{otherwise,}\end{cases}$$

which converges to w in $L^\gamma(S)$. Since W is continuous and satisfies the growth condition (2.2) of order γ, by the dominated convergence theorem, we have that

$$\begin{aligned} 0 &\leqslant \lim_{\varepsilon\to 0}\sum_{i=1}^{k_\varepsilon}\int_{S_i} W(\varphi_i u_\varepsilon + (1-\varphi_i)v_\varepsilon)\,\mathrm{d}x \\ &= \lim_{\varepsilon\to 0}\int_S W(w_\varepsilon(x))\,\mathrm{d}x \\ &= \lim_{\varepsilon\to 0}\int_S W(u_\varepsilon(x))\,\mathrm{d}x \\ &\leqslant \lim_{\varepsilon\to 0}\varepsilon G_\varepsilon(u_\varepsilon, S) \\ &= 0, \end{aligned}$$

which completes the proof. □

THEOREM 3.3 (compactness by Γ-convergence). *For every sequence $(\varepsilon_j)_j$ converging to 0, there exists a subsequence $(\varepsilon_{j_k})_k$ and a functional $G : L^1_{\mathrm{loc}}(\mathbb{R}^n) \times \mathcal{A} \to [0,+\infty]$ such that $(G_{\varepsilon_{j_k}})_k$ Γ-converges to G for every U bounded Lipschitz open set and for every $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that $u \in BV(U;\{0,1\})$, with respect to the strong topology of $L^1(U)$. Moreover, for every $u \in BV_{\mathrm{loc}}(\mathbb{R}^n;\{0,1\})$, $G(u,\cdot)$ is the restriction to $\mathcal{A}$ of a regular Borel measure.*

Proof. By a standard compactness argument (see, for example, [10, § 7.3]), we can assume that $(G_{\varepsilon_{j_k}}(\cdot,R))_k$ Γ-converges to a functional $G_0(\cdot,R)$, with respect to the $L^1(R)$ convergence, for all R belonging to the class $\mathcal{R}$ of all polyrectangles with rational vertices. If $u \in BV_{\mathrm{loc}}(\mathbb{R}^n;\{0,1\})$, we define $G(u,A)$ on all open sets $A \in \mathcal{A}$ by setting

$$G(u,A) = \sup\{G_0(u,R) : R \subset\subset A, R \in \mathcal{R}\}.$$

For every $A, A' \in \mathcal{A}$ with $A' \subset\subset A$, there exists $R \in \mathcal{R}$ such that $A' \subset\subset R \subset\subset A$. Hence we get

$$\begin{aligned} G(u,A) &= \sup\{G'(u,A') : A' \subset\subset A, A' \in \mathcal{A}\} \\ &= \sup\{G''(u,A') : A' \subset\subset A, A' \in \mathcal{A}\} \end{aligned} \tag{3.7}$$

for all $A \in \mathcal{A}$, that is, G is the inner regular envelope of G' and of G''. Hence the set function $G(u,\cdot)$ is inner regular (see [13, remark 16.3]), superadditive (see [13, proposition 16.12]) and, by using the fundamental estimate above, we can prove that $G(u,\cdot)$ is also subadditive (see [13, proposition 18.4]). Hence, by the measure

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property criterion of De Giorgi and Letta, $G(u,\cdot)$ is the restriction to $\mathcal{A}$ of a regular Borel measure (see [10, ch. 10]). Since, by the fundamental estimate, $G'(u,\cdot)$, $G''(u,\cdot)$ are themselves inner regular on the class of bounded Lipschitz open sets U (see [10, propositions 11.5 and 11.6]), then, by (3.7), we deduce that

$$\begin{aligned} G'(u,U) &= \sup\{G'(u,A') : A' \subset\subset U, A' \in \mathcal{A}\} \\ &= G(u,U) \\ &= \sup\{G''(u,A') : A' \subset\subset U, A' \in \mathcal{A}\} \\ &= G''(u,U) \end{aligned}$$

for all such sets U. Hence G is the Γ-limit of $(G_{\varepsilon_{j_k}})_k$ for every U bounded Lipschitz open set and for every $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Remark 3.1 completes the proof. □

In the sequel, we still denote by G the extension of $G(u,\cdot)$ to the family $\mathcal{B}$ of all Borel subsets of $\mathbb{R}^n$.

Remark 3.4. G is a local functional on $\mathcal{A}$, i.e.

$$G(u,A) = G(v,A)$$

for every set $A \in \mathcal{A}$ and every $u,v \in BV_{\mathrm{loc}}(\mathbb{R}^n;\{0,1\})$ such that $u = v$ a.e. in A. This follows directly by applying the definition of Γ-convergence, each G_ε being a local functional too. Moreover, by remark 3.1, we can deduce the following estimate,

$$G(u,U) \leqslant c_0\sqrt{c_2}\Phi(u,U) = c_0\sqrt{c_2}\mathcal{H}^{n-1}(S_u \cap U), \tag{3.8}$$

for every Lipschitz set $U \in \mathcal{A}$ and every $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that $u \in BV(U;\{0,1\})$.

Theorem 3.5 (integral representation). *There exists a Borel function $\varphi : \mathbb{R}^n \times S^{n-1} \to [0,+\infty[$ such that*

$$c_0\sqrt{c_1} \leqslant \varphi(x,\nu) \leqslant c_0\sqrt{c_2} \quad \textit{for a.e. } x \in \mathbb{R}^n,\ \nu \in S^{n-1}, \tag{3.9}$$

$$G(u,B) = \begin{cases} \displaystyle\int_{S_u\cap B} \varphi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} & \textit{if } u \in BV(U;\{0,1\}), \\ +\infty & \textit{otherwise} \end{cases} \tag{3.10}$$

for every Lipschitz set $U \in \mathcal{A}$ and every Borel set $B \subseteq U$. Moreover, φ satisfies the derivation formula

$$\varphi(x,\nu) = \limsup_{\rho\to 0+} \rho^{1-n} \inf\{G(u,\overline{Q^\nu_\rho(x)}) : u = u^\nu_x \textit{ in } \mathbb{R}^n \setminus Q^\nu_\rho(x)\}, \tag{3.11}$$

where u^ν_x is the characteristic function of the half-space $\{y \in \mathbb{R}^n : \langle y-x,\nu\rangle > 0\}$.

Proof. It suffices to notice that G, as defined in theorem 3.3, satisfies the hypotheses of theorem 1.4 of [5] (a direct proof can be also obtained by following that of lemma 3.5 in [9]). □

Remark 3.6. If φ does not depend on x, then, from (3.11),

$$\varphi(\nu) = \inf\{G(u,\overline{Q^\nu}) : u = u^\nu \text{ in } \mathbb{R}^n \setminus Q^\nu\}$$

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where $u^\nu = u_0^\nu$. Moreover, the one-homogeneous extension of φ to $\mathbb{R}^n$ is convex (see [2]), and, in particular, it is continuous. We will use this fact to identify φ by computing it on a dense set in S^{n-1}.

3.1. Boundary conditions

In this section we extend the preceding results to include the case of problems with some types of boundary conditions.

Let $w : \mathbb{R} \to [0,1]$ be such that

$$w(-\infty) := \lim_{t\to-\infty} w(t) = 0, \qquad w(+\infty) := \lim_{t\to+\infty} w(t) = 1$$

and

$$\int_{-\infty}^{+\infty} (W(w) + |w'|^2)\,\mathrm{d}t = c < +\infty. \tag{3.12}$$

We define $w_\varepsilon(t) = w(t/\varepsilon)$ and $v_\varepsilon(x) = w_\varepsilon(\langle x,\nu\rangle)$. We easily see that

$$v_\varepsilon \to u^\nu, \tag{3.13}$$

where $u^\nu = u_0^\nu$ is defined in theorem 3.5.

With fixed $x \in \mathbb{R}^n$ and $\rho > 0$, we define

$$\tilde F_\varepsilon(u, Q_\rho^\nu(x)) = \begin{cases} F_\varepsilon(u, Q_\rho^\nu(x)) & \text{if } u(y) = v_\varepsilon(y-x) \text{ on } \mathbb{R}^n \setminus Q_\rho^\nu(x),\ u \in H^1_{\mathrm{loc}}(\mathbb{R}^n), \\ +\infty & \text{otherwise.} \end{cases}$$

THEOREM 3.7. *Let F_ε be defined by (3.2) and suppose that (F_ε) Γ-converges to F as in theorem 3.3 (upon passing to a subsequence). Let the function φ given by the integral representation theorem 3.5 be independent of x. Then*

$$\Gamma\text{-}\lim_{\varepsilon\to 0} \tilde F_\varepsilon(u, Q_\rho^\nu(x)) = F(u, \bar Q_\rho^\nu(x)), \tag{3.14}$$

where we extend u by setting $u(y) = u^\nu(y-x)$ on $\mathbb{R}^n \setminus Q_\rho^\nu(x)$.

Proof. It clearly suffices to prove the theorem with $x = 0$. We begin by proving the Γ-liminf inequality.

