

SURVEY

Partially hyperbolic dynamics in dimension three

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Abstract. Partial hyperbolicity appeared in the 1960s as a natural generalization of hyperbolicity. In the last 20 years, there has been great activity in this area. Here we survey the state of the art in some related topics, focusing especially on partial hyperbolicity in dimension three. The reason for this is not only that it is the smallest dimension in which non-degenerate partial hyperbolicity can occur, but also that the topology of 3-manifolds influences the dynamics in revealing ways.

1. Introduction

Partial hyperbolicity was introduced in the late 1960s as a generalization of the classical notion of hyperbolicity. In hyperbolic systems, the tangent bundle splits into two directions that are invariant under the derivative: one, the stable direction, is contracted, and the other, the unstable direction, is expanded. More precisely, a diffeomorphism of a compact manifold $f : M \rightarrow M$ is *hyperbolic* if the tangent bundle splits as $TM = E^s \oplus E^u$, where $Df(x)E_x^s = E_{f(x)}^s$ and $Df(x)E_x^u = E_{f(x)}^u$ and, for each pair of unit vectors $v^s \in E_x^s$ and $v^u \in E_x^u$,

$$\|Df(x)v^s\| < 1 < \|Df(x)v^u\|.$$

The simplest examples of this behavior are the hyperbolic (also known as Anosov) automorphisms on tori.

Partially hyperbolic diffeomorphisms, in turn, allow one extra, center direction, which is neither as expanded as the unstable one nor as contracted as the stable one. Again the

simplest examples are certain automorphisms of the torus, for instance

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This matrix has three eigenvalues: $\lambda > 1$, λ^{-1} and 1. Thus, the associated automorphism of \mathbb{T}^3 has three invariant bundles parallel to its three eigendirections. Two of these bundles, the ones associated to λ and λ^{-1} , have a hyperbolic behavior and the one associated to 1 corresponds to the center direction.

Another classical (and more interesting) example in dimension three comes from the action of the diagonal subgroup on quotients of $G = \text{PSL}(2, \mathbb{R})$. Consider a left invariant Riemannian metric on G .

Statement. The right multiplication by $d_1 = \begin{pmatrix} e^{1/2} & 0 \\ 0 & e^{-1/2} \end{pmatrix}$ induces a partially hyperbolic diffeomorphism of G , $\psi := R_{d_1} : G \rightarrow G$, $\psi(g) := R_{d_1}(g) = gd_1$.

Let

$$u_t^- = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad u_t^+ = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad d_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$$

be, respectively, the stable horocycle, the unstable horocycle and the diagonal 1-parameter groups. Consider the foliations generated by these 1-parameter groups

$$\mathcal{F}^s(g) = \{u_t^- g; t \in \mathbb{R}\}, \quad \mathcal{F}^u(g) = \{u_t^+ g; t \in \mathbb{R}\}, \quad \mathcal{F}^c(g) = \{d_t g; t \in \mathbb{R}\}.$$

ψ , respectively, contracts, expands and is an isometry on the leaves of these foliations. Notice that

$$u_s^\pm d_t = d_t u_{e^{\pm t} s}^\pm d_t. \tag{1.1}$$

This equation shows that ψ intertwines leaves of the foliation \mathcal{F}^s and, similarly, with \mathcal{F}^u . Moreover, ψ preserves the leaves of \mathcal{F}^c . We also get from equation (1.1) that

$$\psi(gu_s^\pm) = gu_s^\pm d_1 = gd_1 u_{e^{\pm 1} s}^\pm = \psi(g)u_{e^{\pm 1} s}^\pm.$$

Remember that we choose a left invariant metric on G , so, using its distance function, we obtain that

$$d(\psi(gu_s^\pm), \psi(g)) = d(gd_1 u_{e^{\pm 1} s}^\pm, gd_1) = d(u_{e^{\pm 1} s}^\pm, e) = d(gu_{e^{\pm 1} s}^\pm, g) \tag{1.2}$$

and

$$d(\psi(gd_t), \psi(g)) = d(gd_t d_1, gd_1) = d(gd_1 d_t, gd_1) = d(d_t, e) = d(gd_t, g). \tag{1.3}$$

From equation (1.2), we obtain that the leaves of \mathcal{F}^s are exponentially contracted in the future and the leaves of \mathcal{F}^u are exponentially contracted in the past by ψ . Equation (1.3) shows that ψ acts isometrically on the leaves of \mathcal{F}^c . Define $E_g^\sigma = T_g \mathcal{F}^\sigma$, $\sigma = s, u, c$ and observe that we obtain a splitting

$$E_g^s \oplus E_g^u \oplus E^c = T_g G,$$

where the first direction is exponentially contracted, the second is exponentially expanded and the third one is isometric.

Let us give a different description of the invariant bundles. Let us describe first what would be the partially hyperbolic splitting for the tangent space to the identity element $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The tangent space to the identity element $T_e G$ is naturally identified with the Lie algebra of G , $\mathfrak{g} = \mathfrak{sl}_2$. On \mathfrak{g} , we have three distinguished elements

$$U^- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad U^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

Observe that

$$[D, U^+] = U^+, \quad [D, U^-] = -U^-, \quad [U^+, U^-] = 2D. \quad (1.4)$$

Let $E_e^u = \text{span}\{U^+\}$, $E_e^s = \text{span}\{U^-\}$ and $E_e^c = \text{span}\{D\}$. Clearly, U^+ , U^- , D are linearly independent and hence

$$E_e^s \oplus E_e^u \oplus E_e^c = \mathfrak{g} = T_e G.$$

Given $g \in G$, let us define $R_g : G \rightarrow G$ by $R_g(h) = hg$. Let $D_e R_g : T_e G \rightarrow T_g G$ be the derivative of R_g over the identity and define $E_g^\sigma = D_e R_g(E_e^\sigma)$, $\sigma = s, c, u$. Let us show that these bundles are, accordingly, the unstable, stable and center bundle. To this end, we shall use $L_g : G \rightarrow G$, $L_g(h) = gh$, the left translation. Since the Riemannian metric we choose is a left invariant metric, we get that L_g is an isometry for every g . Observe that $(L_g)^{-1} = L_{g^{-1}}$.

Both ψ and the right group actions remain well defined if we take a quotient of $\text{PSL}(2, \mathbb{R})$ under the left action of a lattice Γ and, since the Riemannian metric we choose is left invariant, we obtain a Riemannian metric on the quotient $\Gamma \backslash \text{PSL}(2, \mathbb{R})$. The contraction and expansion properties are hence preserved on this quotient also. Recall that the 3-manifold $\Gamma \backslash \text{PSL}(2, \mathbb{R})$ can be naturally identified with the unit tangent bundle of a closed surface (compact in the case when Γ is co-compact) and ψ with the time-one map of the geodesic flow of a metric of constant negative curvature on this surface (again the extra direction is given by the direction of the diagonal flow); see [KH95].

Precisely, we say that a diffeomorphism f of a closed manifold M is partially hyperbolic if the tangent bundle splits as $TM = E^s \oplus E^c \oplus E^u$, where $Df(x)E_x^u = E_{f(x)}^u$, $Df(x)E_x^c = E_{f(x)}^c$ and $Df(x)E_x^s = E_{f(x)}^s$ and if, for each unit vector $v^s \in E_x^s$, $v^c \in E_x^c$ and $v^u \in E_x^u$,

$$\|Df(x)v^s\| < 1 < \|Df(x)v^u\| \quad \text{and} \quad (1.5)$$

$$\|Df(x)v^s\| < \|Df(x)v^c\| < \|Df(x)v^u\|. \quad (1.6)$$

The set of partially hyperbolic diffeomorphisms is \mathcal{C}^1 -open in $\text{Diff}^1(M)$ (see, for instance, Theorem 2.15 in [HPS77]). In other words, a \mathcal{C}^1 -perturbation of a partially hyperbolic diffeomorphism is partially hyperbolic.

1.1. *More examples.* The examples given in the Introduction fall into more general classes.

1.1.1. *Time-one maps of Anosov flows.* Consider an Anosov flow in a 3-manifold $\phi_t : M \rightarrow M$ such that the tangent bundle of M splits into three sub-bundles invariant under $D\phi_t: TM = E^s \oplus X \oplus E^u$, where X is the direction tangential to the flow, and such that, for each unit vector $v^s \in E^s$ and $v^u \in E^u$,

$$\|D\phi_t v^s\| < 1 < \|D\phi_t v^u\|.$$

Then the time-one map of the flow ϕ_t is a partially hyperbolic diffeomorphism (exercise).

Examples of this kind of partially hyperbolic diffeomorphism are the ones induced by the diagonal action on $\mathrm{PSL}(2, \mathbb{R})$ mentioned above.

As a different type of example, consider the suspension of an Anosov map $A : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ by the constant roof function one. Let A be a hyperbolic automorphism on \mathbb{T}^2 and consider in $\mathbb{T}^2 \times \mathbb{R}$ the equivalence relation $(x, t + 1) \sim (Ax, t)$. Then $M = \mathbb{T}^2 \times \mathbb{R} / \sim$ is a smooth manifold and $f([x, t]) = [x, t + 1]$ is a partially hyperbolic diffeomorphism.

We remark that these examples are truly different: for the second one the distribution $E^u \oplus E^s$ is integrable whereas for the first one it is not.

Note that, in both examples, the corresponding Anosov flow is transitive, but there also exist non-transitive Anosov flows [FW80a]. There is, in fact, a huge zoo of Anosov flows (see, for instance, [Bar98, BL94, Fen94, Fri83, HT80]); so classification of time-one maps of these flows is naturally a difficult task.

1.1.2. *Skew products.* Another example of a partially hyperbolic diffeomorphism is a certain kind of skew product that is a circle extension over the 2-torus of the form $f(x, \theta) = (Bx, h(x, \theta))$ where B is a hyperbolic automorphism of the 2-torus and $h(x, \cdot)$ are circle rotations. The resulting ambient manifold is a 3-nilmanifold. In the case when the product is direct, the ambient manifold is the 3-torus. To this class belongs A , the 3-toral automorphism defined at the beginning of this introduction.

1.1.3. *DA-diffeomorphisms.* A DA ('Derived from Anosov') partially hyperbolic diffeomorphism is one that is isotopic to an Anosov one. By a result of Franks, DA-diffeomorphisms are semi-conjugate to Anosov diffeomorphisms. The prototypical example in this class is Mañé's [Mañ78]. It is obtained by taking a linear Anosov map in \mathbb{T}^3 with eigenvalues $\lambda^{ss} < 1 < \lambda^u < \lambda^{uu}$ and making a bifurcation of the origin into three points along the weak unstable direction (see Figure 1). The resulting map is a dynamically coherent (robustly) transitive partially hyperbolic diffeomorphism.

2. Some open problems

Partial hyperbolicity appeared as a natural generalization of hyperbolicity. One main motivation for studying partially hyperbolic systems was ergodicity. A system $f : M \rightarrow M$ preserving a measure m is ergodic when, on average, any two events tend to be independent: that is, for any two measurable sets A and B ,

$$\lim_{n \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} m(f^n(A) \cap B) = m(A)m(B).$$

Ergodicity is therefore an interesting property from the physical point of view. Since conservative hyperbolic systems are ergodic [AS67], it was asked whether adding some

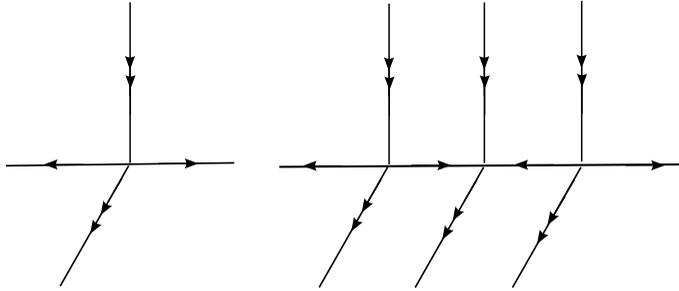


FIGURE 1. Mañé's DA partially hyperbolic diffeomorphism.

hyperbolicity to a system and then perturbing would generate ergodicity in the whole system. Concretely, Pugh and Shub [PS97] asked whether, for any system, taking the product of it by a sufficiently strong hyperbolic system (and hence obtaining a partially hyperbolic system) and then making a small perturbation would yield ergodicity in a robust way. They went further to state Conjecture 2.2 below. The area of partial hyperbolicity has become very active since then, and other aspects have attracted interest as well. These are also stated below.

2.1. Ergodicity. As stated above, one problem in partially hyperbolic dynamics is studying the ergodicity of conservative systems; to be more precise, of C^r -diffeomorphisms preserving a smooth volume. We shall denote conservative systems by $\text{Diff}_m^r(M)$, where m denotes the probability measure arising from this smooth volume. An equivalent definition of an ergodic system f is that any measurable set A satisfying $f(A) = A$ must also satisfy either $m(A) = 1$ or $m(A) = 0$. In other words, an ergodic system is one not admitting an invariant measurable set with intermediate measure.

An example of a non-ergodic partially hyperbolic diffeomorphism is the toral automorphism A defined at the beginning of the introduction. This automorphism can be seen in the following way: $A = B \times \text{id}$, where B is the automorphism on the 2-torus generated by the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and id is the identity on the circle. Indeed, any interval of the circle times the 2-torus is an invariant set with intermediate measure.

One can easily perturb this system to obtain an *ergodic* partially hyperbolic one. For example, $g = B \times (\text{irrational rotation})$. However, this new system g can be easily perturbed to obtain, again, non-ergodic diffeomorphisms. It is, in fact, approximated by diffeomorphisms $g' = B \times (\text{rational rotation})$, which are non-ergodic.

