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ON BIRATIONALLY TRIVIAL FAMILIES AND ADJOINT QUADRICS

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Abstract Let $\pi\colon \mathcal{X} \to B$ be a family whose general fibre X_b is a $(d_1, \dots, d_a)$-polarization on a general abelian variety, where $1 \le d_i \le 2$, $i = 1, \dots, a$ and $a \ge 4$. We show that the fibres are in the same birational class if all the $(m, 0)$-forms on X_b are liftable to $(m, 0)$-forms on $\mathcal{X}$, where $m = 1$ and $m = a - 1$. Actually, we show a general criteria to establish whether the fibres of certain families belong to the same birational class.

Keywords: extension class of a vector bundle; holomorphic forms; Albanese variety; families of varieties; infinitesimal invariant

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1. Introduction

A family of n-dimensional complex varieties is a flat, smooth proper morphism $\pi\colon \mathcal{X} \to B$ such that the *fibre* $X_b := \pi^{-1}(b)$ over a point b on the base B has dimension n. In this paper, we assume that *B is a smooth connected open complex variety of dimension* 1. We will also assume that X_b is an irregular smooth variety of general type such that its Albanese morphism $\mathrm{alb}(X_b) : X_b \to \mathrm{Alb}(X_b)$ is of degree 1. We want to study conditions which ensure that the fibres of $\pi\colon \mathcal{X} \to B$ are of the same birational type.

It is well known that, up to base change, we can associate with $\pi\colon \mathcal{X} \to B$ the family of corresponding Albanese varieties. In fact, we can work in the more general setup of families of Albanese type (cf. [23, Definition 1.1.1]), for which we recall the basic definition.

Let $p\colon \mathcal{A} \to B$ be a family of abelian varieties; that is, the fibre $A_b := p^{-1}(b)$ is an abelian variety of dimension $a > 0$. We say that a morphism $\Phi\colon \mathcal{X} \to \mathcal{A}$ is a *family of Albanese type* over B if:

1. Φ fits into the commutative diagram:

$$\begin{array}{ccc} & \mathcal{X} & \\ {}^{\Phi}\swarrow & & \searrow^{\pi} \\ \mathcal{A} & \xrightarrow{p} & B \end{array}$$

2. the induced map $\phi_b \colon X_b \to A_b$ of Φ on X_b is birational onto its image Z_b for general b;

3. the cycle Z_b generates the fibre A_b as a group for general b.

A family of Albanese type comes equipped with a global object: its relative homologically trivial cycle. Indeed, let $-\mathrm{Id}_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{A}$ be the natural involution induced on $p : \mathcal{A} \to B$ by the multiplication by (-1) on the fibres. The composition $(-\mathrm{Id}_{\mathcal{A}}) \circ \Phi \colon \mathcal{X} \to \mathcal{A}$ is an Albanese type family. We set $(-\mathrm{Id}_{\mathcal{A}}) \circ \Phi := \Phi^-$. Then, we can construct two cycles $[\Phi \colon \mathcal{X} \to \mathcal{A}]$ and $[\Phi^- : \mathcal{X} \to \mathcal{A}]$ in the relative group $Z^{a-n}(\mathcal{A}/B)$, which we denote, respectively, by $[\mathcal{X}]^+$ and $[\mathcal{X}]^-$. The following cycle will be called the *basic cycle of the Albanese type family* $\Phi \colon \mathcal{X} \to \mathcal{A}$:

$$[\mathcal{Z}] = [\mathcal{X}]^+ - [\mathcal{X}]^-. \tag{1.1}$$

It is well known that the cycle $[\mathcal{Z}]$ is relatively homologically trivial; that is $[\mathcal{Z}] \in Z_h^{a-n}(\mathcal{A}/B)$. By the theory of normal functions and its infinitesimal invariant $\delta_{\mathcal{Z}}$, see [13, 14, 31], we know that Albanese type families come into two types: those whose infinitesimal invariant is non-zero and those which have $\delta_{\mathcal{Z}} = 0$. The latter are called *Nori trivial families.*

Another piece of information carried by the morphism $\phi_b \colon X_b \to A_b$ is a splitting of $H^{n,0}(X_b)$. Indeed let $\phi_b^* \colon H^{n,0}(A_b) \to H^{n,0}(X_b)$ and set $V_b := \mathrm{Im}(\phi_b^*)$. Inside the dual $H^{0,n}(X_b)$ of $H^{n,0}(X_b)$, we can define:

$$\mathrm{Ann}(V_b) := \{\tau \in H^{0,n}(X_b) \mid \int_{X_b} \phi_b^*(\mu) \wedge \tau = 0,\ \forall \mu \in H^{n,0}(A_b)\}$$

and we know that

$$H^{0,n}(X_b) = \overline{V_b} \oplus \mathrm{Ann}(V_b) \tag{1.2}$$

where $\overline{V_b} \subset H^{0,n}(X)$ is the conjugate space of V_b. It also holds:

$$H^{n,0}(X_b) = V_b \oplus \overline{\mathrm{Ann}(V_b)} \tag{1.3}$$

which in turns gives a decomposition of the symmetric product

$$\mathrm{Sym}^2 H^{n,0}(X_b) = V_b \odot H^{n,0}(X_b) \oplus \mathrm{Sym}^2(\overline{\mathrm{Ann}(V_b)}). \tag{1.4}$$

The standard multiplication map $H^{n,0}(X_b) \otimes H^{n,0}(X_b) \to H^0(X, \omega_{X_b}^{\otimes 2})$ factors on the symmetric product

$$\mu_{X_b} \colon \mathrm{Sym}^2 H^{n,0}(X_b) \to H^0(X, \omega_{X_b}^{\otimes 2}) \tag{1.5}$$

and induces homomorphisms

$$\nu_{X_b}\colon \mathrm{Sym}^2\overline{\mathrm{Ann}(V_b)} \to H^0(X, \omega_{X_b}^{\otimes 2}) \tag{1.6}$$

and

$$\gamma_{X_b}\colon V_b \odot H^{n,0}(X_b) \to H^0(X, \omega_{X_b}^{\otimes 2}). \tag{1.7}$$

Finally, we say that a family of relative dimension n satisfies the *extremal liftability assumptions* if the natural restriction homomorphisms $H^0(\mathcal{X}, \Omega^1_{\mathcal{X}}) \to H^0(X_b, \Omega^1_{X_b})$ and $H^0(\mathcal{X}, \Omega^n_{\mathcal{X}}) \to H^0(X_b, \Omega^n_{X_b})$ are surjective for every b on B. The case where the map $H^0(X_b, \Omega^n_{\mathcal{X}|X_b}) \to H^0(X_b, \Omega^n_{X_b})$ is surjective is deeply studied in [19, 24]. Hence, our results must be read and applied jointly to the results in [19, 24]. We show:

Theorem [A]. *Let $\Phi\colon \mathcal{X} \to \mathcal{A}$ be a Nori family. If it satisfies extremal liftability assumptions and for each non-zero element η in $\overline{\mathrm{Ann}(V_b)}$ its square is not contained in the image of γ_{X_b}, then its fibres belong to the same birational class.*

The proof is a direct consequence of the new notion of adjoint quadric introduced in [28]. By the extremal liftability assumptions, we are actually concerned with families of varieties equipped with a morphism to a fixed abelian variety; see Proposition 3.4. Nevertheless, our result should be considered in light of the theory of families of varieties of general type as described in [15]; for this reason, we present the theorem in the above general set up, which strongly relies on the theory exposed in [23]. In particular, we use the Volumetric Theorem [23, Theorem 1.5.3] recalled in Theorem 3.7.

As an immediate consequence of Theorem [A] and of the well-known fact that, if C is a hyperelliptic curve then the Ceresa's cycle $C - C^-$ is trivial in its Jacobian, we have the well-known Torelli Theorem for hyperelliptic deformations of hyperelliptic curves, see [18]; this is a case where $\overline{\mathrm{Ann}(V_b)} = 0$.

More deeply, by a famous Theorem of Nori [17, pp. 372] (see also [8]), Theorem [A] applies to the case where the family $\mathcal{X}$ is given by a family of cycles inside a general abelian variety of dimension $a \geq 4$ such that for each element η contained in $\overline{\mathrm{Ann}(V_b)}$, its square is not contained in the image of γ_{X_b}.

In the particular case of families of divisors, the above-mentioned theorem by Nori does not apply. However, by using techniques which are analogous to those we use to prove Theorem [A], we can show another result.

Let $(A, \mathcal{L})$ be a $(d_1, d_2, \ldots, d_a)$ polarized abelian variety which is general inside its moduli space and $1 \leq d_i \leq 2$, $i = 1, \ldots, a$. Let $X \subset A$ be a smooth divisor in $|\mathcal{L}|$. Consider the incidence variety inside $|\mathcal{L}| \times A$ and $\hat{B}$ a curve inside $|\mathcal{L}|$ passing and smooth through X. Possibly after resolution of singularities, we obtain a fibration $\mathcal{Y} \to \hat{B}$ with $\mathcal{Y}$ smooth. We say that a (local) family $\pi\colon \mathcal{X} \to B$ is *induced by embedded deformations* if it is obtained by the restriction of $\mathcal{Y} \to \hat{B}$ over an open contractible set $B \subset \hat{B}$ contained in the subset of smooth elements of $|\mathcal{L}|$.

Theorem [B]. *Let $(A, \mathcal{L})$ be a $(d_1, d_2, \ldots, d_a)$ polarized abelian variety which is general inside its moduli space and $1 \leq d_i \leq 2$, $i = 1, \ldots, a$. Let $\pi\colon \mathcal{X} \to B$ be a family induced by embedded deformations. If $\pi\colon \mathcal{X} \to B$ satisfies the extremal liftability conditions then the fibres belong to the same birational class.*

Theorem [B], see § (4.5), follows by a careful study of the map

$$\mu_{X_b}\colon \mathrm{Sym}^2 H^{n,0}(X_b) \to H^0(X, \omega_{X_b}^{\otimes 2}) \tag{1.8}$$

where X_b is an element of the linear system associated with a $(d_1, d_2, \ldots, d_a)$-polarization, and $b \in B$.

This problem has turned out to be quite involved. Actually, the well-known problem to determine conditions for the surjectivity of μ_{X_b}, as well as of the maps

$$\mu_n\colon H^0(A, \mathcal{L}^n) \otimes H^0(A, \mathcal{L}) \to H^0(A, \mathcal{L}^{n+1}) \tag{1.9}$$

are still of research interest. For instance, the surjectivity of μ_{a-1} implies that the Infinitesimal Torelli Theorem holds for the smooth hypersurfaces of A of the linear system $|\mathcal{L}|$, and it has been recently proved (see [4]) that μ_n is surjective, provided that $h^0(A, \mathcal{L}) > (\frac{n+1}{n})^a \cdot a!$ and A is simple.

As a first step in this direction, in § 4, we study the loci in the moduli space of polarized abelian varieties of type $(1, \ldots, 1, 2, \ldots, 2)$ where

$$\mu_b\colon \mathrm{Sym}^2 H^0(A_b, \mathcal{L}_b) \to H^0(A_b, \mathcal{L}_b^2)$$

is not injective. Note that the injectivity of μ_b is equivalent to the injectivity of ν_{X_b}, since there is the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Sym}^2 H^0(A_b, \mathcal{L}_b) & \xrightarrow{\mu_b} & H^0(A_b, \mathcal{L}_b^2) \\ \big\downarrow {\scriptstyle |_{X_b}} & & \big\downarrow {\scriptstyle |_{X_b}} \\ \mathrm{Sym}^2 \overline{\mathrm{Ann}(V_b)} & \xrightarrow{\nu_{X_b}} & H^0(X_b, \omega_{X_b}^2). \end{array} \tag{1.10}$$

We prove:

Theorem [C]. *Let $(A, \mathcal{L})$ be a general polarized abelian variety of type $\underline{(1, \ldots, 1}, 2, \ldots, 2)$ and $\mathcal{D}$ a general element in $|\mathcal{L}|$. For each element η contained in $\overline{\mathrm{Ann}(V)}$, its square is not contained in the image of $V \odot H^0(\mathcal{D}, \omega_{\mathcal{D}})$ with respect to the multiplication map*

$$\mu_{\mathcal{D}}\colon \mathrm{Sym}^2(H^0(\mathcal{D}, \omega_{\mathcal{D}})) \longrightarrow H^0(\mathcal{D}, \omega_{\mathcal{D}}^{\otimes 2}). \tag{1.11}$$

Moreover, the natural map

$$\nu_{\mathcal{D}}\colon \mathrm{Sym}^2 \overline{\mathrm{Ann}(V)} \longrightarrow H^0(\mathcal{D}, \omega_{\mathcal{D}}^{\otimes 2}) \tag{1.12}$$

is injective.