Let $\rho_1 > \rho$. With fixed $\varepsilon > 0$, for all v with $v = v_\varepsilon$ on $\mathbb{R}^n \setminus Q_\rho^\nu(x)$, we have

$$\tilde F_\varepsilon(v, Q_\rho^\nu) = F_\varepsilon(v, Q_{\rho_1}^\nu) - F_\varepsilon(v_\varepsilon, Q_{\rho_1}^\nu \setminus Q_\rho^\nu). \tag{3.15}$$

We define

$$Q_{\rho_1,n-1}^\nu = \{x \in Q_{\rho_1}^\nu : \langle x,\nu\rangle = 0\},$$
$$Q_{\rho,n-1}^\nu = \{x \in Q_\rho^\nu : \langle x,\nu\rangle = 0\}$$

and

$$A_1 = Q_{\rho_1,n-1}^\nu \setminus Q_{\rho,n-1}^\nu \times (-\tfrac12\rho_1, \tfrac12\rho_1),$$
$$A_2 = Q_{\rho,n-1}^\nu \times (-\tfrac12\rho_1, -\tfrac12\rho) \cup (\tfrac12\rho, \tfrac12\rho_1).$$

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We now compute

$$\begin{aligned}F_\varepsilon(v_\varepsilon, Q^\nu_{\rho_1}\setminus Q^\nu_\rho) &= F_\varepsilon(v_\varepsilon, A_1) + F_\varepsilon(v_\varepsilon, A_2)\\ &\leqslant \bar c(\rho_1^{n-1}-\rho^{n-1})\int_{-\infty}^{+\infty}(W(w)+|w'|^2)\,\mathrm{d}t\\ &\quad + \bar c\rho^{n-1}\biggl(\int_{-\infty}^{-\rho/2\varepsilon}\Bigl(W(w)+|w'|^2\Bigr)\,\mathrm{d}t + \int_{\rho/2\varepsilon}^{+\infty}(W(w)+|w'|^2)\,\mathrm{d}t\biggr),\end{aligned} \tag{3.16}$$

where $\bar c = \max\{1, c_2\}$. Hence, by (3.12) and (3.16), for every sequence u_ε converging to u such that $u_\varepsilon = v_\varepsilon$ on $\mathbb{R}^n\setminus Q^\nu_\rho$ and $\liminf_{\varepsilon\to 0}\tilde F_\varepsilon(u_\varepsilon, Q^\nu_\rho) < +\infty$, we get that

$$\begin{aligned}\liminf_{\varepsilon\to 0}\tilde F_\varepsilon(u_\varepsilon, Q^\nu_\rho) &\geqslant \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, Q^\nu_{\rho_1}) - O(\rho_1^{n-1}-\rho^{n-1})\\ &\geqslant F(u, Q^\nu_{\rho_1}) - O(\rho_1^{n-1}-\rho^{n-1}).\end{aligned} \tag{3.17}$$

Passing to the limit as ρ_1 tends to ρ, we have the liminf inequality

$$F(u, \bar Q^\nu_\rho) \leqslant \liminf_{\varepsilon\to 0}\tilde F_\varepsilon(u_\varepsilon, Q^\nu_\rho). \tag{3.18}$$

We now prove the Γ-limsup inequality. Let $u \in BV(Q^\nu_\rho;\{0,1\})$ be such that $u = u^\nu$ on $\mathbb{R}^n\setminus Q^\nu_\rho$.

(a) We first assume that $u = u^\nu$ on $\mathbb{R}^n\setminus Q^\nu_{\rho_1}$ with $\rho_1 < \rho$. Let u_ε be a sequence converging to u such that

$$F(u, Q^\nu_\rho) = \lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, Q^\nu_\rho).$$

In particular, u_ε converges to u^ν on $\mathbb{R}^n\setminus Q^\nu_{\rho_1}$. Let φ_ε be a cut-off function between $U = Q^\nu_{(\rho+\rho_1)/2}$ and $U' = Q^\nu_\rho$ and let $V = Q^\nu_\rho\setminus\bar Q^\nu_{\rho_1}$. By the fundamental estimate,

$$F_\varepsilon(u_\varepsilon\varphi_\varepsilon + (1-\varphi_\varepsilon)v_\varepsilon, Q^\nu_\rho) \leqslant F_\varepsilon(u_\varepsilon, Q^\nu_\rho) + F_\varepsilon(v_\varepsilon, Q^\nu_\rho\setminus\bar Q^\nu_{\rho_1}) + \delta_\varepsilon(u_\varepsilon, v_\varepsilon, U, U', V). \tag{3.19}$$

By the assumptions on u_ε and (3.13), we also have

$$u_\varepsilon \to u^\nu, \quad v_\varepsilon \to u^\nu \quad \text{on } V.$$

Hence we get

$$\lim_{\varepsilon\to 0}\delta_\varepsilon(u_\varepsilon, v_\varepsilon, U, U', V) = 0$$

and, by (3.15), (3.16) and (3.19),

$$\Gamma\text{-}\limsup_{\varepsilon\to 0}\tilde F_\varepsilon(u, Q^\nu_\rho) \leqslant F(u, Q^\nu_\rho).$$

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(b) In the general case, we consider $\rho_1 < \rho$ and we define $u_{\rho_1}(x) = u((\rho/\rho_1)x)$. By the previous case (a) and (3.10),

$$\begin{aligned}\Gamma\text{-}\limsup_{\varepsilon\to 0} \tilde{F}_\varepsilon(u_{\rho_1}, Q^\nu_\rho) &\leqslant F(u_{\rho_1}, Q^\nu_\rho) \\ &= \int_{Q^\nu_\rho \cap S(u_{\rho_1})} \varphi(\nu_{u_{\rho_1}})\,\mathrm{d}\mathcal{H}^{n-1} \\ &\leqslant \int_{\bar{Q}^\nu_\rho \cap S(u)} \varphi(\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + O(\rho^{n-1} - \rho_1^{n-1}) \\ &= F(u, \bar{Q}^\nu_\rho) + O(\rho^{n-1} - \rho_1^{n-1}). \end{aligned} \tag{3.20}$$

Since u_{ρ_1} converges to u as ρ_1 tends to ρ, if we denote

$$\tilde{F}''(u_{\rho_1}, Q^\nu_\rho) = \Gamma\text{-}\limsup_{\varepsilon\to 0} \tilde{F}_\varepsilon(u_{\rho_1}, Q^\nu_\rho),$$

then, by the lower semicontinuity of the Γ-upper limit (see, for example, [10, remark 7.8]) and (3.20),

$$\Gamma\text{-}\limsup_{\varepsilon\to 0} \tilde{F}_\varepsilon(u, Q^\nu_\rho) \leqslant \liminf_{\rho_1\to\rho} \tilde{F}''(u_{\rho_1}, Q^\nu_\rho) \leqslant F(u, \bar{Q}^\nu_\rho). \tag{3.21}$$

Hence, by (3.21) and (3.18), we get the required equality (3.14).

□

COROLLARY 3.8. *Let the function φ given by the integral representation theorem 3.5 be independent of x. Then*

$$\begin{aligned}\varphi(\nu) &= \min\{F(u, \bar{Q}^\nu) u = u^\nu \text{ on } \mathbb{R}^n \setminus Q^\nu\} \\ &= \lim_{j\to+\infty} \min\{\tilde{F}_{\varepsilon_j}(u, Q^\nu) u = v_{\varepsilon_j} \text{ on } \partial Q^\nu\}.\end{aligned} \tag{3.22}$$

Proof. The first equality follows from remark 3.6, while the convergence of minima comes from the Γ-convergence of $\tilde{F}_{\varepsilon_j}$ and the fact that we may find a sequence of minimizers that is compact in $L^1(Q^\nu)$. This can be proved by following [18, proposition 3] on noticing that we may assume that minimizers take values in $[0, 1]$ by a truncation argument. □

REMARK 3.9. If the function u in theorem 3.7 satisfies $u(y) = u^\nu(y-x)$ on a neighbourhood of $\partial Q^\nu_\rho(x)$ (i.e. the support of $y \mapsto u(y) - u^\nu(y-x)$ is compactly contained in $Q^\nu_\rho(x)$), then the hypothesis that φ be independent of x may be removed. In fact, that hypothesis is used in step (b) of the Γ-limsup inequality only.

REMARK 3.10. We want to show by a simple example that if φ explicitly depends on x, then theorem 3.7 is not true. Consider

$$F_\varepsilon(u, U) = \int_U \left(\frac{1}{\varepsilon} W(u) + \varepsilon a(x)|Du|^2\right) \mathrm{d}x,$$

where

$$a(x) = \begin{cases} 1 & \text{if } x \in Q, \\ \frac{1}{4} & \text{otherwise.} \end{cases}$$

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It can be easily checked that

$$F(u,U)=\Gamma\text{-}\lim_{\varepsilon\to 0}F_\varepsilon(u,U)=c_0\int_{S(u)\cap U}\sqrt{a(x)}\,\mathrm{d}\mathcal{H}^{n-1}. \tag{3.23}$$

Now we want to show that there exists u such that

$$\Gamma\text{-}\liminf_{\varepsilon\to 0}\tilde F_\varepsilon(u,Q)>F(u,\bar Q). \tag{3.24}$$

Such a u can be chosen as

$$u=\begin{cases}1 & \text{if } x_n>\frac{1}{2},\\ 0 & \text{otherwise.}\end{cases}$$

In fact, let u_ε be a sequence converging to u such that $u_\varepsilon=v_\varepsilon$ on ∂Q. Then

$$\begin{aligned}\tilde F_\varepsilon(u_\varepsilon,Q)&=\int_Q\left(\frac{1}{\varepsilon}W(u_\varepsilon)+\varepsilon|Du_\varepsilon|^2\right)\mathrm{d}x\\ &=\int_{(1+\eta)Q}\left(\frac{1}{\varepsilon}W(u_\varepsilon)+\varepsilon|Du_\varepsilon|^2\right)\mathrm{d}x\\ &\qquad-\int_{(1+\eta)Q\setminus Q}\left(\frac{1}{\varepsilon}W(v_\varepsilon)+\varepsilon|Dv_\varepsilon|^2\right)\mathrm{d}x\end{aligned}$$

and

$$\liminf_{\varepsilon\to 0}\tilde F_\varepsilon(u_\varepsilon,Q)\geqslant c_0\mathcal{H}^{n-1}(S(u)\cap(1+\eta)Q)-c((1+\eta)^{n-1}-1).$$

Passing to the limit as η tends to 0, we get

$$\begin{aligned}\liminf_{\varepsilon\to 0}\tilde F_\varepsilon(u_\varepsilon,Q)&\geqslant c_0\mathcal{H}^{n-1}(S(u)\cap\bar Q)\\ &>c_0\mathcal{H}^{n-1}(S(u)\cap Q)+\tfrac{1}{2}c_0\mathcal{H}^{n-1}(S(u)\cap\partial Q).\end{aligned}$$

By (3.23), we get the required inequality (3.24).

