THE JOURNAL OF NAVIGATION (2007), **60**, 158–170.

Forum

doi:10.1017/S0373463307214122

The Vector Function for Distance Travelled in Great Circle Navigation

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Traditionally, on a great circle, the latitude or longitude of a waypoint is found by inspection. In this paper, using an elementary knowledge of vector algebra including linear combination of vectors and vector basis, we provide an easy method for finding the equation of a great circle path as a parameterized curve. By use of this vector function of distance travelled, the latitude and longitude of waypoints can be found based on the distance from departure point along a great circle. The approach is intended to appeal to the navigator who is interested in the mathematics of navigation and who, nowadays, solves his navigation problems with a personal computer.

KEY WORDS

1. Great Circle. 2. Linear Combination. 3. Geodesic.

1. INTRODUCTION. An equation for the geodesic on spherical surface, namely the great circle equation [1] [5] [10], could be found in many textbooks of variation calculus [4] [6] [11] and a well-known mathematics website [12]. Their great circle equation is ingenious and creative. In some articles [5] [10], they discuss the great circle equation determined by a given departure and destination and use the results to find the waypoint on a great circle. Their results encourage us to make an effort to discover a great circle equation. We give a concise method for finding the equation of great circle paths (on spherical surface) as vector functions of distance travelled.

Suppose one wishes to travel along the shortest path between two points on the Earth's surface with given latitude and longitude coordinates. What are the equations governing this route and how are they derived? What are the applications of these equations? The representations of the equations would dictate the complexity in the applications of these equations. The aim of this paper is to propose an equation of a great circle path as a parameterized curve so that the computation in various applications is straightforward.

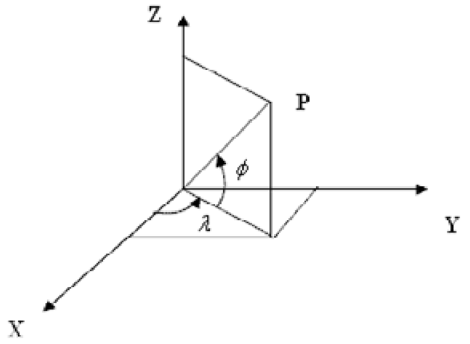

Figure 1. Spherical Polar Co-ordinates.

2. THE GREAT CIRCLE SAILING. For centuries, navigators have been interested in great circle paths because it is the shortest path between two points on the surface of the Earth provided that the Earth is assumed to be a perfect sphere. Given two points on the Earth, there are infinite planes containing these two points. Each of these planes intersects with the Earth and results in an intersection with the Earth as a circle. The great circle refers to the largest circle among these circles. A straight line is the shortest distance between two points on a plane. However when two points lie on the Earth's surface, the shortest arc joining them is the lesser arc on the great circle. The great circle has the greatest radius and the least curvature [2] [9] [11].

A standard reference for great circles and their algebraic formulation is the classic text by Bowditch [2] [7]. He teaches the concept of spherical navigation by using the laws of spherical trigonometry. By definition, a great circle is the intersection of a sphere with a plane that passes through the centre of the sphere. We assume that the Earth is a perfect sphere, and the path cut between two non-antipodal points on the surface of the Earth is the smaller portion of the great circle's arc joining the two points. Although a rigorous proof of the cut (shortest-distance) property follows from the calculus of variation [4] [6], one can visualize this by imagining a part of string along the great circle with its both ends attached at two ends. Then if the string is to lie on a different path on the surface of the earth with same ends location, it must be stretched.

3. THE VECTOR FUNCTION ON A SPHERE. The Earth's coordinate system can be replaced by a spherical coordinate. Therefore, the vector of a point on the surface of the Earth can be represented by latitude ϕ and longitude λ in the Cartesian coordinate system as Equation (1). We supposed that the radius is equal to 1 (Figure 1).

$$\vec{P} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T = \begin{bmatrix} cos\ \phi\ cos\ \lambda \\ cos\ \phi\ sin\ \lambda \\ sin\ \phi \end{bmatrix}^T, 0 < \lambda \leqslant 2\pi,\ -\pi/2 \leqslant \phi \leqslant \pi/2 \tag{1}$$

In mathematics, parametric functions are a bit like functions. They allow someone to fill in some variables, called parameters or independent variables, with any values they wish. When this is done, the equations then tell the values of dependent variables. A simple example is, in kinematics, filling in a time parameter to get the position, velocity, and other facts about a moving object.

