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Cubic Functional Equations on Restricted Domains of Lebesgue Measure Zero

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Abstract. Let X be a real normed space, Y a Banach space, and $f\colon X \to Y$. We prove the Ulam–Hyers stability theorem for the cubic functional equation

$$f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)=0$$

in restricted domains. As an application we consider a measure zero stability problem of the inequality

$$\|f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)\| \le \epsilon$$

for all (x,y) in $\Gamma \subset \mathbb{R}^2$ of Lebesgue measure 0.

1 Introduction

Throughout this paper, we denote by $\mathbb{R}$, X, and Y the set of real numbers, a real normed space, and a real Banach space, respectively. A mapping $f\colon X \to Y$ is called cubic if f satisfies the equation

$$f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)=0 \tag{1.1}$$

for all $x, y \in X$. It is known [12, Theorem 2.1] that the general solutions f of (1.1) are given by $f(x) = B(x,x,x)$ for all $x \in X$, where $B\colon X \times X \times X \to Y$ is a symmetric function that is additive for each variable when the other two variables are fixed. The following is a particular result of Jun and Kim [12, Theorem 3.1] when $\phi(x,y) = \epsilon$ for all $x, y \in X$.

Theorem 1.1 *Let $\epsilon \ge 0$ be fixed. Suppose that $f\colon X \to Y$ satisfies the cubic functional inequality*

$$\|f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)\| \le \epsilon \tag{1.2}$$

for all $x, y \in X$. Then there exists a unique cubic mapping $C\colon X \to Y$ such that

$$\|f(x)-C(x)\| \le \frac{\epsilon}{14}$$

for all $x \in X$.

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It is a very natural subject to study functional equations or inequalities satisfied on restricted domains or satisfied under restricted conditions [1–8, 10, 13–15, 17–20]. Among the results, Jung (see [14]) and Rassias (see [18]) proved the Hyers–Ulam stability of the quadratic functional equations in a restricted domain. Here we state a slight modified version of the results in [14, 18].

Theorem 1.2 *Let $d > 0$. Suppose that $f: X \to Y$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \le \delta \tag{1.3}$$

for all $x, y \in D := \{(x,y) \in X \times X : \|x\| + \|y\| \ge d\}$. Then there exists a unique mapping $q: X \to Y$ satisfying

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.4}$$

for all $x, y \in X$ such that

$$\|f(x) - q(x)\| \le \frac{7}{2}\delta \tag{1.5}$$

for all $x \in X$.

Also, it is very natural to ask whether the restricted domain D in Theorem 1.2 can be replaced by a smaller subset $\Omega \subset D$ (*e.g.*, a subset of measure 0 if X is a measure space). In [9], the stability of (1.4) was considered in a set $\Omega \subset \{(x,y) \in \mathbb{R}^2 : |x| + |y| \ge d\}$ of Lebesgue measure $m(\Omega) = 0$ when $f: \mathbb{R} \to Y$. As a result, it was proved that if $f: \mathbb{R} \to Y$ satisfies (1.3) for all $(x,y) \in \Omega$, then there exists a unique quadratic mapping $q: \mathbb{R} \to Y$ satisfying (1.5).

In this paper, we consider the Ulam–Hyers stability of the functional equation (1.1) in some restricted domains $\Omega \subset X \times X$. First, as an abstract approach, imposing a condition (**C**) on Ω (see Section 2) we prove that if $f: X \to Y$ satisfies the inequality (1.2) for all $(x,y) \in \Omega$, then there exists a unique cubic mapping C such that

$$\left\| f(x) - C(x) - 48f(0) \right\| \le \frac{79}{14}\epsilon$$

for all $x \in X$. Since $\Omega = \{(x,y) \in X \times X : \|x\| + \|y\| \ge d\}$ satisfies condition (**C**), we obtain the parallel result for cubic functional equation as Theorem 1.2 for quadratic functional equation.

Secondly, when $X = \mathbb{R}$, constructing a subset $\Gamma_d \subset \mathbb{R}^2$ of measure 0 satisfying the condition (**C**) we consider a measure zero stability problem of the inequality (1.2); *i.e.*, we consider the inequality

$$\|f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x)\| \le \epsilon$$

for all $x, y \in \Gamma_d$, where $f: \mathbb{R} \to Y$ and $\Gamma_d \subset \{(x,y) \in \mathbb{R}^2 : |x| + |y| \ge d\}$ has 2-dimensional Lebesgue measure 0.