4. Homogenization

In this section we treat the case of highly oscillating coefficients. Comparing with the compactness result in the previous sections, we make an additional hypothesis of positive homogeneity of the integrands in order to simplify the notation of the results, while keeping the main phenomenon of the different behaviours in the presence of different regimes of oscillations.

Let $f:\mathbb{R}^n\times\mathbb{R}^n\to[0,+\infty)$ be a Borel function satisfying the following conditions. There exist $0<c_1\leqslant c_2$ such that

$$c_1|\xi|^2\leqslant f(y,\xi)\leqslant c_2|\xi|^2; \tag{4.1}$$

$$\text{for all } y\in\mathbb{R}^n,\ f(y,\cdot) \text{ is positively homogeneous of degree two;} \tag{4.2}$$

$$\text{for all } \xi\in\mathbb{R}^n,\ f(\cdot,\xi) \text{ is 1-periodic,} \tag{4.3}$$

i.e. $f(x+e_i,\xi)=f(x,\xi)$ for all $x\in\mathbb{R}^n$, and $i=1,\dots,n$.

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Let $\delta : (0, +\infty) \to (0, +\infty)$ and let W be as in § 2. For all $\varepsilon > 0$, we consider the functional $F_\varepsilon : L^1_{\mathrm{loc}}(\mathbb{R}^n) \times \mathcal{A} \to [0, +\infty]$ defined by

$$F_\varepsilon(u, A) = \begin{cases} \displaystyle\int_A \left(\frac{W(u)}{\varepsilon} + \varepsilon f\left(\frac{x}{\delta(\varepsilon)}, Du \right) \right) \mathrm{d}x & \text{if } u \in W^{1,\gamma}(A), \\ +\infty & \text{otherwise.} \end{cases} \tag{4.4}$$

With a fixed sequence (ε_j) of positive numbers converging to 0, by applying theorem 3.3 with

$$f_\varepsilon(x, \xi) = f\left(\frac{x}{\delta(\varepsilon)}, \xi \right),$$

we conclude that, upon extracting a subsequence (not relabelled), the functionals F_{ε_j} Γ-converge on all Lipschitz bounded open subsets of $\mathbb{R}^n$. Their limit F can be represented as an integral by theorem 3.5, with an energy density φ given by (3.11). In this section we will characterize this function φ and hence also F. We begin by remarking that φ is independent of x.

PROPOSITION 4.1. *Let*

$$\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0.$$

Then the function φ *is independent of* x.

Proof. To prove that φ is independent of x, we show that $\varphi(x, \nu) = \varphi(y, \nu)$ for all $x, y \in \mathbb{R}^n$.

We first remark that, besides (3.11), $\varphi(x, \nu)$ is equivalently described as

$$\varphi(x, \nu) = \limsup_{\rho \to 0^+} \frac{1}{\rho^{n-1}} \inf\{F(v, Q^\nu_\rho(x)) : v = u^\nu_x \text{ in a neighbourhood of } \partial Q^\nu_\rho(x)\}. \tag{4.5}$$

In order to prove this representation, with fixed $x \in \mathbb{R}^n$ and $\rho > 0$, we consider the minimum problems

$$m(\rho) = \inf\{F(u, \overline{Q^\nu_\rho(x)}) : u = u^\nu_x \text{ in } \mathbb{R}^n \setminus Q^\nu_\rho(x)\},$$

and we denote by $u^{x,\rho}$ one of its minimizers.

Let (ρ_j) be a sequence converging to 0 such that

$$\lim_j \frac{m(\rho_j)}{\rho_j^{n-1}} = \limsup_{\rho \to 0^+} \frac{m(\rho)}{\rho^{n-1}} = \varphi(x, \nu),$$

by (3.11). Let $\rho'_j = (1 - 1/j)\rho_j$. By using the function u^{x,ρ'_j} (a fixed minimizer of $m(\rho'_j)$) as a test function in the minimum problem defining $m(\rho_j)$ and remarking that $F(u^{x,\rho'_j}, \overline{Q^\nu_\rho(x)}) = F(u^{x,\rho'_j}, Q^\nu_\rho(x))$, we get

$$\frac{m(\rho_j)}{\rho_j^{n-1}} \leqslant \frac{1}{\rho_j^{n-1}} F(u^{x,\rho'_j}, Q^\nu_\rho(x)) \leqslant \frac{1}{\rho_j^{n-1}} (m(\rho'_j) + c(\rho_j^{n-1} - {\rho'_j}^{n-1})),$$

and, taking the limit as $j \to +\infty$,

$$\varphi(x, \nu) = \lim_j \frac{m(\rho_j)}{\rho_j^{n-1}} \leqslant \lim_j \frac{1}{\rho_j^{n-1}} F(u^{x,\rho'_j}, Q^\nu_\rho(x)) \leqslant \limsup_j \frac{m(\rho'_j)}{{\rho'_j}^{n-1}} \leqslant \varphi(x, \nu).$$

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Hence all inequalities in this last formula are equalities. This shows that we may replace $m(\rho_j)$ by

$$\inf\{F(u, Q^\nu_{\rho_j}(x)) : u = u^\nu_x \text{ in } \mathbb{R}^n \setminus Q^\nu_{\rho'_j}(x)\}$$

in the computation of $\varphi(x, \rho)$ and, in particular, we obtain the representation (4.5).

We now fix $x \in \mathbb{R}^n$ and $\rho > 0$ and a function $u = u^{x,\rho}$ with $u = u^\nu_x$ on a neighbourhood of $\partial Q^\nu_\rho(x)$. By remark 3.9 and theorem 3.7, there exists a sequence $u^{x,\rho}_j$ converging to $u^{x,\rho}$, with $u^{x,\rho}_j(z) = v_{\varepsilon_j}(z - x)$ on $\mathbb{R}^n \setminus Q^\nu_\rho(x)$, such that

$$\lim_j F_{\varepsilon_j}(u^{x,\rho}_j, Q^\nu_\rho(x)) = F(u^{x,\rho}, \overline{Q^\nu_\rho(x)}).$$

We define $\tau_j \in \mathbb{Z}^n$ by

$$(\tau_j)_i = \left[\frac{y_i - x_i}{\varepsilon_j}\right],$$

and $u^{y,\rho}_j(z) = u^{x,\rho}_j(z - \varepsilon_j\tau_j)$. Note that $\lim_j \varepsilon_j\tau_j = y - x$, $u^{y,\rho}_j$ converges to $u^{y,\rho}$ given by $u^{y,\rho}(z) = u^{x,\rho}(z - y + x)$ and

$$u^{y,\rho}_j(z) = w_j(z) \quad \text{on } \mathbb{R}^n \setminus (\varepsilon_j\tau_j + Q^\nu_\rho(x)),$$

where

$$w_j(z) = v_{\varepsilon_j}(z - x - \varepsilon_j\tau_j).$$

By plugging $u^{y,\rho}_j$ into F_{ε_j}, we get

$$F_{\varepsilon_j}(u^{y,\rho}_j, \varepsilon_j\tau_j + Q^\nu_\rho(x)) = F_{\varepsilon_j}(u^{x,\rho}_j, Q^\nu_\rho(x)),$$

so that, for fixed $r > 1$, we get

$$\begin{aligned}
&F(u^{y,\rho}, \overline{Q^\nu_\rho(y)}) \\
&\quad\leqslant F(u^{y,\rho}, Q^\nu_{r\rho}(y)) \leqslant \liminf_j F_{\varepsilon_j}(u^{y,\rho}_j, Q^\nu_{r\rho}(y)) \\
&\quad= \liminf_j (F_{\varepsilon_j}(u^{y,\rho}_j, \varepsilon_j\tau_j + Q^\nu_\rho(x)) + F_{\varepsilon_j}(u^{y,\rho}_j, Q^\nu_{r\rho}(y) \setminus (\varepsilon_j\tau_j + Q^\nu_\rho(x)))) \\
&\quad= \liminf_j (F_{\varepsilon_j}(u^{x,\rho}_j, Q^\nu_\rho(x)) + F_{\varepsilon_j}(w_j, Q^\nu_{r\rho}(y) \setminus (\varepsilon_j\tau_j + Q^\nu_\rho(x)))) \\
&\quad= \lim_j F_{\varepsilon_j}(u^{x,\rho}_j, Q^\nu_\rho(x)) + \lim_j F_{\varepsilon_j}(w_j, Q^\nu_{r\rho}(y) \setminus (\varepsilon_j\tau_j + Q^\nu_\rho(x))) \\
&\quad\leqslant F(u^{x,\rho}, \overline{Q^\nu_\rho(x)}) + c\rho^{n-1}(r^{n-1} - 1),
\end{aligned}$$

with

$$c = \int_{-\infty}^{+\infty} (W(w) + |w'|^2)\,\mathrm{d}t.$$

By the arbitrariness of $r > 1$, we get

$$F(u^{y,\rho}, \overline{Q^\nu_\rho(y)}) \leqslant F(u^{x,\rho}, \overline{Q^\nu_\rho(x)})$$

and, by symmetry, the equality, so that $\varphi(x, \nu) = \varphi(y, \nu)$ by letting $\rho \to 0$, by formula (4.5) and the arbitrariness of u. ◻

