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QUANTUM LOGIC ASSOCIATED TO FINITE DIMENSIONAL INTERVALS OF MODULAR ORTHOLATTICES

R. GIUNTINI, H. FREYTES, AND G. SERGIOLI

Abstract. In this work we study an abstract formulation of a problem posed by J.M. Dunn, T.J. Hagge et al. about the inclusion of varieties generated by the modular ortholattice of subspaces of $\mathbb{C}^n$. We shall prove that, this abstract formulation is equivalent to the direct irreducibility for atomic complete modular ortholattices.

§1. Introduction. In their 1936 seminal paper [1], Birkhoff and von Neumann introduced a suitable model for the logic of quantum mechanics based on the lattice $L(\mathcal{H})$ of all closed subspaces of a Hilbert space $\mathcal{H}$. The lattice $L(\mathcal{H})$, equipped with the orthogonal complement, can be described as an ortholattice. In the case of a finite-dimensional Hilbert space, the ortholattice of its closed subspaces is modular. In this way, they provided the first notion of quantum logic.

However this notion can assume several meanings according to the different authors. In this work we refer to the terminology used in [4] i.e., the *quantum logic associated to a Hilbert space* $\mathcal{H}$, denoted by $\mathcal{QL}(\mathcal{H})$, is identified with the class of all models of the set of true equations in $L(\mathcal{H})$ formulated in the language of ortholattices. In terms of the universal algebra, $\mathcal{QL}(\mathcal{H})$ is the subvariety of ortholattices generated by $L(\mathcal{H})$.

In [4], J.M. Dunn, T.J. Hagge et al. show that, for any $n \geq 0$, $\mathcal{QL}(\mathbb{C}^n)$ is a proper subvariety of $\mathcal{QL}(\mathbb{C}^{2n+1})$ and they raise the question whether this result could be extended to any finite-dimensional complex Hilbert space $\mathbb{C}^n$. In other words:

is $\mathcal{QL}(\mathbb{C}^n)$ *a proper subvariety of* $\mathcal{QL}(\mathbb{C}^m)$ *whenever* $n < m$?

It should be noticed that, an explicit positive solution to this question was given by T.J. Hagge in [5].

The aim of this paper is to study this problem in a general algebraic framework. More precisely, taking into account that the modular ortholattice $L(\mathbb{C}^n)$ can be thought as an interval of $L(\mathbb{C}^m)$ whenever $n \leq m$, the problem posed in [4] can be generalized by studying inclusion relations among varieties generated by finite-dimensional intervals in modular ortholattices. We also see that, this abstract form of the problem is closely related to the direct irreducibility of atomic complete modular ortholattices.

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The paper is organized as follows. In Section 1, we summarize some basic notions about universal algebra and modular ortholattices. In Section 2, we outline some properties concerning the dimension on direct irreducibility atomic complete modular ortholattices. In Section 3, we introduce and study varieties generated by finite-dimensional intervals in modular ortholattices as a generalization of $\mathcal{QL}(\mathbb{C}^m)$. In this framework we reformulate, in an abstract way, the problem posed in [4]. Finally, we prove that it turns out to be equivalent to the direct irreducibility of atomic complete modular ortholattices.

§2. Basic notions. We first recall from [2, 6, 7] some notions about universal algebra and ortholattices that play an important role throughout the paper. A *variety* is a class of algebras of the same type defined by a set of equations.

Let $\mathcal{A}$ be a variety of algebras of type σ. If $A \in \mathcal{A}$, $\mathcal{V}_{\mathcal{A}}(A)$ denotes the *subvariety of $\mathcal{A}$ generated by A* i.e., the smallest subvariety of $\mathcal{A}$ containing A. We denote by $Term_{\mathcal{A}}$ the *absolutely free algebra* of type σ built from the set of variables $V = \{x_1, x_2, \dots\}$. Each element of $Term_{\mathcal{A}}$ is referred to as a *term*. We denote by $Comp(t)$ the complexity of the term t.

Let $A \in \mathcal{A}$. If $t \in Term_{\mathcal{A}}$ and $a_1, \dots, a_n \in A$, by $t^A(a_1, \dots, a_n)$ we denote the result of the application of the term operation t^A to the elements $a_1, \dots, a_n$. A *valuation* in A is a map $v : V \to A$. Of course, any valuation v in A can be uniquely extended to an $\mathcal{A}$-homomorphism $v : Term_{\mathcal{A}} \to A$ in the usual way, i.e., if $t_1, \dots, t_n \in Term_{\mathcal{A}}$ then $v(t(t_1, \dots, t_n)) = t^A(v(t_1), \dots, v(t_n))$. Thus, valuations are identified with $\mathcal{A}$-homomorphisms from the absolutely free algebra. If $t, s \in Term_{\mathcal{A}}$, $A \models t = s$ means that for each valuation v in A, $v(t) = v(s)$ and $\mathcal{A} \models t = s$ means that for each $A \in \mathcal{A}$, $A \models t = s$. An algebra $A \in \mathcal{A}$ is *directly irreducible* iff A is not isomorphic to a direct product of two nontrivial algebras in $\mathcal{A}$.

An *ortholattice* [6] is an algebra $\langle L, \wedge, \vee, ', 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ that satisfies the following conditions:

1. $\langle L, \wedge, \vee, 0, 1\rangle$ is a bounded lattice,
2. $(x')' = x$,
3. $(x \vee y)' = x' \wedge y'$,
4. $x \wedge x' = 0$.

It is not difficult to see that the equation $(x \wedge y)' = x' \vee y'$ holds in any ortholattice. Boolean algebras are distributive ortholattices. More precisely, if $\mathcal{B}$ is the variety of Boolean algebras and $\mathcal{OL}$ is the variety of ortholattices then $\mathcal{B} = \mathcal{OL} + \{x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)\}$.

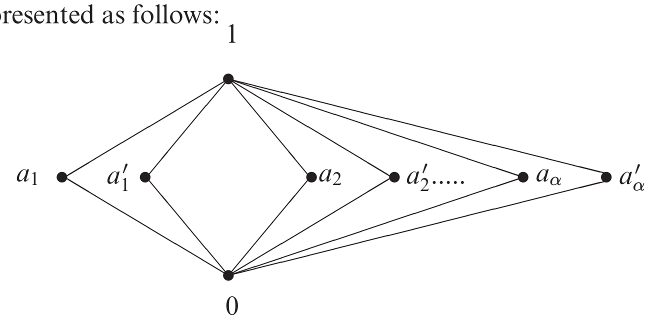
Let L be an ortholattice. If $a, b \in L$, we say that b *covers* a (and we write $a \prec b$) iff, $a < b$ and does not exist $x \in L$ such that $a < x < b$. An element $p \in L$ is called an *atom* of L iff $0 \prec p$. We denote by $\Omega(L)$ the set of all atoms of L. L is said to be *atomic* iff for each $x \in L - \{0\}$, $x = \bigvee\{p \in L : p \leq x, \ \ p \in \Omega(L)\}$. Two atoms p_1, p_2 in $\Omega(L)$ are said to be *strongly perspective* iff there exists $x \in \Omega(L)$ such that $0 < x < p_1 \vee p_2$ and $p_1 \vee x = p_2 \vee x = p_1 \vee p_2$.