As an application we consider an asymptotic behavior of $f: \mathbb{R} \to Y$ satisfying the weak condition

$$\|f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x)\| \to 0$$

as $|x| + |y| \to \infty$ only for $(x,y) \in \Gamma_d$, where Γ has 2-dimensional Lebesgue measure 0.

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2 Stability of the Cubic Functional Equation in Restricted Domain

Given $x, y, z \in X$, define

$$P_{x,y,t} := \Big\{ (x-t, y+2t), (x-t, -y+2t), (x+t, y+2t), (x+t, -y+2t), (x, y+4t), (x, -y+4t), (-t, x+y+2t), (-t, x-y+2t), (t, x+y-2t), (t, x-y-2t), (-t, x+y+t), (-t, x-y+t), (t, x+y-t), (t, x-y-t), (x-t, y), (x+t, y), (x, y+2t), (x, y-2t), (0, x+y+4t), (0, x+y-4t), (0, x-y+4t), (0, x-y-4t), (0, x+y+3t), (0, x+y-3t), (0, x-y+3t), (0, x-y-3t), (0, x+y+2t), (0, x+y-2t), (0, x-y+2t), (0, x-y-2t), (0, t) \Big\}.$$

Then throughout this section, we assume $\Omega \subset X \times X$ satisfies the following condition:

(C) $$P_{x,y,t} \subset \Omega.$$

Theorem 2.1 *Let $\epsilon \ge 0$ be fixed. Suppose that $f: X \to Y$ satisfies the cubic functional inequality*

$$\|f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x)\| \le \epsilon$$

for all $(x, y) \in \Omega$. Then there exists a unique cubic mapping $C: X \to Y$ such that

$$\|f(x) - C(x) - 48f(0)\| \le \frac{79}{14}\epsilon \tag{2.1}$$

for all $x \in X$.

Proof Let

$$D(x,y) = f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x)$$

for all $x, y \in X$. Then we have

$$\begin{aligned}
D(x-t, y+2t) &= f(2x+y) + f(2x-y-4t) - 2f(x+y+t) \\
&\quad - 2f(x-y-3t) - 12f(x-t) \\
D(x-t, -y+2t) &= f(2x-y) + f(2x+y-4t) - 2f(x-y+t) \\
&\quad - 2f(x+y-3t) - 12f(x-t) \\
D(x+t, y+2t) &= f(2x+y+4t) + f(2x-y) - 2f(x+y+3t) \\
&\quad - 2f(x-y-t) - 12f(x+t) \\
D(x+t, -y+2t) &= f(2x-y+4t) + f(2x+y) - 2f(x-y+3t) \\
&\quad - 2f(x+y-t) - 12f(x+t)
\end{aligned}$$

$$\begin{aligned}
D(x, y+4t) &= f(2x+y+4t) + f(2x-y-4t) - 2f(x+y+4t) \\
&\quad - 2f(x-y-4t) - 12f(x) \\
D(x, -y+4t) &= f(2x-y+4t) + f(2x+y-4t) - 2f(x-y+4t) \\
&\quad - 2f(x+y-4t) - 12f(x)
\end{aligned}$$

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$$\begin{aligned}
D(-t, x+y+2t) &= f(x+y) + f(-x-y-4t) - 2f(x+y+t) \\
&\quad - 2f(-x-y-3t) - 12f(-t) \\
D(-t, x-y+2t) &= f(x-y) + f(-x+y-4t) - 2f(x-y+t) \\
&\quad - 2f(-x+y-3t) - 12f(-t) \\
D(t, x+y-2t) &= f(x+y) + f(-x-y+4t) - 2f(x+y-t) \\
&\quad - 2f(-x-y+3t) - 12f(t) \\
D(t, x-y-2t) &= f(x-y) + f(-x+y+4t) - 2f(x-y-t) \\
&\quad - 2f(-x+y+3t) - 12f(t)
\end{aligned}$$

$$\begin{aligned}
D(-t, x+y+t) &= f(x+y-t) + f(-x-y-3t) - 2f(x+y) \\
&\quad - 2f(-x-y-2t) - 12f(-t) \\
D(-t, x-y+t) &= f(x-y-t) + f(-x+y-3t) - 2f(x-y) \\
&\quad - 2f(-x+y-2t) - 12f(-t) \\
D(t, x+y-t) &= f(x+y+t) + f(-x-y+3t) - 2f(x+y) \\
&\quad - 2f(-x-y+2t) - 12f(t) \\
D(t, x-y-t) &= f(x-y+t) + f(-x+y+3t) - 2f(x-y) \\
&\quad - 2f(-x+y+2t) - 12f(t)
\end{aligned}$$