4. THE VECTOR FUNCTIONS OF A GREAT CIRCLE.

There are two scenarios for determining a great circle; (1) a great circle can be determined by two points on the sphere, (2) a great circle can be determined by one point on the sphere and the angle at this point between the great circle and its own meridian.

4.1. *Scenario 1: A great circle determined by two points on the sphere.* If the departure and the destination are given no antipodal point, by using the relative longitude concept and replacing the Greenwich meridian by the meridian of departure point, the vector of the departure A and the destination B on a great circle can be expressed as:

$$\vec{A} = [\cos\phi_a, 0, \sin\phi_a] \tag{2}$$

where ϕ_a denotes the latitude of the departure.

$$\vec{B} = [\cos\phi_b \cos\lambda_d, \cos\phi_b \sin\lambda_d, \sin\phi_b] \tag{3}$$

where ϕ_b denotes the latitude of the destination and $\lambda_d = \lambda_b - \lambda_a$ denotes the longitude difference between the departure and the destination.

The great circle distance (d) is the angle between the vector $\vec{A}$ and the vector $\vec{B}$. The inner product of the two vectors gives the angle between two vectors. Therefore

$$\vec{A} \cdot \vec{B} = |\vec{A}| \cdot |\vec{B}| \cos d \tag{4}$$

$$\cos d = \cos\phi_a \cos\phi_b \cos\lambda_d + \sin\phi_a \sin\phi_b \tag{5}$$

The two vectors $\vec{A}$ and $\vec{B}$ are linearly independent, so the two vectors form a basis for the set of all vectors R^2 for the plane of the great circle. In the sense that every $\vec{P} \in R^2$ containing the great circle is a linear combination of $\vec{A}$ and $\vec{B}$ with a real coefficient.

The two decompositions of the vector $\vec{P}$ along a great circle are $\vec{U}$ and $\vec{V}$ (Figure 2) respectively. The vector $\vec{U}$ is parallel to $\vec{A}$ being from $\vec{B}$ to the end of $\vec{P}$ and the vector $\vec{V}$ lies in $\vec{B}$ being from origin point to intersection between $\vec{B}$ and $\vec{P}$. Using triangular sine formula gives Equation (6).

$$\frac{|\vec{U}|}{\sin(d-s)} = \frac{|\vec{V}|}{\sin(s)} = \frac{1}{\sin(2\pi - d)} \tag{6}$$

where s is the distance travelled from departure along the great circle.

After manipulating Equation (6), we get

$$\vec{U} = \frac{\sin(d-s)}{\sin(d)} \vec{A} \tag{7}$$

$$\vec{V} = \frac{\sin(s)}{\sin(d)} \vec{B} \tag{8}$$

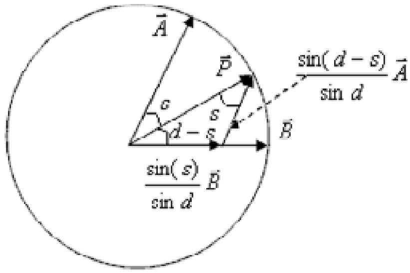

Figure 2. The linear combination of two vectors in a great circle.

The vector function $\vec{P}$ is the addition of the two vectors as Equation (9).

$$\vec{P}(s)=\begin{bmatrix} x(s) \\ y(s) \\ z(s) \end{bmatrix}^T=\frac{\sin(d-s)}{\sin d}\vec{A}+\frac{\sin(s)}{\sin d}\vec{B} \tag{9}$$

Substituting equation (2) (3) into Equation (9) yields

$$\vec{P}(s)=\begin{bmatrix} \cos\phi_a\sin(d-s)+\cos\phi_b\cos\lambda_d\sin(s) \\ \cos\phi_b\sin\lambda_d\sin(s) \\ \sin\phi_a\sin(d-s)+\sin\phi_b\sin(s) \end{bmatrix}^T \tag{10}$$