See Corollary 4.4 and Theorem 4.5. Theorem [B] is a direct consequence of Theorem [C]. Actually, Theorem [C] is interesting on its own because it proves a condition on the kernel of $\mu_{\mathcal{D}}$ which is strictly stronger that the simple injectivity of $\nu_{\mathcal{D}}$. As the proof of Theorem 4.5 clearly shows the condition on the squares of the elements η is independent from the injectivity of $\nu_{\mathcal{D}}$ and in the explicit cases that one can consider they are indeed not true at the same time. They must both hold, however, on the general abelian variety described in Theorem [C].

Finally, we have an application of the above circle of ideas to the case of fibrations with maximal relative irregularity. Let S, B be, respectively, a smooth surface and a smooth curve. A fibration $f\colon S \to B$ is said to be of maximal relative irregularity if $q(S) - g(B) = g(F) - 1$ where $q(S)$ is the irregularity of S and $g(B)$, $g(F)$ are, respectively, the genus of B and of a general fibre F. There are many papers on this topic. Here, we can quote [16, 20] and [2], which also contains basic references to this problem. In this particular case, it occurs that $\overline{\mathrm{Ann}(V_b)} \simeq \mathbb{C}$. Indeed, if the natural morphism $F \to \mathrm{Alb}(S)$ has degree 1, we can find a suitable open subset $U \subset B$ contained in the locus where $f\colon S \to B$ is smooth, and we can form a family of Albanese type $\Phi_U\colon S_U \to \mathcal{A}_U$ where $p\colon \mathcal{A} \to U$ is such that its fibres are isomorphic to a fixed abelian variety A of dimension $g(F) - 1$ and $S_U := f^{-1}(U)$. In Theorem 5.2, which does not depend on Theorem [A], we show that the infinitesimal invariant associated with the basic cycle associated with $f_{|S_U}\colon S_U \to U$ is not zero.

2. Adjoint quadrics

We recall some of the results of [23]. See also [26–28].

2.1. The Adjoint theorem

2.1.1. The Gauss-type homomorphism

Let X be a compact complex smooth variety of dimension m and let $\mathcal{F}$ be a locally free sheaf of rank n. Fix an extension class $\xi \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{O}_X)$ associated with the exact sequence:

$$0 \to \mathcal{O}_X \xrightarrow{d\epsilon} \mathcal{E} \xrightarrow{\rho_1} \mathcal{F} \to 0. \tag{2.1}$$

By the Koszul resolution associated with the section $d\epsilon \in H^0(X, \mathcal{E})$ and by the isomorphisms

$$\mathrm{Ext}^1(\mathcal{F}, \mathcal{O}_X) \cong \mathrm{Ext}^1\left(\bigwedge^{i} \mathcal{F}, \bigwedge^{i-1} \mathcal{F}\right)$$

we see that the coboundary homomorphisms

$$\partial^i_\xi\colon H^0\left(X, \bigwedge^{i} \mathcal{F}\right) \to H^1\left(X, \bigwedge^{i-1} \mathcal{F}\right)$$

are computed by cup product and interior product with ξ, $i = 1, \ldots, n$.

Denote by $H^n_{d\epsilon}\colon \det \mathcal{E} \to \det \mathcal{F}$ the natural isomorphism and by Λ^{n+1} the natural map

$$\Lambda^{n+1}\colon \bigwedge^{n+1} H^0(X, \mathcal{E}) \to H^0(X, \det \mathcal{E}). \tag{2.2}$$

By composition, we define a Gauss-type homomorphism:

$$\Lambda := H^n_{d\epsilon} \circ \Lambda^{n+1}\colon \bigwedge^{n+1} H^0(X, \mathcal{E}) \to H^0(X, \det \mathcal{F}). \tag{2.3}$$

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2.1.2. Adjoint forms

Let $W \subset \mathrm{Ker}\,(\partial^1_\xi) \subset H^0(X,\,\mathcal{F})$ be a vector subspace of dimension $n+1$ and let $\mathcal{B} := \{\eta_1,\,\ldots,\,\eta_{n+1}\}$ be a basis of W. By definition, we can take liftings $s_1,\,\ldots,\,s_{n+1} \in H^0(X,\,\mathcal{E})$ such that $\rho_1(s_i) = \eta_i$, $i = 1,\,\ldots,\,n+1$.

Definition 2.1. The section

$$\omega_{\xi,W,\mathcal{B}} := \Lambda(s_1 \wedge \ldots \wedge s_{n+1}) \in H^0(X, \det \mathcal{F}).$$

is called *an adjoint form* of ξ, W, $\mathcal{B}$.

If we consider the natural map

$$\lambda^n\colon \bigwedge^n H^0(X,\mathcal{F}) \to H^0(X, \det \mathcal{F}),$$

we can define the subspace $\lambda^n W \subset H^0(X,\,\det \mathcal{F})$ generated by

$$\omega_i := \lambda^n(\eta_1 \wedge \ldots \wedge \widehat{\eta_i} \wedge \ldots \wedge \eta_{n+1})$$

for $i = 1,\,\ldots,\,n+1$.

Definition 2.2. The class

$$[\omega_{\xi,W,\mathcal{B}}] \in \frac{H^0(X, \det \mathcal{F})}{\lambda^n W}$$

is called *the Massey product* of W along ξ.

In the literature, $[\omega_{\xi,W,\mathcal{B}}]$ is also called the adjoint image of W by ξ. For the main properties of Massey products in our context see [7, 23, 26–28]. Here we only recall that while the section $\omega_{\xi,W,\mathcal{B}}$ depends on the choice of the liftings s_i, the class $[\omega_{\xi,W,\mathcal{B}}]$ does not. Furthermore, if we choose another basis $\mathcal{B}' := \{\eta'_1,\,\ldots,\,\eta'_{n+1}\}$ of W then $[\omega_{\xi,W,\mathcal{B}}] = k[\omega_{\xi,W,\mathcal{B}'}]$ where k is the determinant of the matrix of the change of basis. Since we will be mostly concerned with the condition $[\omega_{\xi,W,\mathcal{B}}] = 0$, the choice of the basis of W is also not essential and that is why in the above definition, we only highlight the dependence on W and ξ.

Definition 2.3. If $\lambda^n W$ is nontrivial we denote by $|\lambda^n W| \subset \mathbb{P}(H^0(X,\,\det \mathcal{F}))$ the induced sublinear system. We call D_W the fixed divisor of this linear system and Z_W the base locus of its moving part $|M_W| \subset \mathbb{P}(H^0(X,\,\det \mathcal{F}(-D_W)))$.

From the natural map $\epsilon_{D_W} : \mathcal{F}(-D_W) \to \mathcal{F}$, we have the induced homomorphism in cohomology:

$$H^1(X, \mathcal{F}^\vee) \xrightarrow{\epsilon_{D_W}} H^1(X, \mathcal{F}^\vee(D_W)).$$

We set

$$\xi_{D_W} := \epsilon_{D_W}(\xi).$$

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Definition 2.4. We say that $\xi \in H^1(X, \mathcal{F}^\vee)$ *is supported on* D_W if $\xi_{D_W} = 0$

In [23, Theorem 1.5.1], see also [28], we have shown:

Theorem 2.5 (Adjoint Theorem). *Let X be a compact m-dimensional complex smooth variety. Let $\mathcal{F}$ be a rank n locally free sheaf on X and $\xi \in H^1(X, \mathcal{F}^\vee)$ the extension class of the exact sequence (2.1). Let W be a $n+1$-dimensional subspace of* $\mathrm{Ker}\,(\partial^1_\xi) \subset H^0(X, \mathcal{F})$ *and ω one of its adjoint forms. If the Massey product $[\omega] = 0$ then ξ is supported on D_W.*

For interesting applications of this theory different from those presented in this paper, we refer to [2, 9–12, 22, 25, 29, 30].

2.2. The notion of Adjoint quadric

We denote by $\lambda^n H^0(X, \mathcal{F})$ the image of

$$\lambda^n \colon \bigwedge^n H^0(X, \mathcal{F}) \to H^0(X, \det \mathcal{F})$$

and we consider the linear subsystem $\mathbb{P}(\lambda^n H^0(X, \mathcal{F}))$ of $|\det \mathcal{F}|$. Denote by $D_{\mathcal{F}}$ its fixed component and by $|M_{\mathcal{F}}|$ its associated mobile linear system. Moreover, we denote $D_{\det \mathcal{F}}$, $M_{\det \mathcal{F}}$, respectively, the fixed and the movable part of $|\det \mathcal{F}|$; that is, $|\det \mathcal{F}| = D_{\det \mathcal{F}} + |M_{\det \mathcal{F}}|$.

Take $W = \langle \eta_1, \ldots, \eta_{n+1} \rangle$ and ω_i, $i = 1, \ldots, n+1$ as above and let $\omega \in H^0(X, \det \mathcal{F})$ be a ξ-adjoint of $W \subset H^0(X, \mathcal{F})$. Let

$$\mu_{\det \mathcal{F}} \colon \mathrm{Sym}^2(H^0(X, \det \mathcal{F})) \to H^0(X, \det \mathcal{F}^{\otimes 2})$$

be the natural multiplication homomorphism. The basic definition of this paper is:

Definition 2.6. An ω-*adjoint quadric* is an element $Q \in \mathrm{Sym}^2(H^0(X, \det \mathcal{F}))$ such that

1. $Q := \omega \odot \omega - \sum_{i=1}^{n+1} \omega_i \odot L_i$ for some $L_i \in H^0(X, \det \mathcal{F})$, $i = 1, \ldots, n+1$;
2. $\mu_{\det \mathcal{F}}(Q) = 0$.

The condition (2) of this Definition means that Q gives an element of $\mathrm{Sym}^2(H^0(X, \det \mathcal{F}))$ which vanishes on the schematic image $\phi_{|M_{\det \mathcal{F}}|}(X)$. The study of ω-adjoint quadrics is useful to find extension classes supported on a divisor.

Theorem 2.7. *Let X be a compact complex smooth variety. Let $\mathcal{F}$ be a locally free sheaf of rank n such that $h^0(X, \mathcal{F}) \geq n+1$. Let $\xi \in H^1(X, \mathcal{F}^\vee)$ and W an $n+1$-dimensional subspace $W \subset \mathrm{Ker}\,\partial_\xi \subset H^0(X, \mathcal{F})$. If ξ is such that $\partial^n_\xi(\omega) = 0$, where ω is an adjoint form associated with W and ξ, then $[\omega] = 0$, providing that there are no ω-adjoint quadrics.*

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Proof. Let $\mathcal{B} = \{\eta_1, \ldots, \eta_{n+1}\}$ be a basis of W. Set ω_i for $i = 1, \ldots, n+1$ as above and denote by $\tilde{\omega}_i \in H^0(\det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W})$ the corresponding sections via

$$0 \to H^0(X, \det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W}) \to H^0(X, \det \mathcal{F}).$$

Recall that $\lambda^n W := \langle \omega_1, \ldots, \omega_{n+1} \rangle \subset H^0(X, \det \mathcal{F})$ is the vector space generated by the sections ω_i. Note also that the sheaf $\bigwedge^n W \otimes \mathcal{O}_X$ is trivial and choosing $\eta_1 \wedge \ldots \wedge \widehat{\eta_i} \wedge \ldots \wedge \eta_{n+1}$, $i = 1, \ldots, n+1$, as a basis for $\bigwedge^n W$ we obtain an isomorphism to $\mathcal{O}_X^{n+1}$.

The standard evaluation map $\bigwedge^n W \otimes \mathcal{O}_X \to \det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W}$ given by $\tilde{\omega}_1, \ldots, \tilde{\omega}_{n+1}$ gives the following exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \bigwedge^n W \otimes \mathcal{O}_X \longrightarrow \det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W} \longrightarrow 0 \tag{2.4}$$

which is associated with a class $\xi' \in \mathrm{Ext}^1(\det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W}, \mathcal{K})$. The sequence (2.4) fits into the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \bigwedge^n W \otimes \mathcal{O}_X & \longrightarrow & \det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle g} & & \\
0 & \longrightarrow & \mathcal{F}^\vee & \longrightarrow & \mathcal{E}^\vee & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0,
\end{array} \tag{2.5}$$

where f is the map given by the contraction by the sections $(-1)^{n+1-i} s_i$, for $i = 1, \ldots, n+1$, and g is given by the global section $\sigma \in H^0(X, \det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W})$ corresponding to the adjoint form ω. We have the standard factorization

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \bigwedge^n W \otimes \mathcal{O}_X & \longrightarrow & \det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W} & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle g} & & \\
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & \mathcal{F}^\vee & \longrightarrow & \mathcal{E}^\vee & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0
\end{array} \tag{2.6}$$

where the sequence in the middle is associated with the class $\xi'' \in H^1(X, \mathcal{K})$ which is the image of $\xi \in H^1(X, \mathcal{F}^\vee)$ through the map $H^1(X, \mathcal{F}^\vee) \to H^1(X, \mathcal{K})$. In particular, we obtain the commutative square:

$$\begin{array}{ccc}
H^0(X, \det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W}) & \longrightarrow & H^1(X, \mathcal{K}) \\
\uparrow & & \| \\
H^0(X, \mathcal{O}_X) & \longrightarrow & H^1(X, \mathcal{K}).
\end{array} \tag{2.7}$$

By commutativity, we immediately have that the image of $\sigma \in H^0(X, \det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W})$ through the coboundary map $H^0(X, \det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W}) \to H^1(X, \mathcal{K})$ is ξ''.