REMARK 4.2. The formula

$$\varphi(\nu) = \lim_{j\to+\infty} \min\biggl\{\varepsilon_j^{n-1}\int_{(1/\varepsilon_j)Q^\nu}\biggl(W(u)+f\biggl(\frac{\varepsilon_j}{\delta(\varepsilon_j)}x, Du\biggr)\biggr)\,\mathrm{d}x : u = v^\nu \text{ on } \partial\biggl(\frac{1}{\varepsilon_j}Q^\nu\biggr)\biggr\} \quad (4.6)$$

holds, where $v^\nu(x) = w(\langle x,\nu\rangle)$. To check this, it suffices to use the previous proposition and corollary 3.8.

4.1. Oscillations on the scale of the transition layer

In this section we treat the case when the scale of oscillation δ and the scale of the transition layer ε are comparable.

THEOREM 4.3. *Let F_ε be defined by (4.4). Let $f(x,\xi)$ be a Borel function, 1-periodic in x, positively homogeneous of degree two in ξ and satisfying the growth conditions (4.1), and let $W(z)$ be a continuous function satisfying conditions (2.1) and (2.2). Let $\delta : (0,+\infty) \to (0,+\infty)$ be such that*

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{\delta(\varepsilon)} = c,$$

where c is a positive constant. Then there exists the Γ-limit

$$\Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(u,U) = \int_{S(u)\cap U}\varphi(\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}$$

for every $u \in BV(U;\{0,1\})$, where

$$\varphi(\nu) = \lim_{T\to+\infty}\frac{1}{T^{n-1}}\inf\biggl\{\int_{TQ^\nu}(W(u)+f(cx,Du))\,\mathrm{d}x : u = v^\nu \text{ on } \partial(TQ^\nu)\biggr\}. \quad (4.7)$$

Proof. First, we prove the theorem when $\delta(\varepsilon) = \varepsilon$.

STEP 1. It is sufficient to prove the formula for a dense set Ξ of ν. In fact, since φ is convex, it is also continuous. Hence, if the formula is true for every $\nu \in \Xi$, then φ is independent from ε_j for every $\nu \in \Xi$. By the continuity of φ, it is also independent from ε_j for every ν. Hence we can conclude that there exists the Γ-limit of $F_{\varepsilon,\delta(\varepsilon)}$ and, by the convergence of minima, the formula is true for every ν.

STEP 2. Let Ξ be the set of unit rational vectors, i.e.

$$\Xi = \{\nu \in S^{n-1} : \exists\lambda \in \mathbb{R},\ \lambda\nu \in \mathbb{Q}^n\}.$$

In can be easily seen that Ξ is dense in S^{n-1}. Now, for simplicity of notation, we develop the proof only in the case $\nu = e_n$, but the same arguments clearly work for any $\nu \in \Xi$, up to a change of variables and of the periodicity cell. We define, for $T > 0$, $TQ = (-\frac{1}{2}T, \frac{1}{2}T)^n$ and

$$g(T) = \inf\biggl\{\frac{1}{T^{n-1}}\int_{TQ}(W(u)+f(x,Du))\,\mathrm{d}x : u = w(x_n) \text{ on } \partial TQ\biggr\}.$$

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We have to prove that the limit exists as T tends to $+\infty$. Let u_T be such that

$$\int_{TQ}(W(u_T)+f(x,Du_T))\,\mathrm{d}x \leqslant T^{n-1}g(T)+1.$$

Let $S>T$. We define

$$Q_{Tz}=z([T]+1)+TQ \quad \text{for } z\in\mathbb{Z}^{n-1}\times\{0\}$$

and

$$I_S=\{z\in\mathbb{Z}^{n-1}\times\{0\}: Q_{Tz}\subseteq SQ\}.$$

We can construct

$$u_S(x)=\begin{cases} u_T(x-z([T]+1)) & \text{if } x\in Q_{Tz},\ z\in I_S,\\ w(x_n) & \text{otherwise.}\end{cases}$$

We can proceed as in the proof of [10, proposition 14.4]. Plugging u_S into the definition of $g(S)$, we obtain the inequality

$$g(S)\leqslant g(T)+r(S,T),$$

with

$$\limsup_{T\to+\infty}\limsup_{S\to+\infty} r(S,T)=0,$$

so that

$$\limsup_{S\to+\infty} g(S)\leqslant \liminf_{T\to+\infty} g(T).$$

Hence we conclude the proof of the case $\delta(\varepsilon)=\varepsilon$.

If $\varepsilon=\delta c$, by a change of variables, we can apply the previous case.

Finally, if $\lim_{\varepsilon\to0}(\varepsilon/\delta(\varepsilon))=c$, then, by the change of variables $(\varepsilon/\delta(\varepsilon))x=cy$, it can be easily checked that

$$\begin{aligned}\varepsilon^{n-1}&\int_{(1/\varepsilon)Q^\nu}\left(W(u)+f\left(\frac{\varepsilon}{\delta}x,Du\right)\right)\mathrm{d}x\\ &=\frac{1}{\varepsilon T}\frac{1}{T^{n-1}}\int_{TQ^\nu}\left(W\left(u\left(\frac{c\delta(\varepsilon)}{\varepsilon}y\right)\right)+\frac{1}{(\varepsilon T)^2}f\left(cy,Du\left(\frac{c\delta(\varepsilon)}{\varepsilon}y\right)\right)\right)\mathrm{d}y, \quad (4.8)\end{aligned}$$

where $T=1/\delta c$ and $\lim_{\varepsilon\to0}1/\varepsilon T=1$. Hence, for every $\eta>0$, there exists $\varepsilon_0>0$ such that, for every $\varepsilon<\varepsilon_0$,

$$\begin{aligned}(1-\eta)&\inf\left\{\frac{1}{T^{n-1}}\int_{TQ^\nu}(W(u)+(1-\eta)^2f(cy,Du))\,\mathrm{d}y : u=v^\nu \text{ on } \partial TQ^\nu\right\}\\ &\leqslant \inf\left\{\frac{1}{\varepsilon T}\frac{1}{T^{n-1}}\int_{TQ^\nu}\left(W(u)+\frac{1}{(\varepsilon T)^2}f(cy,Du)\right)\mathrm{d}y : u=v^\nu \text{ on } \partial TQ^\nu\right\}\\ &\leqslant (1+\eta)\inf\left\{\frac{1}{T^{n-1}}\int_{TQ^\nu}(W(u)+(1+\eta)^2f(cy,Du))\,\mathrm{d}y : u=v^\nu \text{ on } \partial TQ^\nu\right\}.\end{aligned} \quad (4.9)$$

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By the previous case $\varepsilon = \delta c$, we can conclude that, for every sequence ε_j converging to 0, there exists the limit

$$\lim_{j\to+\infty} \inf\left\{\varepsilon_j^{n-1}\int_{(1/\varepsilon_j)Q^\nu}\left(W(u)+f\left(\frac{\varepsilon_j}{\delta(\varepsilon_j)}x, Du\right)\right)\mathrm{d}x : u = v^\nu \text{ on } \partial\left(\frac{1}{\varepsilon_j}Q^\nu\right)\right\}$$
$$= \lim_{T\to+\infty}\inf\left\{\frac{1}{T^{n-1}}\int_{TQ^\nu}(W(u)+f(cy,Du))\,\mathrm{d}y : u = v^\nu \text{ on } \partial TQ^\nu\right\}. \tag{4.10}$$

Hence, by remark 4.2, φ is independent from ε_j for every ν and satisfies (4.7). □

4.2. Oscillations on a larger scale than the transition layer

In this section we treat the case when the scale of oscillation δ is much larger that the scale of the transition layer ε.