4.2. *Scenario 2: A great circle determined by one point on the sphere and an angle at this point between the great circle and the meridian.* Take a departure point on the Earth, along with an initial bearing course α, and then find the tangent vector $\vec{V}$ (velocity vector) of point A along great circle. However, navigational directions are given in the terms of a bearing angle α measured clockwise in degrees from north (Figure 3). If the departure point is A, then define the orthonormal vectors as Equation (11) and (12).

$$\vec{E}=[0 \quad 1 \quad 0] \tag{11}$$

$$\vec{N}=[-\sin\phi_a \quad 0 \quad \cos\phi_a] \tag{12}$$

At the departure, $\vec{E}$ is tangent to the latitude line, and $\vec{N}$ is tangent to the longitude line. Since the unit normal vector is $\vec{A}$ to the plane spanned by $\vec{E}$ and $\vec{N}$, and follow the definition of $\vec{A}$ given Equation (2), we have $\vec{A}\cdot\vec{E}=\vec{A}\cdot\vec{N}=0$. ($\vec{E}\times\vec{N}=\vec{A}$.) [3] [8]

Since the two normal vectors $\vec{E}$ and $\vec{N}$ can form an orthonormal basis for the set of all vectors in the tangent plane at a point A on a sphere, we may present velocity vector $\vec{V}$ by a linear combination of $\vec{E}$ and $\vec{N}$ (Figure 4), which is shown in Equation (13).

$$\vec{V}=\sin\alpha\cdot\vec{E}+\cos\alpha\cdot\vec{N}=\begin{bmatrix} -\sin\phi_a\cos\alpha \\ \sin\alpha \\ \cos\phi_a\cos\alpha \end{bmatrix}^T \tag{13}$$

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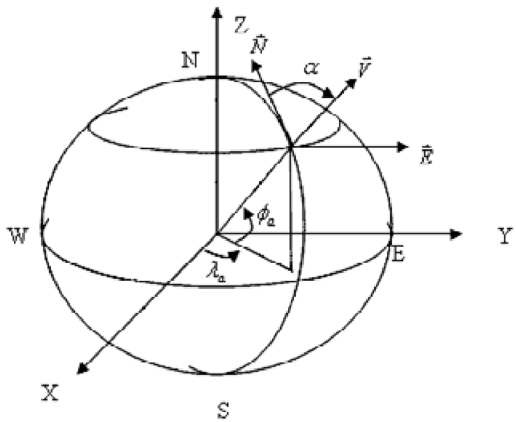

Figure 3. Latitude, longitude, and bearing angle.

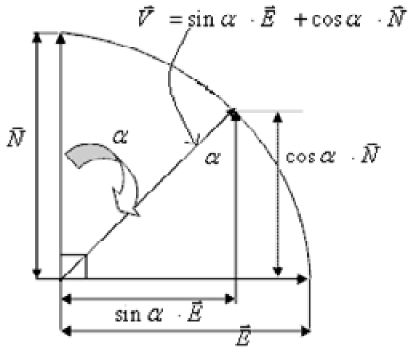

Figure 4. The linear combination of two orthonormal vectors.

Since the velocity vector $\vec{V}$ lies in the plane of the great circle, the vector $\vec{V}$ can be considered as the destination vector $\vec{B}$ and thus scenario 2 becomes scenario 1.

The vector $\vec{V}$ is orthogonal to $\vec{A}$ and on the plane of the great circle, so the great circle distance is $d=\pi/2$ between $\vec{A}$ and $\vec{V}$. Substitute the identity $\sin(\pi/2)=1$ into Equation (9) to obtain Equation (14).

$$\vec{P}(s) = \cos(s)\cdot\vec{A} + \sin(s)\cdot\vec{V} \tag{14}$$

Substituting equation (2) and (13) into equation (14) yields

$$\vec{P}(s) = \begin{bmatrix} \cos\phi_a\cos(s) - \sin\phi_a\cos\alpha\sin(s) \\ \sin\alpha\sin(s) \\ \sin\phi_a\cos(s) + \cos\phi_a\cos\alpha\sin(s) \end{bmatrix} \tag{15}$$

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Table 1. Longitude, latitude, and course – given distance to travel.