Tensoring by $\det \mathcal{F}$, the map $\mathcal{F}^\vee \to \mathcal{K}$ gives

$$\begin{array}{ccc} \mathcal{F}^\vee \otimes \det \mathcal{F} & \longrightarrow & \mathcal{K} \otimes \det \mathcal{F} \\ \| & \overset{\Gamma}{\nearrow} & \\ \bigwedge^{n-1} \mathcal{F} & & \end{array} \tag{2.8}$$

and, since $\xi \cdot \omega \in H^1(X, \mathcal{F}^\vee \otimes \det \mathcal{F})$ is sent to $\xi'' \cdot \omega \in H^1(X, \mathcal{K} \otimes \det \mathcal{F})$, we have that

$$H^1(\Gamma)(\xi \cdot \omega) = \xi'' \cdot \omega, \tag{2.9}$$

where $\xi \cdot \omega$ is the cup product.

By hypothesis $\partial^n_\xi(\omega) = \xi \cdot \omega = 0 \in H^1(X, \bigwedge^{n-1} \mathcal{F})$, so also $\xi'' \cdot \omega = 0 \in H^1(X, \mathcal{K} \otimes \det \mathcal{F})$, hence, the global section $\sigma \cdot \omega \in H^0(X, \det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W} \otimes \det \mathcal{F})$ is in the kernel of the coboundary map $H^0(X, \det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W} \otimes \det \mathcal{F}) \to H^1(X, \mathcal{K} \otimes \det \mathcal{F})$ associated to the sequence

$$0 \longrightarrow \mathcal{K} \otimes \det \mathcal{F} \longrightarrow \bigwedge^n W \otimes \det \mathcal{F} \longrightarrow \det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W} \otimes \det \mathcal{F} \longrightarrow 0. \tag{2.10}$$

This occurs if there exist $L^\sigma_i \in H^0(X, \det \mathcal{F})$, $i = 1, \dots, n+1$ such that

$$\sigma \cdot \omega = \sum_{i=1}^{n+1} \tilde{\omega}_i \cdot L^\sigma_i. \tag{2.11}$$

This relation gives the following relation in $H^0(X, \det \mathcal{F}^{\otimes 2})$:

$$\omega \cdot \omega = \sum_{i=1}^{n+1} L^\sigma_i \cdot \omega_i. \tag{2.12}$$

Then, equation (2.12) gives an adjoint quadric. By contradiction, the claim follows. □

Corollary 2.8. *In the hypothesis of Theorem 2.7 it holds that ξ is supported on D_W; that is, ξ_{D_W} is trivial. Moreover, if we further assume that $H^0(X, \mathcal{F}) = \operatorname{Ker} \partial_\xi$ and W is generic inside $H^0(X, \mathcal{F})$ it follows that ξ is supported on $D_\mathcal{F}$.*

Proof. The first claim follows by Theorem 2.7 and by Theorem 2.5. To show the second claim, we recall that by [23, Proposition 3.1.6] $D_\mathcal{F} = D_W$ since W is a generic $n+1$-dimensional subspace of $H^0(X, \mathcal{F})$. Then, the claim follows. □

Note that in the rest of the paper, $\mathcal{F}$ will be Ω^1_X and therefore, the assumption $H^0(X, \mathcal{F}) = \operatorname{Ker} \partial_\xi$ of the Corollary comes from the extremal liftability assumptions.

3. Nori families

We apply the notion of adjoint quadrics to the case where $\mathcal{F}$ is the cotangent sheaf Ω^1_X of a smooth variety. We stress that we want to find conditions on a family $\pi\colon \mathcal{X} \to B$ which ensure that the fibres are in the same birational class.

3.1. A notion of equivalence among families of Albanese type

The notion of Albanese type family behaves well under base change and we can introduce a notion of equivalence for this kind of families. Consider a family of Albanese type $\Phi\colon \mathcal{X}\to\mathcal{A}$ as in § 1.

3.1.1. Translation equivalence

If $s\colon B\to\mathcal{A}$ is a section of $p:\mathcal{A}\to B$, we define the translated family $\Phi_s:\mathcal{X}\to\mathcal{A}$ of Φ by the formula:

$$\Phi_s(x)=\Phi(x)+s(\pi(x)).$$

Notice that $\Phi_s:\mathcal{X}\to\mathcal{A}$ is a family of Albanese type. Two families Φ and Ψ over B are said to be *translation equivalent* if there exists a section σ of $p\colon\mathcal{A}\to B$ such that the *images* of Φ_σ and Ψ (fibrewise) coincide.

We recall also the following definition given in [23, definition 1.1.2]:

Definition 3.1. Two families of Albanese type $\Phi\colon\mathcal{X}\to\mathcal{A}$, $\Phi'\colon\mathcal{X}^{'}\to\mathcal{A}^{'}$ over, respectively, B and B' will be said *locally translation equivalent*, if there exist an open set $U\subset B$ an open set $U'\subset B'$ and a biregular map $\mu\colon U'\to U:=\mu(U')\subset B$ such that the pull-back families $\mu^*(\Phi_U)$ and $\Phi^{'}_{U'}$ are translation equivalent where U, U' are dense with respect to the classical topology on B, respectively, B'. We will say that Φ is *trivial* if $\mathcal{X}=X\times B$, $\mathcal{A}=A\times B$ and $\pi_A(\Phi(X_b))=\pi_A(\Phi(X_{b_0}))$ for all b where $\pi_A\colon A\times B\to A$ is the natural projection.

We will use the following:

Proposition 3.2. *An Albanese type family $\Phi\colon\mathcal{X}\to\mathcal{A}$ is locally translation equivalent to a trivial family if and only if the fibres X_b are birationally equivalent.*

Proof. See [23, Proposition 1.1.3]. □

3.2. Liftability assumptions

The following conditions are natural in order to find families locally translation equivalent to trivial families.

Definition 3.3. We say that a family $\pi\colon\mathcal{X}\to B$ of relative dimension n satisfies the extremal liftability conditions over a one-dimensional variety B if

1. $H^0(\mathcal{X},\,\Omega^1_{\mathcal{X}})\twoheadrightarrow H^0(X_b,\,\Omega^1_{X_b})$;
2. $H^0(\mathcal{X},\,\Omega^n_{\mathcal{X}})\twoheadrightarrow H^0(X_b,\,\Omega^n_{X_b})$

where the symbol $\twoheadrightarrow$ means that the homomorphism is surjective.

The above definition says that all the 1-forms and all the n-forms of the fibre X_b are obtained by restriction of forms defined on the family $\mathcal{X}$. Comparing the two conditions with the hypotheses of Theorem 2.7, we see that they ensure that $\partial^1_{\xi_b}=0$ and $\partial^n_{\xi_b}=0$,

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where $\xi_b \in H^1(X_b, \Theta_{X_b})$ is an infinitesimal deformation in the image of the Kodaira–Spencer map associated with $\pi\colon \mathcal{X}\to B$.

Proposition 3.4. *Let $\Phi\colon \mathcal{X} \to \mathcal{A}$ be an Albanese type family such that for every $b \in B$ it holds that $H^0(\mathcal{X}, \Omega^1_{\mathcal{X}}) \twoheadrightarrow H^0(X_b, \Omega^1_{X_b})$. Then up to shrinking B, the fibres of $p\colon \mathcal{A} \to B$ are isomorphic.*

Proof. Let $\mu_b \in \mathrm{Ext}^1(\Omega^1_{A_b}, \mathcal{O}_{A_b})$ be the class given by the family $p\colon \mathcal{A} \to B$, that is the class of the following extension:

$$0 \to \mathcal{O}_{A_b} \to \Omega^1_{\mathcal{A}|A_b} \to \Omega^1_{A_b} \to 0.$$

Now $\phi_b^*\mathcal{O}_{A_b} = \mathcal{O}_{X_b}$ and the map $\phi_b^*\mathcal{O}_{A_b} \to \phi_b^*\Omega^1_{\mathcal{A}|A_b}$ is generically injective; hence, it is injective because otherwise the kernel would be a torsion subsheaf of $\mathcal{O}_{X_b}$. Thus, we have the following exact sequence

$$0 \to \phi_b^*\mathcal{O}_{A_b} \to \phi_b^*\Omega^1_{\mathcal{A}|A_b} \to \phi_b^*\Omega^1_{A_b} \to 0$$

which fits into the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \phi_b^*\mathcal{O}_{A_b} & \longrightarrow & \phi_b^*\Omega^1_{\mathcal{A}|A_b} & \longrightarrow & \phi_b^*\Omega^1_{A_b} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{X_b} & \longrightarrow & \Omega^1_{\mathcal{X}|X_b} & \longrightarrow & \Omega^1_{X_b} & \longrightarrow & 0.
\end{array}$$

In cohomology, we have

$$\begin{array}{ccc}
H^0(X_b, \phi_b^*\Omega^1_{A_b}) & \longrightarrow & H^1(X_b, \mathcal{O}_{X_b}) \\
\downarrow & & \| \\
H^0(X_b, \Omega^1_{X_b}) & \longrightarrow & H^1(X_b, \mathcal{O}_{X_b})
\end{array}$$

so, by commutativity and by the hypothesis $H^0(\mathcal{X}, \Omega^1_{\mathcal{X}}) \twoheadrightarrow H^0(X_b, \Omega^1_{X_b})$, we immediately obtain $H^0(X_b, \phi_b^*\Omega^1_{\mathcal{A}|A_b}) \twoheadrightarrow H^0(X_b, \phi_b^*\Omega^1_{A_b})$ and hence the coboundary $\partial_{\mu_b}\colon H^0(A_b, \Omega^1_{A_b}) \to H^1(A_b, \mathcal{O}_{A_b})$ is trivial. Then we can apply the argument in cf. [7, Page 77, 78] on a conveniently small B. □

3.3. Nori families

Let $\Phi : \mathcal{X} \to \mathcal{A}$ be an Albanese type family over the unitary disc Δ. From $\Phi(\mathcal{X}) \hookrightarrow \mathcal{A}$, we obtain the basic cycle $[\mathcal{Z}] = [\mathcal{X}]^+ - [\mathcal{X}]^-$ as in § 1; see (1.1). Note that $[\mathcal{Z}] \in Z_h^{a-n}(\mathcal{A}/B)$. To the normal function defined by $\mathcal{Z}$, it is associated with its infinitesimal invariant $\delta_{\mathcal{Z}}$; see cf.[31].

Definition 3.5. An Albanese type family Φ is called Nori trivial if the infinitesimal invariant $\delta_{\mathcal{Z}}$ induced by the cycle $[\mathcal{Z}]$ is zero for the generic $b \in B$ (hence for all b).

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Transversality. Fix $s_1, \ldots, s_{n+1} \in H^0(\mathcal{A}, \Omega^1_{\mathcal{A}})$ such that $s_1 \wedge \cdots \wedge s_{n+1}$ induces, by fibre restriction, a non-trivial form $\Omega \in H^0(A_b, \Omega^{1+n}_{\mathcal{A}|A_b})$. Let $\xi_b \in H^1(X_b, T_{X_b})$ be an infinitesimal deformation given by the Kodaira–Spencer map. We remind the reader that, as in the previous Proposition 3.4, ϕ_b is the map $\phi_b \colon X_b \to A_b$ and we have the diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\Phi} & \mathcal{A} \\ {\scriptstyle j_b}\uparrow & \overset{\Phi_b}{\nearrow} & \uparrow{\scriptstyle i_b} \\ X_b & \xrightarrow{\phi_b} & A_b \end{array} \tag{3.1}$$

Let

$$r : \Phi_b^* \Omega^1_{\mathcal{A}} \to \Omega^1_{X_b}$$

be the restriction map and set $\eta_i = r(\Phi_b^*(s_i))$, $i = 1, \ldots, n+1$. In our case, the set $\mathcal{B} = \{\eta_i\}_{i=1}^{n+1}$ is a basis of a vector space $W \subset H^0(X_b, \Omega^1_{X_b})$. Suppose that $W \subset \mathrm{Ker}\partial_{\xi_b}$. Then the element $\Phi_b^*(s_1 \wedge \cdots \wedge s_{n+1})$ gives precisely an adjoint form $\omega_{\xi_b, W, \mathcal{B}}$ once it is restricted to X_b. In [23, Theorem 5.2.5], it is proved:

Theorem 3.6 (Transversality Criteria.). *If $\delta_{\mathcal{Z}}(b) = 0$ then for every $\sigma \in H^0(A_b, \Omega^n_{A_b})$ it holds*:

$$\int_X \omega_{\xi_b, W, \mathcal{B}} \wedge \overline{\phi_b^* \sigma} = 0.$$

3.4. Proof of Theorem [A]

The main tool for the proof is the Volumetric Theorem [23, Theorem 1.5.3] that we recall here for convenience.