A *modular* ortholattice (or *MOL*, for short) is an ortholattice that satisfies the *modular law*, i.e.,

$$x \vee (y \wedge (x \vee z)) = (x \vee y) \wedge (x \vee z). \tag{1}$$

We denote by $\mathcal{MOL}$ the variety of modular ortholattices. Two examples of atomic MOL are the following:

a. if $\mathcal{H}$ is a finite-dimensional Hilbert space then $L(\mathcal{H})$ is an atomic complete MOL. In particular if $\dim(\mathcal{H}) > 1$ then any two atoms in $L(\mathcal{H})$ are strongly perspectives;
b. the lattice MO_α where α is an ordinal number. The Hasse diagram of MO_α can be represented as follows:



For each ordinal number α, MO_α is a complete lattice and if $\alpha > 1$ then any two atoms in MO_α are strongly perspectives.

Note that $MO_0 = \mathbf{2}$ and $MO_1 = \mathbf{2} \times \mathbf{2}$ where $\mathbf{2}$ is the boolean algebra of two elements. Thus,

$$\mathcal{B} = \mathcal{V}_{\mathcal{MOL}}(MO_0) = \mathcal{V}_{\mathcal{MOL}}(MO_1). \tag{2}$$

An important characterization of the equations $t = s$ that hold in $\mathcal{MOL}$ is given by:

$$\mathcal{MOL} \models t = s \quad \textit{iff} \quad \mathcal{MOL} \models (t \wedge s) \vee (t' \wedge s') = 1. \tag{3}$$

Therefore, we can assume, without loss of generality, that all $\mathcal{MOL}$-equations are of the form $t = 1$, where $t \in Term_{\mathcal{MOL}}$. Let L be an atomic MOL. An element $a \in L$ is called *finite* iff, $a = 0$ or there exists $p_1, \ldots, p_n$ atoms in L such that $a = p_1 \vee \cdots \vee p_n$. A finite set of atoms $\{p_1, \ldots, p_n\}$ is a *base* iff $(p_1 \vee \cdots \vee p_{i-1}) \wedge p_i = 0$ for $i = 2, \ldots, n$; in this case, if $a = p_1 \vee \cdots \vee p_n$ then we say that $\{p_1, \ldots, p_n\}$ is a *base of* a. It is well known that if $a \in L - \{0\}$ is a finite element then a admits a base $(p_i)_{1 \leq i \leq n}$, where the number n is uniquely determinated by the element a [7, Lemma 7.6 and Theorem 8.4]. The number n is called the *dimension* of a and it is denoted by $d(a)$. In particular, $d(0) = 0$. If 1 is finite then $d(1)$ is called the *dimension of* L.

PROPOSITION 2.1. *Let L be an atomic MOL and let $a, b \in L$ be finite elements. Then we have*:

1. *If $a < b$ then $d(a) < d(b)$.*
2. *$d(a \vee b) = d(a) + d(b) - d(a \wedge b)$.*
3. *Let S be a base. Then, for any pair of finite subsets F_1, F_2 of S we have that*: $(\bigvee_{x \in F_1} x) \wedge (\bigvee_{x \in F_2} x) = \bigvee \{x \in F_1 \cap F_2\}$.
4. *If 1 is finite then L is complete. Moreover $d(x') = d(1) - d(x)$.*

PROOF. 1) and 2) See [7, Lemma 8.8, Theorem 8.14]. 3) See [7, Lemma 3.3]. 4) By [7, Lemma 8.10] L is a complete lattice. $d(1) = d(x \vee x') = d(x) + d(x') - d(x \wedge x')$, so that $d(x') = d(1) - d(x)$. ⊣

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The following proposition is a lattice theoretical version of the Gram-Schmidt procedure.

Proposition 2.2. *Let L be an atomic MOL and let n, r be natural numbers such that $0 < n < r \leq d(1)$. If $\{e_1, \ldots, e_n\}$ is a base, then there exists $e_{n+1}, \ldots, e_r$ atoms in L such that $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_r\}$ is a base. Therefore, if $a \in L$ and $d(a) = n$, then there exists $a_r \in L$ such that $a_n < a_r$ and $d(a_r) = r$.*

Proof. If e is an atom such that $e \notin \{e_1, \ldots, e_n\}$ and $e \wedge \bigvee_{1 \leq i \leq n} e_i \neq 0$ then $e \leq \bigvee_{1 \leq i \leq n} e_i$, since $0 \prec e$. Consequently, there exists an atom e_{n+1} such that $e_{n+1} \wedge \bigvee_{1 \leq i \leq n} e_i = 0$ (otherwise $d(1) = n$ and this is a contradiction). Thus, we can extend $\{e_1, \ldots, e_n\}$ to a base $\{e_1, \ldots, e_n, e_{n+1}\}$. Finally, in $r - (n+1)$-steps we obtain a base of r atoms $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_r\}$. ⊣

Let L be a MOL and let $a \in L$. The *commutator* of L is the map $k : L \times L \to L$ such that for any $x, y \in L$:

$$k(x, y) = (x \vee y) \wedge (x \vee y') \wedge (x' \vee y) \wedge (x' \vee y').$$

Since $'$ is an involution in $\mathcal{MOL}$, it is clear that $k(x, y) = k(x', y) = k(x, y') = k(x', y')$. It is not very hard to see that a MOL L is a Boolean algebra iff for any $x, y \in L$, $k(x, y) = 0$.

Remark 2.3. For the sake of simplicity, the set $Term_{\mathcal{MOL}}$ will be denoted by $Term$, and $k(x, y \mid z)$ will be used in place of $k(k(x, y), z)$.

§3. Dimension on directly irreducible atomic complete *MOL*s. Let L be a MOL. A reflexive and symmetric binary relation can be defined on L. This is the *compatibility relation* referred as *a is compatible with b in L* iff $a = (a \wedge b) \vee (a \wedge b')$. An element $a \in L$ is called *central* iff it is compatible with any $x \in L$. The set of all central elements of L is said to be the *center of L* and denoted by $Z(L)$. In [7, Theorem 4.15] it is proved that $Z(L)$ is a Boolean sub algebra of L. Direct irreducibility in $\mathcal{MOL}$ is closely related to $Z(L)$. In fact, L is directly irreducible MOL iff $Z(L) = \{0, 1\}$. In [3] it is shown that the direct irreducibility of an atomic complete MOL can be equivalently characterized as follows:

Proposition 3.1. *Let L be an atomic, complete MOL such that $L \neq MO_1$. Then L is directly irreducible iff for each pair of distinct atoms $p_1, p_2 \in L$ there exists an atom e in $L - \{p_1, p_2\}$ such that $e \leq p_1 \vee p_2$.* ⊣

The modular ortholattices $L(\mathcal{H})$, with $1 < dim(\mathcal{H}) < \infty$ and MO_α for $\alpha > 1$, are examples of directly irreducible atomic complete MOLs.