Time(hrs)	Distance(nm)	Longitude	Latitude	Course
0	0	121·0000E	35·0000N	47·7716
24	600	130·8603E	41·3352N	53·8835
48	1200	142·6537E	46·6590N	62·0993
72	1800	156·6252E	50·5341N	72·6076
96	2400	172·4178E	52·4918N	85·0014
120	3000	171·1785W	52·2255N	98·0238
144	3600	155·7350W	49·7801N	110·0577
164·6853	4117·133	144·0000W	46·2000N	118·7950

Now, a parameterized curve on the earth is represented by the vector function.

$$\vec{P}(s)=\mathbf{X}(s)=\begin{bmatrix} x(s) \\ y(s) \\ z(s) \end{bmatrix}=\begin{bmatrix} \cos\phi\cos\lambda \\ \cos\phi\sin\lambda \\ \sin\phi \end{bmatrix} \tag{16}$$

To obtain longitude and latitude from Equation (10) or (15) for a given distance along the great circle, where the support is $0 \leqslant s \leqslant 2\pi$.

$$\lambda(s)=\begin{cases} \tan^{-1}(y/x), x>0, y>0 \\ \tan^{-1}(y/x)+\pi, x<0 \\ \tan^{-1}(y/x), x>0, y<0 \end{cases} \tag{17}$$

$$\phi(s)=\sin^{-1}(z) \tag{18}$$

Further expansion of Equation (13) into trigonometric terms is unnecessary for computer evaluation.

In conclusion, we remark that in Scenario 1, if the two points are antipodal, then there are infinitely many great circles through them. In order to choose a particular one, it is necessary to specify (as in Scenario 2) an initial direction at one of the points.

5. ILLUSTRATIVE EXAMPLE FOR VALIDATION.

A ship is about to depart from one position to another. The navigator wishes to use the great circle sailing from A, latitude 35°00·0′N, longitude 121°00·0′E to B, latitude 46°12·0′N, longitude 144°00·0′W at 25 knots. Find the position of the ship on the great circle every 24 hours.

The longitudes, latitudes, and courses of waypoints from A with successive distance of 0 miles, 600 miles, 1200 miles, …, 3600 miles, and 4117·133 miles are found and are shown in Table 1.

6. CONCLUSION.

In this paper, we have provided an easy method for finding the vector function of a great circle on a spherical surface as parameterized. The method uses some basic vector analysis. The latitudes and longitudes along a great circle are directly calculated without recourse for the rules of spherical trigonometry. The parametric vector function is developed to calculate the latitude and longitude of the waypoints along a great circle path after sailing for some distance. The results have been verified for their correctness. For those proficient in computer programming languages, the expressions presented lend themselves readily to any particular discipline.

ACKNOWLEDGMENT

The authors express their appreciation for Professor Tai-Sheng Lee's and Professor Ho-ping Chou's many helpful and insightful suggestions.

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doi:10.1017/S0373463307224129

Building the Latitude Equation of the Mid-longitude

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We were amused by the Cross Track Distance at Mid-Longitude problem posed by Paul Hickley in The Journal of Navigation **57**, 320. Two professors, John Ponsonby and Peter Hoare, replied to the invitations immediately. Both their solutions to the original article give superb accuracy. The two solutions are certainly ingenious and creative and encouraged us to develop new formula for building the Mid-Longitude Equation on great circle. Regrettably, the original author doesn't think that the two were the solution that ATPL examiners were looking for. I also think that the two solutions would be beyond the capacity of the average undergraduate. Our method gives a good understanding logically and easily to be mnemonic, and the derivation process is found without any need to appeal to any formula of spherical trigonometry.

KEY WORDS

4. Great Circle Sailing. 5. Mid-Longitude Equation.

1. INTRODUCTION.

Using the properties of the great circle to construct the mid-longitude equation can provide the navigator and students with a more comprehensive and mnemonic formula. The final formula is that the sum of the tangents of two latitudes divided by the double cosine of mid-longitude is equal to the tangent of latitude at mid-longitude, that is:

$$tan\,\phi_m = \frac{tan\,\phi_1 + tan\,\phi_2}{2\,cos\,\lambda_m}$$

This is a smart mid-longitude equation! The equation is similar to an arithmetical average but additionally divided by the cosine of mid-longitude. An equation for the geodesic on spherical surface, namely "the great circle equation", could be found in many textbooks of variation calculus [2] [3] [4] and some well-known mathematics websites [8]. But the proofs of the great circle equation are all rigorous and would be beyond the capacity of the average student. We apply the fundamental knowledge of junior high school mathematics and the concepts of spherical and Cartesian coordinates to construct the great circle equation easily. Our proposed method gives a good logical understanding so the formula is easily remembered, and the derivation is found without any need for spherical trigonometry formulae.