THEOREM 4.4. *Let F_ε be defined by (4.4). Let $f(x,\xi)$ be a continuous function, 1-periodic in x, positively homogeneous of degree two and locally Lipschitz in ξ, satisfying the growth conditions (4.1), and let $W(z)$ be a continuous function satisfying conditions (2.1) and (2.2). Let $\delta : (0,+\infty)\to(0,+\infty)$ be such that*

$$\lim_{\varepsilon\to 0}\delta(\varepsilon) = 0, \qquad \lim_{\varepsilon\to 0}\frac{\varepsilon}{\delta(\varepsilon)} = 0.$$

Then there exists the Γ-limit

$$\Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(u,U) = \int_{S(u)\cap U}\varphi(\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}$$

for every Lipschitz set $U\in\mathcal{A}$ and every $u\in BV(U;\{0,1\})$, where

$$\varphi(\nu) = c_0\inf_{T>0}\inf\left\{\frac{1}{T^{n-1}}\int_{\overline{TQ^\nu}\cap S(u)}\sqrt{f(x,\nu_u)}\,\mathrm{d}\mathcal{H}^{n-1} : u\in BV(\Omega;\{0,1\})u = u^\nu \textit{ on } \mathbb{R}^n\setminus TQ^\nu\right\}.$$

Proof. We recall that

$$c_0 = 2\int_0^1\sqrt{W(z)}\,\mathrm{d}z = \min\left\{\int_{-\infty}^{+\infty}(W(v)+|v'|^2)\,\mathrm{d}t : v(-\infty)=0, v(+\infty)=1\right\} \tag{4.11}$$

(see, for example, [1,6]) and we denote

$$\psi_{\mathrm{hom}}(\nu) = \inf_{T>0}\inf\left\{\frac{1}{T^{n-1}}\int_{\overline{TQ^\nu}\cap S(u)}\psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} : u\in BV(\Omega;\{0,1\})u = u^\nu \text{ on } \mathbb{R}^n\setminus TQ^\nu\right\}, \tag{4.12}$$

where $\psi(x,\xi) = \sqrt{f(x,\xi)}$.

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STEP 1 (Γ-liminf inequality). Let $u \in BV(\Omega; \{0,1\})$ and let u_ε be a sequence converging to u in $L^1(\Omega)$. We can always assume that $u_\varepsilon \in H^1(\Omega, [0,1])$. With fixed $N \in \mathbb{N}$, we divide $[0,1]$ in intervals of length $1/N$. If we define

$$I_k = \left\{ x \in \Omega : \frac{k-1}{N} \leqslant u_\varepsilon \leqslant \frac{k}{N} \right\} \quad \text{and} \quad u_\varepsilon^k = \left(u_\varepsilon \vee \frac{k-1}{N} \right) \wedge \frac{k}{N}$$

for $k = 1, \dots, N$, then u_ε^k converges to

$$u^k = \left(u \vee \frac{k-1}{N} \right) \wedge \frac{k}{N} = \frac{k-1}{N} + \frac{u}{N}$$

and

$$\begin{aligned} F_{\varepsilon,\delta(\varepsilon)}(u_\varepsilon, \Omega) &\geqslant 2 \int_\Omega \sqrt{W(u_\varepsilon) f\left(\frac{x}{\delta(\varepsilon)}, Du_\varepsilon\right)}\,\mathrm{d}x \\ &= 2 \sum_{k=1}^N \int_{I_k} \sqrt{W(u_\varepsilon) f\left(\frac{x}{\delta(\varepsilon)}, Du_\varepsilon\right)}\,\mathrm{d}x \\ &= 2 \sum_{k=1}^N \int_\Omega \sqrt{W(u_\varepsilon^k) f\left(\frac{x}{\delta(\varepsilon)}, Du_\varepsilon^k\right)}\,\mathrm{d}x \\ &\geqslant 2 \sum_{k=1}^N \min_{z \in [(k-1)/N, k/N]} \sqrt{W(z)} \int_\Omega \sqrt{f\left(\frac{x}{\delta(\varepsilon)}, Du_\varepsilon^k\right)}\,\mathrm{d}x. \end{aligned} \tag{4.13}$$

By [8, theorem 5.1], we have that the Γ-limit as $\eta \to 0$ of the functionals

$$u \mapsto \int_\Omega \sqrt{f\left(\frac{x}{\eta}, Du\right)}\,\mathrm{d}x$$

takes the value

$$\int_{S(u)\cap\Omega} \psi_{\mathrm{hom}}((u^+ - u^-)\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}$$

if $u = u^k$. Then, since ψ_{hom} is a positively one-homogeneous function, we get that

$$\begin{aligned} \liminf_{\varepsilon\to 0} \int_\Omega \sqrt{f\left(\frac{x}{\delta(\varepsilon)}, Du_\varepsilon^k\right)}\,\mathrm{d}x &\geqslant \int_{S(u^k)\cap\Omega} \psi_{\mathrm{hom}}((u^{k+} - u^{k-})\nu_{u^k})\,\mathrm{d}\mathcal{H}^{n-1} \\ &= \frac{1}{N} \int_{S(u)\cap\Omega} \psi_{\mathrm{hom}}(\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}, \end{aligned} \tag{4.14}$$

so that

$$\liminf_{\varepsilon\to 0} F_{\varepsilon,\delta(\varepsilon)}(u_\varepsilon, \Omega) \geqslant \sum_{k=1}^N \frac{2}{N} \min_{z \in [(k-1)/N, k/N]} \sqrt{W(z)} \int_{S(u)\cap\Omega} \psi_{\mathrm{hom}}(\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}$$

and, passing to the limit as N tends to $+\infty$, we get

$$\liminf_{\varepsilon\to 0} F_{\varepsilon,\delta(\varepsilon)}(u_\varepsilon, \Omega) \geqslant 2 \int_0^1 \sqrt{W(z)}\,\mathrm{d}z \int_{S(u)\cap\Omega} \psi_{\mathrm{hom}}(\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}$$

(we have used the Riemann integrability of $\sqrt{W}$).

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STEP 2 (Γ-limsup inequality). We can consider the case $\nu = e_n$. By (4.12), if we fix $\eta > 0$, there exist $k > 0$ and $\bar{u} \in BV(kQ; \{0,1\})$ such that $\bar{u} = u^{e_n}$ on $\mathbb{R}^n \setminus kQ$ and

$$\frac{1}{k^{n-1}} \int_{S(\bar{u}) \cap \overline{kQ}} \psi(x, \nu_{\bar{u}}) \, \mathrm{d}\mathcal{H}^{n-1} \leqslant \psi_{\mathrm{hom}}(e_n) + \eta. \tag{4.15}$$

We extend by periodicity $\bar{u}$ so that it is k-periodic in $(x_1, \dots, x_{n-1})$ and $\bar{u} = u^{e_n}$ when $|x_n| > \frac{1}{2}k$.

Let v be such that

$$\int_{-\infty}^{+\infty} (W(v) + |v'|^2) \, \mathrm{d}t = \min\biggl\{ \int_{-\infty}^{+\infty} (W(v) + |v'|^2) \, \mathrm{d}t : v(-\infty) = 0, v(+\infty) = 1 \biggr\}.$$

If we define

$$v^\eta = 0 \vee (((1 + 2\eta)v - \eta) \wedge 1),$$

then there exists R such that $v^\eta(t) \in \{0, 1\}$ if $|t| > R$, and

$$\int_{-\infty}^{+\infty} (W(v^\eta) + |Dv^\eta|^2) \, \mathrm{d}t \to c_0 \quad \text{as } \eta \to 0. \tag{4.16}$$

We can always assume that $\bar{u}$ is such that $S(\bar{u})$ is of class C^2. Hence, for $\alpha > 0$ small enough, there exists a unique projection of class C^2,

$$p : \{x \in \Omega : \operatorname{dist}(x, S(\bar{u})) < \alpha\} \to S(\bar{u}).$$

We set

$$\bar{\nu}(x) = \begin{cases} \nu(p(x)) & \text{if } \operatorname{dist}(x, S(\bar{u})) < \alpha, \\ e_n & \text{otherwise} \end{cases}$$

and

$$d(x) = \begin{cases} \operatorname{dist}(x, \{u = 0\}) & \text{if } u(x) = 1, \\ -\operatorname{dist}(x, \{u = 1\}) & \text{if } u(x) = 0. \end{cases}$$

We define

$$\bar{u}_{\varepsilon,\delta}(x) = v^\eta \biggl(\frac{\delta d(x)}{\varepsilon \psi(x, \bar{\nu}(x))} \biggr)$$

and

$$u_\varepsilon(x) = \bar{u}_{\varepsilon,\delta}\biggl(\frac{x}{\delta}\biggr) = v^\eta(d_\varepsilon(x)),$$

where

$$d_\varepsilon(x) = \frac{\delta d(x/\delta)}{\varepsilon \psi(x/\delta, \bar{\nu}(x/\delta))}.$$

Hence

$$Du_\varepsilon(x) = Dv^\eta \biggl(\frac{\delta Dd(x/\delta)}{\varepsilon \psi(x/\delta, \bar{\nu}(x/\delta))} - \frac{\delta d(x/\delta) D\psi(x/\delta, \bar{\nu}(x/\delta))}{\varepsilon \psi(x/\delta, \bar{\nu}(x/\delta))^2} \biggr),$$

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but $Dv^\eta \neq 0$ on $D = \{x \in \Omega : |d(x/\delta)| \leqslant R(\varepsilon/\delta)\sqrt{c_1}\}$ and $D(\delta d(x/\delta)) = \bar\nu(x/\delta)$, so that

$$\begin{aligned} F_{\varepsilon,\delta(\varepsilon)}&(u_\varepsilon, Q) \\ &= \int_{D\cap Q}\biggl(\frac{1}{\varepsilon}W(u_\varepsilon) + \varepsilon f\biggl(\frac{x}{\delta}, Dv^\eta\biggl(\frac{\bar\nu(x/\delta)}{\varepsilon\psi(x/\delta,\bar\nu(x/\delta))} \\ &\qquad\qquad - \frac{\delta d(x/\delta)D\psi(x/\delta,\bar\nu(x/\delta))}{\varepsilon\psi(x/\delta,\bar\nu(x/\delta))^2}\biggr)\biggr)\biggr)\,\mathrm{d}x \\ &= \frac{1}{\varepsilon}\int_{D\cap Q}\biggl(W(u_\varepsilon) + \biggl(\frac{Dv^\eta}{\psi(x/\delta,\bar\nu(x/\delta))} - \frac{\delta d(x/\delta)D\psi(x/\delta,\bar\nu(x/\delta))}{\bar\nu(x/\delta)\psi(x/\delta,\bar\nu(x/\delta))^2}\biggr)^2 \\ &\qquad\qquad \times f\biggl(\frac{x}{\delta},\bar\nu\biggl(\frac{x}{\delta}\biggr)\biggr)\biggr)\,\mathrm{d}x \\ &= \frac{1}{\varepsilon}\int_{D\cap Q}\biggl(W(u_\varepsilon) + \biggl(Dv^\eta - \frac{\delta d(x/\delta)D\psi(x/\delta,\bar\nu(x/\delta))}{\bar\nu(x/\delta)\psi(x/\delta,\bar\nu(x/\delta))}\biggr)^2\biggr)\,\mathrm{d}x. \end{aligned} \tag{4.17}$$