Pugh and Shub were the first to conjecture that ergodicity is, in fact, very abundant in the partially hyperbolic world; a conjecture that remains open today (Conjecture 2.2). As a matter of fact, this conjecture was made public for the first time in Montevideo, in 1995 [PS96]. We thank Burns for recalling this fact, and Mike Shub for confirming it. Pugh and Shub considered not only ergodicity, but a stronger concept, as given by the following definition.

Definition 2.1. A conservative C^2 diffeomorphism $f : M \rightarrow M$ is stably ergodic (in $\text{Diff}_m^1(M)$) if there exists $U \subset \text{Diff}_m^1(M)$, a neighborhood of f , such that every $g \in U$ of class C^2 is ergodic.

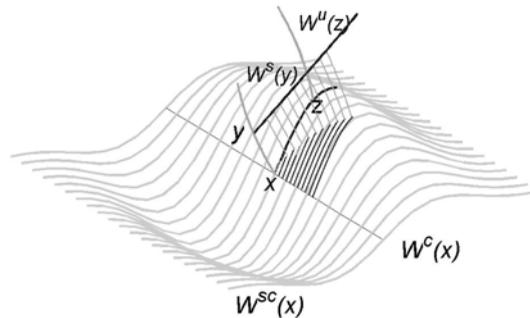
Until 1994, the only known examples of stably ergodic diffeomorphisms were Anosov diffeomorphisms, that is, hyperbolic ones. In 1994, Grayson, Pugh and Shub found the first non-hyperbolic examples. A year later, Pugh and Shub made the following conjecture.

CONJECTURE 2.2. (Pugh and Shub (1995) [PS96]) *Stable ergodicity is C^r -dense among volume-preserving partially hyperbolic diffeomorphisms, for all $r > 1$.*

Pugh and Shub suggested a program in order to prove their conjecture. It involves accessibility, as defined below.

Definition 2.3. Two points x and y are in the same *accessibility class* if there is a path that is piecewise tangential to E^s or E^u joining them. The partially hyperbolic diffeomorphism f has the *accessibility property* if there is only one accessibility class. The diffeomorphism has the *essential accessibility property* if any measurable set that is a union of accessibility classes has either full or null measure.

In the picture below, the points x , y and z are in the same accessibility class.



Exercise 2.4. Prove that the example of the action of the diagonal subgroup defined on page 2802 has the accessibility property.

Obviously, accessibility implies essential accessibility, but the converse is not true.

Exercise 2.5. Show that the automorphism on \mathbb{T}^3 defined by

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & -1 & 5 \end{pmatrix}$$

has the essential accessibility property, but does not have the accessibility property.

CONJECTURE 2.6. (Pugh and Shub) *If f is a C^r conservative diffeomorphism, with $r > 1$, (essential) accessibility implies ergodicity.*

CONJECTURE 2.7. (Pugh and Shub) *Stable accessibility is C^r -dense among volume-preserving partially hyperbolic diffeomorphisms, for $r > 1$.*

In [BW10], Burns and Wilkinson proved Conjecture 2.6 under the additional condition of *center bunching* (which means that the hyperbolicity of the $E^s \oplus E^u$ -bundle is stronger than the non-conformality of E^c). Previously, Dolgopyat and Wilkinson [DW03] had

proved that stable accessibility is C^1 -dense. Recently, Avila, Crovisier and Wilkinson have shown that stable ergodicity is C^1 -dense among C^r partially hyperbolic diffeomorphisms, with $r > 1$.

THEOREM 2.8. (Avila, Crovisier and Wilkinson [ACW14]) *Stable ergodicity is C^1 -dense among C^r volume-preserving partially hyperbolic diffeomorphisms, $r > 1$.*

In their work, they build on an approach by [HHTU11] in which this result was proved for the case of two-dimensional center bundles. In [HHTU11], the *Pesin homoclinic classes* were introduced, which were shown to be hyperbolic ergodic components of m ; these classes were made into one by the use of blenders. The use of this technique for center bundles of higher dimensions is a non-trivial fact, which was overcome by [ACW14] by using a different kind of blender, introduced by Moreira and Silva [MS12].

Other important advance was made by Burns–Dolgopyat–Pesin [BDP02], who proved a version of 2.6: namely, that essential accessibility and positive center Lyapunov exponents imply stable ergodicity.

In [HHU08b], Conjecture 2.2 was proved for one-dimensional center bundles. In particular, this gives the following theorem.

THEOREM 2.9. (Hertz, Hertz and Ures [HHU08b]) *Stable ergodicity is C^∞ -dense among volume-preserving partially hyperbolic diffeomorphisms on 3-manifolds.*

In summary, the vast majority of three-dimensional partially hyperbolic diffeomorphisms are ergodic and, moreover, are stably ergodic. Hence one could ask the following question: can we classify the *non-ergodic* partially hyperbolic diffeomorphisms? And also: are there manifolds where *all* partially hyperbolic diffeomorphisms are ergodic? Can we classify all the 3-manifolds admitting non-ergodic partially hyperbolic diffeomorphisms?

Initial evidence in this direction was obtained in [HHU08a].

THEOREM 2.10. (Hertz, Hertz and Ures) *If N is a 3-nilmanifold other than \mathbb{T}^3 , then all conservative partially hyperbolic diffeomorphisms are ergodic.*

The proof of this theorem involves the study of accessibility classes defined above. It follows from [BW10] and [HHU08b] that, for C^2 -partially hyperbolic diffeomorphisms in 3-dimensional manifolds, accessibility implies ergodicity. This fact is very interesting since it allows us to convert an ergodic problem into a geometric problem: the study of the set of accessibility classes. In other words, if the system has only one accessibility class, then it is ergodic. In §3, a better description of these sets in 3-manifolds is given.

While proving Theorem 2.10, we got the impression that the only obstruction to ergodicity is the existence of a proper *compact* accessibility class.

CONJECTURE 2.11. (Ergodic conjecture: Hertz, Hertz and Ures (2008)) *If a conservative partially hyperbolic diffeomorphism of a 3-manifold is non-ergodic, then there is a 2-torus tangential to $E^s \oplus E^u$.*

The importance of this kind of hyperbolic sub-dynamics will become apparent later on (see §6). As we will see, these tori seem to be ‘behind’ a lot of interesting behavior

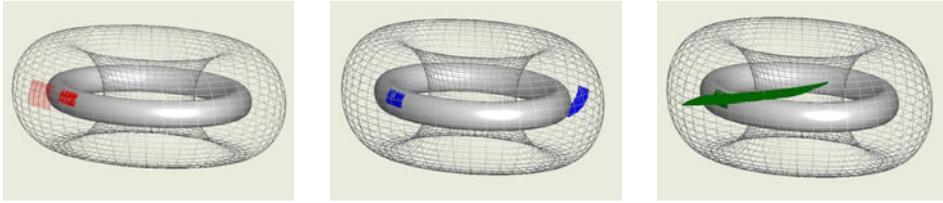


FIGURE 2. (1) The 3-torus (2) The mapping torus of $-\text{Id}$ (3) The mapping torus of a hyperbolic automorphism.

in partially hyperbolic dynamics. Moreover, not every orientable manifold can support any such sub-dynamics. In order to describe the 3-manifolds admitting these 2-tori, let us recall the concept of *mapping torus*. If N is a closed manifold and $g : N \rightarrow N$ is a diffeomorphism, we define the mapping torus of g as the manifold obtained by identifying in $N \times [0, 1]$ the points $(x, 1)$ with $(g(x), 0)$.

As examples of these, we can mention the 3-nilmanifolds, which can be seen as the mapping tori of automorphisms of the form $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, with $k \in \mathbb{Z}$. The particular case when $k = 0$ gives the 3-torus. Examples of solvmanifolds are the mapping tori of hyperbolic automorphisms $A : \mathbb{T}^2 \rightarrow \mathbb{T}^2$.

THEOREM 2.12. [HHU11] *If a partially hyperbolic diffeomorphism $f : M^3 \rightarrow M^3$ has a 2-dimensional embedded torus T which is tangential either to $E^s \oplus E^u$, $E^c \oplus E^u$ or $E^c \oplus E^s$, then the ambient manifold M^3 can only be one of the following possibilities:*

- (1) *the 3-torus \mathbb{T}^3 ;*
- (2) *the mapping torus of $-\text{id} : \mathbb{T}^2 \rightarrow \mathbb{T}^2$; or*
- (3) *the mapping tori of hyperbolic automorphisms on 2-tori.*

See Figure 2.

In the three cases, it is possible to find partially hyperbolic dynamics with embedded tori tangential to $E^s \oplus E^u$ or $E^c \oplus E^u$ ($E^c \oplus E^s$ is analogous). We refer the reader to §4 to see how a torus tangential to $E^c \oplus E^u$ can be built in any of these examples. Finding a partially hyperbolic diffeomorphism with a torus tangential to $E^s \oplus E^u$ in cases (1) and (3) is trivial; in case (2), it is enough to consider a hyperbolic automorphism on \mathbb{T}^2 , B , and take the diffeomorphism $f([x, t]) = [Bx, t]$. This is the desired partially hyperbolic diffeomorphism.

If Conjecture 2.11 is true, there would be very few 3-manifolds supporting non-ergodic partially hyperbolic dynamics. We state this explicitly as a weaker conjecture.

CONJECTURE 2.13. (Weak ergodic conjecture: Hertz, Hertz and Ures (2008)) *The only orientable 3-manifolds that admit a non-ergodic conservative partially hyperbolic diffeomorphism are:*

- (1) *the 3-torus \mathbb{T}^3 ;*
- (2) *the mapping torus of $-\text{id} : \mathbb{T}^2 \rightarrow \mathbb{T}^2$; and*
- (3) *the mapping tori of hyperbolic automorphisms on 2-tori.*

The following problem is also open.

Problem 2.14. Let f be a conservative non-ergodic partially hyperbolic diffeomorphism in any of the manifolds (1), (2), (3) stated in Theorem 2.12. Prove that there exists a torus tangential to $E^s \oplus E^u$.

In §3, we collect the advances towards proving this conjecture, to the best of our knowledge, and a description of the set of accessibility classes.

2.2. Dynamical coherence. In partially hyperbolic dynamics, the strong bundles E^s and E^u are always integrable: that is, there are invariant foliations W^s and W^u , the *stable* and *unstable* foliations, that are tangential to each of the strong bundles (see, for instance, [BP74, HPS77]). However, the center bundle, E^c , may or may not be integrable. In fact, as Wilkinson noticed in [Wil98], the Anosov example of A. Borel, mentioned by Smale in [Sma67], when viewed as a partially hyperbolic diffeomorphism, has a non-integrable center bundle. The Borel–Smale example is explained in detail in [BW08]. For the sake of completeness, let us briefly mention what it is about.

2.2.1. A non-dynamically coherent example. The aforementioned example is, in fact, a non-toral Anosov automorphism on a six-dimensional nilmanifold. Let G_1 and G_2 be copies of the three-dimensional simply connected non-abelian nilpotent Lie group. Consider bases X_i, Y_i, Z_i of the corresponding Lie algebras \mathcal{G}_i , $i = 1, 2$, with the bracket condition

$$[X_1, Y_1] = Z_1, \quad [X_2, Y_2] = Z_2. \quad (2.7)$$

Now consider a hyperbolic automorphism $A \in SL(2, \mathbb{Z})$ and let $\lambda > 1$ be one of its eigenvalues. Consider f such that the derivative acts over the Lie algebra as given by

$$\begin{aligned} X_1 &\mapsto \lambda X_1, & X_2 &\mapsto \lambda^{-1} X_2, \\ Y_1 &\mapsto \lambda^2 Y_1, & Y_2 &\mapsto \lambda^{-2} Y_2, \\ Z_1 &\mapsto \lambda^3 Z_1, & Z_2 &\mapsto \lambda^{-3} Z_2. \end{aligned}$$

The next step (which we will not explain) is to find a lattice Γ of $G = G_1 \times G_2$ such that it is invariant, so that the whole construction yields a diffeomorphism over the six-dimensional nilmanifold G/Γ (the coset space).

This example is, as a matter of fact, an Anosov diffeomorphism, but we may also look at it as a partially hyperbolic diffeomorphism such that E^s is the space generated by Z_2 , E^u is the space generated by Z_1 and the center bundle, E^c , is the space generated by X_1, X_2, Y_1 and Y_2 .

Note that X_1, X_2, Y_1 and Y_2 do not satisfy the Frobenius condition, due to (2.7), and therefore E^c is not integrable: that is, there is no invariant foliation tangential to E^c .

After examining this example, we may ask: Is the lack of the Frobenius condition the only reason for non-integrability of the center bundle? What about the case of one-dimensional E^c , where the Frobenius condition is always trivially satisfied? Notice that E^c is *a priori* only Hölder, so Picard's Theorem does not necessarily hold. The question of whether a partially hyperbolic diffeomorphism exists with a one-dimensional non-integrable center bundle has remained open since the 1970s. Let us define a stronger concept of integrability of the center bundle.

Definition 2.15. A partially hyperbolic diffeomorphism is *dynamically coherent* if the following two conditions are satisfied.

- (1) There is an invariant foliation tangential to the distribution $E^c \oplus E^u$.
- (2) There is an invariant foliation tangential to the distribution $E^s \oplus E^c$.

