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COEXISTENCE SOLUTIONS FOR A PERIODIC COMPETITION MODEL WITH SINGULAR–DEGENERATE DIFFUSION

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Abstract We investigate a system of singular–degenerate parabolic equations with non-local terms, which can be regarded as a spatially heterogeneous competition model of Lotka–Volterra type. Applying the Leray–Schauder fixed-point theorem, we establish the existence of coexistence periodic solutions to the problem, which, together with the existing literature, gives a complete picture for such a system for all parameters.

Keywords: coexistence solutions; periodic competition model; singular–degenerate diffusion

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1. Introduction

This paper is concerned with the system

$$\left.\begin{aligned} L^{m,p}[u] &= u^{p-1}\left(a(x,t) - \int_\Omega K_1(\xi,t)u^2(\xi,t-\tau_1)\,\mathrm{d}\xi - \int_\Omega K_2(\xi,t)v^2(\xi,t-\tau_2)\,\mathrm{d}\xi\right),\\ L^{n,q}[v] &= v^{q-1}\left(b(x,t) - \int_\Omega K_3(\xi,t)u^2(\xi,t-\tau_3)\,\mathrm{d}\xi - \int_\Omega K_4(\xi,t)v^2(\xi,t-\tau_4)\,\mathrm{d}\xi\right)\end{aligned}\right\} \tag{1.1}$$

in $Q = \Omega \times \mathbb{R}$, under Dirichlet boundary conditions

$$u(x,t) = v(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}, \tag{1.2}$$

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and periodic conditions

$$u(x,t+\omega)=u(x,t), \quad v(x,t+\omega)=v(x,t), \quad (x,t)\in Q, \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geqslant 1)$ with smooth boundary $\partial\Omega$, $\omega > 0$, $\tau_i \in (0,\infty)$, $i = 1,\dots,4$, the functions K_i, a, b: $\Omega \times \mathbb{R} \to \mathbb{R}$ are ω-periodic with respect to time t, $p,q \in (1,2)$, $m > 1$, $n > 1$, and $L^{s,r}$ is the nonlinear operator defined by

$$L^{s,r}[w] = \frac{\partial w}{\partial t} - \operatorname{div}(|\nabla w^s|^{r-2}\nabla w^s).$$

The system (1.1), (1.2) models the interactions between two competition species, with population densities $u(x,t)$ and $v(x,t)$, inhabiting the region Ω. We are therefore only interested in non-negative solutions $u \geqslant 0$, $v \geqslant 0$. Moreover, we are assuming that Ω is fully surrounded by a lethal environment, because both population densities are subject to homogeneous Dirichlet boundary conditions. For a detailed description of the model, we refer readers to [**12**] and the references therein.

In recent years, coexistence solutions, namely those solutions (u,v) with $u \gneqq 0$, $v \gneqq 0$, have received considerable attention. Most of the works in the earlier literature were devoted to studying the linear diffusion case: that is, $m = n = 1$, $p = q = 2$ (see, for example, [**1**, **7**, **13**, **16**, **20**, **21**]). We also mention that, as a special case, the coexistence steady state for biological community system has also been investigated in some references: see [**3**, **5**, **6**, **8**, **18**]. The biological background makes it very interesting to study the coexistence periodic solutions of a nonlinear diffusion system of the same type as (1.1). In this respect, the case with double degeneracy, that is, $m,n > 1$, $p,q > 2$, has been discussed extensively: see [**2**, **9**–**11**, **23**] and the references therein. For the case with gradient singularity, that is, $1 < p,q < 2$, it was Fragnelli *et al*. who established the existence of coexistence periodic solutions to the problem (1.1)–(1.3) in the case of $m > p$, $n > q$, by applying the theory of Leray–Schauder degree (see [**12**]).