If we set $x = y + t\nu(y)$ with $t = \delta d(x/\delta)$ and $y \in S(\bar u)$, then $\bar\nu(x/\delta) = \nu(y/\delta)$ and, by (4.17) and the co-area formula, using the fact that $|D(\delta d(x/\delta))| = 1$, we get

$$\begin{aligned} F_{\varepsilon,\delta(\varepsilon)}&(u_\varepsilon, Q) \\ &\leqslant \frac{1}{\varepsilon}\int_{-\varepsilon R\sqrt{c_1}}^{\varepsilon R\sqrt{c_1}}\int_{S(\bar u)\cap Q}\biggl(W\biggl(v^\eta\biggl(\frac{t}{\varepsilon\psi((y+t\nu(y))/\delta,\nu(y/\delta))}\biggr)\biggr) \\ &\qquad + \biggl|Dv^\eta\biggl(\frac{t}{\varepsilon\psi((y+t\nu(y))/\delta,\nu(y/\delta))}\biggr) \\ &\qquad - \frac{tD\psi((y+t\nu(y))/\delta,\nu(y/\delta))}{\nu(y/\delta)\psi((y+t\nu(y))/\delta,\nu(y/\delta))}\biggr|^2\biggr)\,\mathrm{d}\mathcal{H}^{n-1}(y)\,\mathrm{d}t \\ &= \int_{-R\sqrt{c_1}}^{R\sqrt{c_1}}\int_{S(\bar u)\cap Q}\biggl(W\biggl(v^\eta\biggl(\frac{s}{\psi((y+\varepsilon s\nu(y))/\delta,\nu(y/\delta))}\biggr)\biggr) \\ &\qquad + \biggl|Dv^\eta\biggl(\frac{s}{\psi((y+\varepsilon s\nu(y))/\delta,\nu(y/\delta))}\biggr) \\ &\qquad - \frac{\varepsilon s}{\nu(y/\delta)}\frac{D\psi((y+\varepsilon s\nu(y))/\delta,\nu(y/\delta))}{\psi((y+t\nu(y))/\delta,\nu(y/\delta))}\biggr|^2\biggr)\,\mathrm{d}\mathcal{H}^{n-1}(y)\,\mathrm{d}s. \end{aligned} \tag{4.18}$$

Since ψ is a Lipschitz function, by (4.18), we get

$$\begin{aligned} F_{\varepsilon,\delta(\varepsilon)}&(u_\varepsilon, Q) \\ &\leqslant \int_{-R\sqrt{c_1}}^{R\sqrt{c_1}}\int_{S(\bar u)\cap Q}\biggl(W\biggl(v^\eta\biggl(\frac{s}{\psi}\biggr)\biggr) + \biggl|Dv^\eta\biggl(\frac{s}{\psi}\biggr)\biggr|^2 + (\varepsilon R)^2 c_1\biggl|\frac{D\psi}{\psi}\biggr|^2\biggr)\,\mathrm{d}\mathcal{H}^{n-1}(y)\,\mathrm{d}s \\ &\qquad + \varepsilon c_2 \\ &\leqslant \int_{-R\sqrt{c}_1}^{R\sqrt{c}_1}\int_{S(\bar u)\cap Q}\biggl(W\biggl(v^\eta\biggl(\frac{s}{\psi}\biggr)\biggr) + \biggl|Dv^\eta\biggl(\frac{s}{\psi}\biggr)\biggr|^2\biggr)\,\mathrm{d}\mathcal{H}^{n-1}(y)\,\mathrm{d}s + \varepsilon\tilde c. \end{aligned} \tag{4.19}$$

By the change of variable

$$t = \frac{s}{\psi((y + \varepsilon s\nu(y))/\delta, \nu(y/\delta))},$$

we obtain

$$\begin{aligned} &F_{\varepsilon,\delta(\varepsilon)}(u_\varepsilon, Q) \\ &\quad \leqslant \int_{S(\bar u)\cap Q} \int_{-\infty}^{+\infty} (W(v^\eta(t)) + |Dv^\eta(t)|^2)\,\mathrm{d}t \\ &\qquad\qquad\qquad \times \frac{\psi(y/\delta, \nu(y/\delta)) + O(\varepsilon/\delta)}{1 + O(\varepsilon/\delta)}\,\mathrm{d}\mathcal{H}^{n-1}(y) + \varepsilon\tilde c \\ &\quad = \int_{-\infty}^{+\infty} (W(v^\eta(t)) + |Dv^\eta(t)|^2)\,\mathrm{d}t \\ &\qquad\qquad \times \biggl(\frac{\delta^{n-1}}{1 + O(\varepsilon/\delta)} \int_{S(\bar u)\cap(1/\delta)Q} \psi(x, \nu(x))\,\mathrm{d}\mathcal{H}^{n-1}(x) \\ &\qquad\qquad\qquad + \frac{O(\varepsilon/\delta)}{1 + O(\varepsilon/\delta)} \delta^{n-1}\mathcal{H}^{n-1}\biggl(S(\bar u) \cap \frac{1}{\delta}Q\biggr)\biggr) + \varepsilon\tilde c. \end{aligned} \tag{4.20}$$

By (4.20), (4.15) and (4.16), we get

$$\limsup_{\varepsilon\to 0} F_{\varepsilon,\delta(\varepsilon)}(u_\varepsilon, Q) \leqslant c_0\psi_{\mathrm{hom}}(e_n),$$

as desired. □

Remark 4.5. Note that the Γ-liminf inequality does not depend on the behaviour of δ with respect to ε and we do not use the assumption of f being locally Lipschitz.

4.3. Oscillations on a finer scale than the transition layer

Finally, in this section we treat the case when the scale of oscillation δ is much smaller that the scale of the transition layer ε.

In order to prove the liminf inequality, we make the following two additional technical hypotheses:

(H1) (*Lipschitz continuity* of W)

$$|W(u) - W(v)| \leqslant C|u - v|$$

if $0 \leqslant u$, $v \leqslant 1$;

(H2) we have

$$\delta \ll \varepsilon\sqrt{\varepsilon}.$$

These hypotheses will be used in the proof of proposition 4.10 only, and will not be needed for the limsup inequality.

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THEOREM 4.6. *Let F_ε be defined by (4.4). Let $f(x,\xi)$ be a Borel function, 1-periodic in x, positively homogeneous of degree two in ξ and satisfying the growth conditions (4.1), and let W be a continuous function satisfying conditions (H1), (2.1) and (2.2). Let $\delta:(0,+\infty)\to(0,+\infty)$ be such that*

$$\lim_{\varepsilon\to 0}\frac{\varepsilon\sqrt{\varepsilon}}{\delta(\varepsilon)}=+\infty.$$

Then there exists the Γ-limit

$$\Gamma\text{-}\lim_{\varepsilon\to 0}F_\varepsilon(u,U)=\int_{S(u)\cap U}\varphi(\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}$$

for every Lipschitz set $U\in\mathcal{A}$ and every $u\in BV(U;\{0,1\})$, where

$$\varphi(\nu)=c_0\sqrt{f_{\mathrm{hom}}(\nu)}$$

and f_{hom} is the homogenized integrand *of f defined by*

$$f_{\mathrm{hom}}(\xi)=\inf\biggl\{\int_{(0,1)^n}f(y,Du+\xi)\,\mathrm{d}y:u\in H^1_{\mathrm{loc}}(\mathbb{R}^n),u\ 1\text{-}periodic\biggr\}$$

for all $\xi\in\mathbb{R}^n$.

The proof of the theorem will be obtained from the results in the rest of the section.

The liminf inequality will be proved if we show that, for every sequence (u_ε) such that

$$\sup_\varepsilon F_\varepsilon(u_\varepsilon)<+\infty,\quad u_\varepsilon\to u,$$

and for every $\eta>0$, there exists a sequence (u_ε^η) converging to u such that

$$\liminf_{\varepsilon\to 0}F_\varepsilon(u_\varepsilon)\geqslant\liminf_{\varepsilon\to 0}\int_\Omega\biggl(\frac{W(u_\varepsilon^\eta)}{\varepsilon}+\varepsilon f_{\mathrm{hom}}(Du_\varepsilon^\eta)\biggr)\,\mathrm{d}x-\eta C.\tag{4.21}$$

The conclusion will then follow, since we already know the Γ-limit of the functionals on the right-hand side of (4.21) (see proposition 4.11 below). Such (u_ε^η) will be obtained from (u_ε) by averaging on a intermediate scale between δ and ε. Before defining such functions, we prove a preliminary proposition.