Proposition 3.2. *Let L be an atomic MOL. The following conditions are satisfied*:

1. *If $p_1, p_2, e \in \Omega(L)$ and $e \leq p_1 \vee p_2$ then $e \vee p_1 = e \vee p_2 = p_1 \vee p_2$.*
2. *If L has dimension 2 then $L = MO_\alpha$ for some ordinal α.*

Proof. 1) Let $p_1, p_2, e \in \Omega(L)$ and $e \leq p_1 \vee p_2$. Suppose that $z = e \vee p_1 < p_1 \vee p_2$. We first note that $p_2 \wedge z = 0$. In fact: if $p_2 \wedge z \neq 0$, $p_2 \wedge z = p_2$ and then $p_2 \leq z$, since p_2 is an atom. Hence, $p_1 \vee p_2 \leq p_1 \vee z = p_1 \vee e = z$, which is a contradiction. Then $p_1 \vee (p_2 \wedge (p_1 \vee z)) = p_1 \vee (p_2 \wedge z) = p_1$ and $(p_1 \vee p_2) \wedge (p_1 \vee z) = z$ which is again a contradiction since $z \neq p_1$ and L is modular. By the same argument we can prove that $e \vee p_2 = p_1 \vee p_2$.

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2) Since L has dimension 2, there exist $p_1, p_2 \in \Omega(L)$ such that $p_1 \vee p_2 = 1$. Suppose that there exists a chain $0 \prec e \prec z < 1$ in L. We can assume that $e \neq p_2$. We first note that $p_2 \vee e = 1$; otherwise, if $e \vee p_2 < 1$, by Proposition 2.1-1, we would have $2 = d(p_2 \vee e) < d(1) = 2$, which is a contradiction. We can also see that $p_2 \wedge z = 0$. In fact, if $p_2 \wedge z \neq 0$, then $p_2 \leq z$, since p_2 is an atom. Therefore, $1 = p_2 \vee e \leq z \vee e = z$, which is a contradiction. Thus, $e \vee (p_2 \wedge (e \vee z)) = e$ and $(e \vee p_2) \wedge (e \vee z) = z$, which is a contradiction since L is modular. Hence, L is formed by $0, 1$ and a string of atoms $p_1, p_1', p_2, p_2', \ldots$, whence $L = MO_\alpha$ for some ordinal α. ⊣

Proposition 3.3. *Let L be an atomic directly irreducible MOL having finite dimension. Then, all pairs of atoms in L are strongly perspective and $2d(1) \leq$ Card $(\Omega(L))$.*

Proof. By Proposition 2.1-4, L is a complete lattice. Therefore, by Proposition 3.1 and Proposition 3.2-1, every pair of atoms is strongly perspective. We now prove (by induction on the dimension of L) that $2d(1) \leq Card(\Omega(L))$. By Proposition 3.2-2, if L has dimension 2 then $L = MO_\alpha$ for some ordinal $\alpha > 1$. Hence, $Card(\Omega(L)) \geq 4 = 2d(1)$. Assume that the proposition holds for each MOL L such that $d(L) < n$. Suppose that $d(L) = n$ and let $e \in \Omega(L)$. Then $d(e') = d(1) - 1 = n - 1$ and $e \notin L_{e'}$. By inductive hypothesis $2d(e') = 2d(1) - 2 \leq Card(\Omega(L_{e'}))$. Let $q_1 \in \Omega(L_{e'})$. Since every pair of atoms in L are strongly perspective, then there exists $q_2 \in \Omega(L) - \{e, q_1\}$ such that $q_2 \leq e \vee q_1$ and $e \vee q_1 = e \vee q_2 = q_1 \vee q_2$. We now prove that $q_2 \notin \Omega(L_{e'})$. If $q_2 \in \Omega(L_{e'})$ then $e \leq q_1 \vee q_2 \leq e'$ which is a contradiction. Thus, $2d(1) \leq Card(\Omega(L_{e'}) \cup \{e, q_2\}) \leq Card(\Omega(L))$. ⊣

Proposition 3.4. *Let L be an atomic MOL such that $d(1) = n$. If $x, y \in L$ then we have*:

1. $d(k(x, y)) = 2(d(x) - d(x \wedge y) - d(x \wedge y')) = 2(d(y) - d(x \wedge y) - d(x' \wedge y))$.
2. *If $d(k(x, y)) = n$, then n is even and $d(x) = d(y) = n/2$.*

Proof. 1) Since L is modular, $(x \wedge y) \vee (x \wedge y') = x \wedge (y \vee (x \wedge y'))$ and $(x' \wedge y) \vee (x' \wedge y') = x' \wedge (y \vee (x' \wedge y'))$. Therefore,

$$((x \wedge y) \vee (x \wedge y')) \wedge ((x' \wedge y) \vee (x' \wedge y')) = 0.$$

We first note that:

$$d(k(x, y)) = n - d(x \wedge y) - d(x \wedge y') - d(x' \wedge y) - d(x' \wedge y'). \tag{4}$$

In fact, by Proposition 2.1, $d(k(x, y)) = n - d(k(x, y)') = n - d(((x \wedge y) \vee (x \wedge y')) \vee ((x' \wedge y) \vee (x' \wedge y'))) = n - d((x \wedge y) \vee (x \wedge y')) - d((x' \wedge y) \vee (x' \wedge y')) = d(x \wedge y) - d(x \wedge y') - d(x' \wedge y) - d(x' \wedge y')$. Moreover:

i. $d(x' \wedge y') = d(x') + d(y') - d(x' \vee y') = (n - d(x)) + (n - d(y)) - (n - d(x \wedge y)) = n - d(x) - d(y) + d(x \wedge y)$,
ii. $d(x' \wedge y) = d(x') + d(y) - d(x' \vee y) = (n - d(x)) + d(y) - (n - d(x \wedge y')) = -d(x) + d(y) + d(x \wedge y')$,
iii. $d(x \wedge y') = d(x) - d(y) + d(x' \wedge y)$.

By Eq. 4 and items i, iii, we obtain $d(k(x, y)) = 2(d(y) - d(x \wedge y) - d(x' \wedge y))$.