The aim of this paper is to supplement the existing results with the case with gradient singularity. More specifically, concerning the existence of coexistence periodic solutions, the results in [**12**] together with our results will give a complete picture of the system (1.1)–(1.3) for all $1 < p,q < 2$, $m > 1$, $n > 1$. In fact, the theory of topological degree has been used to deal with the case $1 < p,q < 2$, $m > p$, $n > q$ in [**12**], but it is difficult to apply this method to the general case due to the complicated, or even impossible, calculation of the topological degree. Indeed, in order to obtain the topological degree of semi-non-trivial solutions such as $(u,0)$ and $(0,v)$ to the approximate problem of (1.1)–(1.3), some estimates on the gradient of convenient powers of the solution are involved, and the technical restriction $m > p$, $n > q$ is therefore imposed in [**12**] (see Proposition 2.3 and Lemma 2.4 therein). To overcome these difficulties, we apply the Leray–Schauder fixed-point theorem and propose a different approximate problem from which we are able to obtain the crucial *a priori* lower-bound estimates.

The paper is organized as follows. In § 2 we introduce some necessary preliminaries and give the statement of our main result. Section 3 is devoted to proofs of our main result. More precisely, first we prove the existence of coexistence solution $(u_\varepsilon, v_\varepsilon)$ for a suitable approximation of problem (1.1)–(1.3), which has a lower bound that is independent of ε, by using the Leray–Schauder fixed-point theorem; and then, by applying the compactness and monotonicity arguments, we perform some *a priori* estimates that allow us to pass to the limit.

2. Preliminaries and our main result

Throughout this paper $C_\omega(\bar{Q}_\omega)$ denotes the set of functions that are continuous in $\bar{\Omega} \times \mathbb{R}$ and are ω-periodic with respect to t, and B_R is a ball centred at the origin with radius R in $L^\infty(Q_\omega)$. We assume that $a(x,t), b(x,t), K_i(x,t) \in C_\omega(\bar{Q}_\omega)$, $i = 1, \dots, 4$, and

$$\left\{x \in \Omega \colon \frac{1}{\omega}\int_0^\omega a(x,t)\,\mathrm{d}t > 0\right\} \neq \emptyset, \qquad \left\{x \in \Omega \colon \frac{1}{\omega}\int_0^\omega b(x,t)\,\mathrm{d}t > 0\right\} \neq \emptyset.$$

Observe that there is no loss of generality if we choose $K_i \geqslant 0$, $i = 1, \dots, 4$, which corresponds to the competition case. In fact, all of our results can be easily extended to the cooperative case of (1.1), provided that L^2-estimates of the periodic solutions to a cooperative model are available.

Due to the singularity or degeneracy of the equations in (1.1), problem (1.1)–(1.3) might not have classical solutions in general. We therefore consider its weak solutions in the following sense.

Definition 2.1. A pair of non-negative functions (u,v) defined in $\Omega \times (0,\omega)$ is called a weak solution of problem (1.1)–(1.3) if $u, v \in C_\omega(\bar{Q}_\omega)$, $u^m \in L^p((0,\omega); W_0^{1,p}(\Omega))$, $v^n \in L^q((0,\omega); W_0^{1,q}(\Omega))$ and (u,v) satisfies

$$\begin{aligned} 0 = \iint_{Q_\omega} \biggl(&-u\frac{\partial \varphi}{\partial t} + |\nabla u^m|^{p-2}\nabla u^m \nabla\varphi - au^{p-1}\varphi \\ &+ u^{p-1}\varphi\biggl[\int_\Omega K_1(\xi,t)u^2(\xi,t-\tau_1)\,\mathrm{d}\xi + \int_\Omega K_2(\xi,t)v^2(\xi,t-\tau_2)\,\mathrm{d}\xi\biggr]\biggr)\,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

and

$$\begin{aligned} 0 = \iint_{Q_\omega} \biggl(&-v\frac{\partial \varphi}{\partial t} + |\nabla v^n|^{q-2}\nabla v^n \nabla\varphi - bv^{q-1}\varphi \\ &+ v^{q-1}\varphi\biggl[\int_\Omega K_3(\xi,t)u^2(\xi,t-\tau_3)\,\mathrm{d}\xi + \int_\Omega K_4(\xi,t)v^2(\xi,t-\tau_4)\,\mathrm{d}\xi\biggr]\biggr)\,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

for any $\varphi \in C^1(\bar{Q}_\omega)$ with $\varphi(x,0) = \varphi(x,\omega)$ for $x \in \Omega$ and $\varphi(x,t) = 0$ for any $(x,t) \in \partial\Omega \times [0,\omega]$.