Theorem 3.7. *Let $\Phi \colon \mathcal{X} \to \mathcal{A}$ be an Albanese type family such that $p \colon \mathcal{A} \to B$ has fibres isomorphic to a fixed abelian variety A. Let $W \subset H^0(A, \Omega^1_A)$ be a generic $(n+1)$-dimensional subspace and $W_b \subset H^0(X_b, \Omega^1_{X_b})$ its pull-back over the fibre X_b. Assume that for every point $b \in B$ it holds that $\omega_{\xi_b, W_b, \mathcal{B}_b} \in \lambda^n W_b$ where $\xi_b \in H^1(X_b, \Theta_{X_b})$ is the class given on X_b by $\pi \colon \mathcal{X} \to B$, then the fibres of $\pi \colon \mathcal{X} \to B$ are birational.*

We now prove Theorem [A] of the introduction.

Proof. By Proposition, 3.4 we can assume that $p \colon \mathcal{A} \to B$ is trivial, that is $\mathcal{A} \simeq A \times B$ and $p \colon \mathcal{A} \to B$ is the first projection. Up to base change, the Albanese family $\mathrm{alb}(\mathcal{X}) \colon \mathcal{X} \to \mathrm{Alb}(\mathcal{X})$ exists and by Proposition 3.2, our claim is equivalent to show that the Albanese family $\mathrm{alb}(\mathcal{X}) \colon \mathcal{X} \to \mathrm{Alb}(\mathcal{X})$ is locally translation equivalent to the trivial family. Hence, it is not restrictive to assume that $\mathrm{Alb}(\mathcal{X}) = A \times B$ too. In particular, we can restrict to consider only the case where $\mathrm{Alb}(X_b) = A$ and the map $\phi_b \colon X_b \to A$ is of degree one for every $b \in B$.

Denote by $\xi_b \in H^1(X_b, \Theta_{X_b})$ a class associated with an infinitesimal deformation of X_b induced by the fibration $\pi \colon \mathcal{X} \to B$. We know that $q \geq n+1$ where $q = \dim_{\mathbb{C}} A$. Let $\mathcal{B} := \{dz_1, \ldots, dz_{n+1}\}$ be a basis of an $n+1$-dimensional generic subspace W of $H^0(A, \Omega^1_A)$,

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(if $q = n+1$ we can take $H^0(A,\, \Omega^1_A) = W$). For every $b \in B$ let $\eta_i(b) := \mathrm{alb}(X_b)^* dz_i$, $i = 1, \ldots, n+1$. By standard theory of the Albanese morphism, it holds that $\mathcal{B}_b := \{\eta_1(b), \ldots, \eta_{n+1}(b)\}$ is a basis of the pull-back W_b of W inside $H^0(X_b,\, \Omega^1_{X_b})$. Let

$$\omega_i(b) := \lambda^n(\eta_1(b) \wedge \ldots \wedge \eta_{i-1}(b) \wedge \widehat{\eta_i(b)} \wedge \ldots \wedge \eta_{n+1}(b))$$

for $i = 1, \ldots, n+1$. Note that if $\omega'_i := dz_1 \wedge \ldots \wedge dz_{i-1} \wedge \widehat{dz_i} \wedge \ldots \wedge dz_{n+1}$ then $\omega_i(b) := \phi_b \omega'_i$, $i = 1, \ldots, n$. Since $\Phi\colon \mathcal{X} \to \mathcal{A}$ is a family of Albanese type, $\dim \lambda^n W_b \geq 1$, (actually if $q > n+1$ by [23, Theorem 1.3.3] it follows that $\lambda^n W_b$ has dimension $n+1$), and we can write: $\lambda^n W_b = \langle \omega_1(b), \ldots, \omega_{n+1}(b) \rangle$.

By extremal liftability assumptions, we can form the Massey class for every $[W] \in \mathbb{G}(n+1,\, q)$ where we denote by $\mathbb{G}(n+1,\, q)$ the Grassmannian of $n+1$-dimensional subspaces of $H^0(X_b,\, \Omega^1_{X_b})$.

Consider the following diagram in Dolbeaut's cohomology:

$$\begin{array}{ccc} H^0(X_b, \phi_b^* \Omega^n_A \otimes \omega_{X_b}) & \xrightarrow{\mu \circ j} & H^0(X_b, \omega_{X_b} \otimes \omega_{X_b}) \\ \downarrow{\scriptstyle j} & \nearrow{\scriptstyle \mu} & \\ H^{n,0}(X_b) \otimes H^{n,0}(X_b) & & \end{array} \tag{3.2}$$

By the identification $H^0(X_b,\, \phi_b^*(\Omega^n_A) \otimes \omega_{X_b}) = H^{n,0}(A) \otimes H^{n,0}(X_b)$, it follows that

$$j := \phi_b^* \otimes \mathrm{id}\colon H^{n,0}(A) \otimes H^{n,0}(X_b) \to H^{n,0}(X_b) \otimes H^{n,0}(X_b).$$

Now, we can consider the induced diagram of the symmetric part:

$$\begin{array}{ccc} H^{n,0}(A) \odot H^{n,0}(X_b) & \xrightarrow{\mu \circ j} & H^0(X_b, \omega_{X_b} \otimes \omega_{X_b}) \\ \downarrow{\scriptstyle j} & \nearrow{\scriptstyle \mu} & \\ \mathrm{Sym}^2 H^{n,0}(X_b) & & \end{array} \tag{3.3}$$

We set $V_b := \mathrm{Im}\phi_b^* \subset H^{n,0}(X_b)$. Recall the decomposition (1.4) in § 1

$$\mathrm{Sym}^2 H^{n,0}(X_b) = V_b \odot H^{n,0}(X_b) \oplus \mathrm{Sym}^2(\overline{\mathrm{Ann}(V_b)})$$

and that this direct sum induces homomorphisms

$$\nu_{X_b}\colon \mathrm{Sym}^2 \overline{\mathrm{Ann}}(V_b) \to H^0(X, \omega_{X_b}^{\otimes 2}) \tag{3.4}$$

and

$$\gamma_{X_b}\colon V_b \odot H^{n,0}(X_b) \to H^0(X, \omega_{X_b}^{\otimes 2}). \tag{3.5}$$

Now assume that for the generic W_b, the generic adjoint form ω has an adjoint quadric

$$Q := \omega \odot \omega - \sum_{i=1}^{n+1} \omega_i(b) \odot L_i \in \mathrm{Sym}^2 H^{n,0}(X_b). \tag{3.6}$$

By Definition 2.6, Q is in $\mathrm{Ker}(\mu)$.

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By the Transversality Theorem 3.6, ω vanishes on $\overline{V_b}$, hence $\omega \odot \omega$ is an element of $\mathrm{Sym}^2(\overline{\mathrm{Ann}(V_b)})$. On the other hand, recall that by definition the forms $\omega_i(b)$ are in V, hence $\sum_{i=1}^{n+1} \omega_i(b) \odot L_i$ is an element of $V_b \odot H^{n,0}(X_b)$. By the hypothesis that for each element η contained in $\overline{\mathrm{Ann}(V_b)}$, its square is not contained in the image of γ_{X_b}, we get that $\omega \odot \omega = 0$. By the Volumetric Theorem 3.7, the claim follows easily. □

Remark 3.8. We stress that the argument above is a local argument applied to the differential forms $\omega_i(b)$.

Corollary 3.9. *Let $\Phi : \mathcal{X} \to \mathcal{A}$ be a family of Albanese type where the general fibre A_b of $p : \mathcal{A} \to B$ is a generic abelian variety of dimension $a \geq 4$, the general fibre X_b of $\pi \colon \mathcal{X} \to B$ is such that $a - \dim X_b > 1$, and for each element η contained in $\overline{\mathrm{Ann}(V_b)}$, its square is not contained in the image of γ_{X_b}. If it satisfies the liftability conditions then the fibres of $\pi \colon \mathcal{X} \to B$ are in the same birational class. In particular, the fibres of a family $\pi \colon \mathcal{X} \to B$ of smooth varieties of general type all contained inside a fixed generic abelian variety of dimension ≥ 4 as cycles of codimension ≥ 2 are in the same birational class if $\pi \colon \mathcal{X} \to B$ satisfies the liftability assumptions and for each element η contained in $\overline{\mathrm{Ann}(V_b)}$, its square is not contained in the image of γ_{X_b}.*

Proof. By [23, Proposition 6.2.2], we know that $\Phi : \mathcal{X} \to \mathcal{A}$ is equivalent to a Nori trivial family. By Theorem [A], the claim follows. □

Remark 3.10. The necessary 'genericity conditions' on the abelian variety A_b of Corollary 3.9 are those which make it possible to apply the fundamental theorem which claims that the Abel–Jacobi map of a homologically trivial cycle of a generic abelian variety of dimension ≥ 4 is torsion modulo the largest abelian subvariety of the intermediate Jacobian; see [17, section 7.5] or the very clear exposition [21]. Actually, we can understand that genericity here means that A_b is outside a countable union of proper Zariski's closed sets.

Remark 3.11. We point out to the reader that since the Ceresa's cycle of an hyperelliptic curve C is trivial, Theorem [A] implies that a family $\pi \colon \mathcal{C} \to B$ of hyperelliptic curves satisfying liftability assumptions is a locally trivial family.

4. Families of divisors of a polarized abelian variety

We cannot use Nori's theorem in the case of Albanese type families of divisors. Nevertheless a statement as the one of Theorem [A] holds also in this case. First, we review some facts on divisors of an abelian variety.

4.1. Theta functions

Let $(A, \mathcal{L})$ be a $(d_1 \cdots d_a)$-polarized abelian variety, where A is a complex torus defined as a quotient of a vector space V of rank a by a lattice Λ, and $\mathcal{L}$ an ample line bundle on it. The algebraic equivalence class of line bundles of $\mathcal{L}$ is defined by a non-degenerate hermitian bilinear form H on V, whose imaginary part E is a bilinear form integer-valued on Λ. Since we are interested in the algebraic equivalence class of line bundles on A defined by $\mathcal{L}$, we may assume the characteristic of $\mathcal{L}$ to be 0. We also recall that $\mathcal{L}$ determines an isogeny

$$\phi_{\mathcal{L}}\colon A \longrightarrow \mathrm{Pic}^0(A)$$

which is defined as follows:

$$\phi_{\mathcal{L}}(z) := t_z^*(\mathcal{L}) \otimes \mathcal{L}^{-1}$$

A decomposition of V for $\mathcal{L}$ is a decomposition of $V = V_1 \oplus V_2$ into real vector spaces of rank a which induces a decomposition for $\Lambda = \Lambda_1 \oplus \Lambda_2$ into E-isotropic free $\mathbb{Z}$-modules of rank a. Such a decomposition of V for $\mathcal{L}$ induces moreover a decomposition of the lattice

$$\Lambda(\mathcal{L}) := \{v \in V \ : \ t_z^*\mathcal{L} \cong \mathcal{L}, [v] = z\} \tag{4.1}$$

into E-isotropic free $\mathbb{Z}$-modules of rank a, which we, respectively, denote by $\Lambda(\mathcal{L})_1$ and $\Lambda(\mathcal{L})_2$. The latter decomposition naturally induces a decomposition of the kernel of $\phi_{\mathcal{L}}$, which we denote by K. It is known, see cf. [3, Theorem 2.7 p.55], that

$$\{\theta_x^{\mathcal{L}} \ : \ x \in K_1\} \tag{4.2}$$

is a basis for $H^0(A, \mathcal{L})$, where

$$\theta_0^{\mathcal{L}}(z) := \sum_{\lambda \in \Lambda_1} e^{\pi(H-B)(z,\lambda) - \frac{\pi}{2}(H-B)(\lambda,\lambda)}$$