2) By Eq. 4, if $d(k(x, y)) = n$ then $d(x \wedge y) = d(x \wedge y') = d(x' \wedge y) = d(x' \wedge y') = 0$. Hence, $n = d(k(x, y)) = 2d(x) = 2d(y)$. ⊣

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Proposition 3.5. *Let L be a directly irreducible atomic MOL of dimension n. If $x \in L - \{0\}$, then there exists an element $y \in L$ satisfying the following conditions*:

1. $d(k(x, y)) = 2d(y)$,
2. $d(y) = \begin{cases} d(x), & if\, d(x) \leq Int(n/2), \\ d(x'), & if\, d(x) > Int(n/2), \end{cases}$

where $Int(n/2)$ is the integer part of $(n/2)$.

Proof. Suppose that $m = d(x) \leq Int(n/2)$. Let $\{e_1, \ldots, e_m\}$ be a base of x. By Proposition 2.1-4 we can consider a base $\{e_{m+1}, \ldots, e_n\}$ of x'. It is clear that $\{e_1, \ldots, e_m, e_{m+1}, \ldots, e_{2m}\}$ is a base. By Proposition 3.3, every pair of atoms e_i, e_{m+i} is strongly perspective for $i \in \{1, \ldots, m\}$. Thus, there exists $a_1, \ldots, a_m \in \Omega(L)$ such that $0 < a_i < e_i \vee e_{m+i}$ and $a_i \vee e_i = a_i \vee e_{m+i} = e_i \vee e_{m+i}$. We now prove that $a_i \neq a_j$ for $i \neq j$ in $\{1, \ldots, m\}$. In fact: if $a_i = a_j$ for some $i \neq j$, then $a_i \leq (e_i \vee e_{m+i}) \wedge (e_j \vee e_{m+j})$, which is a contradiction since, by Proposition 2.1-3, $\{e_i, e_j, e_{m+i}, e_{m+j}\}$ is a base. Now we prove that $\{e_1, \ldots, e_m, a_1, \ldots, a_m\}$ is a base. Suppose that $(e_1 \vee \cdots \vee e_m \vee a_1 \vee \cdots \vee a_{i-1}) \wedge a_i \neq 0$ for some $i \in \{1, \ldots, m\}$. Then $a_i \leq e_1 \vee \cdots \vee e_m \vee a_1 \vee \cdots \vee a_{i-1}$ since a_i is an atom. Therefore we have that $e_{i+m} = (a_i \vee e_i) \wedge e_{i+m} \leq ((e_1 \vee a_1) \vee \cdots \vee (e_{i-1} \vee a_{i-1}) \vee e_i \vee \cdots \vee e_m) \wedge e_{i+m} = ((e_1 \vee e_{m+1}) \vee \cdots \vee (e_{i-1} \vee e_{m+i-1}) \vee e_i \vee \cdots \vee e_m) \wedge e_{i+m}$. Thus, $e_{i+m} \leq e_1 \vee \cdots \vee e_m \vee e_{m+1} \vee \cdots \vee e_{m+i-1}$ which is a contradiction since $\{e_1, \ldots, e_m, e_{m+1}, \ldots, e_{i+m}\}$ is a base. In a similar way we can prove that $\{e_{m+1}, \ldots, e_{2m}, a_1, \ldots, a_m\}$ is a base. Let $y = a_1 \vee \cdots \vee a_m$. Consequently, by Proposition 2.1-3, we have that $x \wedge y = 0$ and $x' \wedge y = 0$. By Proposition 3.4 we obtain $d(k(x, y)) = 2(d(x) - d(x \wedge y) - d(x \wedge y')) = 2d(x) = 2d(y)$.

Suppose that $d(x) > Int(n/2)$. Clearly, $d(x') \leq Int(n/2)$. Similarly to the previous case, we can show that $d(k(x, y)) = 2d(y)$. Since $k(x', y) = k(x, y)$, we can conclude that $d(k(x, y)) = 2d(x') = 2d(y)$. ⊣

Corollary 3.6. *Let L be a directly irreducible atomic MOL of dimension $n > 0$. Then, n is odd iff $d(k(x, y)) \neq n$ for any $x, y \in L$.*

Proof. $\Rightarrow$) It directly follows from Proposition 3.4.

$\Leftarrow$) Assume that n is even. Then, by Proposition 2.2, there exists an element $x \in L$ such that $d(x) = n/2$. By Proposition 3.5, there exists an element $y \in L$ such that $d(y) = n/2$. Hence, $d(k(x, y)) = 2d(y) = n$. ⊣

Corollary 3.7. *Let L be a directly irreducible atomic MOL of dimension $n > 0$. If n is odd then the following conditions are satisfied*:

1. *there exists $x, y \in L$ such that $d(k(x, y)) = n - 1$,*
2. *there exists $x, y, z \in L$ such that $d(k(x, y \mid z)) = n - 1$.*

Proof. 1) Let $x \in L$ such that $d(x) = (n-1)/2$. By Proposition 3.5, there exists $y \in L$ such that $d(k(x, y)) = n - 1$.

2) We consider two cases:

Case i: $n = 3 + 4i$ ($i \in \{0, 1, 2, \ldots\}$). Let $x \in L$ such that $d(x) = 1 + i$. Since $1 + i \leq Int((3 + 4i)/2)$, by Proposition 3.5, there exists $y \in L$ satisfying $d(k(x, y)) = 2d(y)$, where $d(y) = d(x) = 1 + i$. Thus, $d(k(x, y)) = 2 + 2i$. Since $2 + 2i > Int((3 + 4i)/2)$, there exists an element $z \in L$ such that $d(k(x, y \mid z)) = d(k(k(x, y), z)) = 2d(z)$ where $d(z) = d((k(x, y))') = 3+4i-d(k(x, y)) = 1+2i$. Therefore, $d(k(x, y \mid z)) = 2(1 + 2i) = (3 + 4i) - 1 = n - 1$.

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Case ii: $n = 5+4i$ ($i \in \{0, 1, 2, \dots\}$). Let $x \in L$ such that $d(x) = (n-1)/4$. Since $(n-1)/4 \leq Int(n/2)$, by Proposition 3.5, there exists an element $y \in L$ such that $d(k(x, y)) = 2d(y)$, where $d(y) = d(x) = (n-1)/4$. Thus $d(k(x, y)) = (n-1)/2$. Since $(n-1)/2 \leq Int(n/2)$, by Proposition 3.5, there exists an element $z \in L$ satisfying $d(k(x, y \mid z)) = d(k(k(x, y), z)) = 2d(z)$, where $d(z) = d(k(x, y)) = (n-1)/2$. Therefore, $d(k(x, y \mid z)) = n - 1$. ⊣