In 2009, Hertz, Hertz and Ures [HHU15b] provided the first example of a partially hyperbolic diffeomorphism with a non-integrable one-dimensional center bundle. It is a non-dynamically coherent example in a 3-torus. Let us mention that this example strongly contrasts with a result by Brin, Burago and Ivanov [BBI09], which proves that all *absolutely* partially hyperbolic diffeomorphisms of the 3-torus are dynamically coherent. Absolute partial hyperbolicity is more restrictive than partial hyperbolicity and requires the bound in (1.6) to be uniform: namely, that there exist $\lambda < 1 < \mu$ such that, for all $x \in M$ and unit vectors $v^\sigma \in E_x^\sigma$, $\sigma = s, c, u$,

$$\|Df(x)v^s\| \leq \lambda \leq \|Df(x)v^c\| \leq \mu \leq \|Df(x)v^u\|. \quad (2.8)$$

In [HHU15a], it was shown that if there is an invariant foliation tangential to the distribution $E^c \oplus E^u$, then this foliation cannot contain a compact leaf, that is, a 2-torus tangential to $E^c \oplus E^u$ (see Theorem 4.4). Note that, in principle, a *cu*-torus could coexist with a *cu*-foliation, but cannot be *part* of it.

In the example mentioned below, in fact, the distribution $E^c \oplus E^u$ is uniquely integrable at every point of \mathbb{T}^3 minus a 2-torus, which is tangential to $E^c \oplus E^u$. This shows that no invariant foliation exists which is tangential to $E^c \oplus E^u$, and hence the example is non-dynamically coherent. We refer the reader to Figure 7 and the caption below.

We will now prove the following theorem.

THEOREM 2.16. [HHU15b] *There exists a partially hyperbolic diffeomorphism $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ such that:*

- (1) E^c is not integrable;
 - (2) there is no invariant foliation tangential to the distribution $E^c \oplus E^u$; and
 - (3) there is an invariant two-dimensional torus T tangential to the distribution $E^c \oplus E^u$.
- Moreover, there is a C^1 -open neighborhood \mathcal{U} of f such that all g in \mathcal{U} satisfy conditions (1) and (2).

Again, a two-dimensional periodic torus appears. How relevant is this object? We conjecture that these tori might be the only obstructions to dynamical coherence.

CONJECTURE 2.17. (Dynamical coherence conjecture: Hertz, Hertz and Ures (2009)) *If a partially hyperbolic diffeomorphism of a 3-manifold is not dynamically coherent, then there is a 2-torus tangent to either $E^c \oplus E^u$ or $E^s \oplus E^c$.*

Again, if this conjecture were true, the only manifolds supporting non-dynamically coherent dynamics would be the ones listed in Theorem 2.12. We state this explicitly as a weaker conjecture.

CONJECTURE 2.18. (Weak dynamical coherence conjecture: Hertz, Hertz and Ures (2009)) *The only orientable 3-manifolds supporting non-dynamically coherent dynamics are:*

- (1) the 3-torus \mathbb{T}^3 ;
- (2) the mapping torus of $-\text{id} : \mathbb{T}^2 \rightarrow \mathbb{T}^2$; and
- (3) the mapping tori of hyperbolic automorphisms on 2-tori.

Conjecture 2.17 was proved to be true in the 3-torus by Potrie in his PhD Thesis [Pot12]. This result was extended to manifolds with solvable fundamental group by Hammerlindl and Potrie in [HP13].

THEOREM 2.19. (Hammerlindl and Potrie [HP13]) *Let f be a partially hyperbolic diffeomorphism of a 3-manifold with solvable fundamental group. If f is not dynamically coherent, then there exists a 2-torus tangential to either $E^c \oplus E^u$ or $E^s \oplus E^c$.*

Hammerlindl and Potrie have hence proved that if the weak Conjecture 2.18 is true, then the strong Conjecture 2.17 also holds. After Hammerlindl and Potrie, in order to prove Conjecture 2.17, it would be enough to show that all partially hyperbolic diffeomorphisms in 3-manifolds with non-solvable fundamental group are dynamically coherent.

Hammerlindl and Potrie's result is the strongest and most complete one in this direction up to this date.

In §4, we sketch a proof of this fact for the simplest case (see also Hammerlindl and Potrie's survey [HP15] for a more complete explanation on this and classification topics).

2.3. Classification. The third main topic in the study of partially hyperbolic dynamics in 3-manifolds is their classification. As early as 2001, Enrique Pujals proposed the following conjecture.

CONJECTURE 2.20. (Classification conjecture: Pujals (2001); see [BW05a]) *If a partially hyperbolic diffeomorphism of a 3-manifold is transitive, then is (finitely covered by) one of the following:*

- (1) a perturbation of the time-one map of an Anosov flow;
- (2) a skew product; or
- (3) a DA-diffeomorphism.

In 2009, Hammerlindl showed in his PhD Thesis [Ham13] that every absolutely partially hyperbolic diffeomorphism of \mathbb{T}^3 is *leafwise conjugate* to its linearization. Let us define this concept.

Definition 2.21. (Leaf conjugacy) Two dynamically coherent partially hyperbolic diffeomorphisms $f, g : M \rightarrow M$ are *leafwise conjugate* if there exists a homeomorphism $h : M \rightarrow M$ carrying center leaves of f to center leaves of g : that is, $h(W_f^c(x)) = W_g^c(h(x))$, such that

$$h(f(W_f^c(x))) = g(h(W_f^c(x))).$$

Note that, under the hypothesis of Hammerlindl's thesis (absolute partial hyperbolicity in \mathbb{T}^3), there is always dynamical coherence, due to a result by Brin, Burago and Ivanov [BBI09].

Hammerlindl's thesis suggested that perhaps dynamical coherence was a more suitable hypothesis for a classification of three-dimensional partially hyperbolic dynamics. This, together with our example [HHU15b], led us to propose the following conjecture.

CONJECTURE 2.22. (Classification conjecture: Hertz, Hertz and Ures (2009)) *Let f be a partially hyperbolic diffeomorphism of a 3-manifold.*

If f is dynamically coherent, then it is (finitely covered by) one of the following:

- (1) *a perturbation of a time-one map of an Anosov flow, in which case it is leafwise conjugate to an Anosov flow;*
- (2) *a skew product, in which case it is leafwise conjugate to a skew product with linear base; or*
- (3) *a DA, in which case it is leafwise conjugate to an Anosov diffeomorphism of \mathbb{T}^3 .*

If f is not dynamically coherent, then there are a finite number of 2-tori tangential either to $E^c \oplus E^u$ or to $E^s \oplus E^c$, and the rest of the dynamics is trivial, as in the non-dynamically coherent example [HHU15b] (see also §4).

Both conjectures are true in certain manifolds, as was proved by Hammerlindl and Potrie [HP13].

THEOREM 2.23. (Hammerlindl and Potrie [HP13]) *Conjectures 2.20 and 2.22 are true for partially hyperbolic diffeomorphisms in 3-manifolds with solvable fundamental group.*

The classification conjectures have motivated much work. However, very recently, Bonatti, Gogolev, Parwani and Potrie found both a dynamically coherent example and a transitive example that are not leafwise conjugate to any of the above models, proving both classification conjectures wrong [BPP14, BGP15] (see §5).

Question 2.24. Is it possible to classify partially hyperbolic dynamics in 3-manifolds, modulo leaf conjugacies?

In the next sections, we shall develop more deeply the concepts mentioned above. We have tried to make each section as self-contained as possible, which may mean that some definitions have been repeated.

3. Ergodicity

To establish the ergodicity of partially hyperbolic maps, the most general method available is the so-called *Hopf method*. To explain it, we first recall the following theorem.

THEOREM 3.1. (Stable manifold theorem) *Let M be a 3-manifold and let $f \in \text{Diff}^r(M)$ be partially hyperbolic. Then there exist continuous foliations $\mathcal{W}^s = \{W^s(x)\}_{x \in M}$ and $\mathcal{W}^u = \{W^u(x)\}_{x \in M}$ tangential to E^s and E^u , respectively, called the stable and the unstable foliations. Their leaves are C^r -immersed lines.*

See [HPS77, Theorem 4.1]. We point out that, while their leaves are as smooth as f , the foliations \mathcal{W}^s , \mathcal{W}^u are seldom differentiable [Ano67, p. 201]. Their transverse regularity is only Hölder [PSW97], in general.

Using the Birkhoff ergodic theorem, it is not hard to see that a conservative diffeomorphism f is ergodic if and only if, for every continuous observable $\varphi : M \rightarrow \mathbb{R}$, the Birkhoff average

$$\tilde{\varphi}(x) = \lim_{|n| \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ f^k(x) \quad (3.9)$$

is almost everywhere constant. But $\tilde{\varphi}(x)$ coincides almost everywhere with

$$\varphi^+(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ f^k(x), \quad (3.10)$$

which is constant on stable manifolds (using uniform continuity of φ). Analogously, $\tilde{\varphi}(x)$ coincides almost everywhere with

$$\varphi^-(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ f^{-k}(x), \quad (3.11)$$

which is constant on unstable manifolds.

Suppose that f is not ergodic. Then there would be a continuous observable φ for which $\tilde{\varphi}$ is not almost everywhere constant, and thus neither are φ^+ nor φ^- . Hence there exist two positive measure-invariant sets A and B and $\alpha \in \mathbb{R}$ such that $\varphi^+(x) \geq \alpha$ for all $x \in A$ and $\varphi^-(x) < \alpha$ for all $x \in B$. Note that A is saturated by stable manifolds while B is saturated by unstable manifolds.

Assume that f is an Anosov diffeomorphism. Let x be a point of A such that almost all points $w \in W^s(x)$ satisfy $\varphi^-(w) = \varphi^+(w) = \varphi^+(x)$. Such an x exists because the stable foliation is *absolutely continuous* [Ano67] and $\varphi^+(w) = \varphi^-(w)$ almost everywhere. Consider y , a point belonging to the support of B , and V , a product neighborhood containing y . On the one hand, observe that $W^s(x)$ intersects V and let W_V be a connected component of $W^s(x) \cap V$. On the other hand, $m(V \cap B) > 0$, and then the absolute continuity of the stable foliation implies that there is a local stable manifold $T \subset V$ such that $m_T(T \cap B) > 0$, where m_T is the Lebesgue measure of T . If we call h^u the unstable holonomy in V sending T to W_V , $m_{W_V}(h^u(T \cap B)) > 0$. Since B is u -saturated, $h^u(T \cap B) \subset B$. We obtain a contradiction with the fact that almost every point in W_V satisfies $\varphi^-(w) = \varphi^+(w)$. This is basically the Hopf argument.

In brief, there are two fundamental ingredients in the Hopf argument for an Anosov map.

- (1) There is a way of joining any pairs of points through a curve that is piecewise either a stable or an unstable leaf.
- (2) The stable and unstable foliations are absolutely continuous and completely transversal.

3.1. Accessibility, a property that implies ergodicity. We would like to apply the previous method to a general partially hyperbolic system, that is, when there is some non-trivial center direction. To begin with, observe that, in general, it is not true that any two pairs of points can be joined by a concatenation of stable and unstable leaves. For example, if we consider the partially hyperbolic diffeomorphism $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \times \text{id}$ in $\mathbb{T}^2 \times \mathbb{S}^1$, then any path consisting of a concatenation of stable and unstable leaves would be contained in a single 2-torus. We fix $f : M \rightarrow M$ \mathcal{C}^2 to be conservative partially hyperbolic and call $c : [0, 1] \rightarrow M$ an *su-path* if it is piecewise \mathcal{C}^1 and, for every t , where defined, $c'(t) \in E^s \cup E^u$.

Definition 3.2. For a point $x \in M$, its accessibility class is the set

$$AC(x) := \{y : \exists c : [0, 1] \rightarrow M \text{ su-path such that } c(0) = x, c(1) = y\}.$$

The map f is accessible if the partition by accessibility classes is trivial and is essentially accessible if the partition by accessibility classes is ergodic (that is, any Borel set saturated by accessibility classes has either volume zero or one).

In the example above, each invariant 2-torus is an accessibility class. When there is only one accessibility class, we will say that f has the *accessibility property*. From now on, let us suppose this is our case. As for (2), absolute continuity of the strong foliations is also satisfied [PS72], but complete transversality is not (due to the presence of the center direction).

This problem can be overcome if the holonomies are *regular* enough. For instance, Sacksteder used accessibility and Lipschitzness of the stable and unstable holonomies to prove ergodicity of linear partially hyperbolic automorphisms of nilmanifolds [Sac70]. More generally, Brin and Pesin proved that accessibility and Lipschitzness of the stable and unstable foliations imply ergodicity (in fact, Kolmogorov), in the following way [BP74, Theorem 5.2, p. 204]; see also [GPS94]. If A and B are defined as before, consider a density point x in A and a density point y in B . Take an su -path joining x and y . Call h a global holonomy map from x to y : that is, h is a local homeomorphism that takes points in a neighborhood U of x , slides them first along a stable segment, then along an unstable, then along a stable again, etc. until reaching a neighborhood V of y , where all the su -paths are near the original su -path joining x and y . Since A is essentially su -saturated, $h(A \cap U) = A \cap V$ modulo a zero set. Since h can be chosen to be Lipschitz, there exists a constant $C > 1$ such that, for each measurable set $E \subset U$, and for each sufficiently small $r > 0$,

$$\frac{1}{C}m(E) < m(h(E)) < Cm(E), \tag{3.12}$$

$$B_{r/C}(y) \subset h(B_r(x)) \subset B_{Cr}(y). \tag{3.13}$$

This implies that

$$\frac{m(B_{Cr}(y) \cap A)}{m(B_{Cr}(y) \setminus A)} \geq \frac{m(h(B_r(x) \cap A))}{m(h(B_{C^2r}(x) \setminus A))} \geq \frac{1}{C^2 \cdot C'} \frac{m(B_r(x) \cap A)}{m(B_r(x) \setminus A)} \rightarrow \infty$$

since $m(B_{C^2r}(x) \setminus A) \leq C'm(B_r(x) \setminus A)$ for some positive constant C' . From this we get that y is also a density point of A . This is absurd, since y was a density point of B , complementary to A modulo a zero set.