Here B denotes the $\mathbb{C}$-linear extension of $H|_{V_2 \times V_2}$, and for every x in K_1

$$\theta_x^{\mathcal{L}}(z) := \psi_x^{\mathcal{L}}(z)^{-1} \theta_0^{\mathcal{L}}(z + x)$$

where $\{\psi_\lambda\}_\lambda$ is the cocycle in $Z^1(\Lambda, \mathcal{O}_V)$ such that, for every λ in the lattice Λ and z in V, we have

$$\theta_0^{\mathcal{L}}(z + \lambda) = \psi_\lambda(z) \theta_0^{\mathcal{L}}(z)$$

Proposition 4.1. *Let $(A, \mathcal{L})$ be an abelian variety, and $\mathcal{D}$ be a divisor in the linear system $|\mathcal{L}|$. Then there is a commutative diagram*

$$\begin{array}{ccc} H^0(A, \Theta_A) & \xrightarrow{d_0\phi_{\mathcal{L}}} & H^1(A, \mathcal{O}_A) \\ \cong \big\downarrow & & \big\uparrow f \\ H^0(A, \Omega_A^{a-1}) & \xrightarrow{|_{\mathcal{D}}} & H^0(\mathcal{D}, \omega_{\mathcal{D}}) \end{array} \tag{4.3}$$

where the arrow on the right side of diagram 4.3 is the connecting homomorphism in the long exact cohomology sequence of the fundamental sequence of $\mathcal{D}$

$$0 \longrightarrow \mathcal{O}_A \longrightarrow \mathcal{O}_A(\mathcal{D}) \longrightarrow \omega_{\mathcal{D}} \longrightarrow 0 \tag{4.4}$$

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Proof. We assume that $\mathcal{D}$ is the zero locus of a holomorphic section s of $\mathcal{L}$. The complex space V can be naturally identified with the space of holomorphic vector fields Θ_A on A. Fixed ω a non-zero $(a,\,0)$-form on A, recall that the map

$$V \cong H^0(A, \Theta_A) \longrightarrow H^0(A, \Omega_A^{a-1}) \tag{4.5}$$

sends under this identification a vector v of V to the holomorphic $(a-1)$-form w obtained by contracting the $(a,\,0)$-form ω with the vector field $\frac{\partial}{\partial v}$.

The holomorphic function $\frac{\partial s}{\partial v}$ can be seen by adjunction as a holomorphic section of the canonical bundle of $\mathcal{D}$, which coincides with the restriction to $\mathcal{D}$ of the $(a-1)$-form w defined above. On the other side, the connecting homomorphism can be computed by using the fact that there is a canonical isomorphism of cohomology groups sequences

$$H^p(\Lambda, H^0(V, \pi^*(\cdot)) \cong H^p(A, \cdot)$$

where π denotes the projection of V onto A, and it holds (see also: [6] Proposition 1.1, p. 4):

$$f\left(\frac{\partial s}{\partial v}\right) = [\pi H(v, \lambda)_{\lambda\in\Lambda}] \tag{4.6}$$

It remains to compute $d_0\phi_{\mathcal{L}}(\frac{\partial}{\partial v})$. Let us consider $S := Spec(\mathbb{C}[\epsilon]/\epsilon^2)$ the scheme of dual numbers over $\mathbb{C}$ and A_S the base change. We have the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_A \longrightarrow \mathcal{O}^*_{A_S} \longrightarrow \mathcal{O}^*_A \longrightarrow 0$$

Its long cohomology sequence identifies $H^1(A,\,\mathcal{O}_A)$ with the kernel of the map $\mathrm{Pic}(A_S) \longrightarrow \mathrm{Pic}(A)$, which to a line bundle on A_S whose transition functions $g_{\alpha\beta} = g'_{\alpha\beta} + \epsilon g''_{\alpha\beta}$ associates the line bundle on A with transition functions $g'_{\alpha\beta}$. Moreover, under the identification $H^1(A,\,\mathcal{O}^*_A) \cong H^1(\Lambda,\,H^0(V,\,\mathcal{O}^*_V))$

$$\mathrm{Pic}(A_S) = H^1(\Lambda, H^0(V, \mathcal{O}^*_V) \otimes_{\mathbb{C}} \mathbb{C}[\epsilon]) \tag{4.7}$$

since A_S is defined through a flat base change. Now, for every z on A, $\phi_{\mathcal{L}}(z)$ is the line bundle of degree 0 with cocycles $[\{e^{2\pi i E(z,\lambda)}\}_\lambda]$. Hence, $d_0\phi_{\mathcal{L}}(\frac{\partial}{\partial v})$ is the line bundle on A_S whose cocycles, according to identification 4.7, are precisely

$$[\{e^{2\pi i\epsilon E(v,\lambda)}\}_\lambda] = [\{1 + 2\pi i E(v,\lambda)\epsilon\}_\lambda] \in H^1(\Lambda, H^0(V, \mathcal{O}^*_V) \otimes_{\mathbb{C}} \mathbb{C}[\epsilon])$$

In conclusion, we have

$$d_0\phi_{\mathcal{L}}\left(\frac{\partial}{\partial v}\right) = [\{2\pi i E(v,\lambda)\}_\lambda] \tag{4.8}$$

It is now easy to see that the two elements in the cohomology group $H^1(\Lambda,\,H^0(V,\,\mathcal{O}^*_V))$ are the same. Indeed, it is enough to show, by the definitions of group cohomology, that there exists a holomorphic function F on V such that, for every z on V and every λ on

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Λ, it holds that

$$\pi H(v,\lambda) = 2\pi i E(v,\lambda) + F(z+\lambda) - F(z) \tag{4.9}$$

But E is defined as the imaginary part of H, which is an alternating $\mathbb{R}$-bilinear form on V, and H can be recovered by E. Indeed, for every z and w on V it holds:

$$H(z,w) = iE(z,w) + E(iz,w)$$

In conclusion, with $F(z) := -\pi(iE(v,\, z) - E(iv,\, z))$, it is easily seen that F is $\mathbb{C}$-linear on V and that 4.9 holds true. □

4.2. The multiplication map

From diagram (4.3) and the long cohomology sequence of (4.4), it follows easily that

$$\begin{aligned} &Im(|_{\mathcal{D}}\colon H^0(A,\Omega_A^{a-1}) \longrightarrow H^0(\mathcal{D},\omega_{\mathcal{D}})) \cong H^1(A,\mathcal{O}_A) \cong V \\ &\overline{\mathrm{Ann}(V)} \cong Im(|_{\mathcal{D}}\colon H^0(A,\mathcal{L}) \longrightarrow H^0(\mathcal{D},\omega_{\mathcal{D}})). \end{aligned} \tag{4.10}$$

Moreover, we have clearly a commutative diagram

$$\begin{array}{ccc} \mathrm{Sym}^2 H^0(A,\mathcal{L}) & \xrightarrow{\ \mu\ } & H^0(A,\mathcal{L}^2) \\ \big\downarrow{\scriptstyle |_{\mathcal{D}}} & & \big\downarrow{\scriptstyle |_{\mathcal{D}}} \\ \mathrm{Sym}^2\overline{\mathrm{Ann}(V)} & \xrightarrow{\ \nu\ } & H^0(\mathcal{D},\omega_{\mathcal{D}}^2). \end{array} \tag{4.11}$$

Note that $\mathrm{Sym}^2H^0(A,\,\mathcal{L}) \cong s^{\otimes 2}\mathbb{C} \oplus s\otimes \overline{\mathrm{Ann}(V)} \oplus \mathrm{Sym}^2\overline{\mathrm{Ann}(V)}$. In particular, when the divisor $\mathcal{D} = (s=0)$ is reduced and irreducible, the map ν is injective if and only if the multiplication map μ is injective. Indeed if μ is not injective, then there exists a non-zero element $w = s\otimes t + \sum_j u_j \otimes v_j$ in $\mathrm{Sym}^2H^0(A,\,\mathcal{L})$ in the kernel of the multiplication map μ, and by the above decomposition, we can assume $(\sum_j u_j\otimes v_j)_{|\mathcal{D}} \neq 0$ in $\mathrm{Sym}^2\overline{\mathrm{Ann}(V)}$. Since the diagram (4.11) is commutative, this implies $\nu(\sum_j u_j \otimes v_j)_{|\mathcal{D}} = 0$, and thus, ν is not injective. On the other side, let us assume that $\sum_j u_j|_{\mathcal{D}} \otimes v_j|_{\mathcal{D}}$ is non-zero and belong to the kernel of ν, where u_j and v_j are non-zero holomorphic sections of $\mathcal{L}$. Then we have that $\mu(\sum_j u_j\otimes v_j) = \sum_j u_jv_j$ vanishes along $\mathcal{D}$. Hence, there exists $t\in H^0(A,\,\mathcal{L})$ such that $st = \sum_j u_jv_j$. It follows that μ is not injective.

4.3. On the injectivity of the multiplication map

Given now an abelian variety $(A,\,\mathcal{L})$, we want to give conditions which ensure the injectivity of the multiplication map μ. We begin by fixing a decomposition of V for $\mathcal{L}^2$ which, according to our previous discussion, induces a decomposition $K_1 \oplus K_2$ of $K := Ker(\phi_{\mathcal{L}^2})$. In particular, the same decomposition induces a decomposition $2K_1 \oplus 2K_2$ for the kernel of $\phi_{\mathcal{L}}$.

Let us assume that H is the non-degenerate hermitian form which corresponds to $\mathcal{L}$ according to Appell–Humbert theorem. We recall that, by [3, Lemma 1.2 p. 48],

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$K(\mathcal{L}) = \Lambda(\mathcal{L})/\Lambda$ and $K(\mathcal{L}^2) = \Lambda(\mathcal{L}^2)/\Lambda$, where

$$\Lambda(\mathcal{L}) = \{v \in V \;:\; \Im m\; H(v, \Lambda) \subseteq \mathbb{Z}\}$$
$$\Lambda(\mathcal{L}^2) = \{v \in V \;:\; 2\Im m\; H(v, \Lambda) \subseteq \mathbb{Z}\}$$

are lattices in V, and $K(\mathcal{L})_i \cong \mathbb{Z}_{d_1} \oplus \cdots\cdots \mathbb{Z}_{d_a}$ $(i = 1, 2)$, where $(d_1 \cdots d_a)$ is the polarization type of $\mathcal{L}$; see [3, Lemma 1.4 p. 50].

Moreover, $\Lambda(\mathcal{L}^2)$ contains the sublattice $\Lambda(\mathcal{L})$, and the quotient is isomorphic to $\mathbb{Z}_2^{2a}$, with $2\Lambda(\mathcal{L}^2) = \Lambda(\mathcal{L})$. On the other side, $K(\mathcal{L}^2)_i \cong \mathbb{Z}_{2d_1} \oplus \cdots\cdots \mathbb{Z}_{2d_a}$ it contains $K(\mathcal{L})_i$, and the quotient is isomorphic to $\mathbb{Z}_2^a$.

Hence, the following is a basis for $H^0(A, \mathcal{L})$:

$$\{\theta_x^{\mathcal{L}} \;\;:\;\; x \in 2K_1\} \tag{4.12}$$

Let us denote by $Z_2 := A[2] \cap K_1 \cong \mathbb{Z}_2^a$. For every $(x_1, x_2) \in 2K_1 \oplus 2K_1$, and $(y_1, y_2) \in K_1 \oplus K_1$ such that $y_1 + y_2 = x_1$ and $y_1 - y_2 = x_2$, it holds the following multiplication formula cf. [3, 1.3 Multiplication Formula, p. 182]:

$$\mu(\theta_{x_1}^{\mathcal{L}} \otimes \theta_{x_2}^{\mathcal{L}}) = \sum_{z \in Z_2} \theta_{y_2+z}^{\mathcal{L}^2}(0)\theta_{y_1+z}^{\mathcal{L}^2}$$

Let us denote $Z_2' := Z_2 \cap 2K_1 \cong \mathbb{Z}_2^{a-s}$, where s is the number of odd indexes among $(d_1 \cdots d_a)$. For a character $\rho\colon Z_2' \to \mathbb{C}^*$ of Z_2' we can define:

$$\theta_{(x_1,x_2),\rho} := \sum_{z \in Z_2'} \rho(z)\theta_{x_1+z}^{\mathcal{L}} \otimes \theta_{x_2+z}^{\mathcal{L}} \tag{4.13}$$

This is an element of $H^0(A, \mathcal{L}) \otimes H^0(A, \mathcal{L})$. We point out to the reader that the diagonal action of Z_2' on $2K_2 \times 2K_2$ leaves every holomorphic section in 4.13 invariant up to the multiplication by a non-zero constant, since

$$\theta_{(x_1+z,x_2+z),\rho} = \rho(z)\theta_{(x_1,x_2),\rho}$$

Hence, we have the following basis for the vector space $H^0(A, \mathcal{L}) \otimes H^0(A, \mathcal{L})$:

$$\left\{\theta_{(x_1,x_2),\rho} \;\;:\;\; [(x_1, x_2)] \in 2K_1 \oplus 2K_1 \Big/ \Delta_{Z_2'} \;\;\rho \in \widehat{Z_2'}\right\} \tag{4.14}$$

where $\Delta_{Z_2'}$ denotes the diagonal subgroup of $Z_2' \times Z_2' \subseteq 2K_1 \oplus 2K_1$.