Corollary 3.8. *Let L be a directly irreducible atomic MOL of dimension $n > 0$. Let $x, z \in L$ such that $d(x) = n - 1$ and $0 < z \leq x$. Then, there exists an element $y \in L$ such that*:

$$0 < z \wedge k(x, y).$$

Proof. Let $\{e_1, \dots, e_{n-1}\}$ be a base of x. By Proposition 2.1, $x' = e_n$ where e_n is an atom. Let $z = e_1 \vee \cdots \vee e_k$ with $k \leq n-1$. By Proposition 3.3, there exists an atom $y \in L - \{e_k, x'\}$ such that $y \leq e_k \vee x'$ and $e_k \vee y = e_k \vee x' = y \vee x'$. We claim that $x \wedge y = 0$. In fact, if $x \wedge y \neq 0$ then $y \leq x$ since y is an atom. Thus $x' \leq e_k \vee y \leq x$, which is a contradiction. Then we have that $z \wedge k(x, y) = z \wedge (x \vee y) \wedge (x \vee y') \wedge (x' \vee y) \wedge (x' \vee y') = z \wedge (x' \vee y) \wedge (x \wedge y)' = z \wedge (x' \vee y) \wedge 1 = z \wedge (x' \vee e_k) \geq e_k > 0$. ⊣

§4. Interval quantum logics. Let L be a MOL and let $a \in L$. Let us consider the interval $[0, a] = \{x \in L : 0 \leq x \leq a\}$ and the unary operation on $[0, a]$ defined as $\neg_a x = x' \wedge a$. One can easily see that the structure

$$L_a = \langle [0, a], \wedge, \vee, \neg_a, 0, a \rangle$$

is a MOL. In particular, if L is atomic then L_a is atomic too and the dimension of the elements of L_a is preserved.

Whenever $n \leq m$, $\mathbb{C}^n$ is a Hilbert subspace of $\mathbb{C}^m$. It allows us to interpret $L(\mathbb{C}^n)$ as an interval of $L(\mathbb{C}^m)$. In fact, by considering the top element $1_{\mathbb{C}^n} = \mathbb{C}^n$ in $L(\mathbb{C}^n)$, we have that

$$L(\mathbb{C}^n) = [0, \mathbb{C}^n] = L_{\mathbb{C}^n}. \tag{5}$$

It suggests that the problem posed by J.M. Dunn, T.J. Hagge et al. in [4] can be generalized by studying proper inclusions of subvarieties of modular ortholattices generated by intervals. More precicely, let L be a MOL, $x \in L$ and let us consider

$$\mathcal{QL}(L_x) = \mathcal{V}_{\mathcal{MOL}}(L_x)$$

i.e., the subvariety of $\mathcal{MOL}$ generated by L_x. Then,

Give conditions under which $\mathcal{QL}(L_a) \subset \mathcal{QL}(L_b)$ whenever $a < b$ in L.

In this section we establish some conditions that guarantee the proper inclusion of the mentioned varieties. As consequence of this, there will follow a positive solution to the question posed in [4].

Proposition 4.1. *Let L be a MOL and let $a, b \in L$ such that $a < b$. Let $v_a : Term \to L_a$ be a valuation. Then, there exists a valuation $v_b : Term \to L_b$ such that $v_a(t) = a \wedge v_b(t)$.*

Proof. We define $v_b : Term \to L_b$ as follows: $v_b(0) = 0$, $v_b(1) = b$, and $v_b(x) = v_a(x)$ for each variable x. By induction on the complexity of terms, we prove that $v_a(t) = a \wedge v_b(t)$. Suppose that $Comp(t) = n > 0$. If t has the form u'

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then $v_a(t) = v_a(u') = \neg_a v_a(u) = \neg_a v_b(u)$. By induction hypothesis, $\neg_a v_b(u) = a \wedge \neg_b v_b(u) = a \wedge v_b(u') = a \wedge v_b(t)$. Thus $v_a(u') = a \wedge v_b(t)$. If t has the form $u_1 \wedge u_2$ then $v_a(t) = v_a(u_1 \wedge u_2) = v_a(u_1) \wedge v_a(u_2)$. Again, by induction hypothesis $v_a(u_1) \wedge v_a(u_2) = (a \wedge v_b(u_1)) \wedge (a \wedge v_b(u_2)) = a \wedge v_b(u_1 \wedge u_2) = a \wedge v_b(t)$. Thus $v_a(t) = a \wedge v_b(t)$. ⊣

THEOREM 4.2. *Let L be a MOL and let $a, b \in L$ such that $a < b$. Then, we have that*:

$$\mathcal{QL}(L_a) \subseteq \mathcal{QL}(L_b).$$

PROOF. By Eq.3, we study equations of the form $t = 1$. By using induction on the complexity of terms, we prove that if $L_b \models t = 1$ then $L_a \models t = 1$. Suppose that $L_b \models t = 1$. Let v_a be a L_a-valuation. By Proposition 4.1 there exists an L_b-valuation v_b such that $v_a(\cdot) = a \wedge v_b(\cdot)$. Thus $v_a(t) = a \wedge v_b(t) = a \wedge 1^{L_b} = a \wedge b = a = 1^{L_a}$. Hence $L_a \models t = 1$. Consequently $\mathcal{QL}(L_a) \subseteq \mathcal{QL}(L_b)$. ⊣

Basically, Theorem 4.2 is an expected result. In the rest of the section we study the proper inclusion $\mathcal{QL}(L_a) \subset \mathcal{QL}(L_b)$ when L is an atomic complete MOL.

Let $s \in Term$. Let us define the map $\tau_s : Term \to Term$ in the following way:

$$\tau_s(t) = \begin{cases} x \wedge s, & \text{if } t \text{ is the variable } x, \\ (\tau_s(u))' \wedge s, & \text{if } t = u', \\ \tau_s(u_1) \wedge \tau_s(u_2), & \text{if } t = u_1 \wedge u_2. \end{cases}$$

Let L be a MOL and let $v : Term \to L$ be a valuation. Given a term s, we denote by v_s the valuation $v_s : Term \to L_{v(s)}$ such that, for any variable x:

$$v_s(x) = v(x) \wedge v(s).$$

PROPOSITION 4.3. *Let L be a MOL. Let $v : Term \to L$ be a valuation and $s \in Term$. Then, $v_s(t) = v(\tau_s(t))$ for any $t \in Term$.*

PROOF. Since v_s is a valuation in $L_{v(s)}$, it is clear that $v_s(t') = \neg_{v(s)} v_s(t) = (v_s(t))' \wedge v(s)$. We prove the proposition by induction on the complexity of term t. If t is a variable then the proof is trivial. If t has the form u' then $v_s(t) = v_s(u') = (v_s(u))' \wedge v(s) = (v(\tau_s(u)))' \wedge v(s) = v((\tau_s(u))' \wedge s) = v(\tau_s(t))$. Finally, if t has the form $u_1 \wedge u_2$ then $v_s(t) = v_s(u_1 \wedge u_2) = v_s(u_1) \wedge v_s(u_2) = v(\tau_s(u_1)) \wedge v(\tau_s(u_2)) = v(\tau_s(u_1) \wedge \tau_s(u_2)) = v(\tau_s(t))$. ⊣