This is essentially how the Hopf argument would work in the partially hyperbolic setting. However, Lipschitzness of the holonomy maps is a very strong hypothesis, not satisfied for most of the partially hyperbolic diffeomorphisms.

The idea of Grayson, Pugh and Shub [GPS94], and later improved by [Wil98, PS00, HHU08b, BW10], is to show that the stable and unstable holonomies do preserve density points according to another basis, different from intervals, called *juliennes*. These sets are dynamically defined and constitute Vitali bases. Burns and Wilkinson made an improvement of this ergodicity argument in [BW05b]. We will roughly sketch it. Consider for a point x a small center segment, and saturate by local unstable leaves; to gain better control of the size of these unstable segments we pre-iterate n times the local unstable manifold of $f^n(x)$ (of a convenient size). The resulting set is then saturated by locally

stable manifolds. This small prism is called *s-julienne* and is denoted by $J_n^{\text{suc}}(x)$. The subscript n essentially tells the size of the julienne and, in particular, everything is chosen so that $m(J_n^{\text{suc}}(x)) \xrightarrow{n \rightarrow \infty} 0$. An *s-julienne density point* of a set E is a point x such that

$$\lim_{n \rightarrow \infty} \frac{m(J_n^{\text{suc}}(x) \cap E)}{m(J_n^{\text{suc}}(x))} = 1. \quad (3.14)$$

The scheme is to consider the sets A and B that we considered above, and prove that:

- (1) the *s-julienne density points* of A (and of any essentially u -saturated set) coincide with the Lebesgue density points of A ; and
- (2) the *s-julienne density points* of A (and of any essentially s -saturated set) are preserved by stable holonomies.

An analogous statement is proved for A with respect to *u-julienne density points*, which are defined with respect to the local basis obtained by locally saturating a small center segment first in a dynamic way by stable leaves, and then by unstable leaves. As the stable and unstable holonomies preserve the Lebesgue density points of A , if the diffeomorphism has the accessibility property, then A is equal to M modulo a zero set. This proves that the system is ergodic.

THEOREM 3.3. [BW10, HHU08b] *If $f \in \text{Diff}_m^2(M^3)$ is partially hyperbolic and satisfies the accessibility property, then it is ergodic.*

In fact, Burns and Wilkinson proved a much more general result.

THEOREM 3.4. [BW10] *If f is a partially hyperbolic diffeomorphism (with any center dimension) satisfying the accessibility property and the center bunching property, then f is ergodic.*

A diffeomorphism is said to satisfy the *center bunching* property if

$$\|Df(x)|E^s\| < \frac{m(Df(x)|E^c)}{\|Df(x)|E^c\|} \leq \frac{\|Df(x)|E^c\|}{m(Df(x)|E^c)} < m(Df(x)|E^u), \quad (3.15)$$

where $m(T) = \|T^{-1}\|^{-1}$.

Hence Theorem 3.4 implies Theorem 3.3. Also, when the center bundle is one-dimensional, it is always locally integrable to center curves, so the fake foliations are not necessary to build local juliennes. Hertz, Hertz and Ures, in [HHU08b], show an alternative way of proving this result in this particular case.

There are two innovations in [BW10]. One is the argument outlined above Theorem 3.3, which uses a much weaker center bunching than in [PS00] paper. This innovation had already been published in [BW05b], and was essential in the proof of Theorem 3.3 in [HHU08b]. In [BW05b], however, dynamical coherence was still a hypothesis, and the task in [HHU08b, Theorem 3.3] consisted of removing this for the center dimension one case. The second innovation in [BW10] was precisely to remove the dynamical coherence hypothesis for any center dimension. This was accomplished by means of ‘fake foliations’, which is a very delicate technical tool.

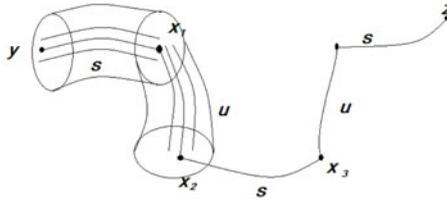


FIGURE 3. An su -path from y to z

3.2. *Properties of accessibility classes.* We want to describe precisely non-ergodic partially hyperbolic diffeomorphisms, and it is possible that this only occurs when there is a compact accessibility class (see Conjecture 2.11): that is, when there is a torus tangential to $E^s \oplus E^u$ (in fact, we conjecture that there must be at least two such tori).

Since accessibility implies ergodicity, in order to describe non-ergodic partially hyperbolic diffeomorphisms, it seems reasonable to look at the non-accessible ones. And, even more precisely, we will study the structure of the set of non-open accessibility classes.

THEOREM 3.5. [HHU08b] *For each $x \in M^3$, its accessibility class $AC(x)$ is either an open set or an immersed surface. Moreover, $\Gamma(f)$, the set of non-open accessibility classes of f , is a compact codimension-one laminated set whose laminae are the accessibility classes.*

Remark 1. This theorem still holds for partially hyperbolic diffeomorphisms with center dimension one.

Let us begin with a local description of open accessibility classes.

PROPOSITION 3.6. *For any point $x \in M$, the following statements are equivalent.*

- (1) $AC(x)$ is open.
- (2) $AC(x)$ has non-empty interior.
- (3) $AC(x) \cap W_{loc}^c(x)$ has non-empty interior for any choice of $W_{loc}^c(x)$.

Proof. (2) \Rightarrow (1) Let y be in the interior of $AC(x)$ and consider any point z in $AC(x)$. Then there is an su -path from y to z with points $y = x_0, x_1, \dots, x_N = z$ such that x_n and x_{n+1} are either in the same s -leaf or in the same u -leaf (see Figure 3). Let U be a neighborhood of y contained in $AC(x)$ and suppose that, for instance, $y = x_0$ and x_1 belong to the same s -leaf. Then $U_1 = W^s(U)$ is an open set contained in $AC(x)$ which contains x_1 , so x_1 is in the interior of $AC(x)$. Indeed, W^s is a C^0 -foliation, so the s -saturation of an open set is open.

Now x_1 and x_2 belong to the same u -leaf. If we consider $U_2 = W^u(U_1)$, then U_2 is an open set contained in $AC(x)$ and containing x_2 in its interior. Defining inductively U_n as $W^s(U_{n-1})$ or $W^u(U_{n-1})$ according to whether x_n belongs to the s - or the u -leaf of x_{n-1} , we obtain that all x_n belong to the interior of $AC(x)$; in particular, z . This proves that $AC(x)$ is open.

(1) \Rightarrow (3) Follows directly from the definition of relative topology.

(3) \Rightarrow (2) Let V be an open set in $AC(x) \cap W_{loc}^c(x)$, relative to the topology of $W_{loc}^c(x)$. Then $W^s(V)$ is contained in $AC(x)$ and contains a disc D^{s+c} of dimension $s + c$ transverse

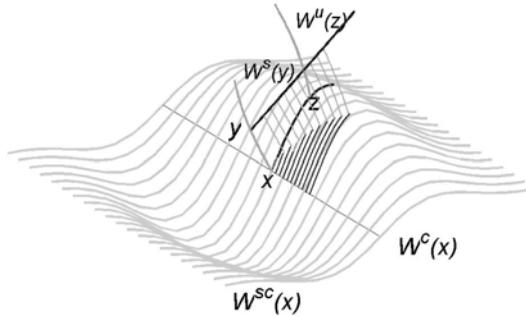


FIGURE 4. An open accessibility class.

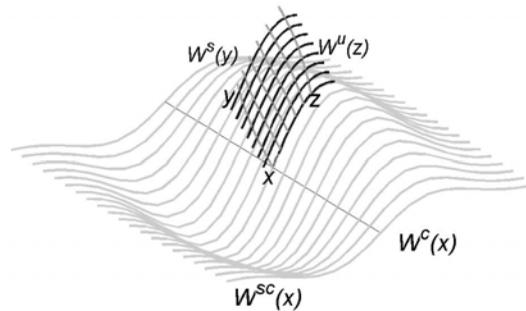


FIGURE 5. An accessibility class in $\Gamma(f)$.

to E^u . This implies that $W^u(D^{sc})$ is contained in $AC(x)$ and contains an open set. Therefore $AC(x)$ has non-empty interior. \square

Let $O(f)$ be the set of open accessibility classes, which is, obviously, an open set. Then its complement, $\Gamma(f)$, is a compact set. Let us see that it is laminated by the accessibility classes of its points.

For any point $x \in M$, consider a local center leaf $W_{loc}^c(x)$. Locally saturate it by stable leaves: that is, take the local stable manifolds of all points $y \in W_{loc}^c(x)$, to obtain a small $(s + c)$ -disc $W_{loc}^{sc}(x)$. Now locally saturate $W_{loc}^{sc}(x)$ by unstable leaves to obtain a small neighborhood $W_{loc}^{usc}(x)$ (see Figure 4).

On $W_{loc}^{usc}(x)$, consider the map

$$p_{us} : W_{loc}^{usc}(x) \rightarrow W_{loc}^c(x), \tag{3.16}$$

which is defined in the following way. Given $y \in W_{loc}^{usc}(x)$, there exists a unique point $p_u(y)$ in the disc $W_{loc}^{sc}(x)$ that belongs to the local unstable manifold of y . Since $W_{loc}^{sc}(x)$ is the local stable saturation of $W_{loc}^c(x)$, then $p_u(y) \in W_{loc}^{sc}(x)$ is in the local stable manifold of a unique point $p^{us}(y)$ in $W_{loc}^c(x)$: that is, $p_{us}(y)$ is the point obtained by first projecting along unstable manifolds onto $W_{loc}^{sc}(x)$ and then projecting along stable manifolds onto $W_{loc}^c(x)$. Since the local stable and unstable foliations are continuous, p_{su} is continuous.

Let $AC_x(y)$ be the connected component of $AC(y) \cap W_{loc}^{usc}(x)$ that contains y . The points of $AC_x(y)$ are the points that can be accessed by su -paths from y without getting

out from $W_{loc}^{usc}(x)$ (see Figures 4 and 5). Then we have the following local description of accessibility classes of points in $\Gamma(f)$.

LEMMA 3.7. *For any $y \in W_{loc}^c(x)$ such that $y \in \Gamma(f)$, $AC_x(y) = p_{su}^{-1}(y)$.*

Proof. Let y be a point in $W_{loc}^c(x)$. Then $p_{su}^{-1}(y) = W_{loc}^u(W_{loc}^s(y))$, which is clearly contained in $AC_x(y)$. But also, $p_{su}(AC_x(y)) = y$. Indeed, if $p_{su}(z)$ were different from y , for some $z \in AC_x(y)$, we would have a situation as described in Figure 4, because, since p_{su} is continuous and $AC_x(y)$ is connected, $p_{su}(AC_x(y))$ is connected. If $p_{su}(AC_x(y))$ contained another point, then it would contain a segment, which has non-empty interior in $W_{loc}^c(x)$. Proposition 3.6 then would imply that $AC(y)$ is open, which is absurd, since $y \in \Gamma(f)$. This proves that also $AC_x(y)$ is contained in $p_{su}^{-1}(y)$. \square

Hence, due to Lemma 3.7 above, for each $x \in \Gamma(f)$,

$$AC_x(x) = p_{su}^{-1}(x) = W_{loc}^u(W_{loc}^s(x)) \approx W_{loc}^u(x) \times W_{loc}^s(x).$$

$W_{loc}^s(x)$ and $W_{loc}^u(x)$ are (evenly sized) embedded segments that vary continuously with respect to $x \in M$ (see Hirsch, Pugh, Shub [HPS77, Chapters 4 and 5]). This implies that $\Gamma(f) \ni x \mapsto AC_x(x)$ is a continuous map that assigns to each x an evenly sized 2-disc. To finish the description of accessibility classes, let us introduce the following definition.

Definition 3.8. The foliations W^s and W^u are *jointly integrable* at a point $x \in M$ if there exists $\delta > 0$ such that, for each $z \in W_\delta^s(x)$ and $y \in W_\delta^u(x)$,

$$W_{loc}^u(z) \cap W_{loc}^s(y) \neq \emptyset.$$

See Figure 5 for an illustration of a point of joint integrability of W^s and W^u .

Then Lemma 3.7 and the discussion above imply the following proposition.

PROPOSITION 3.9. *A point x belongs to $\Gamma(f)$ if and only if W^s and W^u are jointly integrable at all points of $AC(x)$.*

Indeed, if x belongs to $\Gamma(f)$, then, for all $y \in AC(x) \subset \Gamma(f)$, $p_{su}(AC_y(x)) = \{y\}$. In particular, if $z \in W_\delta^u(y)$ and $w \in W_\delta^s(y)$, then $W_{loc}^s(z) \cap W_{loc}^u(w) \neq \emptyset$. On the other hand, if W^s and W^u are jointly integrable at all points of $AC(x)$, then $AC(x)$ is a lamina, due to the explanation above (the coherence of the charts ϕ_x defined above depends only on the joint integrability of W^s and W^u). Moreover, this two-dimensional lamina is transverse to $W_{loc}^c(x)$, and so $AC(x) \cap W_{loc}^c(x)$ cannot be open. Proposition 3.6 implies that $AC(x)$ is not open, so $x \in \Gamma(f)$.