2. METHODOLOGY.

The earth coordinate system can be replaced by spherical coordinates. Therefore, the vector of a point on the surface of the earth can be represented by latitude ϕ and longitude λ in the Cartesian coordinate system as Equation (1) (Figure 1) [5], and we assumed the spherical earth of radius unity (radius = 1).

$$\vec{P} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T = \begin{bmatrix} cos\,\phi\,cos\,\lambda \\ cos\,\phi\,sin\,\lambda \\ sin\,\phi \end{bmatrix}^T, \quad 0 < \lambda \leqslant 2\pi,\ -\pi/2 \leqslant \phi \leqslant \pi/2 \qquad (1)$$

By definition, a great circle is the intersection of a sphere with a plane that passes through the centre of the sphere. We assume that the earth is a perfect sphere, and the cut path between two non-antipodal points on the surface of the earth is the smaller portion of the great circle's arc joining the two points [1] [6] [7].

If the departure point and the destination point are given, the unit vector of the departure WP1 (A), the destination point WP2 (B) on the great circle can be expressed as follows. By using the relative longitude concept replace the Greenwich

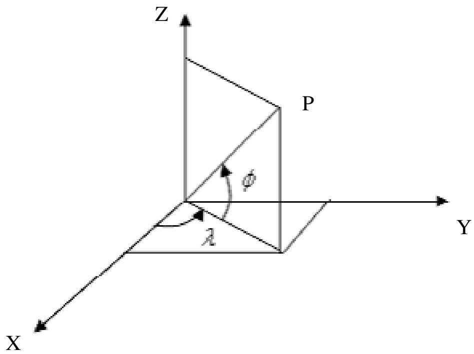

Figure 1. Spherical Coordinates.

meridian as the meridian of departure point.

$$\vec{A}=\begin{bmatrix}\cos\phi_a\\0\\\sin\phi_a\end{bmatrix}\quad\vec{B}=\begin{bmatrix}\cos\phi_b\cos d\lambda\\\cos\phi_b\sin d\lambda\\\sin\phi_b\end{bmatrix}\tag{2}$$

where ϕ_a, ϕ_b, and $d\lambda=\lambda_b-\lambda_a$ denotes the latitude of the departure, the latitude of the destination, and the longitude difference between the departure and the destination, respectively.

We propose two methods that can determine the equation of mid-longitude, which are (1) derived by the equation of the plane determined by the two points and the centre of the sphere, (2) derived by the equation of straight line on the Polar Gnomonic.

2.1. *Method 1. The equation of the plane determined by the two points and the centre of the sphere.* By definition, a great circle is the intersection of a sphere with a plane that passes through the centre of the sphere [1] [6] [7]. We assume that the earth is a perfect sphere, and the cut path between two non-antipodal points on the surface of the earth is the smaller portion of the great circle's arc joining the two points [7]. If given two points on the sphere, the equation of the plane determined by the two points and the centre of the sphere (Figure 2). The equation of the plane ΔOAB that passes through the centre of the sphere can be represented in Equation (3)

$$z=ax+by\tag{3}$$

Converting spherical coordinates to Cartesian coordinates, substituting point P into Equation (3) yields:

$$\sin\phi=a\cos\lambda\cos\phi+b\sin\lambda\cos\phi\tag{4}$$

Rearranging Equation (4) yields:

$$\tan\phi=a\cos\lambda+b\sin\lambda\tag{5}$$

The equation includes the two points A and B, substituting the spherical coordinates of A and B yields the simultaneous equations as follows:

$$\begin{cases}\tan\phi_a=a\cos 0^\circ+b\sin 0^\circ\\\tan\phi_b=a\cos d\lambda+b\sin d\lambda\end{cases}\tag{6}$$