PROPOSITION 4.7. *Let U be a connected bounded open set. For every $\eta>0$, there exists $K\in\mathbb{N}$ such that, for all $u\in H^1(U)$ and for all $h\geqslant K$, $h\in\mathbb{N}$, we have*

$$⨍_U f(hx,Du)\,\mathrm{d}x\geqslant f_{\mathrm{hom}}\biggl(⨍_U Du\,\mathrm{d}x\biggr)-\eta\biggl|⨍_U Du\,\mathrm{d}x\biggr|^2.$$

Proof. Suppose, by contradiction, that $\eta>0$, (h_k) an increasing sequence of integers and functions $u_k\in H^1(U)$ exist such that

$$\biggl|⨍_U Du_k\,\mathrm{d}x\biggr|=1$$

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and

$$\fint_U f(h_k x, Du_k)\,\mathrm{d}x < f_{\mathrm{hom}}\Big(\fint_U Du_k\Big) - \eta$$

(we use a scaling argument by positive homogeneity). Upon a translation argument and a passage to a subsequence, we may assume that $u_k \rightharpoonup \bar{u}$ in $H^1(U)$. In particular, we have

$$\fint_U Du_k\,\mathrm{d}x \to \fint_U D\bar{u}\,\mathrm{d}x,$$

and hence

$$\left|\fint_U D\bar{u}\,\mathrm{d}x\right| = 1,$$

from which we obtain, by the classical homogenization theorem (see, for example, [10, § 14]) and Jensen's inequality,

$$\liminf_k \fint_U f(h_k x, Du_k)\,\mathrm{d}x \geqslant \fint_U f_{\mathrm{hom}}(D\bar{u})\,\mathrm{d}x \geqslant f_{\mathrm{hom}}\Big(\fint_U D\bar{u}\,\mathrm{d}x\Big)$$

and a contradiction easily follows. □

Note preliminarily that, by the compactness and representation theorem, we may limit our analysis in (4.21) to the case $u = u^\nu$ with $\nu = e_n$, $\Omega = Q = (-\frac{1}{2}, \frac{1}{2})^n$. Moreover, by applying, if necessary, a truncation argument, we may suppose that $u_\varepsilon \in L^\infty(\mathbb{R}^n) \cap H^1_{\mathrm{loc}}(\mathbb{R}^n)$, $0 \leqslant u_\varepsilon \leqslant 1$, and that

$$u_\varepsilon(x) = w(x_n/\varepsilon)$$

on $\mathbb{R}^n \setminus Q$ by theorem 3.7.

With fixed $\eta > 0$, let K be given by proposition 4.7. We define

$$u^\eta_\varepsilon(x) = \fint_{Q(x,K\delta)} u_\varepsilon(y)\,\mathrm{d}y.$$

Note that $u^\eta_\varepsilon \in C^\infty(\mathbb{R}^n)$ and that

$$Du^\eta_\varepsilon(x) = \fint_{Q(x,K\delta)} Du_\varepsilon(y)\,\mathrm{d}y.$$

For each $\eta > 0$, we then have $u^\eta_\varepsilon \to u$ in $H^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Proposition 4.8. *Let $\varphi \in C^0(\mathbb{R}^n)$ and let η, K and u^η_ε be as above. Then there exists $y \in Q(0, K\delta)$ such that, if we set*

$$x^K_i = y + iK\delta, \qquad Q^K_i = Q(x^K_i, K\delta)$$

and

$$I^\delta_K = \{i \in \mathbb{Z}^n : Q^K_i \cap Q \neq \emptyset\},$$

we have

$$\int_Q \varphi(Du^\eta_\varepsilon)\,\mathrm{d}x \leqslant \sum_{i \in I^\delta_K} (K\delta)^n \varphi(Du^\eta_\varepsilon(x^K_i)).$$

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Proof. The thesis follows immediately from the mean-value theorem, upon remarking that

$$\int_Q \varphi(Du_\varepsilon^\eta)\,\mathrm{d}x \leqslant \int_{Q(0,K\delta)} \sum_{i\in I_\delta^K} \varphi(Du_\varepsilon^\eta(z+iK\delta))\,\mathrm{d}z.$$

□

PROPOSITION 4.9. *Let* (u_ε) *and* (u_ε^η) *be as above. Then we have*

$$\liminf_{\varepsilon\to 0} \varepsilon \int_Q f\left(\frac{x}{\delta}, Du_\varepsilon\right)\mathrm{d}x \geqslant \liminf_{\varepsilon\to 0} \varepsilon \int_Q f_{\mathrm{hom}}(Du_\varepsilon^\eta)\,\mathrm{d}x - c\eta.$$

Proof. Let y be given by proposition 4.8, with $\varphi(\xi) = f_{\mathrm{hom}}(\xi) - \eta|\xi|^2$. Then we have, using both propositions above (in addition to proposition 4.7, we have to use a change of variable and the positive homogeneity of f),

$$\begin{aligned}\varepsilon \int_Q f\left(\frac{x}{\delta}, Du_\varepsilon\right)\mathrm{d}x + O\left(\frac{K\delta}{\varepsilon}\right) &= \varepsilon \sum_{i\in I_\delta^K} \int_{Q_i^K} f\left(\frac{x}{\delta}, Du_\varepsilon\right)\mathrm{d}x \\ &\geqslant \varepsilon \sum_{i\in I_\delta^K} (K\delta)^n (f_{\mathrm{hom}}(Du_\varepsilon^\eta(x_i^K)) - \eta|Du_\varepsilon^\eta(x_i^K)|^2) \\ &\geqslant \varepsilon \int_Q (f_{\mathrm{hom}}(Du_\varepsilon^\eta) - \eta|Du_\varepsilon^\eta|^2)\,\mathrm{d}x.\end{aligned}$$

The thesis follows by remarking that, by the growth conditions on W and the definition of u_ε^η, we have

$$\sup_\varepsilon \varepsilon \int_Q |Du_\varepsilon^\eta|^2\,\mathrm{d}x \leqslant c < +\infty,$$

with c independent of $\eta \in (0,1)$. □

Note that we have not yet used hypotheses (H1) and (H2).

PROPOSITION 4.10. *Let* (u_ε) *and* (u_ε^η) *be as above. Then we have*

$$\liminf_{\varepsilon\to 0} \int_Q \frac{W(u_\varepsilon)}{\varepsilon}\,\mathrm{d}x \geqslant \liminf_{\varepsilon\to 0} \int_Q \frac{W(u_\varepsilon^\eta)}{\varepsilon}\,\mathrm{d}x$$

Proof. By Poincaré's inequality applied in each $Q(x, K\delta)$ and the Lipschitz continuity of translations in Sobolev spaces (see, for example, [23, theorem 2.1.6]), setting $x_i = K\delta i$,

$$Q_i = Q(x_i, K\delta) \quad \text{and} \quad I = \{i \in \mathbb{Z}^n : Q_i \cap Q \neq \emptyset\},$$

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we have

$$\int_Q |u_\varepsilon(x) - u_\varepsilon^\eta(x)|\,\mathrm{d}x \leqslant \sum_{i\in I}\int_{Q_i}\Big|u_\varepsilon(x) - \fint_{Q_i} u_\varepsilon(y)\,\mathrm{d}y\Big|\,\mathrm{d}x + \sum_{i\in I}\int_{Q_i}\fint_{Q_i}|u_\varepsilon(y+(x-x_i)) - u_\varepsilon(y)|\,\mathrm{d}y\mathrm{d}x \leqslant cK\delta\Big(\int_Q |Du_\varepsilon|^2\,\mathrm{d}x\Big)^{1/2}.$$

Hence, by (H1), we get

$$\begin{aligned}\int_Q W(u_\varepsilon)\,\mathrm{d}x &\geqslant \int_Q W(u_\varepsilon^\eta)\,\mathrm{d}x - C\int_Q |u_\varepsilon - u_\varepsilon^\eta|\,\mathrm{d}x \\ &\geqslant \int_Q W(u_\varepsilon^\eta)\,\mathrm{d}x - cK\delta\Big(\int_Q |Du_\varepsilon|^2\,\mathrm{d}x\Big)^{1/2} \\ &= \int_Q W(u_\varepsilon^\eta)\,\mathrm{d}x - \varepsilon cK\frac{\delta}{\varepsilon\sqrt{\varepsilon}}\Big(\varepsilon\int_Q |Du_\varepsilon|^2\,\mathrm{d}x\Big)^{1/2},\end{aligned}$$

and the thesis follows by (H2). □

The Γ-liminf inequality reads as follows.

PROPOSITION 4.11. *For all $u \in BV(U;\{0,1\})$, we have*

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon(u,U) \geqslant \int_{S(u)\cap U}\sqrt{f_{\mathrm{hom}}(\nu_u)}\,\mathrm{d}\mathcal{H}^{n-1}.$$

Proof. It suffices to use the two previous propositions and recall that the Γ-limit of the functionals

$$u \mapsto \int_\Omega \Big(\frac{W(u)}{\varepsilon} + \varepsilon f_{\mathrm{hom}}(Du)\Big)\,\mathrm{d}x$$

is given by

$$\int_{S(u)\cap\Omega}\sqrt{f_{\mathrm{hom}}(\nu_u)}\,\mathrm{d}\mathcal{H}^{n-1}$$

on $BV(\Omega;\{0,1\})$ (see [6, § 4.2]). □

It remains to prove the Γ-limsup inequality, which completes the proof of theorem 4.6.