$$\begin{aligned}
D(x-t, y) &= f(2x+y-2t) + f(2x-y-2t) - 2f(x+y-t) \\
&\quad - 2f(x-y-t) - 12f(x-t) \\
D(x+t, y) &= f(2x+y+2t) + f(2x-y+2t) - 2f(x+y+t) \\
&\quad - 2f(x-y+t) - 12f(x+t)
\end{aligned}$$

$$\begin{aligned}
D(x, y+2t) &= f(2x+y+2t) + f(2x-y-2t) - 2f(x+y+2t) \\
&\quad - 2f(x-y-2t) - 12f(x) \\
D(x, y-2t) &= f(2x+y-2t) + f(2x-y+2t) - 2f(x+y-2t) \\
&\quad - 2f(x-y+2t) - 12f(x)
\end{aligned}$$

$$\begin{aligned}
D(0, x+y+4t) &= -f(x+y+4t) - f(-x-y-4t) - 12f(0) \\
D(0, x+y-4t) &= -f(x+y-4t) - f(-x-y+4t) - 12f(0) \\
D(0, x-y+4t) &= -f(x-y+4t) - f(-x+y-4t) - 12f(0) \\
D(0, x-y-4t) &= -f(x-y-4t) - f(-x+y+4t) - 12f(0)
\end{aligned}$$

$$\begin{aligned}
D(0, x+y+3t) &= -f(x+y+3t) - f(-x-y-3t) - 12f(0) \\
D(0, x+y-3t) &= -f(x+y-3t) - f(-x-y+3t) - 12f(0) \\
D(0, x-y+3t) &= -f(x-y+3t) - f(-x+y-3t) - 12f(0) \\
D(0, x-y-3t) &= -f(x-y-3t) - f(-x+y+3t) - 12f(0)
\end{aligned}$$

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$$D(0, x+y+2t) = -f(x+y+2t) - f(-x-y-2t) - 12f(0)$$
$$D(0, x+y-2t) = -f(x+y-2t) - f(-x-y+2t) - 12f(0)$$
$$D(0, x-y+2t) = -f(x-y+2t) - f(-x+y-2t) - 12f(0)$$
$$D(0, x-y-2t) = -f(x-y-2t) - f(-x+y+2t) - 12f(0)$$

$$D(0, t) = -f(t) - f(-t) - 12f(0).$$

Thus, we obtain the functional identity

(2.2)
$$\begin{aligned}
&f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x) + 672f(0) = \\
&\tfrac{1}{2}D(x-t, y+2t) + \tfrac{1}{2}D(x-t, -y+2t) + \tfrac{1}{2}D(x+t, y+2t) \\
&\quad + \tfrac{1}{2}D(x+t, -y+2t) - \tfrac{1}{2}D(x, y+4t) - \tfrac{1}{2}D(x, -y+4t) \\
&\quad + D(-t, x+y+2t) + D(-t, x-y+2t) + D(t, x+y-2t) + D(t, x-y-2t) \\
&\quad + D(-t, x+y+t) + D(-t, x-y+t) + D(t, x+y-t) + D(t, x-y-t) \\
&\quad - D(x+t, y) - D(x-t, y) + D(x, y+2t) + D(x, y-2t) - 48D(0, t) \\
&\quad + D(0, x+y+4t) + D(0, x+y-4t) + D(0, x-y+4t) + D(0, x-y-4t) \\
&\quad - D(0, x+y+3t) - D(0, x+y-3t) - D(0, x-y+3t) - D(0, x-y-3t) \\
&\quad - 2D(0, x+y+2t) - 2D(0, x+y-2t) \\
&\quad - 2D(0, x-y+2t) - 2D(0, x-y-2t)
\end{aligned}$$

for all $x, y, t \in X$. Since Ω satisfies the condition (**C**), for given $x, y \in X$, there exists $t \in X$ such that

(2.3)
$$\|D(u, v)\| \le \epsilon$$

for all $(u, v) \in P_{x,y,t}$. Thus, from (2.2) and (2.3) and using the triangle inequality we have

(2.4)
$$\begin{aligned}
&\|f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x) + 672f(0)\| \\
&\quad \le 79\epsilon
\end{aligned}$$

for all $x, y \in X$. Let $F(x) = f(x) - 48f(0)$ for all $x \in X$. Then from (2.4) we have

(2.5)
$$\|F(2x+y) + F(2x-y) - 2F(x+y) - 2F(x-y) - 12F(x)\| \le 79\epsilon$$

for all $x, y \in X$. Using Theorem 1.1 with (2.5), we get (2.1). This completes the proof. ∎

Remark Letting $x = 0$ in (2.1) and dividing the result by 47, we have $\|f(0)\| \le \frac{79\epsilon}{14\times 47}$. Thus, inequality (2.1) implies

$$\|f(x) - C(x)\| \le 48\|f(0)\| + \frac{79}{14}\epsilon \le \frac{7505}{658}\epsilon$$

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for all $x \in X$.