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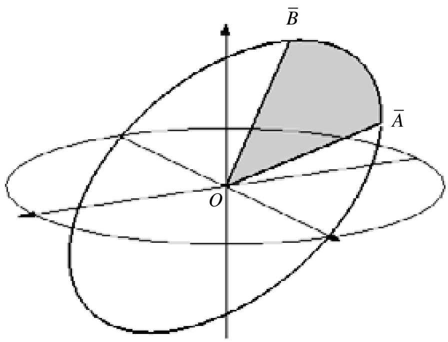

Figure 2. Plane determined by the two points and the centre of the sphere.

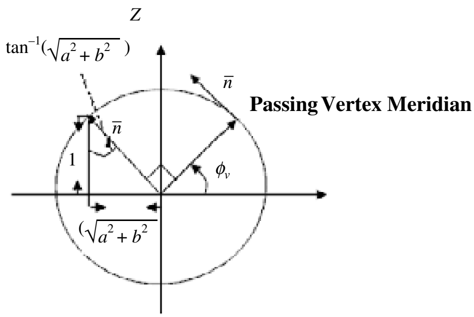

Figure 3. The vertex latitude presented by normal vector of the great circle.

The solutions of Equation (6) is given in Equation (7).

$$\begin{cases} a = \tan\phi_a \\ b = \dfrac{\tan\phi_b - \tan\phi_a \cos d\lambda}{\sin d\lambda} \end{cases} \tag{7}$$

The normal vector of the plane of great circle can be determined by those coefficients in Equation (3).

$$\vec{n} = (a, b, -1) \tag{8}$$

Additionally, by using Equation (8) we can find a concise way to give the points of vertex in the great circle. The latitude of vertex is 90 degrees minus the angle between the altitude of the normal vector, the meaning is shown in Figure 3, it yields:

$$\tan\phi_v = \sqrt{a^2 + b^2} \tag{9}$$

2.2. *Method 2. The equation of straight line along the Polar Gnomonic.* The Polar Gnomonic is the simplest of all projects to construct and understand. On this projection great circles are straight lines and at the pole, angles are correct. The accompanying Figure 4 illustrates the equation of straight line, namely the great circle equation, on Polar Gnomonic.

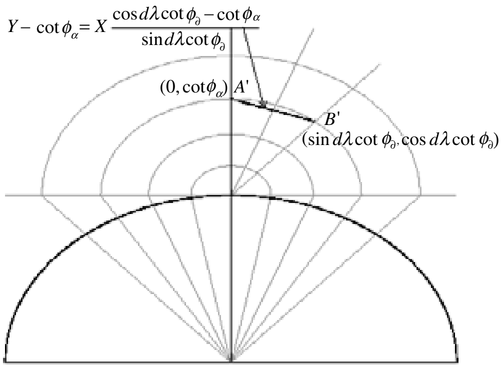

Figure 4. The straight line equation of great circle on the Polar Gnomonic.

The positions of the two points A, B on the earth projected onto the Polar Gnomonic are:

$$A' = (0, \cot\phi_a) \tag{10}$$

$$B' = (\sin d\lambda \cot\phi_b, \cos d\lambda \cot\phi_b) \tag{11}$$

The equation of straight line $\overline{A'B'}$ is:

$$Y - \cot\phi_a = X\frac{\cos d\lambda \cot\phi_b - \cot\phi_a}{\sin d\lambda \cot\phi_b} \tag{12}$$

Any point P on the earth projects onto Polar Gnomonic, the coordinate of positions is:

$$P' = (\sin\lambda \cot\phi, \cos\lambda \cot\phi) \tag{13}$$

Substituting Equation (13) into Equation (12) gives us:

$$\cos\lambda \cot\phi - \cot\phi_a = \sin\lambda \cot\phi \frac{\cos d\lambda \cot\phi_b - \cot\phi_a}{\sin d\lambda \cot\phi_b} \tag{14}$$

Rearranging Equation (14) yields:

$$\tan\phi = \left(\frac{\cos\lambda}{\cot\phi_a} - \frac{\sin\lambda \cos d\lambda}{\cot\phi_a \sin d\lambda} + \frac{\sin\lambda}{\sin d\lambda \cot\phi_b}\right) \tag{15}$$