In order to obtain the existence of coexistence solutions for the problem (1.1)–(1.3), we add some viscosity terms to the degenerate parabolic equation and then consider the

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following regularized problem:

$$\left.\begin{aligned} L_\varepsilon^{m,p}[u] &= u^{p-1}\bigg[a(x,t) - \int_\Omega K_1(\xi,t)u^2(\xi,t-\tau_1)\,\mathrm{d}\xi \\ &\qquad - \int_\Omega K_2(\xi,t)v^2(\xi,t-\tau_2)\,\mathrm{d}\xi\bigg] + \varepsilon, \quad (x,t)\in Q_\omega, \\ L_\varepsilon^{n,q}[v] &= v^{q-1}\bigg[b(x,t) - \int_\Omega K_3(\xi,t)u^2(\xi,t-\tau_3)\,\mathrm{d}\xi \\ &\qquad - \int_\Omega K_4(\xi,t)v^2(\xi,t-\tau_4)\,\mathrm{d}\xi\bigg] + \varepsilon, \quad (x,t)\in Q_\omega, \\ u(x,t) &= v(x,t) = 0, \quad (x,t)\in\partial\Omega\times[0,\omega], \\ u(x,0) &= u(x,\omega), \quad v(x,0) = v(x,\omega), \quad x\in\Omega, \end{aligned}\right\} \tag{2.1}$$

where $L_\varepsilon^{m,p}$ is the nonlinear operator defined by

$$L_\varepsilon^{m,p}[w] = \frac{\partial w}{\partial t} - \varepsilon\,\mathrm{div}(|\nabla w|^{p-2}\nabla w) - \mathrm{div}(|\nabla w^m|^{p-2}\nabla w^m)$$

with sufficiently small $\varepsilon > 0$, and $L_\varepsilon^{n,q}$ can be similarly defined. The coexistence solution (u, v) of problem (1.1)–(1.3) will be obtained as the limit point of the weak solution $(u_\varepsilon, v_\varepsilon)$ for the problem (2.1), which has a lower bound independent of ε.

In order to apply Leray–Schauder's fixed-point theorem to obtain the existence of the coexistence solutions for the problem (2.1), we introduce a map $G_\varepsilon\colon [0,1]\times C_\omega(\bar{Q}_\omega)\times C_\omega(\bar{Q}_\omega)\to C_\omega(\bar{Q}_\omega)\times C_\omega(\bar{Q}_\omega)$ as follows:

$$(\sigma, f, g)\mapsto (u_\varepsilon, v_\varepsilon) = G_\varepsilon(\sigma, f, g),$$

where $(u_\varepsilon, v_\varepsilon)$ is the solution of the following uncoupled periodic problem:

$$\begin{aligned} L_\varepsilon^{m,p}[u] &= \sigma f, \quad (x,t)\in Q_\omega, \\ L_\varepsilon^{n,q}[v] &= \sigma g, \quad (x,t)\in Q_\omega, \\ u(x,t) &= v(x,t) = 0, \quad (x,t)\in\partial\Omega\times[0,\omega], \\ u(x,0) &= u(x,\omega), \quad v(x,0) = v(x,\omega), \quad x\in\Omega. \end{aligned}$$

As pointed out in [**12**], the operator $A\colon \mathcal{X} = L^p(0,\omega; W_0^{1,p}(\Omega)\cap L^2(\Omega))\mapsto\mathcal{X}'$,

$$Au = \varepsilon\,\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \mathrm{div}(|\nabla u^m|^{p-2}\nabla u^m)$$

is hemi-continuous, strictly monotone (and hence pseudo-monotone), coercive and bounded. The map G_ε is therefore well defined (see [**26**, Theorem 32.D]). Furthermore, by applying similar methods from [**12, 27**] and the classical regularity results from [**14**], one can infer that the map $G_\varepsilon(\sigma, f, g)$ is a compact continuous map. Now let

$$f(u,v) = \bigg[a(x,t) - \int_\Omega K_1(\xi,t)u^2(\xi,t-\tau_1)\,\mathrm{d}\xi - \int_\Omega K_2(\xi,t)v^2(\xi,t-\tau_2)\,\mathrm{d}\xi\bigg]u_+^{p-1} + \varepsilon$$