The following lemma shows that, in fact, the laminae of $\Gamma(f)$, that is, the accessibility classes of points in $\Gamma(f)$, are C^1 .

LEMMA 3.10. [Did03, Lemma 5] *If W^s and W^u are jointly integrable at x , then the set*

$$W_{loc}^{su}(x) = \{W^u(z) \cap W^s(y) : \text{with } z \in W_\delta^s(x) \text{ and } y \in W_\delta^u(x)\},$$

where $\delta > 0$ is as in the definition of joint integrability (Definition 3.8), is a two-dimensional C^1 -disc that is everywhere tangential to $E^s \oplus E^u$.

In order to prove Lemma 3.10, we shall use the following result by Journé.

THEOREM 3.11. [**Jou88**] *Let F^h and F^v be two transverse foliations with uniformly smooth leaves on an open set U . If $\eta : U \rightarrow M$ is uniformly C^1 along F^h and F^v , then η is C^1 on U .*

Proof of Lemma 3.10. Let D be a small, smooth two-dimensional disc containing x and transverse to E_x^c . Consider a one-dimensional smooth foliation of a small neighborhood N of x , transverse to D . If D is sufficiently small, there is a smooth map, $\pi : N \rightarrow D$, defined as projection along this smooth one-dimensional foliation. Note that $W_{\text{loc}}^{su}(x)$ can be seen as the graph of a continuous function $\eta : D \rightarrow N$.

We produce a grid on D in the following way. The horizontal lines are the projections of the stable manifolds $W^s(y)$, with $y \in W_{\delta}^u(x)$: that is, the horizontal lines are of the form $\pi(W_{\text{loc}}^s(\eta(v)))$, with $v \in D$. Analogously, the vertical lines are the projections of the unstable manifolds $W_{\text{loc}}^u(z)$, with $z \in W_{\delta}^s(x)$: that is, the vertical lines are of the form $\pi(W_{\text{loc}}^u(\eta(w)))$, with $w \in D$.

Now $v \mapsto W_{\text{loc}}^s(\eta(v))$ and $w \mapsto W_{\text{loc}}^u(\eta(w))$ are continuous in the C^1 -topology: that is, for close v , we obtain close $W_{\text{loc}}^s(\eta(v))$ in the C^1 -topology (E^s is a continuous bundle). Since π is smooth, we also obtain that $F^h = \{\pi(W_{\text{loc}}^s(\eta(v)))\}_{v \in D}$, the horizontal partition of D , and $F^v = \{W_{\text{loc}}^u(\eta(w))\}_{w \in D}$, the vertical partition of D , are transverse foliations continuous in the C^1 -topology.

But η is uniformly C^1 along F^h , since η along a leaf $F^h(v_0) = \pi(W_{\text{loc}}^s(\eta(v_0)))$ is exactly $W_{\text{loc}}^s(\eta(v_0))$. Indeed, $\eta \circ \pi : W_{\text{loc}}^{su}(x) \rightarrow W_{\text{loc}}^{su}(x)$ is the identity map and $W_{\text{loc}}^s(\eta(v_0))$ is a smooth manifold. Analogously, we obtain that η is uniformly C^1 along F^v . Hence, by Theorem 3.11, η is C^1 . \square

3.3. Properties of the lamination $\Gamma(f)$ of non-open accessibility classes. In order to prove the ergodicity Conjecture 2.11, a possible strategy is to accurately describe the lamination $\Gamma(f)$. More precisely, one would like to see that non-accessibility implies the existence of a compact (toral) accessibility class. However, the state of the art is given by the following theorem.

THEOREM 3.12. [**HHU08a**, Theorem 1.6] *Let $f : M^3 \rightarrow M^3$ be a conservative partially hyperbolic diffeomorphism that is not accessible. Then one of the following possibilities holds.*

- (1) *There is a compact accessibility class (a torus tangential to $E^s \oplus E^u$).*
- (2) *There exists an invariant sub-lamination $\Lambda \subset \Gamma(f)$ of M that trivially extends to a (not necessarily invariant) foliation without compact leaves. Moreover, if $\Lambda \neq M$, the boundary leaves of Λ are periodic, have Anosov dynamics and periodic points are dense in each boundary leaf with the intrinsic topology.*
- (3) *$\Gamma(f)$ is a minimal foliation.*

With respect to item (2), a leaf L of a lamination Λ is a *boundary leaf* if there is a transverse segment to L containing a subsegment α with an endpoint in L and such that $\alpha \cap \Lambda = \emptyset$. In [**HHU08a**], it is proved that boundary leaves are periodic in the

conservative setting and, moreover, that periodic points are dense in each boundary leaf with the intrinsic topology.

If Case (1) holds, then Conjecture 2.11 is true. We conjecture that Case (2) is not possible; more precisely, we conjecture that each boundary leaf should be a torus. Answering the following question in the affirmative would rule out Case (2).

Question 3.13. Let L be a complete immersed surface in a 3-manifold such that there is an Anosov dynamics on L , where:

- (1) each stable and unstable manifold is complete, and angles between stable and unstable manifolds are bounded;
- (2) periodic points are dense with the intrinsic topology; and
- (3) the stable and unstable manifold of each periodic point are dense in L with the intrinsic topology.

Is L the 2-torus?

Case (3) of Theorem 3.12 means that each leaf of $\Gamma(f)$ is dense. We conjecture that, in this case, in fact, f is *essentially accessible*, which means that each set that is a union of accessibility classes has either measure one or zero. Essential accessibility in dimension three implies ergodicity [BW10, HHU08b]. If this could be established, then Conjecture 2.11 would be proved true. Finding a counterexample, however, would be very interesting.

Question 3.14. Is there an example of a partially hyperbolic diffeomorphism in a 3-manifold such that the accessibility classes of f form a minimal foliation, but f is not essentially accessible?

Since in Case (1) of Theorem 3.12, Conjecture 2.11 follows trivially, we would like to better describe what happens in Cases (2) and (3). The following describes the accessibility classes in these cases.

THEOREM 3.15. *If f has no compact accessibility class, then the π_1 of each accessibility class injects in $\pi_1(M)$.*

Proof. The result follows almost directly from the following Theorem by Novikov.

THEOREM 3.16. (Novikov) *Let M be a compact orientable 3-manifold and let \mathcal{F} be a transversely orientable codimension-one foliation without Reeb components. Then, for each leaf L in \mathcal{F} , $\pi_1(L)$ injects in $\pi_1(M)$.*

A *Reeb component* of a foliation is a solid torus subfoliated by planes, as in Figure 6.

Question 3.17. Does Theorem 3.15 hold without assuming that there are no compact accessibility classes?

If $\Gamma(f) = M$, then we are already in the hypothesis of Theorem 3.16, since the fact that $\Gamma(f)$ has no compact leaves precludes the existence of Reeb components. The rest of the theorem follows by proving that if $\Gamma(f) \neq M$ has no compact leaves, then it can be extended to a foliation without Reeb components. This follows almost immediately from [HHU08a, Theorem 4.1].

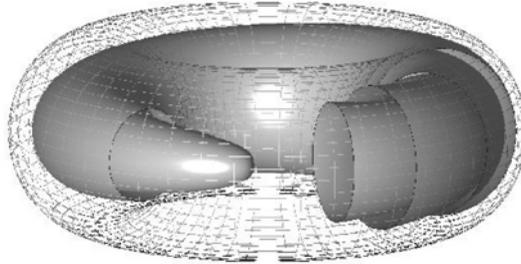


FIGURE 6. Reeb component.

THEOREM 3.18. (Hertz, Hertz and Ures) *If $\Lambda \subset \Gamma(f)$ is an orientable and transversely orientable f -invariant sub-lamination without compact leaves such that $\Lambda \neq M$, then all closed complementary regions of Λ are I -bundles.*

By taking finite coverings, we may assume that $\Gamma(f)$ itself is orientable and transversely orientable. $\Gamma(f) \neq M$ has no compact leaves, therefore its complementary regions are I -bundles. This allows us to extend $\Gamma(f)$ to a foliation in a trivial way, by ‘copying’ the boundary leaves. This means that each complementary region is of the form $L \times [0, 1]$, where L is a boundary leaf of $\Gamma(f)$. We foliate each complementary region by considering leaves of the form $L \times \{t\}$, with $t \in [0, 1]$. \square

4. Dynamical coherence

It turns out that ergodicity in our setting is tightly related to integrability of the invariant bundles. As we explained before, the bundles E^s, E^u are always integrable. The integrability of the center bundle E^c , on the other hand, cannot be always guaranteed, even in our setting. This was a long-standing problem and was recently solved in [HHU15b] (see Theorem 2.16, rewritten below).

Let us recall *dynamical coherence*, which was stated in Definition 2.15. A partially hyperbolic diffeomorphism is dynamically coherent if there is an f -invariant foliation tangential to $E^s \oplus E^c$ (the *center-stable foliation*) and an f -invariant foliation tangential to $E^c \oplus E^u$ (the *center-unstable foliation*). Note that, in this case, the center bundle is also integrable: an f -invariant foliation tangential to E^c is obtained by simply intersecting the center-stable and center-unstable leaves and taking connected components. This is called the *center foliation*.

PROPOSITION 4.1. *If $f : M^3 \rightarrow M^3$ is a partially hyperbolic diffeomorphism whose center bundle is C^1 , then f is dynamically coherent.*

Proof. First, observe that \mathcal{W}^c is f -invariant: if $c : [0, 1] \rightarrow W^c(x)$ is a differentiable curve with $c(0) = x$, then $f \circ c : [0, 1] \rightarrow M$ is a differentiable curve tangential to E^c , and hence, by uniqueness of solutions of differential equations, $f \circ c([0, 1]) \subset W^c(f \circ c(0)) = W^c(f(x))$.

Theorem 6.1 and Theorem 7.6 in [HPS77] imply that, through each leaf L of \mathcal{W}^c , there exist immersed submanifolds $W^s(L)$ and $W^u(L)$ tangential to E^{cs} and E^{cu} , respectively,

saturated by the corresponding strong foliations. Again using uniqueness of solutions of differential equations, one proves that the families $\mathcal{W}^{cs} = \{W^s(L)\}_{L \in \mathcal{W}^c}$, $\mathcal{W}^{cu} = \{W^u(L)\}_{L \in \mathcal{W}^c}$ are pairwise disjoint and, since their tangent spaces vary continuously, they form foliations. Invariance follows, since \mathcal{W}^c , \mathcal{W}^s , \mathcal{W}^u are invariant. \square

More details about this can be found in [BW08]. When the center bundle is not differentiable, we still have curves tangential to it as a consequence of Peano’s theorem. This family of curves, however, is not assembled as a foliation, but it still can contain relevant information (see [HHU15a] and [HHU08c]).

Problem 4.2. Find an example of a dynamically coherent partially hyperbolic diffeomorphism that is not leafwise conjugate to a C^1 dynamically coherent one. Can the examples in [BPP14] be adjusted to get one?

Another condition that guarantees dynamical coherence in \mathbb{T}^3 is *absolute partial hyperbolicity*, a notion stronger than partial hyperbolicity, which was described in Equation (2.8).

THEOREM 4.3. (Brin–Burago–Ivanov [BBI09]) *If $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ is absolutely partially hyperbolic, then f is dynamically coherent.*

However, this is not the general case, as we explain in the next subsection.

4.1. *A non-dynamical coherent example.*

THEOREM 2.13. [HHU15b] *There exists a partially hyperbolic diffeomorphism $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ such that:*

- (1) *there is no invariant foliation tangential to the distribution $E^c \oplus E^u$; and*
 - (2) *there is an invariant two-dimensional torus T tangential to the distribution $E^c \oplus E^u$.*
- Moreover, there is a C^1 -open neighborhood \mathcal{U} of f in $\text{Diff}^1(M)$ such that all g in \mathcal{U} satisfy conditions (1) and (2).*

Sketch. Let $A : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be a hyperbolic linear map with eigenvalues $\lambda < 1 < 1/\lambda$. Take u , a unit eigenvector corresponding to the eigenvalue λ . Consider also a north pole–south pole function $f : \mathbb{T} \rightarrow \mathbb{T}$ such that

$$f(0) = 0, \quad f(1/2) = 1/2,$$

$$f'(1/2) = \sigma < \lambda < 1 < \nu = f'(0) < 1/\lambda$$

and a differentiable function $\phi : \mathbb{T} \rightarrow \mathbb{R}$.

Now construct a perturbation F of the Axiom-A map $A \times f$ by ‘pushing’ in the stable direction of A , namely,

$$F(x, \theta) = (Ax, f(\theta)) + (\phi(\theta)e_s, 0), \quad \phi(1/2) = 0,$$

where e_s is a unit vector in the E^s direction of A .

Note that the strong unstable direction of $A \times f$ is unaltered by this perturbation and, in particular, the strong stable manifold of the perturbation exists and coincides with the

strong stable manifold of the unperturbed map. Observe that the unperturbed map is not partially hyperbolic. Now we study the other invariant directions.