Rearranging Equation (15) gives:

$$\begin{aligned} \tan\phi &= \tan\phi_a \cos\lambda + \left(\frac{\tan\phi_b - \tan\phi_a \cos d\lambda}{\sin d\lambda}\right)\sin\lambda \\ &= a\cos\lambda + b\sin\lambda \end{aligned} \tag{16}$$

where:

$$\begin{cases} a = \tan\phi_a \\ b = \dfrac{\tan\phi_b - \tan\phi_a \cos d\lambda}{\sin d\lambda} \end{cases} \tag{17}$$

The result of equation (16) is the same as equation (7).

3. MID-LONGITUDE EQUATION ON GREAT CIRCLE. We turn to the mid-longitude equation, in which the equation can govern the latitude at mid-longitude averaged the two longitudes of two points on the great circle determined by the two points. By definition, substituting equation (7) into equation (5) yields:

$$\tan(\phi) = \tan(\phi_a)\cos\lambda_m + \frac{\tan\phi_b - \tan\phi_a\cos d\lambda}{\sin d\lambda}\sin\lambda_m \tag{18}$$

where $d\lambda = 2\lambda_m$ is the mid-longitude of the departure and destination.

Now, we can apply the two circular functions in (19) into Equation (18) as follows,

$$\sin(2\lambda_m) = 2\sin\lambda_m\cos\lambda_m,\ \cos(2\lambda_m) = \cos^2\lambda_m - \sin^2\lambda_m \tag{19}$$

Substituting the two circular functions into Equation (18) gives

$$\tan(\phi) = \frac{2\tan(\phi_a)\cos^2\lambda_m + \tan\phi_b - \tan\phi_a(\cos^2\lambda_m - \sin^2\lambda_m)}{2\cos\lambda_m} \tag{20}$$

Rearranging Equation (20) yields:

$$\tan(\phi) = \frac{\tan(\phi_a) + \tan\phi_b}{2\cos\lambda_m} \tag{21}$$

4. NUMERICAL EXAMPLE. Now, we wish to respond the following question proposed by Paul Hickey in his article on "Great Circle Versus Rhumb Line Cross-Track Distance at Mid-Longitude" in the May 2004 Journal of Navigation.

Waypoint One is 60N 30W. Waypoint Two is 60N 20W. Your autopilot is coupled to the INS and you are steering from WP1 to WP2. What will be your latitude on passing 25 W?

This question is merely a special case which is included in equation (21). Write $\phi_a = \phi_b$ and rearrange equation (21) to obtain:

$$tan\,\phi = \frac{tan\,\phi_a}{cos\,\lambda_m} \tag{22}$$

In this specified condition, we also can use the formula of vertex Equation (9), and then get the same result in Equation (22).

$$\tan\phi_v = \pm\frac{\tan\phi_a}{\cos\lambda_m} \tag{23}$$

Substituting the numerical latitude and mid-longitude into Equation (22) gives us the results as follows:

$$\tan\phi = \tan 60^\circ \frac{1}{\cos(25^\circ - 20^\circ)} = \frac{\sqrt{3}}{2} \times \frac{1}{\cos 5^\circ} = 1{\cdot}738667$$

$$\tan^{-1}\left(\frac{\tan 60^\circ}{\cos 5^\circ}\right) = 60^\circ 05{\cdot}66991562303997'$$

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5. CONCLUSION. We give a neater and elegant solution without using spherical trigonometry. We use only the concept of converting spherical coordinates to Cartesian coordinates, fundamental simultaneous equations, and circular functions give us the mid-longitude equation. The scope of this knowledge can be learned from the mathematics of senior high school. Even if students lack this knowledge, the instructor can quickly give them a comprehensive understanding using our proposed processing steps.

The function of mid-longitude is a very easy mnemonic for those average ATPL candidates. Surely, the undergraduate candidates have to possess a fundamental knowledge of linear algebra; if not their qualifications have to be challenged.

ACKNOWLEDGMENT

The authors express their appreciation for Professor Hoping Chou's and Tai-Sheng Lee's many helpful and insightful suggestions.

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