We can now choose a complement W of Z_2' in Z_2; that is $Z_2 = Z_2' \oplus W$ as $\mathbb{Z}_2$-vector spaces. Now choose U a transversal subset for Z_2 in K_1, that is U is a subset of K_1 such that every (right or left) coset of Z_2 contains precisely one element of U. Thus, the set U contains $\prod_{i=j}^a d_j$ elements and the quotient $2K_1/Z_2'$ contains $\prod_{i=j}^a d_j/2^{a-s}$ elements.

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By means of this choice, we can fix the following basis for $H^0(A, \mathcal{L}^2)$:

$$\left\{\theta^{\mathcal{L}^2}_{(y,w,\rho)} \ : \ y \in U, \quad w \in W, \quad \rho \in \widehat{Z'_2}\right\} \tag{4.15}$$

where

$$\theta^{\mathcal{L}^2}_{(y,w,\rho)} := \sum_{z\in Z'_2} \rho(z)\theta^{\mathcal{L}^2}_{y+w+z}$$

We aim to express the multiplication map μ with respect to the basis (4.14) and (4.15). We have:

$$\begin{aligned}
\mu(\theta_{(x_1,x_2),\rho}) &= \sum_{z\in Z'_2} \rho(z)\mu(\theta^{\mathcal{L}}_{x_1+z} \otimes \theta^{\mathcal{L}}_{x_2+z}) \\
&= \sum_{z\in Z'_2}\sum_{t\in Z_2} \rho(z)\theta^{\mathcal{L}^2}_{y_2+t}(0)\cdot\theta^{\mathcal{L}^2}_{y_1+t+z} \\
&= \sum_{z,z'\in Z'_2}\sum_{w\in W} \rho(z)\theta^{\mathcal{L}^2}_{y_2+w+z'}(0)\cdot\theta^{\mathcal{L}^2}_{y_1+w+z+z'}
\end{aligned}$$

where, in the third equality, we decompose the summation variable by using the complement W. If in the last summation above, we replace z by $z'+z$, we get an expression of the multiplication map in terms of the elements of the canonical basis for the vector space $H^0(A, \mathcal{L}^2)$:

$$\begin{aligned}
\mu(\theta_{(x_1,x_2),\rho}) &= \sum_{z,z'\in Z'_2}\sum_{w\in W} \rho(z)\rho(z')\theta^{\mathcal{L}^2}_{y_2+w+z'}(0)\cdot\theta^{\mathcal{L}^2}_{y_1+w+z} \\
&= \sum_{w\in W}\left[\sum_{z'\in Z'_2}\rho(z')\theta^{\mathcal{L}^2}_{y_2+w+z'}(0)\right]\left[\sum_{z\in Z'_2}\rho(z)\theta^{\mathcal{L}^2}_{y_1+w+z}\right] \\
&= \sum_{w\in W} C_{(y_2,w,\rho)}\cdot\theta^{\mathcal{L}^2}_{(y_1,w,\rho)}
\end{aligned}$$

where we set:

$$C_{(t,w,\rho)} := \sum_{z\in Z'_2}\rho(z)\theta^{\mathcal{L}^2}_{t+w+z}(0)$$

where $w \in W$ and $t \in K_1$. We consider moreover the following set:

$$\Omega = \left\{(y,\bar{t}) \in U \times \left(K_1 \middle/ Z'_2\right) \ : \ y+t \in 2K_1\right\}$$

Let us consider the function $\psi\colon \Omega \longrightarrow 2K_1 \times 2K_1 \big/ \Delta_{Z'_2}$, which sends (y, t) to $(y + t,\, y-t)$. We now show that ψ is a bijection.

Indeed let $\overline{(u, v)}$ be a pair in the quotient $2K_1 \times 2K_1/\Delta_{Z'_2}$. We show that there exists a unique element y of the transversal U and a unique class $\bar{t}$ in the quotient group K_1/Z'_2 such that $\overline{(y+t,\, y-t)} = \overline{(u,\, v)}$.

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The sum $u+v$ belongs to $2K_1$. Thus, $u+v=2r$ for a certain r of K_1. However, r belongs to a unique coset of the form $y+Z_2$ with y in U. Thus, y is uniquely determined, and it holds that $u+v=2y$. By the same procedure, we determine a t of K_1 such that $u-v=2t$. Note that t is unique up to a 2-torsion in $2K_1$, since both u and v belong to $2K_1$. Hence, we have:

$$\begin{cases} 2(y+t)=2u \\ 2(y-t)=2v \end{cases} \tag{4.16}$$

In particular there exist two 2-torsion elements of Z_2', p and q, in such that:

$$\begin{cases} y+t=u+p \\ y-t=v+q \end{cases} \tag{4.17}$$

Since $u+v=2y$, we conclude that $p=q$ and our claim that the elements $\overline{(y+t,\, y-t)}$ and $\overline{(u,\, v)}$ coincide is proved. We conclude that:

$$\mu(\theta_{(y+t,y-t),\rho}) = \sum_{w\in W} C_{(t,w,\rho)} \cdot \theta^{\mathcal{L}^2}_{(y,w,\rho)}$$

It can be easily seen that the matrix of multiplication map μ with respect to the basis (4.14) and (4.15) splits into blocks, each for every character ρ inside the group of characters $\widehat{Z_2'}$. Moreover, the multiplication map is injective if and only if the restriction to the subspaces associated with the characters ρ of Z_2' is injective. We can denote the latter subspaces as follows:

$$\mathbb{V}_{y,\rho} := \left\langle \theta_{(y+t,y-t),\rho} \quad : t \in 2K_1/Z_2' \right\rangle$$
$$\mathbb{W}_{y,\rho} := \oplus_{w\in W} \left\langle \theta^{\mathcal{L}^2}_{(y,w,\rho)} \right\rangle$$

and the restrictions, respectively, by

$$\mu|_{\mathbb{V}_{y,\rho}} : \mathbb{V}_{y,\rho} \longrightarrow \mathbb{W}_{y,\rho}$$

We point out to the reader that even if we exchange x_1 and x_2 in definition (4.13), we get the same element inside the image of the projection $H^0(A,\mathcal{L})\otimes H^0(A,\mathcal{L}) \to \mathrm{Sym}^2(H^0(A,\mathcal{L}))$ and this exchange is induced by the change $t\mapsto -t$. Hence, the restriction of the multiplication map on the symmetric part $\mathrm{Sym}(H^0(A,\mathcal{L}))$ can be described on the different blocks, each corresponding to y and ρ, with the matrix:

$$M_\rho := (C_{t,w,\rho})_{t\in \pm 2K_1/Z_2',\, w\in W} \tag{4.18}$$

where the sign can be arbitrarily chosen, since the obtained element of $\mathrm{Sym}^2(H^0(A,\mathcal{L}))$ obtained is independent from this choice.

Remark 4.2. The theorem below describes the loci in the moduli space of polarized abelian varieties where the multiplication map is not injective. Indeed μ is injective precisely when for every y and for every character ρ the matrix M_ρ has maximal rank.

Theorem 4.3. *Let* $(A, \mathcal{L})$ *be a general* $(1, 1, \cdots, 2, \cdots 2)$*-polarized abelian variety of dimension* a*. Then the multiplication map* μ *is injective.*

Proof. Let us begin by the case in which the polarization type is $(2, \cdots 2)$. For such a polarization, the matrix M_ρ in 4.18 with $\rho \in Hom(\mathbb{Z}_2^g, \mathbb{C}^*)$ is just the scalar:

$$C_\rho = \sum_{z \in A[2]} \rho(z)\theta_z^{\mathcal{L}^2}(0)$$

Now inside the moduli space of $(2, 2, \cdots 2)$-polarized abelian variety it is easy to conclude that $C_\rho \neq 0$ holds for the general abelian variety since it holds for the product of 2-polarized elliptic curves. If we now take a general $(1, 1, \cdots 1, 2, \cdots 2)$-polarization we can consider $(A', \mathcal{L}')$ a $(2, \cdots, 2)$-polarized abelian variety with an isogeny $h\colon A' \longrightarrow A$ such that $h^*\mathcal{L} = \mathcal{L}'$. Then the multiplication map μ on the sections of $\mathcal{L}$ is just the restriction of the multiplication map

$$\mu_{A'}\colon \mathrm{Sym}^2 H^0(A', \mathcal{L}') \longrightarrow H^0(A', \mathcal{L}'^2) \tag{4.19}$$

to the symmetric product of the subvector space of the $Ker(h)$-invariant sections of $\mathcal{L}'$. □

Corollary 4.4. *Let* $(A, \mathcal{L})$ *be as above and* $\mathcal{D}$ *a general element in* $|\mathcal{L}|$*. The natural map*

$$\nu\colon \mathrm{Sym}^2\overline{\mathrm{Ann}(V)} \longrightarrow H^0(\mathcal{D}, \omega_{\mathcal{D}}^{\otimes 2}) \tag{4.20}$$

is injective.

Proof. This follows immediately by the above theorem and Diagram (4.11). □

4.4. The divisorial case

The following theorem, together with Corollary 4.4, is Theorem [C] from § 1.

Theorem 4.5. *Let* $\mathcal{D}$ *be a general divisor on a general* $(1, 1, \cdots, 1, 2, \cdots, 2)$*-polarized abelian variety of dimension* a*. Then for each element* η *contained in* $\overline{\mathrm{Ann}(V)}$*, its square is not contained in the image of* $V \odot H^0(\mathcal{D}, \omega_{\mathcal{D}})$ *with respect to the multiplication map*

$$\mu_{\mathcal{D}}\colon \mathrm{Sym}^2(H^0(\mathcal{D}, \omega_{\mathcal{D}})) \longrightarrow H^0(\mathcal{D}, \omega_{\mathcal{D}}^{\otimes 2}) \tag{4.21}$$

Proof. As in the proof of Theorem 4.3, it is enough to work on $(2, \ldots, 2)$-polarizations. First, we prove that the locus in the moduli space of polarized abelian varieties $(A, \mathcal{D})$ such that there is no element η in $\overline{\mathrm{Ann}(V)} \cong H^0(A, \mathcal{D})$ whose square is contained in the image of $V \odot H^0(\mathcal{D}, \omega_{\mathcal{D}})$ is open.

Indeed, the moduli space U of smooth divisors in the polarization of an a-dimensional $(2, \cdots, 2)$-polarized abelian variety is an open subset of a $\mathbb{P}^{d-1}$-bundle on the moduli space $\mathcal{A}_a$ of $(2, \cdots, 2)$-polarized abelian varieties of dimension a, where $d = 2^a$. Note

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that U is a smooth Kuranishi family, in fact

$$\dim U = \dim(\mathcal{A}_a) + d - 1$$

and for every element $[A, \mathcal{D}]$ of U, it is known that

$$\begin{aligned} h^1(\mathcal{D}, \mathcal{T}_{\mathcal{D}}) &= \dim Ext^1_{\mathcal{O}_{\mathcal{D}}}(\Omega^1_{\mathcal{D}}, \mathcal{O}_{\mathcal{D}}) \\ &= \dim(\mathcal{A}_a) + d - 1 \\ &= \dim U. \end{aligned} \tag{4.22}$$

Now for every element $[A, \mathcal{D}]$ of U, the canonical bundle of $\mathcal{D}$ is very ample and we can choose uniformizing coordinates $[X_1, \cdots, X_a, Y_0, Y_1, \cdots, Y_{d-1}]$ on $\mathbb{P}^N \cong \mathbb{P}(H^0(\mathcal{D}, \omega_{\mathcal{D}}))$, where $N = d + a$. Let H be the Hilbert scheme of closed subvarieties of $\mathbb{P}^N$ with Hilbert polynomial $p(n) = (2n)^a - (2n-1)^a$. The natural morphism of schemes $\phi: U \to H$ is smooth and finite onto a locally closed subscheme $Z \subset H$ by (4.22).

The decomposition of $H^0(\mathcal{D}, \omega_{\mathcal{D}})$ for every divisor $\mathcal{D}$ into the direct sum $V \oplus \overline{\mathrm{Ann}(V)}$ induces a decomposition $\mathbb{P}^N = \mathbb{P}(V' \oplus W')$, where V is generated by the forms $X_1 \cdots X_a$, and W by the remaining forms $Y_0 \cdots Y_{d-1}$.