We are seeking invariant directions of the derivative of F ,

$$dF_{(x,\theta)}(v, t) = (Av, f'(\theta)t) + (\phi'(\theta)te_s, 0).$$

An invariant direction (inside the $e_s \times \mathbb{T}$ cylinder) will be generated by a vector field of the form $(a(\theta)e_s, 1)$ for some function a , and hence we need to solve

$$a(f(\theta))f'(\theta) = \lambda a(\theta) + \phi'(\theta). \quad (4.17)$$

We are thus led to find a solution of the cohomological equation

$$b \circ f = \lambda b + \phi$$

(the solution of (4.17) is just $a = b'$). One then checks that the following two functions are solutions, that is,

$$\eta(\theta) = \frac{1}{\lambda} \sum_1^{\infty} \lambda^n \phi(f^{-n}\theta), \quad (4.18)$$

$$\zeta(\theta) = -\frac{1}{\lambda} \sum_0^{\infty} \lambda^{-n} \phi(f^n\theta), \quad (4.19)$$

and that the previous assumptions imply that $\eta \in C^1(T \setminus \{1/2\})$, $\zeta \in C^1(T \setminus \{0\})$.

Let us define

$$E^c(\theta) = \text{span}(\eta'(\theta)e_s, 1) \quad \text{for } \theta \neq \frac{1}{2}$$

and

$$E^s(\theta) = \text{span}(\zeta'(\theta)e_s, 1) \quad \text{for } \theta \neq 0.$$

Back to the invariant directions, note that, for generic ϕ , $\eta'(\theta)$ gets bigger as θ approaches $1/2$, and thus, if we can choose ϕ so that

$$\lim_{\theta \rightarrow 1/2} \eta'(\theta) = \infty, \quad (4.20)$$

we will get continuity for E^c by defining

$$E^c(\theta = \frac{1}{2}) = \text{span}\{(e_s, 0)\} = E_A^s \times 0.$$

Arguing similarly, we define

$$E^s(\theta = 0) = E_A^s \times 0,$$

and we will get a continuous bundle provided that we prove that

$$\lim_{\theta \rightarrow 0} \zeta'(\theta) = \infty. \quad (4.21)$$

Assume for now that we have proved that these bundles are continuous. Now we want to show that $T\mathbb{T}^3 = E^s \oplus E^c \oplus E^u$ or, equivalently, that the angle between E^s and E^c is not zero. What we need to show is that $\eta' \neq \zeta'$ for $\theta \neq 0, 1/2$. Note that for, $\theta = 0, 1/2$,

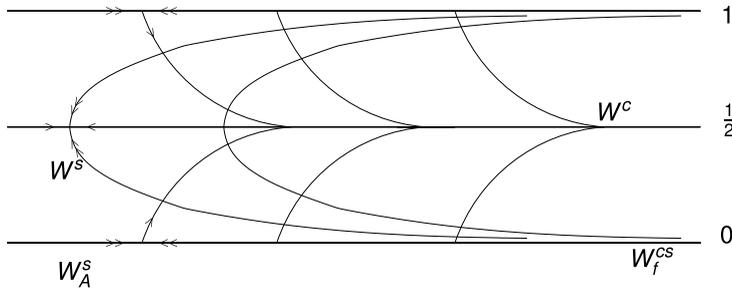


FIGURE 7. The figure above shows a center-stable leaf. The center leaves are unique for $\theta \neq 1/2$ and become tangential to the center leaf at $\theta = 1/2$, making a peak. This precludes the existence of a center foliation.

the angle is not zero, and hence it is not zero in a neighborhood of these points. But, by the cohomological equations,

$$(\eta' - \zeta') \circ f = \lambda(\eta' - \zeta')$$

and, using the form of the dynamics of f , we conclude that the sign of $\eta' - \zeta'$ is constant in $(0, 1/2)$ and $(1/2, 1)$ and is clearly non zero. The following lemma is proved in detail in [HHU15b].

LEMMA 4.3. *There exists $\phi : \mathbb{T} \rightarrow \mathbb{R}$ such that:*

- (1) *the limits in (4.20) and (4.21) hold; and*
- (2) *η' has opposite signs in $(0, 1/2)$ and $(1/2, 1)$.*

In particular, F is partially hyperbolic (but NOT absolutely partially hyperbolic). Finally we prove that it is not dynamically coherent. Observe that, since the bundles only depend on θ we obtain the stable, unstable and center manifolds (provided that this last one exists) by translation.

Consider the function $h : T^3 \rightarrow T^2$ given by

$$h(x, \theta) = x - \eta(\theta)e_s.$$

Then $F \circ h = h \circ A_T$ and h is clearly surjective. Hence it is a semiconjugacy. Note that we have a parametrization $l_x(\theta)$ of $h^{-1}(h(x, 0))$ given by

$$l_x(\theta) = (x, 0) + (\eta(\theta)e_s, \theta),$$

and hence the family of curves $\{l_x(\theta)\}$ is tangential to E^c if $\theta \neq 1/2$. For $\theta \neq 1/2$ the bundle E^c is uniquely integrable and hence its invariant curves are precisely the $l_x(\theta)$. But, for $\theta = 1/2$, $E^c = E_A^s \times \{0\}$, and hence its tangent curves have to be horizontal. Now we use that η' have different signs on the intervals $(0, 1/2)$ and $(1/2, 1)$ to conclude that this family is not a foliation near $\theta = 1/2$ and hence that the bundle E^c is not integrable (see Figure 7). □

The example previously constructed is, in fact, robust, meaning that in a neighborhood of it there are no dynamically coherent partially hyperbolic diffeomorphisms, which is a surprising fact. This is a consequence of the fact that the invariant torus corresponding

to $\theta = 1/2$ is a *cu*-torus and, in particular, that it is a normally hyperbolic torus. The results of [HPS77] imply that this torus persists under perturbations, meaning that any small perturbation of f has a *cu* torus. On the other hand, outside a neighborhood of this *cu*-torus, there is a unique center foliation, which is persistent under perturbations, due also to [HPS77]. This foliation is extended by invariance to all the complements of the *cu*-torus. Hence, if there were a foliation, it would contain a *cu*-torus. This is not possible, due to the following result.

THEOREM 4.4. (Hertz, Hertz and Ures [HHU15b]) *Let $f : M^3 \rightarrow M^3$ be a dynamically coherent partially hyperbolic diffeomorphism. Then the center unstable foliation does not have any closed leaf.*

Question: Does the same result hold in any dimension?

4.2. *Non-dynamical coherence conjecture—state of the art.* It has been proved by A. Hammerlindl and Potrie [HP13] that the dynamical coherence Conjecture 2.17 is true in tori, solvmanifolds and nilmanifolds and their finite covers.

THEOREM 4.5. (Hammerlindl and Potrie [HP13]) *If f is a non-dynamical coherent diffeomorphism in a 3-manifold with (virtually) solvable fundamental group, then there exists a 2-torus, tangential either to $E^s \oplus E^c$ or $E^c \oplus E^u$. In particular, any partially hyperbolic diffeomorphism with $\Omega(f) = M$ in these manifolds is dynamically coherent.*

This is the sharpest result concerning Conjecture 2.17 so far. Let us give a brief sketch of the ideas used to establish this theorem in the simpler case where $M = \mathbb{T}^3$. This result has been proved by Potrie in his thesis (see [Pot12]), and we shall follow his arguments. Consider $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ partially hyperbolic, and, by passing to a finite covering, it is no loss of generality to assume that the bundles E_f^σ are oriented and, furthermore, that f preserves their orientation.

We will rely heavily in the seminal papers of Brin, Burago and Ivanov [BI08, BBI04, BBI09]. The starting point is that the action in homology $f_* : H_1(\mathbb{T}^3) \rightarrow H_1(\mathbb{T}^3)$ is partially hyperbolic [BI08, Theorem 1.2.]: namely, if $A = f_* \in SL(3, \mathbb{R})$ has eigenvalues $\lambda_1, \lambda_2, \lambda_3$, then $|\lambda_1| < 1 < |\lambda_3|$. There are two cases.

- (1) $|\lambda_2| = 1$; this is the skew-product case.
- (2) $|\lambda_2| \neq 1$; in this case f is a DA.

It suffices to show the existence of an f -invariant foliation tangential to E^{cs} (the other case being analogous). It turns out that there is a natural candidate for \mathcal{F}^{cs} .

THEOREM 4.6. [BI08, Theorem 4.1] *There exists a family $\mathcal{B}^{cs} = \{B^{cs}(x)\}_{x \in M}$ such that:*

- (1) *each $B^{cs}(x)$ is an immersed boundaryless surface of class C^1 tangential to E^{cs} ;*
- (2) *for every $x, x' \in M$, $x \neq x'$, the surfaces $B^{cs}(x), B^{cs}(x')$ do not cross topologically, that is, $B^{cs}(x')$ cannot intersect two different connected components of $B \setminus B^{cs}(x)$, for any neighborhood B of x ; and*
- (3) *\mathcal{B}^{cs} is f -invariant.*

The family \mathcal{B}^{cs} is what is called a *branched foliation*, and its elements are called leaves. To prove that \mathcal{B}^{cs} is a genuine foliation, it suffices to show that it is *unbranched*: namely,

that any $x \in M$ is contained in exactly one leaf of \mathcal{B}^{cs} . This fact is a direct consequence of [BW05a, Proposition 1.6] and the remark afterwards.

Another important fact is that E^{cs} is *almost integrable*: that is, there exists a foliation (not necessarily invariant) that is transverse to E^u . This concept of almost integrability has been coined by R. Potrie and has proven useful in this context. Almost integrability of E^{cs} for all three-dimensional partially hyperbolic diffeomorphisms with orientable bundles had been established by Burago–Ivanov.

THEOREM 4.7. ([BI08], Key lemma) *For every $\epsilon > 0$ sufficiently small there exists a true (not necessarily invariant) foliation $\mathcal{T}_\epsilon^{cs}$ such that the angle $\angle(T\mathcal{T}_\epsilon^{cs}, E^{cs})$ is less than ϵ . Moreover, there exists a continuous surjective map $h_\epsilon^{cs} : M \rightarrow M$ that is ϵ -close to the identity and sends the leaves of $\mathcal{T}_\epsilon^{cs}$ into leaves of \mathcal{B}^{cs} .*

Potrie then shows that, for sufficiently small ϵ , the lifted foliations $\widetilde{\mathcal{F}}^u, \widetilde{\mathcal{T}}_\epsilon^{cs}$ to \mathbb{R}^3 have global product structure (that is, any two leaves $F \in \widetilde{\mathcal{F}}^u$ and $T \in \widetilde{\mathcal{T}}_\epsilon^{cs}$) intersect exactly in one point). This has the following consequence.

THEOREM 4.8. [Pot12, Proposition 8.4] *\mathcal{B}^{cs} is unbranched.*

The proof of this theorem relies on two geometric facts.

Fact 1. $\widetilde{\mathcal{F}}^u$ is quasi-isometric: that is, there exist $a, b > 0$ such that, for every $x, y \in \mathbb{R}^3, y \in \widetilde{\mathcal{F}}^u(x)$, the length $l(x, y)$ of the interval with endpoints x, y contained in $\widetilde{\mathcal{F}}^u(x)$ satisfies

$$l(x, y) \leq a|x - y| + b.$$

Fact 2. There exists an A -invariant plane P and $R > 0$ such that every leaf of \mathcal{B}^{cs} is contained in an R -neighborhood of a plane parallel to P , and either:

- (1) the projection of P in \mathbb{T}^3 is dense and the R -neighborhood of every leaf of \mathcal{B}^{cs} contains a plane parallel to P ; or
- (2) the projection of P in \mathbb{T}^3 is a 2-torus. In this case, there exists a 2-torus tangential to $E^s \oplus E^c$.

Note that if f is a DA, only (1) above can hold. Let us see how the proof goes.

Proof. A modification of [BBI04, Proposition 3.7] gives that a codimension-one foliation in \mathbb{T}^3 either has a Reeb component or there exists $R > 0$ and an A -invariant plane $P \subset \mathbb{R}^3$ such that every leaf of the lifted foliation is in a R -neighborhood of P . The projection of a plane in \mathbb{T}^3 is either dense or a 2-torus. In the former case, every leaf of the lifted foliation is parallel to a fixed translate of the plane, while, in the later case, there is a leaf of the foliation in \mathbb{T}^3 which is a torus (see [Pot12, Theorem 5.3 and Proposition 5.6]).

As $\mathcal{T}_\epsilon^{cs}$ is transverse to \mathcal{W}^u , it cannot have a Reeb component and thus we can consider the plane P , as above. Note that, by 4.7, the leaves of $\widetilde{\mathcal{B}}^{cs}$ are ϵ -close to the leaves of $\widetilde{\mathcal{T}}_\epsilon^{cs}$, and hence the first part of the claim follows.

Now it suffices to observe that if $\mathcal{T}_\epsilon^{cs}$ has a torus leaf, by Theorem 4.7 \mathcal{B}^{cs} also has a torus leaf. Therefore, there is a 2-torus tangent to $E^s \oplus E^c$. □

Once the above facts are established, the proof of the theorem is carried out by standard arguments. To finish the proof of the conjecture, one has to analyze the skew product and

the DA case separately. In both cases, with the machinery developed, it is not too hard to check that the absence of tori tangential to $E^s \oplus E^c$ implies the global product structure referred above, and thus implies dynamical coherence.

When the manifold M is a solvmanifold, but not a nilmanifold, the proof of Theorem 4.5 becomes technically harder, although some of the guidelines presented are still valid. It relies on more background on foliation theory (codimension-one foliations of compact solvmanifolds are reasonably well understood). Solvmanifolds of this type are covered by torus bundles over the circle (a reasonable geometric object), and then there is a canonical model (isotopic to the identity) to compare the dynamics. For more details and a complete proof, we refer the reader to [HP13].