and

$$g(u,v)=\left[b(x,t)-\int_{\Omega}K_3(\xi,t)u^2(\xi,t-\tau_3)\,\mathrm{d}\xi-\int_{\Omega}K_4(\xi,t)v^2(\xi,t-\tau_4)\,\mathrm{d}\xi\right]v_+^{q-1}+\varepsilon,$$

where $u_+=\max\{u,0\}$, $v_+=\max\{v,0\}$. It is then observed that if the non-negative functions u_ε, v_ε satisfy $(u_\varepsilon,v_\varepsilon)=G_\varepsilon(1,f(u_\varepsilon,v_\varepsilon),g(u_\varepsilon,v_\varepsilon))$, then $(u_\varepsilon,v_\varepsilon)$ is also a non-negative solution of problem (2.1). The existence of non-negative solutions of problem (2.1) is therefore equivalent to the existence of the fixed point $(u_\varepsilon,v_\varepsilon)$ of the map $(u,v)\mapsto G_\varepsilon(1,f(u,v),g(u,v))$ with $u,v\geqslant 0$.

Let $\mu_k(k=p,q)$ be the Poincaré constant such that $\mu_k\|\varphi\|^k_{L^k(\Omega)}\leqslant\|\nabla\varphi\|^k_{L^k(\Omega)}$ for any $\varphi\in W_0^{1,k}(\Omega)$. The existence result can then be stated as follows.

Theorem 2.2. *Assume that $K_i(x,t)\geqslant 0$ $(i=1,\dots,4)$ and $K_i(x,t)\leqslant k_i$ $(i=2,3)$ for some constants $k_i>0$. If*

$$\left\{x\in\Omega\colon\frac{1}{\omega}\int_0^\omega a(x,t)\,\mathrm{d}t-k_2|\Omega|\left(\frac{\|b\|_{L^\infty(Q_\omega)}}{\mu_q}\right)^{2/((n-1)(q-1))}>0\right\}\neq\emptyset \tag{2.2}$$

and

$$\left\{x\in\Omega\colon\frac{1}{\omega}\int_0^\omega b(x,t)\,\mathrm{d}t-k_3|\Omega|\left(\frac{\|a\|_{L^\infty(Q_\omega)}}{\mu_p}\right)^{2/((m-1)(p-1))}>0\right\}\neq\emptyset, \tag{2.3}$$

then problem (1.1)–(1.3) admits a coexistence solution $(u,v)\in C_\omega(\bar{Q}_\omega)\times C_\omega(\bar{Q}_\omega)$.

Comparing this with the related results in [**13**], conditions (2.2) and (2.3) are both local in space, which means that the nonlinear diffusion can efficiently improve the coexistence ability of the biological populations.

3. Proof of the main result

First, we want to prove the existence of a periodic solution $(u_\varepsilon,v_\varepsilon)$ for the regularized problem (2.1), where $(u_\varepsilon,v_\varepsilon)\in C_\omega(\bar{Q}_\omega)\times C_\omega(\bar{Q}_\omega)$, $u_\varepsilon,v_\varepsilon>0$ in Q_ω and $\varepsilon>0$ is small enough. To do this, we will apply the fixed-point theorem to obtain the positive fixed points of the map $(u,v)\mapsto G_\varepsilon(1,f(u,v),g(u,v))$. As necessary preparation for the proof of Theorem 2.2, we list the following two results.

Lemma 3.1 (Fragnelli *et al.* [12, Proposition 2.1]). *If the non-trivial pair $(u_\varepsilon,v_\varepsilon)$ solves*

$$(u_\varepsilon,v_\varepsilon)=G_\varepsilon(\sigma,f(u_\varepsilon,v_\varepsilon),g(u_\varepsilon,v_\varepsilon))$$

for some $\sigma\in[0,1]$, then

$$u_\varepsilon(x,t)\geqslant 0,\quad v_\varepsilon(x,t)\geqslant 0\quad\text{for any }(x,t)\in Q_\omega.$$

Moreover, if $u_\varepsilon\neq 0$ or $v_\varepsilon\neq 0$, then $u_\varepsilon>0$ or $v_\varepsilon>0$ in Q_ω, respectively.