The set Q of points of Z which are contained in a quadric of the form $w^2 - \sum_i v_i z_i$ in $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}}(2))$, where $w \in W'$ and the elements v_i are forms of V' not all equal to 0, is a proper locally closed subset of Z. Since ϕ is flat, the scheme theoretic counterimage of Q is also a proper locally closed subset of U.

We have shown that the locus in the moduli space of polarized abelian varieties $(A, \mathcal{D})$ such that there is no element η in $\overline{\mathrm{Ann}(V)} \cong H^0(A, \mathcal{D})$ whose square is contained in the image of $V \odot H^0(\mathcal{D}, \omega_{\mathcal{D}})$ is open. Therefore, it suffices to prove that the claim of the Theorem holds true in the case of a smooth divisor $\mathcal{D}$ in a product of a (2)-polarized elliptic curves.

To this purpose, we denote by A the product of $E_1, \cdots, E_a$ elliptic curves, each of them considered as the quotient of the complex plane by the lattice $\tau_i\mathbb{Z} \oplus 2\mathbb{Z}$, where τ_i denotes a certain element in the Siegel upper half-space, and equipped with the polarization $\mathcal{L}_i$ of type (2) induced by the divisor $2 \cdot 0$. For each i, we denote moreover by $\theta_0^{(i)}$ and $\theta_1^{(i)}$ the canonical theta functions which span the vector space of the global holomorphic sections of the polarization $\mathcal{L}_i$ on E_i.

Considering $\mathcal{L}$ to be the induced product polarization on A, we first fix a basis for the vector space of the global holomorphic sections of $\mathcal{L}$ and a convenient notation for its elements. Since $\mathcal{L}$ is the product of all the polarizations on the factors E_i, a basis for the global sections of $\mathcal{L}$ can be easily defined by considering all the possible products of sections on each factor E_i, arising by selecting $\theta_0^{(i)}$ or $\theta_1^{(i)}$ for each index i. Every global section of this basis corresponds to a unique subset of $\{1, \cdots, a\}$ containing the indices i of the factors on which $\theta_1^{(i)}$ has been selected. Hence, if we consider $\mathcal{P}$ to be the power set of $\{1, \cdots, a\}$, we can denote for every $S \in \mathcal{P}$ the following global section of $\mathcal{L}$:

$$\theta_S(z) := \prod_{i=1}^{a} \theta^{(i)}_{\chi_S(i)}(z_i)$$

where $\chi_S(i)$ is the characteristic function of S. It can be now easily seen that $\{\theta_S\}_{S \in \mathcal{P}}$ is a basis for $H^0(A, \mathcal{L})$. For the reader's convenience, we remark that $\theta_\emptyset$ is the element

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of the latter basis obtained by multiplying on every factor E_i the section $\theta_0^{(i)}$:

$$\theta_\emptyset(z) = \prod_{i=1}^{a} \theta_0^{(i)}(z_i)$$

Let us consider now a general non-zero section

$$s := \sum_{S\in\mathcal{P}} a_S \theta_S \in H^0(A, \mathcal{L}) \tag{4.23}$$

for some complex coefficients a_S such that its zero locus $\mathcal{D}$ is smooth.

We prove that the kernel of the multiplication map (4.21) is generated by elements of $\mathrm{Sym}^2\overline{\mathrm{Ann}(V)}$ of the form:

$$\theta_S \otimes \theta_T - \theta_A \otimes \theta_B \tag{4.24}$$

with $S \cup T = A \cup B$ and $S \cap T = A \cap B$. This will immediately imply our thesis that for each element η contained in $\overline{\mathrm{Ann}(V)}$, its square is not contained in the image of $V \odot H^0(\mathcal{D}, \omega_{\mathcal{D}})$, since we recall that the intersection of this space with $\mathrm{Sym}^2\overline{\mathrm{Ann}(V)}$ is trivial.

This statement can be proven on a suitable affine open subset of $\mathcal{D}$.

On each factor, the canonical theta functions $\theta_0^{(i)}$ and $\theta_1^{(i)}$ induce a covering $x_i : E_i \longrightarrow \mathbb{P}^1$, where:

$$x_i(z_i) = \frac{\theta_0^{(i)}(0)}{\theta_1^{(i)}(0)} \frac{\theta_1^{(i)}(z_i)}{\theta_0^{(i)}(z_i)} \tag{4.25}$$

This covering is of degree 2, branched over four distinct points 1, -1, δ_i, $-\delta_i$, which are, respectively, the images of the points 0, 1, $\frac{\tau_i}{2}$, $1 + \frac{\tau_i}{2}$.

Hence, the elliptic curve E_i is the Riemann surface which on an affine neighbourhood $\mathcal{U}_i$ with local coordinates (x_i, y_i) is defined by the equation:

$$h_i(x_i, y_i) := y_i^2 - (x_i^2 - 1)(x_i^2 - \delta_i^2) \tag{4.26}$$

The affine model in (4.26) is called the Legendre normal form of E_i (see also [5]).

Let us consider the affine neighbourhood of $\mathcal{U}$ of $\mathcal{D}$ defined as:

$$\mathcal{U} := \mathcal{D} - div(\theta_\emptyset)$$

Then $\mathcal{U}$ can be described as the Zariski closed subsed of the affine space $\mathbb{A}^{2a}$ with coordinates $x_1 \cdots, x_a, y_1, \cdots, y_a$ with defining equations (4.26) together with the local equation of $\mathcal{D}$, which can we obtain from (4.23) by dividing by $\theta_i^{(0)}$ for each i (according to the definition of x_i in (4.25)). Hence, the local equation of $\mathcal{D}$ on $\mathcal{U}$ is $f(x_1, \cdots, x_a) = 0$,

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where

$$f(x_1, \cdots, x_a) := \frac{s}{\theta_\emptyset} = \sum_{S \in \mathcal{P}} b_S X_S$$
$$X_S := \prod_{j \in S} x_j \tag{4.27}$$

for some complex coefficients b_S. For every j in $\{1, \cdots, a\}$, the tangent vector $\frac{\partial}{\partial z_j}$ is naturally identified with the holomorphic $a-1$-form

$$\omega_j := \frac{dx_1}{y_1} \wedge \cdots \wedge \widehat{\frac{dx_j}{y_j}} \wedge \cdots \wedge \frac{dx_a}{y_a}$$

(see (4.4) in the proof of Proposition 4.1).

Hence, on the affine subset $\mathcal{V}$ of $\mathcal{U}$ where $f_a := \frac{\partial f}{\partial x_a}$ does not vanish, we obtain, up to a sign:

$$\omega_j = \frac{f_j}{f_a \cdot y_1 \cdot y_2 \cdots \widehat{y_j} \cdots y_a} dx_1 \wedge \cdots \wedge dx_{a-1} \tag{4.28}$$

where f_j denotes the derivative of f with respect to x_j.

The global holomorphic differentials on $\mathcal{D}$ obtained by restricting the global sections of the polarization of A to $\mathcal{D}$ can be computed by applying the residue map $H^0(A, \mathcal{O}_A(\mathcal{D})) = H^0(A, \omega_A(\mathcal{D})) \longrightarrow H^0(\mathcal{D}, \omega_{\mathcal{D}})$. When restricted to $\mathcal{V}$ this gives, for each element S of $\mathcal{P}$,

$$\psi_S := (\theta_S \cdot dz_1 \wedge \cdots \wedge dz_a) \neg \left(\frac{1}{\theta_\emptyset f_a} \frac{\partial}{\partial x_a} \right) \tag{4.29}$$

where $\neg$ is the contraction operator. We have in conclusion

$$\begin{aligned} \psi_S &:= (\theta_S \cdot dz_1 \wedge \cdots \wedge dz_a) \neg \left(\frac{1}{\theta_\emptyset f_a} \frac{\partial}{\partial x_a} \right) \\ &= \left(\theta_S \cdot \frac{dx_1}{y_1} \wedge \cdots \wedge \frac{dx_a}{y_a} \right) \neg \left(\frac{1}{\theta_\emptyset f_a} \frac{\partial}{\partial x_a} \right) \\ &= \frac{\theta_S}{\theta_\emptyset} \frac{1}{f_a y_1 \dots y_a} dx_1 \wedge \cdots \wedge dx_{a-1} \end{aligned} \tag{4.30}$$

and since X_S as in (4.27) is equal to $\frac{\theta_S}{\theta_\emptyset}$ (up to a non-zero constant):

$$\psi_S = \frac{X_S}{f_a y_1 \cdots y_a} dx_1 \wedge \cdots \wedge dx_{a-1}. \tag{4.31}$$

Hence, if we multiply the expressions (4.28) and (4.31) by $f_a y_1 \cdots y_a$, we see that the elements in (4.24) become, up to a constant

$$X_S \otimes X_T - X_A \otimes X_B \tag{4.32}$$

which are mapped by the multiplication map to

$$X_S \cdot X_T - X_A \cdot X_B = X_{S \cap T} X_{S \cup T} - X_{A \cap B} X_{A \cup B} = 0$$

by the assumptions that $S \cup T = A \cup B$ and $S \cap T = A \cap B$.

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On the other side, if a linear combination of tensor elements $X_S \otimes X_T$, say

$$K = \sum_{S,T} a_{ST} X_S \otimes X_T$$

is mapped to zero by the multiplication map, then we have

$$\begin{aligned} 0 &= \sum_{S,T} a_{ST} X_S X_T \\ &= \sum_{U \subseteq V} \Bigg(\sum_{\substack{S,T \\ S\cap T=U \\ S\cup T=V}} a_{ST} \Bigg) X_U \cdot X_V \end{aligned}$$

This implies that for every couple of subsets U and V of $\{1, \cdots, a\}$ with $U \subseteq V$, we have:

$$\sum_{\substack{S,T \\ S\cap T=U \\ S\cup T=V}} a_{ST} = 0$$

since x_j appears in every term $X_U \cdot X_V$ with degree 0, 1 or 2, according to whether $j \notin V$, $j \in V - U$ or $j \in U$, which implies that the tensor element

$$K_{UV} := \sum_{\substack{S,T \\ S\cap T=U \\ S\cup T=V}} a_{ST} X_S \otimes X_T$$

is linear combination of elements of the form (4.32) with $S \cap T = U = A \cap B$ and $S \cup T = V = A \cup B$. Finally, also K must be linear combination of such elements, since clearly $K = \sum_{U \subseteq V} K_{UV}$.

Hence, our claim that the kernel of the multiplication map is generated by elements of the form as in (4.24) holds true once we prove that there are no other quadratic relations between the polynomials X_S for all $S \in \mathcal{P}$ and $Q_j := f_j y_j$ for $j \in \{1, \cdots, a\}$ in the quotient ring:

$$R := \mathbb{C}[x_1, \cdots, x_a, y_1, \cdots, y_a] \Big/ (h_1, \cdots, h_a, f)$$

We stress here that V corresponds to the vector space spanned by all polynomials Q_j, while the polynomials X_S span a vector subspace which, according to (4.10), corresponds precisely to the subspace $\overline{\mathrm{Ann}(V)}$. In particular, V is generated by elements which involve the letters y_j.

Since $X_\emptyset = 1$, we can define also the vector space W inside R generated as $\mathbb{C}$-vector space by all monomials in the letters x_j of degree at most 2 in each letter x_j.

A quadratic relation among the polynomials X_S and Q_j would give in R the following relation:

$$\beta + \sum_{j=1}^{a} \gamma_j y_j + \sum_{i,j=1}^{a} \eta_{ij} y_i y_j = 0 \tag{4.33}$$

where β, γ_j, $\eta_{ij} \in W$ since for η_{ij} we have the form

$$\eta_{ij} = c_{ij} f_i f_j$$

where c_{ij} are constants depending on the definition of the polynomials Q_j.

If we rearrange the terms of the equation (4.33) and we express it in the form of an algebraic relation between y_a and the remaining generators of R, we obtain:

$$\left[\beta + \sum_{j=1}^{a-1} \gamma_j y_j + \sum_{i,j=1}^{a-1} \eta_{ij} y_i y_j\right] + \left[\sum_{i=1}^{a-1} \eta_{ia} y_i + \gamma_a\right] y_a + \eta_{aa} y_a^2 = 0 \tag{4.34}$$

However, since by (4.26), the equality $y_a^2 = (x_a^2 - 1)(x_a^2 - \delta_a^2)$ holds in R, we obtain:

$$\left[\beta - \eta_{aa}(x_a^2 - 1)(x_a^2 - \delta_a^2) + \sum_{j=1}^{a-1} \gamma_j y_j + \sum_{i,j=1}^{a-1} \eta_{ij} y_i y_j\right] + \left[\sum_{i=1}^{a-1} \eta_{ia} y_i + \gamma_a\right] y_a = 0 \tag{4.35}$$

Since this holds in R and the polynomial which in the latter equation (4.35) multiplies y_a is of degree at most 1 in x_a, it must be 0 in R, since the only relation involving y_a is h_a, which is of degree 2 in y_a and 4 in x_a. Hence, $\eta_{ia} = 0$ for every $i \neq a$ and $\gamma_a = 0$. By applying the same procedure to each index, it follows that $\eta_{ij} = 0$ for every i and j with $i \neq j$, and $\gamma_i = 0$ for every i.