We also recommend the excellent survey by Hammerlindl and Potrie [HP15].

5. Classification

For many years, the only known examples of partially hyperbolic diffeomorphisms in 3-manifolds were the ones listed in §1.1: namely, time-one maps of Anosov flows, skew products, DA-diffeomorphisms, and their perturbations. As stated in §2.3, this led E. Pujals, in 2001, to conjecture that, for transitive ones, this was the complete list of partially hyperbolic diffeomorphisms.

CONJECTURE 2.20. (Pujals (2001)) *Any transitive partially hyperbolic diffeomorphism in a 3-manifold is finitely covered by a map which is conjugated to:*

- (1) *a perturbation of the time-one map of an Anosov flow;*
- (2) *a perturbation of a skew product; or*
- (3) *a DA.*

Two different particular cases of this conjecture were verified in the transitive setting by Bonatti and Wilkinson [BW05a].

THEOREM 5.1. (Bonatti–Wilkinson) *Let $f : M \rightarrow M$ be a transitive partially hyperbolic diffeomorphism.*

- (1) *Assume that there exists some embedded circle c such that[†] $f(c) = c$, with the property that, for some $\epsilon > 0$, the set*

$$\bigcup_{x \in c} W_\epsilon^s(x) \cap \bigcup_{y \in c} W_\epsilon^u(y) \setminus c$$

contains a connected component that is a circle. Then f is dynamically coherent and finitely covered by a map which is conjugated to a circle extension of an Anosov map (a topological skew product).

- (2) *Assume that f is dynamically coherent and that, for some $\epsilon > 0$, each end of a center leaf contained in*

$$\bigcup_{x \in c} W_\epsilon^s(x)$$

is periodic. Then, the center foliation is fixed under f^n and it supports a continuous flow conjugate to an expansive transitive flow.

[†] Interestingly enough, the proof does not extend to the case where c is merely periodic.

It would be interesting to know if, in case (2), one can, in fact, take an Anosov flow and thus settle Pujals's conjecture for that case. This still remains an open problem.

More recently, a new type of non-dynamically coherent example was presented, which is the one described in §4.1. The examples in §4.1 suggested another possibility.

CONJECTURE 2.22. (Classification conjecture: Hertz, Hertz and Ures (2009)) *Let f be a partially hyperbolic diffeomorphism of a 3-manifold.*

If f is dynamically coherent, then it is (finitely covered by) one of the following:

- (1) *a perturbation of a time-one map of an Anosov flow, in which case it is leafwise conjugate to an Anosov flow;*
- (2) *a skew product, in which case it is leafwise conjugate to a skew product with linear base; or*
- (3) *a DA, in which case it is leafwise conjugate to an Anosov diffeomorphism of \mathbb{T}^3 .*

If f is not dynamically coherent, then there are a finite number of 2-tori tangential either to $E^c \oplus E^u$ or to $E^s \oplus E^c$, and the rest of the dynamics are trivial, as in the non-dynamically coherent example [HHU15b].

Both conjectures have been proved false very recently by Bonatti, Gogolev, Parwani and Potrie (see [BPP14, BGP15] and §5.1 for a description of these examples). However, both conjectures are true in certain manifolds, as was proved by Hammerlindl and Potrie.

THEOREM 5.2. (Hammerlindl and Potrie [HP13]) *Both Conjectures 2.20 and 2.22 are true on 3-manifolds with (virtually) solvable fundamental group.*

Theorem 5.2 was first proved in tori by Hammerlindl in his thesis [Ham13], and it was later extended to 3-manifolds with (virtually) nilpotent groups by Hammerlindl and Potrie in [HP14]. Finally, it was extended to 3-manifolds with (virtually) solvable groups by the same authors in [HP13]. This is still in press.

In [HP13], it is proved that, for solvmanifolds, as stated in Conjecture 2.17, the absence of tori tangential to either $E^s \oplus E^c$ or $E^c \oplus E^u$ implies dynamical coherence. Observe that the existence of such a torus implies the existence of either a repelling or an attracting periodic torus (see more details in §6). Transitivity precludes this possibility. Therefore, for solvmanifolds, we can assume that there is dynamical coherence in both Conjectures 2.20 and 2.22.

Let us give a flavor of how Theorem 5.2 is proved in the case of solvmanifolds with non-virtually nilpotent fundamental group.

THEOREM 5.3. [HP13] *If $f : M \rightarrow M$ is a dynamically coherent partially hyperbolic diffeomorphism on a solvmanifold whose fundamental group is not virtually nilpotent, then a finite cover of a finite iterate of f is leafwise conjugate to the time-one map of a suspension Anosov flow.*

See Definition 2.21 for the definition of leaf conjugacy.

To start the proof of Theorem 5.3, one shows that any such manifold is finitely covered by the mapping torus M_A of a hyperbolic automorphism on \mathbb{T}^2 : that is $M_A = \mathbb{T}^2 \times \mathbb{R} / \sim$ such that $(Ax, t) \sim (x, t + 1)$, where A is a hyperbolic automorphism of

\mathbb{T}^2 . Any diffeomorphism of M_A has a finite iterate that is homotopic to the identity. This is not hard to prove.

Now, on the universal cover of M_A , there are model foliations \mathcal{A}^{cs} and \mathcal{A}^{cu} ; (v_1, t_1) and (v_2, t_2) belong to the same leaf of the foliation \mathcal{A}^{cs} if and only if $v_1 - v_2$ is in the stable eigenspace of the automorphism A . Similarly, (v_1, t_1) and (v_2, t_2) belong to the same leaf of the foliation \mathcal{A}^{cu} if and only if $v_1 - v_2$ is in the unstable eigenspace of the automorphism A . In [HP13], it is seen that the lift to the universal cover of any foliation without compact leaves is *almost parallel* to either \mathcal{A}^{cs} or \mathcal{A}^{cu} . Two foliations \mathcal{F} and \mathcal{F}' are *almost parallel* if there is a uniform bound $R > 0$ such that:

- (1) for each leaf $L \in \mathcal{F}$, there is a leaf $L' \in \mathcal{F}'$ such that $d_H(L, L') < R$; and
- (2) for each leaf $L' \in \mathcal{F}'$, there is a leaf $L \in \mathcal{F}$ such that $d_H(L, L') < R$,

where d_H is the Hausdorff distance: that is

$$d_H(L, L') = \max\left\{\sup_{x \in L} d(x, L'), \sup_{y \in L'} d(y, L)\right\}.$$

Now neither \mathcal{F}^{cs} nor \mathcal{F}^{cu} contain compact leaves [HHU15a], and therefore each one is almost parallel to either \mathcal{A}^{cs} or \mathcal{A}^{cu} . They proceed then to show that if \mathcal{F}^{cs} is almost parallel to \mathcal{A}^{cs} , then \mathcal{F}^{cu} is almost parallel to \mathcal{A}^{cu} . This step is more delicate.

Note that the center leaves of the model foliation (that is, the leaves in \mathcal{A}^c that are intersections of leaves \mathcal{A}^{sc} and \mathcal{A}^{cu}) correspond to trajectories of an Anosov flow, which is *infinitely expansive*. Infinite expansivity means that, for any two different points x and y in the universal cover and any $K > 0$, there will be a time $t \in \mathbb{R}$ such that $X_t(x)$ and $X_t(y)$ are K -apart. Therefore, any two such leaves A_1^c and A_2^c are at an infinite Hausdorff distance. This implies that the almost-parallel relation defined above assigns to each center leaf F^c in the intersection of \mathcal{F}^{sc} and \mathcal{F}^{cu} a unique center leaf in \mathcal{A}^c . Less trivially, there is a unique leaf in \mathcal{F}^{sc} at a finite Hausdorff distance from each leaf in \mathcal{A}^{sc} and a unique leaf in \mathcal{F}^{cu} at a finite distance from each leaf in \mathcal{A}^{cu} [HP13, Lemma 5.3]. Therefore, any two center leaves F_1^c and F_2^c that are at a finite Hausdorff distance from each other must be in the intersection of a single leaf of \mathcal{F}^{sc} and a single leaf of \mathcal{F}^{cu} .

Now let us assume that the center bundle E_f^c is orientable, for, otherwise, we can take a finite cover. Then there exists a field X^c tangential to E^c , defining a flow φ on M_A . We claim that φ is an expansive flow.

Indeed, any two φ -trajectories that are at most ε -apart correspond to two center leaves that are at a finite Hausdorff distance; hence, they are in the intersection of a single leaf of \mathcal{F}^{sc} and a single leaf of \mathcal{F}^{cu} . This implies either that a stable leaf intersects (at least) twice a center unstable leaf of \mathcal{F}^{cu} or that an unstable leaf intersects (at least) twice a center-stable leaf of \mathcal{F}^{sc} . A classical argument, according to Novikov, implies the existence of a compact leaf either in \mathcal{F}^{cu} or in \mathcal{F}^{sc} , which is a situation precluded by [HHU15a].

Finally, Brunella establishes, in [Bru93], that any expansive flow on a torus bundle is leafwise conjugate to a transitive Anosov suspension, which concludes the classification theorem in solvmanifolds.

Again, we refer the reader to the survey of Hammerlindl and Potrie, in order to gain a deeper understanding of this topic.

5.1. *Anomalous partially hyperbolic diffeomorphisms.* Very recently, Bonatti, Gogolev, Parwani and Potrie found both a dynamically coherent example and a transitive example that are not leafwise conjugate to any of the above models, which proved both conjectures wrong [BPP14, BGP15].

The idea behind both constructions is to perform a large ‘perturbation’ by composing the time-one map of certain Anosov flows in an appropriate neighborhood with a map of the form (g, Dg) , where g is a Dehn twist and Dg is the derivative acting on the unitary tangent bundle. This neighborhood has to be large enough so that the effect of the derivative of the Dehn twist is negligible. We will explain this in more detail for the case of the perturbations of the time-one map of the geodesic flow of surfaces of constant negative curvature in Theorem 5.5.

The first family of (non-transitive) examples is obtained by modifying the Franks–Williams’ construction [FW80b] of a non-transitive Anosov flow.

THEOREM 5.4. (Bonatti, Parwani and Potrie [BPP14]) *There is a closed orientable 3-manifold M endowed with a non-transitive Anosov flow X_t and a diffeomorphism $f : M \rightarrow M$ such that:*

- *f is absolutely partially hyperbolic;*
- *f is robustly dynamically coherent;*
- *the restriction of f to its chain recurrent set coincides with the time-one map of the Anosov flow X_t ; and*
- *for any $n > 0$, f^n is not isotopic to the identity.*

The transitive examples are built on time-one maps of two different Anosov flows: the Bonatti–Langevin example [BL94] and the geodesic flow of surfaces of negative constant curvature (a similar construction works for the Handel–Thurston Anosov flow [HT80]). After the statement of the theorem, we will give a brief outline of the construction for the case of geodesic flows.

THEOREM 5.5. (Bonatti, Gogolev and Potrie [BGP15]) *There exist a closed orientable 3-manifold M and an absolutely partially hyperbolic diffeomorphism $f : M \rightarrow M$ that satisfy that:*

- *M admits an Anosov flow;*
- *f^n is not homotopic to the identity map for all $n > 0$;*
- *f is volume preserving; and*
- *f is robustly transitive and stably ergodic.*

5.2. *Unit tangent bundle of surfaces of negative curvature.* Here we give the main ideas of the construction of the examples for this case. Let S be a surface and let g be a Riemannian metric of curvature -1 . Fix a simple closed geodesic γ . It is possible to deform the hyperbolic metric in such a way the length of γ goes to zero. Indeed, there is a sequence g_n of metrics of curvature -1 such that $\text{length}_{g_n} \gamma \rightarrow 0$. Then there are collars of uniform length (for the metric g_n) of γ ; call them C_n . These collars become thinner and thinner as n goes to infinity. This implies that Dehn twists ρ_n on these collars are very close to isometries, for n large enough.

Now consider $h_n = D_{\rho_n} \circ \varphi_n$, where φ_n is the time-one map of the geodesic flow of g_n and D_{ρ_n} is the projectivization of the derivative of ρ . Since the partially hyperbolic structure does not change with the metric (indeed, the partially hyperbolic structure depends on the metric and, for all n , the metric on the universal cover is the same) this will imply that h_n is partially hyperbolic for n large enough. By constructing ρ_n with some care, D_{ρ_n} can be made volume preserving. Known results and techniques imply that there is a stably ergodic and robustly transitive perturbation of h_n .

Bonatti, Hammerlindl, Gogolev and Potrie have announced a generalization of the latter construction (see [BGP15]). There is a natural homomorphism of the mapping class group of a surface of genus greater than one, S , into the mapping class group of its unit tangent bundle given by the projectivization $I : \text{MCG}(S) \rightarrow \text{MCG}(T_1 S)$.

THEOREM 5.6. *Each mapping class of the image of I admits a volume-preserving partially hyperbolic representative.*

There are some open questions about the examples given by Theorem 5.5. The most important is whether they are dynamically coherent.

Many new questions arise regarding the classification. Some of them are the given below.

Question 5.7. Suppose that the fundamental group of M is not (virtually) solvable. If M admits a partially hyperbolic diffeomorphism, does it support an Anosov flow?

Hammerlindl, Potrie and Shannon have announced that the answer is positive for Seifert manifolds having fundamental group with exponential growth.