Lemma 3.2 (Cirmi and Porzio [4, Lemma 2.2]; Wu *et al.* [24, Lemma 4.1.1]). *Let $\varphi(t)$ be a non-negative and non-increasing function on $[k_0,+\infty)$, satisfying*

$$\varphi(h)\leqslant\left(\frac{M}{h-k}\right)^{\alpha}[\varphi(k)]^{\beta}\quad\text{for any } h>k\geqslant k_0$$

with some constants $M>0$, $\alpha>0$, $\beta>1$. Then $\varphi(k_0+d)=0$ with $d=2^{\beta/(\beta-1)}M[\varphi(k_0)]^{(\beta-1)/\alpha}$.

In order to use the Leray–Schauder fixed-point theorem, we next give an *a priori* L^∞-estimate on the fixed point of the map $G_\varepsilon(\sigma,f(\cdot),g(\cdot))$ for all $\sigma\in[0,1]$. The main tool is De Giorgi's iteration, which is different from that in [**12**].

Lemma 3.3. *If $(u_\varepsilon,v_\varepsilon)$ is the solution of $(u_\varepsilon,v_\varepsilon)=G_\varepsilon(\sigma,f(u_\varepsilon,v_\varepsilon),g(u_\varepsilon,v_\varepsilon))$ for some $\sigma\in[0,1]$, then there exists a positive constant R independent of σ and ε such that*

$$\max\{\|u_\varepsilon\|_{L^\infty(Q_\omega)},\|v_\varepsilon\|_{L^\infty(Q_\omega)}\}\leqslant R. \tag{3.1}$$

Proof. From Lemma 3.1 it follows that $u_\varepsilon\geqslant0$, $v_\varepsilon\geqslant0$. Multiplying the equation

$$L_\varepsilon^{m,p}[u_\varepsilon]=\sigma u_+^{p-1}\left[a(x,t)-\int_\Omega K_1(\xi,t)u_\varepsilon^2(\xi,t-\tau_1)\,\mathrm{d}\xi-\int_\Omega K_2(\xi,t)v_\varepsilon^2(\xi,t-\tau_2)\,\mathrm{d}\xi\right]+\sigma\varepsilon \tag{3.2}$$

by u_ε^m and then integrating it over Ω, we have

$$\frac{1}{m+1}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon(t)\|_{m+1}^{m+1}+\|\nabla u_\varepsilon^m(t)\|_p^p\leqslant\|a\|_{L^\infty(Q_\omega)}\|u_\varepsilon(t)\|_{m+p-1}^{m+p-1}+\varepsilon\|u_\varepsilon(t)\|_m^m.$$

By using the Poincaré inequality and the Young inequality, we get

$$\begin{aligned}\frac{1}{m+1}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon(t)\|_{m+1}^{m+1}&+\mu_p|\Omega|^{(1-m(p-1))/(m+1)}\|u_\varepsilon(t)\|_{m+1}^{mp}\\&\leqslant\|a\|_{L^\infty(Q_\omega)}|\Omega|^{(2-p)/(m+1)}\|u_\varepsilon(t)\|_{m+1}^{m+p-1}+\varepsilon|\Omega|^{1/(m+1)}\|u_\varepsilon(t)\|_{m+1}^m.\end{aligned}$$

Due to the periodicity of u_ε with respect to time t, there is a $t_0\in[0,\omega]$ such that

$$\int_\Omega u_\varepsilon^{m+1}(t_0)\,\mathrm{d}x=\sup_{t\in[0,\omega]}\|u_\varepsilon(t)\|_{m+1}^{m+1},$$

and thus

$$\sup_{t\in[0,\omega]}\|u_\varepsilon(t)\|_{m+1}\leqslant\left(\frac{\|a\|_{L^\infty(Q_\omega)}}{\mu_p}\right)^{1/((m-1)(p-1))}|\Omega|^{1/(m+1)}+h(\varepsilon),$$

where the function $h(\varepsilon)$ satisfies $\lim_{\varepsilon\to0}h(\varepsilon)=0$. In particular, we have

$$\sup_{t\in[0,\omega]}\|u_\varepsilon(t)\|_2\leqslant\left(\frac{\|a\|_{L^\infty(Q_\omega)}}{\mu_p}\right)^{1/((m-1)(p-1))}|\Omega|^{1/2}+h(\varepsilon)|\Omega|^{(m-1)/(2(m+1))}. \tag{3.3}$$