It follows that the quadratic relation (4.33) can be written in the following form:

$$\beta + \sum_{j=1}^{a} \eta_{jj} y_j^2 = 0.$$

Using again the relations $h_1, \cdots, h_a$, we can write:

$$\beta + \sum_{j=1}^{a} c_{jj} f_j^2 (x_j^2 - 1)(x_j^2 - \delta_j^2) = 0.$$

If at least one of the coefficients c_{jj} is non-zero, say c_{aa}, then there exists a polynomial $u = u(x_1 \cdots x_a)$ such that, the following relation holds in the polynomial ring $\mathbb{C}[x_1, \cdots, x_a]$:

$$\beta + \sum_{j=1}^{a} c_{jj} f_j^2 (x_j^2 - 1)(x_j^2 - \delta_j^2) = uf. \tag{4.36}$$

We recall that the variable x_a occurs in β with an exponent at most 2, hence it occurs in degree 4 in the left side of (4.36). Clearly, if u is equal to zero, then $c_{aa} = 0$ is a contradiction. Hence, assume $u \neq 0$. By definition, we can write $f = p + qx_a$, where x_a does not occur neither in p nor in q. This forces that u must have degree 3 in x_a.

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On the other hand, the left side of (4.36) does not contain monomials of degree 3 in x_a. This implies $p = 0$. Hence, x_a divides f and $\mathcal{D}$ is reducible, which contradicts our hypothesis. $\square$

4.5. The proof of Theorem [B]

We can show Theorem [B] by a direct argument not relying on Theorem 3.6, which we used in the case of Nori's families.

We consider, as in § 1, a family induced by embedded deformations $\pi\colon \mathcal{X} \to B$. By construction, it is obtained by shrinking to B a fibration $\Pi\colon \mathcal{Y} \to \hat{B}$ where $\mathcal{Y}$ is smooth and $\hat{B}$ is a curve. If $X \subset \mathcal{X}$ is a fibre and $U := \mathcal{Y} \setminus X$, by the composition of the residue homomorphism $H^{2n+1}(U, \mathbb{Z}) \to H^{2n}(X, \mathbb{Z})$ with the Gysin homomorphism $l_*\colon H^{2n}(X, \mathbb{Z}) \to H^{2n+2}(\mathcal{Y}, \mathbb{Z})$, we obtain the exact sequence

$$H^{2n+1}(U, \mathbb{Z}) \xrightarrow{\text{Res}} H^{2n}(X, \mathbb{Z}) \xrightarrow{l_*} H^{2n+2}(\mathcal{Y}, \mathbb{Z}) \tag{4.37}$$

We also recall the compatible identifications $\int_X\colon H^{2n}(X, \mathbb{C}) \to \mathbb{C}$, $\int_{\mathcal{Y}}\colon H^{2n+2}(\mathcal{Y}, \mathbb{C}) \to \mathbb{C}$.

4.5.1. The proof

By the assumptions of Theorem [B], $\pi\colon \mathcal{X} \to B$ satisfies extremal liftability conditions. Call $\Phi\colon \mathcal{X} \to A$ the morphism induced by the projection from the incidence variety. We stress that Φ induces the natural inclusion $X_b \hookrightarrow A$. Let $W := \langle \Phi^*\sigma_1, \ldots, \Phi^*\sigma_{n+1}\rangle$, where $\sigma_1, \ldots, \sigma_{n+1}$ are independent 1-forms on A. Set $s_i := \Phi^*\sigma_{n+1}$, $i = 1, \ldots, n+1$. Let $\Omega \in H^0(\mathcal{X}, \Omega^{n+1}_{\mathcal{X}})$ be the form induced by $s_1 \wedge s_2 \wedge \cdots \wedge s_{n+1}$ and $\xi_b \in H^1(X_b, T_{X_b})$ be an infinitesimal deformation given by the Kodaira–Spencer map. Let

$$r\colon \Omega^1_{\mathcal{X}} \to \Omega^1_{X_b}$$

be the restriction map and set $\eta_i = r(s_i)$, $i = 1, \ldots, n+1$. In this case, the set $\mathcal{B} = \{\eta_i\}_{i=1}^{n+1}$ is a basis of $H^0(X_b, \Omega^1_{X_b})$. Now we work on the fibre X and we denote $\phi\colon X \to A$ the morphism induced by Φ. Let t be a local parameter on B such that $(\pi^*(t) = 0)$ is the equation of X. We know that by the sheaf-homomorphism $\Omega^{n+1}_{\mathcal{X}}(\log(X)) \to \omega_X$, the residue of the form locally given by $\frac{\Omega}{\pi^*(t)}$ is the adjoint form $\omega_{\xi_b, W, \mathcal{B}}$, see Definition 2.1. We consider $H^0(\mathcal{X}, \Omega^n_{\mathcal{X}}) \ni \Omega_i$ the forms induced by $s_1 \wedge s_2 \wedge \cdots \wedge \widehat{s_i} \wedge \cdots \wedge s_{n+1}$, $i = 1, \ldots, n+1$. For every $i = 1, \ldots, n+1$ the form locally given by $\frac{\Omega}{\pi^*(t)} \wedge \overline{\Omega_i}$, gives an element of $H^{2n+1}(U, \mathbb{C})$ whose residue in $H^{2n}(X, \mathbb{C})$ is $\omega_{\xi_b, W, \mathcal{B}} \wedge \phi^*\overline{\omega_i}$. By the compatibility of the sequence (4.37) with both $\int_{\mathcal{Y}}$ and $\int_X$ it follows that for every $\sigma \in H^0(A_b, \Omega^n_{A_b})$ it holds:

$$\int_X \omega_{\xi_b, W, \mathcal{B}} \wedge \overline{\phi^*\sigma} = 0;$$

(in other words, the Transversality criteria applies also under the hypothesis of Theorem [B]). Finally, by Theorem 4.5, we have that for each element η contained in $\text{Ann}(V)$, its square is not contained in the image of $V \odot H^0(\mathcal{D}, \omega_{\mathcal{D}})$. Hence, we can conclude as in the proof of Theorem [A].

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Remark 4.6. We stress that the above proof of the Transversality claim in the case of divisors of an abelian variety does not make any generality assumption on the abelian variety.

5. The case of maximal relative irregularity for a fibred surface

We can use the strategy behind the proof of Theorem [A] to study a class of surfaces too. Let S be a smooth surface and let $f\colon S \to B$ be a fibration to a smooth curve B with general fibre F. Let $g(F)$ be the genus of F. In [32, Corollary 3 and 4], it is shown that a non-isotrivial fibration satisfies $q(S) - g(B) \leq \frac{5g(F)+1}{6}$. A very interesting class of fibrations is the one where $q(S) - g(B) = g(F) - 1$. These fibrations are called fibrations with maximal relative irregularity. In the same paper, Xiao showed that they can exist only for $g(F) \leq 7$. These fibrations have received a lot of attention in recent years thanks to the beauty of the interplay between surface theory and the theory of abelian varieties which they help to see [16, 30] and the bibliography there quoted. In order to understand the geometry of any fibration, it is natural to try to obtain information by relating the invariants of B and of (the general fibre) F to those of S. By definition, if $q(S) - g(B) = g(F) - 1$, there exists a hyperplane V of $H^0(F, \omega_F)$ such that the standard restriction homomorphisms $H^0(S, \Omega^1_S) \to H^0(F, \omega_F)$ has V as its own image. We need the following:

Lemma 5.1. *Let $f\colon S \to B$ be a non-isotrivial fibration with a general fibre of genus ≥ 3 and such that $q(S) - g(B) = g(F) - 1$. Then, the sublinear system induced by V is base point free.*

Proof. Take an infinitesimal deformation $\xi \in H^1(F, T_F)$ of F given by the Kodaira–Spencer map. Assume that for the general fibre F the image V of $H^0(S, \Omega^1_S) \to H^0(F, \omega_F)$ has base points. Since V is a hyperplane then Riemann–Roch theorem on curves implies that there exists a unique point $p_F \in F$ which is the base point of the linear system $|V|$. By the viceversa of the Adjoint theorem in the case of one-dimensional varieties, see [7], it follows that ξ is the Shiffer variation supported on p_F. This is a contradiction to [1, Corollary 6.11]; see also [9, Prop. 6.3.9] □

Our basic reference for this last part [23, Section V]. Since $q(S) - g(B) = g(F) - 1$ the Jacobians of the fibres have an abelian variety A' of dimension $g - 1$ in common. Let B^0 the open subscheme of B where $f\colon S \to B$ is smooth. By shrinking to open subsets $U \subset B^0$, the family $\mathrm{Alb}(S) \times_{\mathrm{Alb(B)}} B \to B$ obtained by standard universal properties restricts to a family $p\colon \mathcal{A}_U \to U$ whose fibres are all isomorphic to the dual A of A'. Note that $p\colon \mathcal{A}_U \to U$ is a family of polarized abelian varieties where the fibre is always isomorphic to A but the polarization on $A \times \{b\}$ is given by:

$$\Theta_b(\eta_1, \eta_2) = \int_{F_b} \phi_b^*(\eta_1) \wedge \overline{\phi_b^*(\eta_2)}$$

where $\phi_b\colon F_b \to A_b$ is given by the composition $\mathrm{alb}(S) \circ j_b\colon F_b \to \mathrm{Alb}(S)$, and $j_b\colon F_b \to S$ is the natural inclusion and A_b is a translate of A inside $\mathrm{Alb}(S)$.

Theorem 5.2. *Let $f\colon S \to B$ be a non-isotrivial fibration of maximal relative irregularity > 2 with fixed abelian variety A and let B^0 the open subscheme where it is smooth. If $\phi_b\colon F_b \to A$ has degree 1 then the infinitesimal invariant associated with the basic cycle of the Albanese type family obtained by restricting over open subschemes, $U \subset B^0$ is not zero.*

Proof. The problem is local over B. We can write $H^0(F_b, \Omega^1_{F_b}) = V \oplus^{\perp} s \cdot \mathbb{C}$ where $V = \phi_b^* H^0(A, \Omega^1_A)$ and s is a nontrivial section. The decomposition is an orthogonal one with respect to the standard pairing on F_b. In particular $\overline{\mathrm{Ann}(V)} = s \cdot \mathbb{C}$. By contradiction assume that the infinitesimal invariant associated with an Albanese type family over a neighbourhood U of b is zero. Let $\xi \in H^1(F, T_F)$ be an infinitesimal deformation of F given by the Kodaira–Spencer map of $f_{|f^{-1}(U)}\colon f^{-1}(U) \to U$. By Theorem 3.6, this means that all the adjoints obtained by ξ and by two-dimensional subspaces $W \subset V$ belongs to $V^{\perp} = s \cdot \mathbb{C}$. This means that if $W = \langle \eta_1, \eta_2 \rangle$ then there exists a constant $c \in \mathbb{C}$ such that $c \cdot s$ is an adjoint form associated with W, in other words $[c \cdot s] \in H^0(F, \omega_F)/W$ is the Massey product of η_1 and η_2. This implies that if we take a general $\eta \in V$ and a general two-dimensional subspace $\langle \eta_1, \eta_2 \rangle = W \subset V$ (in particular $\eta \notin W$, here we need $q(S) - g(B) > 2$, that is $g(F) \geq 4$), we can find a $\sigma \in W$ such that the Massey product of the two-dimensional subspace $\langle \eta, \sigma \rangle$ along ξ is zero. Indeed if $[c_i \cdot s] \in H^0(F, \omega_F)/W_i$ is the Massey product of $W_i = \langle \eta, \eta_i \rangle$, $i = 1, 2$ then $\sigma = c_1\eta_2 - c_2\eta_1 \in W$. By the Adjoint theorem, it follows that $\xi = 0$ if the linear system $\langle \eta, \sigma \rangle$ has no base points or that ξ is supported on the base points of $\langle \eta, \sigma \rangle$. By the genericity of η and W it follows that ξ is supported on the base points of the linear subsystem $V \subset H^0(F, \omega_F)$. By Lemma 5.1, we conclude that $\xi = 0$. This means that $f\colon S \to B$ has constant moduli; a contradiction. □

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