PROPOSITION 4.4. *Let L be an atomic MOL and a, b be two elements of L such that $a < b$, $d(a) = n$ and $d(b) = n + 1$. Let $s \in Term$ and $v : Term \to L_b$ be a valuation such that $v(s) \neq 1_{L_b}$. If $L_a \models t_1 = t_2$, then $v(\tau_s(t_1)) = v(\tau_s(t_2))$.*

PROOF. By Proposition 4.3, $v(\tau_s(-))$ is the valuation $v_s : Term \to L_{v(s)}$. Since $v(s) \neq 1_{L_b}$ we have that $d(v(s)) < n + 1$. Taking into account that $L_a \subset L_b$, each $t \in Term$ is interpreted as an element of L_a. Since $L_a \models t_1 = t_2$ we have that $v_s(t_1) = v_s(t_2)$ i.e., $v(\tau_s(t_1)) = v(\tau_s(t_2))$. ⊣

DEFINITION 4.5. Let $(x_i)_{i \in N}$, $(y_i)_{i \in N}$, $(z_i)_{i \in N}$ be three disjoint sequences of variables such that $x_i \neq x_j$, $y_i \neq y_j$ and $z_i \neq z_j$ if $i \neq j$. Let us define the sequence

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of terms $(\alpha_i)_{i\in N}$ as follows:

$$\alpha_i = \begin{cases} k(x_i, y_i), & \text{if } i = 1,\\ \tau_{k(x_i,y_i)}(\alpha_{i-1}), & \text{if } i > 1 \text{ and } i \text{ is odd},\\ \tau_{k(x_i,y_i|z_i)}(\alpha_{i-1}), & \text{if } i \text{ is even and } i/2 \text{ is odd},\\ \tau_{k(x_i,y_i)}(\alpha_{i-1}) \wedge k(x_i, y_i \mid z_i), & \text{if } i \text{ is even and } i/2 \text{ is even} \end{cases}$$

where each term α_i is called the *i-dimensional discriminator*.

The reason for this name will appear more clear in Proposition 4.7 and Proposition 4.8.

LEMMA 4.6. *Let L be a MOL and $v : Term \to L$ be a valuation. Then we have*:

$$v(\tau_{k(x_i,y_i)}(\alpha_{i-1})) \leq v(k(x_i, y_i)).$$

PROOF. By Proposition 4.3, $v(\tau_{k(x_i,y_i)}(\alpha_{i-1})) = v(k(x_i, y_i)) \wedge v(\alpha_{i-1}) \leq v(k(x_i, y_i))$. ⊣

PROPOSITION 4.7. *Let L be an atomic MOL. If $a \in L$ and $0 < d(a) = n$, then*:

$$\mathcal{QL}(L_a) \models \alpha_n = 0.$$

PROOF. Let $a \in L$ such that $0 < d(a) = n$. We prove that $\alpha_n = 0$ in L_a for each positive natural number $n \leq d(1) \leq \infty$. The proof is by induction on n.

Suppose $n = 1$. Then a is an atom and therefore L_a is the Boolean algebra of two elements $\{0, a\}$. Hence $\alpha_1 = k(x_1, y_1)$. Thus, we can conclude that $L_a \models \alpha_1 = 0$. Suppose that the Theorem holds for $n < i$. We want to show that the Theorem holds for $n = i$, also. Three cases are possible:

1. i is odd. In this case the i-dimensional discriminator is given by $\alpha_i = \tau_{k(x_i,y_i)}(\alpha_{i-1})$. By Proposition 2.2, there exists $b < a$ such that $d(b) = i - 1$. By inductive hypothesis $L_b \models \alpha_{i-1} = 0$. By Corollary 3.6, for each valuation $v : Term \to L_a$ we have that $d(v(k(x_i, y_i))) < i$, i.e. $v(k(x_i, y_i)) \neq 1_{L_a} = a$. Thus, by Proposition 4.4 it follows that $v(\alpha_i) = v(\tau_{k(x_i,y_i)}(\alpha_{i-1})) = 0$.
2. i is even and $i/2$ is even. In this case, the i-dimensional discriminator is given by $\alpha_i = \tau_{k(x_i,y_i)}(\alpha_{i-1}) \wedge k(x_i, y_i \mid z_i)$. Let $v : Term \to L_a$ be a valuation. If $v(k(x_i, y_i)) = 1_{L_a} = a$ then $v((k(x_i, y_i \mid z_i)) = 0$. Hence, our claim. Otherwise, suppose that $v(k(x_i, y_i)) < 1_{L_a} = a$. Then $v(\alpha_i) = v(\tau_{k(x_i,y_i)}(\alpha_{i-1}) \wedge k(x_i, y_i \mid z_i)) = v((\tau_{k(x_i,y_i)}(\alpha_{i-1})) \wedge v(k(x_i, y_i \mid z_i)) = v_{k(x_i,y_i)}(\alpha_{i-1}) \wedge v(k(x_i, y_i \mid z_i))$. Since $v(k(x_i, y_i)) < 1_a = a$ then $v_{k(x_i,y_i)}(\alpha_{i-1})$ is a valuation of α_{i-1} in L_c for some $c < a$. By inductive hypothesis $v_{k(x_i,y_i)}(\alpha_{i-1}) = 0$; therefore $v(\alpha_i) = 0$.
3. i is even and $i/2$ is odd. In this case, the i-dimensional discriminator is given by $\alpha_i = \tau_{k(x_i,y_i|z_i)}(\alpha_{i-1})$. Let $v : Term \to L_a$ be a valuation. If $v(k(x_i, y_i)) = 1_{L_a}$ then $v((k(x_i, y_i \mid z_i)) = 0$. Therefore, by Lemma 4.6, $v(\alpha_i) = v(\tau_{k(x_i,y_i|z_i)}(\alpha_{i-1})) = v_{k(x_i,y_i|z_i)}(\alpha_{i-1}) \leq v(k(x_i, y_i \mid z_i)) = 0$. Assume that $v(k(x_i, y_i)) < 1_{L_a}$. We first note that $v(k(x_i, y_i \mid z_i)) \neq 1_{L_a}$. In fact: suppose, by contradiction, that $v(k(x_i, y_i \mid z_i)) = 1_{L_a}$. By Proposition 3.4, $d(v(k(x_i, y_i)) = d(v(z_i)) = i/2$ is even, which contradicts the hypothesis that $i/2$ is odd. By Lemma 4.6 and Proposition 4.3, we have that $v_{\tau_{k(x_i,y_i|z_i)}}(\alpha_{i-1}) = v(\tau_{k(x_i,y_i|z_i)}(\alpha_{i-1})) \leq v(k(x_i, y_i \mid z_i)) < 1_{L_a} = a$. Therefore $v_{\tau_{k(x_i,y_i|z_i)}}(\alpha_{i-1})$ is