By exploiting the similar methods in [**12**, **22**], we can easily conclude that for any $r > 0$ there exists a constant C_r depending only on r, m, p and $\|a\|_{L^\infty(Q_\omega)}$ such that

$$\sup_{t\in[0,\omega]} \|u_\varepsilon(t)\|_r \leqslant C_r.$$

Next, we can employ De Giorgi's iteration to prove (3.1). In fact, multiplying (3.2) by $(u_\varepsilon - k)_+^r \chi_{[t_1,t_2]}(t)$ for any $k \geqslant 1$, where $\chi_{[t_1,t_2]}(t)$ is the characteristic function of the interval $[t_1, t_2]$, and integrating the result over Q_ω, we have

$$\begin{aligned} \frac{1}{r+1}\int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u_\varepsilon - k)_+^{r+1}\,\mathrm{d}x\,\mathrm{d}t + rm^{p-1}\int_{t_1}^{t_2}\int_\Omega (u_\varepsilon - k)_+^{r-1}|\nabla u|^p\,\mathrm{d}x\,\mathrm{d}t \\ \leqslant \|a\|_{L^\infty(Q_\omega)}\int_{t_1}^{t_2}\int_\Omega u_\varepsilon^{p-1}(u_\varepsilon - k)_+^r\,\mathrm{d}x\,\mathrm{d}t + \varepsilon\int_{t_1}^{t_2}\int_\Omega (u_\varepsilon - k)_+^r\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Therefore, as in the proof of Theorem 2.2 in [**25**], by using the Sobolev embedding theorem, the Young inequality and Lemma 3.2, we can conclude that $\|u_\varepsilon\|_{L^\infty(Q_\omega)} \leqslant R$ for some R independent of ε and σ. Analogously, $\|v_\varepsilon\|_{L^\infty(Q_\omega)} \leqslant R$, and the proof of this lemma is therefore complete. □

Corollary 3.4. *The map $(u, v) \mapsto G_\varepsilon(1, f(u, v), g(u, v))$ admits at least one fixed point $(u_\varepsilon, v_\varepsilon)$ with $u_\varepsilon, v_\varepsilon \geqslant 0$.*

Proof. It follows from Lemma 3.3 that there exists a positive constant R independent of σ and ε such that

$$\max\{\|u_\varepsilon\|_{L^\infty(Q_\omega)}, \|v_\varepsilon\|_{L^\infty(Q_\omega)}\} \leqslant R,$$

where $(u_\varepsilon, v_\varepsilon)$ satisfies

$$(u, v) = G_\varepsilon(\sigma, f(u, v), g(u, v)) \quad \text{for some } \sigma \in [0, 1].$$

On the other hand, it is obvious that $G_\varepsilon(0, f(u, v), g(u, v)) = (0, 0)$. Therefore, by applying the Leray–Schauder fixed-point theorem, we can get a fixed point of the map $(u, v) \mapsto G_\varepsilon(1, f(u, v), g(u, v))$. The proof is complete. □

Next, we prove that $(u_\varepsilon, v_\varepsilon)$ obtained in the above corollary has a uniform lower bound in $L^\infty(Q_\omega) \times L^\infty(Q_\omega)$, provided that ε is small enough.

Lemma 3.5. *Let the assumptions (2.2) and (2.3) in Theorem 2.2 hold. There then exist positive constants ε_0 and r_0 such that*

$$\|u_\varepsilon\|_{L^\infty(Q_\omega)} \geqslant r_0, \qquad \|v_\varepsilon\|_{L^\infty(Q_\omega)} \geqslant r_0$$

for any fixed point $(u_\varepsilon, v_\varepsilon)$ of the map $(u, v) \mapsto G_\varepsilon(1, f(u, v), g(u, v))$ with $\varepsilon < \varepsilon_0$.

Proof. If the lemma were not true, we could without loss of generality assume that there existed a pair of functions $(u_\varepsilon, v_\varepsilon)$ such that $(u, v) = G_\varepsilon(1, f(u, v), g(u, v))$ with $\|v_\varepsilon\|_{L^\infty(Q_\omega)} < r_0$. From Lemma 3.1, we have $u_\varepsilon(x, t) > 0$, $v_\varepsilon(x, t) > 0$. Moreover, from inequality (3.3) it follows that

$$\sup_{t\in[0,\omega]} \|u_\varepsilon(t)\|_2^2 \leqslant A + H(\varepsilon),$$

where

$$A = \left(\frac{\|a\|_{L^\infty(Q_\omega)}}{\mu_p}\right)^{2/((m-1)(p-1))} |\Omega|$$

and the non-negative function $H(\varepsilon)$ satisfies $\lim_{\varepsilon\to 0} H(\varepsilon) = 0$.