PROPOSITION 4.12. *For all $u \in BV(U;\{0,1\})$, we have*

$$\Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon(u,U) \leqslant \int_{S(u)\cap U}\sqrt{f_{\mathrm{hom}}(\nu_u)}\,\mathrm{d}\mathcal{H}^{n-1}.$$

Proof. We want to prove that there exists a sequence u_ε converging to u^ν such that

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, Q^\nu) \leqslant c_0\sqrt{f_{\mathrm{hom}}(\nu)}.$$

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By (4.11),

$$c_0\sqrt{f_{\hom}(\nu)} = \min\left\{\int_{-\infty}^{+\infty}(W(v)+f_{\hom}(\nu)|v'|^2)\,\mathrm{d}t : v(-\infty)=0, v(+\infty)=1\right\}$$
$$= \inf_{T\geqslant 0}\inf\left\{\int_{-T}^{T}(W(v)+f_{\hom}(\nu)|v'|^2)\,\mathrm{d}t : \right.$$
$$\left. v(t)=0 \text{ if } t\leqslant -T, v(t)=1 \text{ if } t\geqslant T\right\}. \tag{4.22}$$

Hence, for fixed $\alpha>0$, there exist $T\geqslant 0$ and v_T such that

$$\int_{-T}^{T}(W(v_T)+f_{\hom}(\nu)|v_T'|^2)\,\mathrm{d}t \leqslant c_0\sqrt{f_{\hom}(\nu)}+\alpha. \tag{4.23}$$

We define

$$c_T=\int_{-T}^{T}(W(v_T)+f_{\hom}(\nu)|v_T'|^2)\,\mathrm{d}t$$

and $u^T(x)=v_T(\langle x,\nu\rangle)$. Then there exists a sequence u_η, converging to u^T, such that $u_\eta=u^T$ on $\partial(Q^\nu_{n-1}\times(-T,T))$ and

$$c_T=\int_{Q^\nu_{n-1}\times(-T,T)}(W(u^T)+f_{\hom}(Du^T))\,\mathrm{d}x$$
$$=\lim_{\eta\to 0}\int_{Q^\nu_{n-1}\times(-T,T)}\left(W(u_\eta)+f\left(\frac{x}{\eta},Du_\eta\right)\right)\mathrm{d}x. \tag{4.24}$$

Let $\eta=\delta(\varepsilon)/\varepsilon$. We define a sequence u_ε on $([\varepsilon/\delta]+1)\delta Q^\nu_{n-1}\times\mathbb{R}$ as follows,

$$u_\varepsilon(x)=\begin{cases} u_\eta(x/\varepsilon) & \text{if } x\in\varepsilon Q^\nu_{n-1}\times(-\varepsilon T,\varepsilon T),\\ u^T(x/\varepsilon) & \text{if } x\in(([\varepsilon/\delta]+1)\delta Q^\nu_{n-1}\setminus\varepsilon Q^\nu_{n-1})\times(-\varepsilon T,\varepsilon T),\\ 1 & \text{if } x_n\geqslant\varepsilon T,\\ 0 & \text{if } x_n\leqslant-\varepsilon T,\end{cases} \tag{4.25}$$

and we extend it by periodicity so that u_ε is $([\varepsilon/\delta]+1)\delta$-periodic in the variables $(x_1,\dots,x_{n-1})$. We define

$$I_\varepsilon=\{i\in\mathbb{Z}^{n-1}:\varepsilon Q^\nu_{i,n-1}\times(-\varepsilon T,\varepsilon T)\cap Q^\nu\neq\emptyset\},$$

where

$$Q^\nu_{i,n-1}=i\left(\left[\frac{\varepsilon}{\delta}\right]+1\right)\frac{\delta}{\varepsilon}+Q^\nu_{n-1}$$

and

$$J_\varepsilon=\left\{i\in\mathbb{Z}^{n-1}: i\left(\left[\frac{\varepsilon}{\delta}\right]+1\right)\delta Q^\nu_{n-1}\setminus\varepsilon Q^\nu_{i,n-1}\times(-\varepsilon T,\varepsilon T)\cap Q^\nu\neq\emptyset\right\}.$$

We get

$$
\begin{aligned}
F_{\varepsilon,\delta(\varepsilon)}&(u_\varepsilon, Q^\nu) \\
&= \int_{Q^\nu}\left(\frac{1}{\varepsilon}W(u_\varepsilon) + \varepsilon f\left(\frac{x}{\delta(\varepsilon)}, Du_\varepsilon\right)\right)\mathrm{d}x \\
&= \int_{Q^\nu_{n-1}\times(-\varepsilon T,\varepsilon T)}\left(\frac{1}{\varepsilon}W(u_\varepsilon) + \varepsilon f\left(\frac{x}{\delta(\varepsilon)}, Du_\varepsilon\right)\right)\mathrm{d}x \\
&\leqslant \sum_{i\in I_\varepsilon}\int_{\varepsilon Q^\nu_{i,n-1}\times(-\varepsilon T,\varepsilon T)}\left(\frac{1}{\varepsilon}W(u_\varepsilon) + \varepsilon f\left(\frac{x}{\delta(\varepsilon)}, Du_\varepsilon\right)\right)\mathrm{d}x \\
&\qquad + \sum_{i\in J_\varepsilon}\int_{i([\varepsilon/\delta]+1)\delta Q^\nu_{n-1}\setminus(\varepsilon Q^\nu_{i,n-1}\times(-\varepsilon T,\varepsilon T))}\left(\frac{1}{\varepsilon}W(u_\varepsilon) + \varepsilon f\left(\frac{x}{\delta(\varepsilon)}, Du_\varepsilon\right)\right)\mathrm{d}x
\end{aligned}
\tag{4.26}
$$

and, in particular, by (4.25),

$$
\begin{aligned}
\sum_{i\in J_\varepsilon}&\int_{i([\varepsilon/\delta]+1)\delta Q^\nu_{n-1}\setminus\varepsilon Q^\nu_{i,n-1}\times(-\varepsilon T,\varepsilon T)}\left(\frac{1}{\varepsilon}W(u_\varepsilon) + \varepsilon f\left(\frac{x}{\delta(\varepsilon)}, Du_\varepsilon\right)\right)\mathrm{d}x \\
&= \sum_{i\in J_\varepsilon}\varepsilon^{n-1}\int_{i([\varepsilon/\delta]+1)(\delta/\varepsilon)Q^\nu_{n-1}\setminus Q^\nu_{i,n-1}\times(-T,T)}\left(W(u^T) + f\left(\frac{x}{\eta}, Du^T\right)\right)\mathrm{d}x \\
&\leqslant \sum_{i\in J_\varepsilon}\varepsilon^{n-1}\left(\left(\left[\frac{\varepsilon}{\delta}\right]\frac{\delta}{\varepsilon} + \frac{\delta}{\varepsilon}\right)^{n-1} - 1\right)\int_{-T}^{T}(W(v_T) + c_2|v_T'|^2)\,\mathrm{d}t \\
&\leqslant \sum_{i\in J_\varepsilon}\varepsilon^{n-1}\left(\left(\left[\frac{\varepsilon}{\delta}\right]\frac{\delta}{\varepsilon} + \frac{\delta}{\varepsilon}\right)^{n-1} - 1\right)\left(c_T + c_2\int_{-T}^{T}|v_T'|^2\,\mathrm{d}t\right).
\end{aligned}
\tag{4.27}
$$

Hence, by (4.27), we obtain

$$
\limsup_{\varepsilon\to 0}\sum_{i\in J_\varepsilon}\int_{i([\varepsilon/\delta]+1)\delta Q^\nu_{n-1}\setminus Q^\nu_{i,n-1}\times(-\varepsilon T,\varepsilon T)}\left(\frac{1}{\varepsilon}W(u_\varepsilon) + \varepsilon f\left(\frac{x}{\delta(\varepsilon)}, Du_\varepsilon\right)\right)\mathrm{d}x = 0.
\tag{4.28}
$$

We now estimate the first term in (4.26) as

$$
\begin{aligned}
\sum_{i\in I_\varepsilon}&\int_{\varepsilon Q^\nu_{i,n-1}\times(-\varepsilon T,\varepsilon T)}\left(\frac{1}{\varepsilon}W(u_\varepsilon) + \varepsilon f\left(\frac{x}{\delta(\varepsilon)}, Du_\varepsilon\right)\right)\mathrm{d}x \\
&\leqslant \left(\left[\frac{1}{\varepsilon}\right] + 1\right)^{n-1}\varepsilon^{n-1}\int_{Q^\nu_{n-1}\times(-T,T)}\left(W(u_\eta) + f\left(\frac{x}{\eta}, Du_\eta\right)\right)\mathrm{d}x.
\end{aligned}
\tag{4.29}
$$

Hence, by (4.26), (4.28), (4.29), (4.24) and (4.23), we get

$$
\begin{aligned}
\limsup_{\varepsilon\to 0}F_{\varepsilon,\delta(\varepsilon)}(u_\varepsilon, Q^\nu) &\leqslant \lim_{\eta\to 0}\int_{Q^\nu_{n-1}\times(-T,T)}\left(W(u_\eta) + f\left(\frac{x}{\eta}, Du_\eta\right)\right)\mathrm{d}x \\
&= c_T \leqslant c_0\sqrt{f_{\mathrm{hom}}(\nu)} + \alpha
\end{aligned}
\tag{4.30}
$$

and, by the arbitrariness of α, we obtain the Γ-limsup inequality. □

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