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a valuation of α_{i-1} in L_c for some $c < a$. Then, by inductive hypothesis, we have that $v_{k(x_i,y_i)}(\alpha_{i-1}) = 0$, resulting $v(\alpha_i) = v_{\tau_{k(x_i,y_i|z_i)}}(\alpha_{i-1}) = v(\tau_{k(x_i,y_i|z_i)}(\alpha_{i-1})) \leq v(k(x_i, y_i \mid z_i)) < 1_{L_a} = a$. Since $v_{\tau_{k(x_i,y_i|z_i)}}(\alpha_{i-1})$ is a valuation of α_{i-1} in L_c for some $c < a$, by inductive hypothesis, $v_{k(x_i,y_i)}(\alpha_{i-1}) = 0$ and $v(\alpha_i) = 0$. ⊣

Proposition 4.8. *Let L be a directly irreducible atomic complete MOL. Then, for each $a \in L$ such that $0 < d(a) = n + 1 \leq d(1) \leq \infty$, we have that*:

$$\mathcal{QL}(L_a) \not\models \alpha_n = 0.$$

Proof. We prove the proposition by induction on n. Suppose that $n = 1$. Then, $\alpha_1 = k(x_1, y_1)$ and $d(a) = 2$. By Proposition 3.3, there exist three distinct atoms $e_1, e_2, e_3 \in \Omega(L)$ such that $e_i \vee e_j = a$ if $i \neq j$. It is not very hard to see that $k(e_1, e_2) = a$. If we consider a valuation $v : Term \to L_a$ satisfying $v(x_1) = e_1$ and $v(y_1) = e_2$ then, $v(k(x_1, y_1)) \neq 0$.

Suppose that the proposition holds for $n < i$. We want to show that the proposition holds for $n = i$, also. Three cases are possible:

1. i is odd. In this case the i-dimensional discriminator is given as $\alpha_i = \tau_{k(x_i,y_i)}(\alpha_{i-1})$. By Proposition 2.2, there are two elements a, b such that $b < a$ and $d(b) = i < d(a) = i + 1$. By induction, $L_b \not\models \alpha_{i-1} = 0$. We show that $L_a \not\models \alpha_{i-1} = 0$. Suppose that $L_a \models \alpha_{i-1} = 0$. Let $v_b : Term \to L_b$ be a valuation satisfying $v_b(\alpha_{i-1}) \neq 0$. By Proposition 4.1 there exists a valuation $v : Term \to L_a$ such that $v_b(t) = b \wedge v(t)$ so that $v_b(\alpha_{i-1}) = b \wedge v(\alpha_{i-1}) = 0$, which is a contradiction.
   Thus, there exists a valuation $v : Term \to L_a$ that satisfies $v(\alpha_{i-1}) \neq 0$. Note that $i + 1$ is even. Then, by Corollary 3.6, there are $a_i, b_i \in L_a$ such that $k(a_i, b_i) = 1_{L_a}$. Since x_i, y_i are not variables of α_{i-1}, we can assume that $v(x_i) = a_i$ and $v(y_i) = b_i$. Then $v(k(x_i, y_i)) = 1_{L_a}$. Consequently, by Proposition 4.3, $v(\alpha_i) = v(\tau_{k(x_i,y_i)}(\alpha_{i-1})) = v_{k(x_i,y_i)}(\alpha_{i-1}) = v(\alpha_{i-1}) \neq 0$.
2. i is even and $i/2$ is even. In this case, the i-dimensional discriminator is given by $\alpha_i = \tau_{k(x_i,y_i)}(\alpha_{i-1}) \wedge k(x_i, y_i \mid z_i)$. We first show that there exists a valuation $v : Term \to L_a$ such that

$$v(\tau_{k(x_i,y_i)}(\alpha_{i-1})) \neq 0.$$

   Indeed: since $i + 1$ is odd then, by Corollary 3.7, there are $a_i, b_i \in L_a$ such that $k(a_i, b_i) = b$ where $b < a$ and $d(b) = i < d(a) = i + 1$. By induction hypothesis and by using the same argument as in the previous item, we obtain $L_b \not\models \alpha_{i-1} = 0$ and then $L_a \not\models \alpha_{i-1} = 0$. Consequently, there exists a valuation $v^i : Term \to L_b$ such that $v^i(\alpha_{i-1}) \neq 0$.
   Let us consider a valuation $v : Term \to L_a$ such that:
   - for all j such that $1 \leq j \leq i - 1$, $v(x_j) = v^i(x_j)$; $v(y_j) = v^i(y_j)$; $v(z_j) = v^i(z_j)$,
   - $v(x_i) = a_i$; $v(y_i) = b_i$.

   For any j such that $1 \leq j \leq i - 1$, $v_{k(x_i,y_i)}(x_j) = v(k(x_i, y_i)) \wedge v(x_j) = b \wedge v(x_j) = b \wedge v^i(x_j) = v^i(x_j)$, since v^i is a valuation in L_b ($b = 1_{L_b}$). Similarly, we can prove that $v_{k(x_i,y_i)}(y_j) = v^i(y_j)$ and $v_{k(x_i,y_i)}(z_j) = v^i(z_j)$. Consequently $v(\tau_{k(x_i,y_i)}(\alpha_{i-1})) = v^i(\alpha_{i-1}) \neq 0$.

By Lemma 4.6, we have that $v(\tau_{k(x_i,y_i)}(\alpha_{i-1})) \leq v(k(x_i, y_i)) = k(a_i, b_i)$ where $d(k(a_i, b_i)) = d(b) = i < i+1$. Thus, by Corollary 3.8, there exists an element $c \in L_a$ such that $v(\tau_{k(x_i,y_i)}(\alpha_{i-1})) \wedge k(a_i, b_i \mid c) \neq 0$. Taking $v(z_i) = c$, we have that

$$v(\alpha_i) = v((\tau_{k(x_i,y_i)}(\alpha_{i-1})) \wedge (k(x_i, y_i \mid z_i))) \neq 0.$$

3. i is even and $i/2$ is odd. In this case, the i-dimensional discriminator is given by $\alpha_i = \tau_{k(x_i,y_i|z_i)}(\alpha_{i-1})$. Note that $i+1$ is odd. Then, by Corollary 3.7, there are three elements $a_i, b_i, c_i \in L_a$ such that $k(a_i, b_i \mid c_i) = b$ where, $b < a$ and $d(b) = i < d(a) = i+1$. By induction hypothesis $L_b \not\models \alpha_{i-1} = 0$. Thus, there exists a valuation $v^i : \mathit{Term} \to L_b$ such that $v^i(\alpha_{i-1}) \neq 0$. Now let us define a valuation $v : \mathit{Term} \to L_a$ satisfying the following conditions:
   - $v(x_j) = v^i(x_j)$, $v(y_j) = v^i(y_j)$ and $v(z_j) = v^i(z_j)$ for all j such that $1 \leq j \leq i-1$,
   - $v(x_i) = a_i$; $v(y_i) = b_i$; $v(z_i) = c_i$.