Let $d \geq 0$. It is easy to see that $\{(x, y) \in X \times X : \|x\| + \|y\| \geq d\}$ satisfies condition (**C**) Indeed, for given $x, y \in X$ if we choose $t \geq \|x\| + \|y\| + d$, then $P_{x,y,t} \subset \{(x, y) \in X \times X : \|x\| + \|y\| \geq d\}$. Thus, as a direct consequence of Theorem 2.1 we obtain the following result (see [14, 17, 18] for similar results).

Corollary 2.2 *Let $\epsilon, d \geq 0$ be fixed. Suppose that $f: X \to Y$ satisfies the cubic functional inequality*

$$\|f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x)\| \leq \epsilon$$

for all $x, y \in X$ such that $\|x\| + \|y\| \geq d$. Then there exists a unique cubic mapping $C: X \to Y$ such that

$$\|f(x) - C(x) - 48f(0)\| \leq \frac{79}{14}\epsilon$$

for all $x \in X$.

In particular, if $\epsilon = 0$, we have the following corollary.

Corollary 2.3 *Suppose that $f: X \to Y$ satisfies*

$$f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x) = 0 \tag{2.6}$$

for all $(x, y) \in \Omega$. Then the equation (2.6) holds for all $x, y \in X$.

3 Further Developments

Condition (**C**) is quite complicated, and it is not so easy to see what kind of set $\Omega \subset X \times X$ fulfills the condition. In this section, we show that even a set Ω of Lebesgue measure zero can satisfies the condition (**C**) when $X = \mathbb{R}$. From now on, we identify $\mathbb{R}^2$ with $\mathbb{C}$. The following lemma is a crucial key of our construction [16, Theorem 1.6].

Lemma 3.1 *There exists a set $K \subset \mathbb{R}$ of Lebesgue 0 such that $\mathbb{R} \setminus K$ is of first Baire Category,* i.e., *F is a countable union of nowhere dense subsets of $\mathbb{R}$, and K is of Lebesgue measure* 0*.*

Using Lemma 3.1 we obtain the following lemma.

Lemma 3.2 *Let $K \subset \mathbb{R}$ of Lebesgue measure* 0 *such that $K^c := \mathbb{R} \setminus K$ is of first Baire category. Then, for any countable subsets $U \subset \mathbb{R}$, $V \subset \mathbb{R} \setminus \{0\}$ and $M > 0$, there exists $\lambda \geq M$ such that*

$$U + \lambda V = \{u + \lambda v : u \in U, v \in V\} \subset K.$$

Proof Let

$$U = \{u_1, u_2, u_3, \ldots\}, \qquad V = \{v_1, v_2, v_3, \ldots\},$$
$$K^c_{m,n} = v_m^{-1}(K^c - u_n), \qquad m, n = 1, 2, 3, \ldots.$$

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Then, since K^c is of first Baire category, $K^c_{m,n}$ are also of first Baire category for all $m, n = 1, 2, 3, \ldots$. Since, each $K^c_{m,n}$ consists of a countable union of nowhere dense subsets, by the Baire category theorem, countable many of them cannot cover $[M, \infty)$, *i.e.*,

$$\bigcup_{m,n=1}^{\infty} K^c_{m,n} \not\supseteq [M, \infty).$$

Thus, there exists $\lambda \geq d$ such that $\lambda \notin K^c_{m,n}$ for all $m, n = 1, 2, 3, \ldots$. This means that $u_n + v_m \lambda \in K$ for all $m, n = 1, 2, 3, \ldots$. This completes the proof. ∎

Theorem 3.3 *Let $\Gamma_d = e^{-\frac{\pi}{6} i}(K \times K) \cap \{(x, y) \in \mathbb{R}^2 : |x| + |y| \geq d\}$. Then Γ_d satisfies condition (C) and has two-dimensional Lebesgue measure 0.*