By the assumption (2.3) in Theorem 2.2, there exist $x_0 \in \Omega$, $\delta > 0$ and $b_0 \in (0, 1)$ such that

$$\frac{1}{\omega}\int_0^\omega b(x, t)\,\mathrm{d}t - k_3 A > b_0$$

for all $x \in B_\delta(x_0) \subset \Omega$.

For any given $\phi(x) \in C_0^2(B_\delta(x_0))$, we can choose $\phi^q/v_\varepsilon^{q-1}$ as a test function. Multiplying the equation

$$L_\varepsilon^{n,q}(v_\varepsilon) = v_\varepsilon^{q-1}\biggl(b(x, t) - \int_\Omega K_3(\xi, t)u_\varepsilon^2(\xi, t-\tau_3)\,\mathrm{d}\xi - \int_\Omega K_4(\xi, t)v_\varepsilon^2(\xi, t-\tau_4)\,\mathrm{d}\xi\biggr) + \varepsilon$$

by $\phi^q/v_\varepsilon^{q-1}$ and integrating it over $Q_\omega^* = B_\delta(x_0) \times (0, \omega)$, we obtain

$$\begin{aligned}\iint_{Q_\omega^*} \frac{\phi^q}{v_\varepsilon^{q-1}}\frac{\partial v_\varepsilon}{\partial t}\,\mathrm{d}t\,\mathrm{d}x &+ \iint_{Q_\omega^*} (n^{q-1}v_\varepsilon^{(n-1)(q-1)} + \varepsilon)|\nabla v_\varepsilon|^{q-2}\nabla v_\varepsilon \nabla\left(\frac{\phi^q}{v_\varepsilon^{q-1}}\right)\mathrm{d}t\,\mathrm{d}x \\ &\geqslant \iint_{Q_\omega^*}\Bigl(b(x, t) - k_3\sup_{t\in(0,\omega)}\|u_\varepsilon(t)\|_2^2 - \max_{(x,t)\in Q_\omega} K_4(x, t)\|v_\varepsilon\|^2_{L^\infty(Q_\omega)}\Bigr)\phi^q(x)\,\mathrm{d}t\,\mathrm{d}x. \end{aligned} \tag{3.4}$$

By the periodicity of v_ε, the first term on the left-hand side in (3.4) satisfies

$$\iint_{Q_\omega^*} \frac{\phi^q}{v_\varepsilon^{q-1}}\frac{\partial v_\varepsilon}{\partial t}\,\mathrm{d}t\,\mathrm{d}x = \frac{1}{2-q}\int_{B_\delta(x_0)} \phi^q \int_0^\omega \frac{\partial(v_\varepsilon^{2-q})}{\partial t}\,\mathrm{d}t\,\mathrm{d}x = 0. \tag{3.5}$$

Moreover, since

$$\begin{aligned}\nabla\left(\frac{\phi^q}{v_\varepsilon^{q-1}}\right) &= qv_\varepsilon\left(\frac{\phi}{v_\varepsilon}\right)^{q-1}\nabla\left(\frac{\phi}{v_\varepsilon}\right) + \left(\frac{\phi}{v_\varepsilon}\right)^q\nabla v_\varepsilon \\ &= q\frac{v_\varepsilon}{\phi}\left(\frac{\phi}{v_\varepsilon}\right)^{q-1}\left[\frac{\phi}{v_\varepsilon}\nabla\phi - \left(\frac{\phi}{v_\varepsilon}\right)^2\nabla v_\varepsilon\right] + \left(\frac{\phi}{v_\varepsilon}\right)^q\nabla v_\varepsilon \\ &= q\left(\frac{\phi}{v_\varepsilon}\right)^{q-1}\nabla\phi - (q-1)\left(\frac{\phi}{v_\varepsilon}\right)^q\nabla v_\varepsilon,\end{aligned}$$