Question 5.8. Given a partially hyperbolic diffeomorphism on M , is there some sort of inverse process of the previous construction leading to a partially hyperbolic diffeomorphism isotopic to the identity? Or to a partially hyperbolic diffeomorphism leaf conjugate (up to finite cover and iterate) to the time-one map of an Anosov flow?

6. A tool to better understand some partially hyperbolic dynamics: Anosov tori

Both the ergodicity and the integrability conjectures propose that the existence of a map with some specific dynamical property leads to very rigid restrictions in the topology of the ambient manifold. The reader may wonder why this is case and how one can attempt to prove such types of results. We discuss these issues in this section.

The unifying link is, surprisingly, the existence of certain tori embedded in the manifold.

Definition 6.1. We say that the manifold M admits an *Anosov torus* if there exist a C^1 -embedded torus $T \subset M$ and a diffeomorphism $\phi : M \rightarrow M$ such that:

- (1) $\phi(T) = T$; and
- (2) $\phi|_T$ is a linear hyperbolic automorphism[†].

As we shall see below, not every manifold admits an Anosov torus.

Recall that a three manifold is irreducible if every embedded two-sphere bounds a three-ball. We then have the following topological result.

[†] This is equivalent to the existence of ϕ such that $\phi|_T$ is isotopic to an Anosov diffeomorphism, which holds if and only if the action on the first homology group of the torus $H_1(T)$ is hyperbolic.

THEOREM 6.2. (Hertz, Hertz and Ures [HHU11]) *Assume that M is a compact, irreducible 3-manifold supporting an Anosov torus. Then M is homeomorphic to:*

- (1) *a 3-torus;*
- (2) *the mapping torus of $-\text{Id} : \mathbb{T}^2 \rightarrow \mathbb{T}^2$; or*
- (3) *the mapping torus of an hyperbolic automorphism of \mathbb{T}^2 .*

We remark that partial hyperbolicity is not required in Theorem 6.2. This theorem just shows that there are actually very few 3-manifolds admitting Anosov tori. Now we apply this result to the partially hyperbolic context. First, we note that irreducibility comes for free in this setting.

LEMMA 6.3. *If a 3-manifold M supports a partially hyperbolic diffeomorphism, then M is irreducible.*

Proof. A 3-manifold admitting a partially hyperbolic diffeomorphism has a codimension-one foliation having neither Reeb components nor spherical leaves [BBI04]. This proves the claim, since Rosenberg shows, in [Ros68], that any codimension-one foliation in a reducible 3-manifold must have a Reeb component or a spherical leaf (see also [Rou71]). □

The following proposition describes Anosov tori that arise naturally in partially hyperbolic dynamics (see [HHU15a] for more details).

PROPOSITION 6.4. *Let $f : M \rightarrow M$ be a partially hyperbolic diffeomorphism and assume that there exists an f -invariant embedded torus T tangential to either $E^c \oplus E^u$, $E^s \oplus E^c$ or $E^u \oplus E^s$. Then T is an Anosov torus.*

Proof. Let $g = f|_T$. In each of the different cases, g or g^{-1} preserves an expanding foliation by lines, so, with no loss of generality, we will assume that $\|dg|_{E^u}\| > 1$. It suffices to prove that $g_* : \pi_1(T) \approx \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ is hyperbolic.

By taking g^2 , if necessary, we can suppose that g preserves the orientation of $E^u|_T$. Since g preserves a foliation without compact leaves, the integral matrix g_* has an eigenspace of irrational slope. This implies that either g_* is hyperbolic or $g_* = \text{Id}$. In the second case, g has a lift $\widehat{g} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $\widehat{g} = \text{Id} + \alpha$, where α is a periodic and, in particular, a bounded function. Hence there exists a constant $K > 0$ such that, given any $X \subset \mathbb{R}^2$,

$$\text{diam}(\widehat{g}^n(X)) \leq \text{diam}(X) + nK.$$

Let γ be an arc contained in a leaf of $W^u(x)$, $x \in T$. Then the length of γ grows exponentially under \widehat{g}^n and its diameter grows at most linearly. This implies that, given a small $\epsilon > 0$, there exists an iterate of g that contains a curve of length arbitrarily large and with endpoints at a distance less than ϵ . Using Poincaré–Bendixon we obtain a singularity of the foliation W^u . This is a contradiction and thus g_* is hyperbolic. □

Definition 6.5. Let $f : M \rightarrow M$ be a partially hyperbolic diffeomorphism. An embedded torus tangential to E^{cs} , E^{cu} or E^{su} will be called a *cs*, *cu* or a *su*-torus, respectively. In these cases, we say that f admits the corresponding torus.

LEMMA 6.6. *If f admits an su -torus, then M admits an Anosov torus.*

Proof. Assume that f admits an su -torus and consider the lamination Λ of all su -tori of f . This is a compact lamination [Hae62]. Therefore there is a recurrent leaf: that is, there is a torus T and an iterate n such that $d_H(f^n(T), T) < \varepsilon$ for small ε . There exists a diffeotopy i_t on M taking $f^n(T)$ into T . Then $\phi = f^n \circ i_1$ fixes T and $\phi|_T$ is isotopic to an Anosov diffeomorphism. \square

LEMMA 6.7. *If f admits an sc or cu torus, then it admits an f -periodic sc or a cu torus. Therefore M admits an Anosov torus (by Proposition 6.4).*

Proof. Let T be a cu -torus and consider the sequence $f^{-n}(T)$. Since the family of all compact subsets of M considered with the Hausdorff metric d_H is compact, there is a subsequence $f^{-n_k}(T)$ converging to a compact set $K \subset M$. Therefore, for each $\varepsilon > 0$, there are arbitrarily large $N \gg L > 0$ such that $d_H(f^{-N}(T), f^{-L}(T)) < \varepsilon$.

Since T is transverse to the stable foliation, the union of all local stable leaves of T forms a small tubular neighborhood of T , $U(T)$. Since stable leaves grow exponentially under f^{-1} , if $N \gg L$, as above, are large enough, then $f^{-L}(\overline{U(T)}) \subset f^{-N}(U(T))$. This implies that $f^{N-L}(\overline{U(T)}) \subset U(T)$.

Exercise 6.8. We finish the proof by showing that $\bigcap_{k=0}^{\infty} f^{k(N-L)}(U(T))$ is a periodic cu -torus. \square

COROLLARY 6.9. *Suppose that $f : M \rightarrow M$ is a partially hyperbolic diffeomorphism admitting an sc , cu or su torus. If M is connected, then M is homeomorphic to:*

- (1) a 3-torus;
- (2) the mapping torus of $-\text{Id} : \mathbb{T}^2 \rightarrow \mathbb{T}^2$; or
- (3) the mapping torus of a hyperbolic automorphism of \mathbb{T}^2 .

Observe that an sc or a cu torus cannot appear on the conservative setting. Indeed, by Lemma 6.7 above, it would imply the existence of a *periodic* sc or cu torus. This 2-torus is, respectively, repelling or attracting, a situation that cannot occur in a conservative setting.

In the following subsection, we sketch the proof of Theorem 6.2. We refer the reader to [HHU11] for the complete proof.

6.1. *Manifolds admitting Anosov tori.* The first step in the proof of Theorem 6.2 is to show that Anosov tori are incompressible.

Definition 6.10. An embedded orientable surface $S \subset M$ is *incompressible* if the homomorphism induced by the inclusion map $i_* : \pi_1(S) \rightarrow \pi_1(M)$ is injective.

Equivalently, S is incompressible if every embedded disc, $D^2 \subset M$, such that $D^2 \cap S = \partial D^2$ is contractible in S (see, for instance, [Hat07, p. 10]).

THEOREM 6.11. (Hertz, Hertz and Ures [HHU08a]) *Anosov tori are incompressible.*

Now, let us assume, as in the hypotheses of Theorem 6.2, that the irreducible 3-manifold M admits an Anosov torus T . Since T is incompressible, we can ‘cut’ M along T and

obtain a manifold N having incompressible 2-tori as boundary components. Theorem 6.2 then follows from the following theorem.

THEOREM 6.12. (Hertz, Hertz and Ures [HHU11]) *Let N be a compact orientable irreducible 3-manifold with non-empty boundary such that all the boundary components are incompressible 2-tori. If N admits an Anosov torus, then $N \approx \mathbb{T}^2 \times [0, 1]$.*

In order to prove this, we make use of the Jaco–Shalen–Johannson (JSJ) decomposition, which states that any manifold satisfying the hypotheses of Theorem 6.12 can be cut by a (unique) family of incompressible tori, so that the remaining pieces have certain characteristics: they are either *Seifert* or else *atoroidal* and *acylindrical*. We provide these definitions below (see also Theorem 6.15).

The proof of Theorem 6.12 consists of showing, on the one hand, that any Seifert manifold having incompressible tori as boundary components is $\mathbb{T}^2 \times [0, 1]$ and, on the other hand, that any manifold satisfying the hypotheses of Theorem 6.12 that is atoroidal has an annulus which is properly embedded and is not isotopic to the boundary of the manifold. This last statement contradicts the fact that the manifold is *acylindrical*, and shows that every component in the JSJ-decomposition must be Seifert, which proves the theorem.

Any compact 3-manifold, with or without boundary, supporting a foliation by circles is a *Seifert manifold* (see [Eps81]). This was not the original definition; a more descriptive one is the following.

Definition 6.13. A *Seifert manifold* is a 3-manifold that admits a decomposition into disjoint circles, the fibers, such that each fiber has a neighborhood diffeomorphic, preserving fibers, to either:

- (1) a solid torus foliated by horizontal circles; or
- (2) a solid torus foliated by the fibration obtained by the identification $D^2 \times [0, 1]/x \sim R_{p/q}(x)$, where $R_{p/q} : D^2 \rightarrow D^2$ denotes the rotation of angle p/q and p, q are coprime.

If the manifold has boundary, its connected components are required to be tori, which are also fibered by circles.

The circles of the first type are the *generic fibers*, while the ones of the second type are the *singular fibers*. For an introduction to Seifert spaces see [Bri93].

Definition 6.14. Let N be a 3-manifold with boundary.

- (1) N is *atoroidal* if every incompressible torus is ∂ -parallel, that is, isotopic to a subsurface of ∂N .
- (2) N is *acylindrical* if every incompressible annulus A that is properly embedded (that is, $\partial A \subset \partial N$) is ∂ -parallel by an isotopy fixing ∂A .

As we mentioned above, any irreducible orientable 3-manifold having incompressible tori as boundary components admits a natural decomposition into Seifert pieces on one side and atoroidal and acylindrical components on the other.

THEOREM 6.15. (JSJ-decomposition [Hat07]) *If N is an irreducible, orientable 3-manifold, having incompressible tori as boundary components, then there exists a finite*

collection \mathcal{T} of disjoint incompressible tori such that, for each component N_i of $N \setminus \bigcup \mathcal{T}$, either:

- (1) N_i is a Seifert manifold; or
- (2) N_i is both atoroidal and acylindrical.

Any minimal such collection is unique up to isotopy. This means that if \mathcal{T} is a collection as described above, it contains a minimal subcollection $m(\mathcal{T})$ satisfying the same claim. All collections $m(\mathcal{T})$ are isotopic.

Any minimal family of incompressible tori, as described above, is called a *JSJ-decomposition* of N . Note that if N is either atoroidal and acylindrical or Seifert, then $\mathcal{T} = \emptyset$.

Let us sketch how Theorem 6.12 is proved for the case of Seifert manifolds; the other case is more delicate, and we refer the reader to [HHU11] for the complete proof.

Assume that N is a Seifert manifold, so it admits a foliation by circles, which is called a Seifert fibration. We lose no generality in assuming that one of the incompressible tori of the boundary of N is an Anosov torus T . By the definition of an Anosov torus, there exists a diffeomorphism ϕ on N such that it is a linear hyperbolic automorphism of T . The image of the Seifert fibration by ϕ is another Seifert fibration, which is non-isotopic to the original one on T .

But orientable manifolds admitting two Seifert fibrations that are non-isotopic on its boundary are completely classified.

LEMMA 6.16. [Hat07] *If N admits two Seifert fibrations that are non-isotopic on ∂N , then N is homeomorphic to:*

- (1) *the solid torus;*
- (2) *a twisted I bundle over the Klein bottle; or*
- (3) *the torus cross the interval.*

In the first two cases, ∂N consists of a single torus, while, in last one, it consists of two disjoint tori. To finish the proof, it suffices then to discard the first two cases.

LEMMA 6.17. *If ∂N contains an Anosov torus, then it contains more than one.*

Proof. Assume that ∂N is a torus T and consider the inclusion map $i : H_1(\partial N) \rightarrow H_1(N)$. Let $\ker i$ be the kernel of the map. Then [Hat07, Lemma 3.5],

$$\text{rank}(\ker i) = \frac{1}{2} \text{rank}(H_1(\partial N)),$$

where *rank* denotes the number of \mathbb{Z} summands in a direct sum splitting into cyclic groups. If $\partial \tilde{M} = T$, then $(1/2)\text{rank}(H_1(T)) = 1$, and hence $K = \ker i$ is a one-dimensional subspace of $H_1(T)$. Then $f_*(K) = K$, where $f_* : H_1(T) \rightarrow H_1(T)$ is the isomorphism induced by any diffeomorphism $f : N \rightarrow N$. This implies that f_* has 1 as an eigenvalue. Hence f cannot be isotopic to a hyperbolic automorphism of T . This implies that T cannot be an Anosov torus. \square

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