Proof Let $x, y, t \in \mathbb{R}$ and let $P_{x,y,t}$ be the set in condition (**C**). We first prove that for every $x, y \in \mathbb{R}$, there exists $t \in \mathbb{R}$ such that

$$e^{\frac{\pi}{6} i} P_{x,y,t} \subset K \times K. \tag{3.1}$$

Since

$$e^{\frac{\pi}{6} i} P_{x,y,t} = \Big\{ \Big(\frac{\sqrt{3}}{2} u - \frac{1}{2} v, \ \frac{1}{2} u + \frac{\sqrt{3}}{2} v \Big) : (u, v) \in P_{x,y,t} \Big\},$$

the inclusion (3.1) is equivalent to

$$Q_{x,y,t} := \Big\{ \frac{\sqrt{3}}{2} u - \frac{1}{2} v, \ \frac{1}{2} u + \frac{\sqrt{3}}{2} v : (u, v) \in P_{x,y,t} \Big\} \subset K.$$

It is easy to see that the set $Q_{x,y,t}$ is written in the form $\{a_j x + b_j y + c_j t : j = 1, 2, \ldots, r\}$ for some $a_j, b_j, c_j \in \mathbb{R}$ with $c_j \neq 0$ for all $j = 1, 2, \ldots, r$. Let

$$U = \{a_j x + b_j y : j = 1, 2, \ldots, r\}, \qquad V = \{c_j : j = 1, 2, \ldots, r\}.$$

Then we have $Q_{x,y,t} \subset U + tV \subset K$. By Lemma 3.2, for given $x, y \in \mathbb{R}$ and $M > 0$ there exists $t \geq M$ such that

$$Q_{x,y,t} \subset U + tV \subset K.$$

Now, for given x, y if we choose $M \geq |x| + |y|$, then we have

$$P_{x,y,t} \subset \Big\{ (x, y) \in X \times X : |x\| + |y| \geq d \Big\}$$

for all $t \geq M$. Thus, Γ_d satisfies (**C**). This completes the proof. ∎

Now, as a direct consequence of Theorems 2.1 and 3.3 we have the following corollary.

Corollary 3.4 *Let $\epsilon, d \geq 0$ be fixed. Suppose that $f : \mathbb{R} \to Y$ satisfies the cubic functional inequality*

$$\|f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x)\| \leq \epsilon$$

for all $(x, y) \in \Gamma_d$. Then there exists a unique cubic mapping $C : \mathbb{R} \to Y$ such that

$$\|f(x) - C(x) - 48f(0)\| \leq \frac{79}{14}\epsilon$$

for all $x \in \mathbb{R}$.

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As a consequence of the Corollary 3.4 we obtain an asymptotic behavior of f satisfying the weak condition

$$\|f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)\| \to 0 \tag{3.2}$$

as $|x|+|y| \to \infty$ only for $(x,y) \in \Gamma_d$.

Corollary 3.5 *Suppose that $f:\mathbb{R} \to Y$ satisfies condition (3.2). Then f is a cubic mapping.*

Proof Condition (3.2) implies that for each $n \in \mathbb{N}$, there exists $d_n > 0$ such that

$$\|f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)\| \le \frac{1}{n}$$

for all $(x,y) \in \Gamma_{d_n}$. By Corollary 3.4, there exists a unique cubic mapping $C_n: X \to Y$ such that

$$\|f(x)-C_n(x)-48f(0)\| \le \frac{79}{14n} \tag{3.3}$$

for all $x \in \mathbb{R}$. Replacing n by m in (3.3) and using the triangle inequality we have

$$\|C_m(x)-C_n(x)\| \le \frac{79}{14n}+\frac{79}{14m} \le \frac{79}{7} \tag{3.4}$$

for all $x \in \mathbb{R}$. Let $C_{m,n}(x) = C_m(x) - C_n(x)$ for all $x \in X$. Then by (3.4), $C_{m,n}$ is a bounded cubic mapping. Thus, we have $C_{m,n} = 0$ and hence $C_m = C_n := C$ for all $m, n \in \mathbb{N}$. Letting $n \to \infty$ in (3.3) we have $f(0) = 0$ and hence $f(x) = C(x)$ for all $x \in \mathbb{R}$. This completes the proof. ∎

Remark 3.6 If we define $\Gamma \subset \mathbb{R}^{2n}$ as an appropriate rotation of $2n$-product K^{2n} of K, then Γ has $2n$-dimensional measure 0 and satisfies conditions (**C**). Consequently, we obtain the following theorem.

Theorem 3.7 *Suppose that $f:\mathbb{R}^n \to Y$ satisfies*

$$\|f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)\| \le \epsilon$$

for all $(x,y) \in \Gamma$. Then there exists a unique cubic mapping $C:\mathbb{R}^n \to Y$ such that

$$\|f(x)-C(x)-48f(0)\| \le \frac{79}{14}\epsilon$$

for all $x \in \mathbb{R}^n$.

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