   For any j such that $1 \leq j \leq i-1$, $v_{k(x_i,y_i|z_i)}(x_j) = v(x_j) \wedge v(k(x_i, y_i \mid z_i)) = v^i(x_j) \wedge b = v^i(x_j)$. Similarly $v_{k(x_i,y_i|z_i)}(y_j) = v^i(y_j)$ and $v_{k(x_i,y_i|z_i)}(z_j) = v^i(z_j)$. Accordingly, $v_{k(x_i,y_i|z_i)}(\alpha_{i-1}) = v^i(\alpha_{i-1})$. Therefore, we have that $v(\alpha_i) = v(\tau_{k(x_i,y_i|z_i)}(\alpha_{i-1})) = v_{k(x_i,y_i|z_i)}(\alpha_{i-1}) = v^i(\alpha_{i-1}) \neq 0$. ⊣

THEOREM 4.9. *Let L be an atomic complete MOL such that $L \neq MO_1$. Then the following statements are equivalent*:

1. *L is a directly irreducible MOL,*
2. *for each $a < b \in L$ where b is a finite element, $\mathcal{QL}(L_a) \subset \mathcal{QL}(L_b)$.*

PROOF. 1 $\Longrightarrow$ 2) Let us assume that L is a directly irreducible MOL. Suppose that $d(a) = n < n+1 \leq d(b)$. By Proposition 4.7, we have $\mathcal{QL}(L_a) \models \alpha_n = 0$. By Proposition 2.2, there exists $c \in L_b$ such that $d(c) = n+1$. Then, by Proposition 4.8, $\mathcal{QL}(L_c) \not\models \alpha_n = 0$. Since $\mathcal{QL}(L_c) \subseteq \mathcal{QL}(L_b)$, $\mathcal{QL}(L_b) \not\models \alpha_n = 0$. Hence, $\mathcal{QL}(L_a) \subset \mathcal{QL}(L_b)$.

2 $\Longrightarrow$ 1) Suppose that L is not directly irreducible MOL. Then, by Proposition 3.1, there exists $u_1, u_2 \in \Omega(L)$ such that

$$[0, u_1 \vee u_2] = \{0, u_1, u_2, u_1 \vee u_2\}.$$

Let $s = u_1 \vee u_2$. We will see that $L_s = \langle [0, s], \vee, \wedge, \neg_s, 0, s \rangle$ is the four elements boolean algebra i.e. $L_s = MO_1$. For this, we have to prove that $\neg_s u_i = u_i' \wedge s = u_j$ where $i, j \in \{1, 2\}$ and $i \neq j$. Clearly $0 \leq \neg_s u_1, \neg_s u_2 \leq s$.

Suppose that $\neg_s u_i = s$ or equivalently $u_i' \wedge s = s$. Then, $s \leq u_i'$ and $u_i \leq s \leq u_i'$. Consequently $u_i = u_i \wedge u_i' = 0$ which is a contradiction because $u_i \in \Omega(L)$.

Suppose that $\neg_s u_i = 0$. Therefore $u_i' \wedge s = 0$ and $u_i \vee s' = 1$. By Eq. 1 we have $s = 1 \wedge s = (u_i \vee s') \wedge (u_i \vee s) = u_i \vee (s' \wedge (u_i \vee s)) = u_i \vee (s' \wedge s) = u_i$ which is a contradiction because $u_i < s$.

Consequently the only possibility is $\neg_s u_i = u_j$ for $i, j \in \{1, 2\}$ and $i \neq j$. Hence, $L_s = MO_1$. Since $L_{u_i} = MO_0$, by Eq 2, $\mathcal{QL}(L_s) = \mathcal{QL}(L_{u_i})$. ⊣

THEOREM 4.10. *Let L be an atomic complete directly irreducible MOL. Then $d(L) = n$ iff, $L \models \alpha_n = 0$ and $L \not\models \alpha_{n+1} = 0$.*

PROOF. Suppose that $L \models \alpha_n = 0$ and $L \not\models \alpha_{n+1} = 0$. By Proposition 4.8 and Theorem 4.2, it is clear that $d(L) < n+1$. By the same argument, if $d(L) < n$

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then $L \not\models \alpha_n = 0$ which is a contradiction. Thus, $d(L) = n$. The other direction is trivial. ⊣

For each $n \in \mathbb{N}$, $L(\mathbb{C}^n)$ is an atomic complete directly irreducible MOL. Then, by Theorem 4.10, the equation $\alpha_n = 0$ together with $\alpha_{n+1} \neq 0$ in $L(\mathbb{C}^n)$, characterize the usual dimension of $\mathbb{C}^n$. Thus, we can establish the following corollary providing a positive answer to the question posed by J.M. Dunn, T.J. Hagge et al. in [4].

COROLLARY 4.11. *$\mathcal{QL}(\mathbb{C}^n) \subset \mathcal{QL}(\mathbb{C}^m)$ whenever $n < m$.* ⊣

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REFERENCES

[1] G. BIRKHOFF and J. VON NEUMANN, *The logic of quantum mechanics*. ***Annals of Mathematics***, vol. 37 (1936), pp. 823–843.

[2] S. BURRIS, H. P. SANKAPPANAVAR, ***A Course in Universal Algebra***, Graduate Text in Mathematics, vol. 78, Springer-Verlag, New York, Heidelberg, Berlin, 1981.

[3] D. E. CATLIN, *Irreducibility conditions in orthomodular lattices*. ***Journal of Natural Sciences and Mathematics***, vol. 8 (1968), pp. 81–87.

[4] J. M. DUNN, T. J. HAGGE, L. S. MOSS and Z. WANG, *Quantum logic as motived by quantum computing*, this JOURNAL, vol. 70 (2005), pp. 353–359.

[5] T. J. HAGGE, *$QL(\mathbb{C}^n)$ determines n*, this JOURNAL, vol. 72 (2007), pp. 1194–1196.

[6] G. KALMBACH, ***Orthomodular Lattices***, Academic Press, New York, 1983.

[7] F. MAEDA and S. MAEDA, ***Theory of Symmetric Lattices***, Springer-Verlag, Berlin, 1970.

UNIVERSITÁ DI CAGLIARI
VIA IS MIRRIONIS 1, 09123
CAGLIARI, ITALIA
E-mail: giuntini@unica.it
E-mail: giuseppe.sergioli@gmail.com

DEPARTAMENTO DE MATEMÁTICA UNR-CONICET
AV. PELLEGRINI 250
CP 2000, ROSARIO, ARGENTINA
E-mail: hfreytes@gmail.com