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we have

$$
\begin{aligned}
|\nabla v_\varepsilon|^{q-2}\nabla v_\varepsilon \nabla\left(\frac{\phi^q}{v_\varepsilon^{q-1}}\right) &\leqslant q\left(\frac{\phi}{v_\varepsilon}|\nabla v_\varepsilon|\right)^{q-1}|\nabla\phi| - (q-1)\left(\frac{\phi}{v_\varepsilon}\right)^q|\nabla v_\varepsilon|^q \\
&= \left(\frac{\phi}{v_\varepsilon}\right)^q|\nabla v_\varepsilon|^q + q\left(\frac{\phi}{v_\varepsilon}|\nabla v_\varepsilon|\right)^{q-1}\left(|\nabla\phi| - \frac{\phi}{v_\varepsilon}|\nabla v_\varepsilon|\right) \\
&\leqslant |\nabla\phi|^q \qquad (3.6)
\end{aligned}
$$

by the convexity of function $|\xi|^q$ for $\xi \in \mathbb{R}$.

Combining (3.4) with (3.5) and (3.6) yields

$$
\begin{aligned}
\iint_{Q_\omega^*}\Big(b(x,t) - k_3\sup_{t\in(0,\omega)}\|u_\varepsilon(t)\|_2^2 - \max_{(x,t)\in Q_\omega}K_4(x,t)\|v_\varepsilon\|^2_{L^\infty(Q_\omega)}\Big)\phi^q(x)\,\mathrm{d}t\,\mathrm{d}x \\
\leqslant \iint_{Q_\omega^*}(n^{q-1}v_\varepsilon^{(n-1)(q-1)} + \varepsilon)|\nabla\phi|^q\,\mathrm{d}t\,\mathrm{d}x \\
\leqslant \omega(n^{q-1}r_0^{(n-1)(q-1)} + \varepsilon)\int_{B_\delta(x_0)}|\nabla\phi|^q\,\mathrm{d}x. \quad (3.7)
\end{aligned}
$$

Now, by an approximating process we can let $\phi = \phi_q$, where ϕ_q is the eigenfunction corresponding to the first eigenvalue $\tilde{\mu}_q$ for the eigenvalue problem

$$
\begin{aligned}
-\operatorname{div}(|\nabla\phi|^{q-2}\nabla\phi) &= \mu|\phi|^{q-2}\phi, \quad x \in B_\delta(x_0), \\
\phi &= 0, \quad x \in \partial B_\delta(x_0),
\end{aligned}
$$

with $\|\phi_q\|_{L^q(B_\delta(x_0))} = 1$ [**15**], and (3.7) then becomes

$$
\begin{aligned}
\iint_{Q_\omega^*}\Big(b(x,t) - k_3\sup_{t\in(0,\omega)}\|u_\varepsilon(t)\|_2^2 - \max_{(x,t)\in Q_\omega}K_4(x,t)\|v_\varepsilon\|^2_{L^\infty(Q_\omega)}\Big)\phi_q^q(x)\,\mathrm{d}t\,\mathrm{d}x \\
\leqslant \omega\tilde{\mu}_q(n^{q-1}r_0^{(n-1)(q-1)} + \varepsilon).
\end{aligned}
$$

Therefore, we have

$$
b_0 \leqslant \tilde{\mu}_q(n^{q-1}r_0^{(n-1)(q-1)} + \varepsilon) + \max_{(x,t)\in Q_\omega}K_4(x,t)r_0^2 + k_3H(\varepsilon).
$$

Since there is some $\varepsilon_0 > 0$ such that $\tilde{\mu}_q\varepsilon + k_3H(\varepsilon) \leqslant \frac{1}{2}b_0$ for $\varepsilon \leqslant \varepsilon_0$, we get

$$
\tfrac{1}{2}b_0 \leqslant \tilde{\mu}_q n^{q-1}r_0^{(n-1)(q-1)} + \max_{(x,t)\in Q_\omega}K_4(x,t)r_0^2,
$$

and thus a contradiction follows provided that the constant r_0 is sufficiently small. The proof of the lemma is complete. □

Now we are ready to prove the main result of our paper.

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Proof of Theorem 2.2. From the results above it follows that (2.1) admits the coexistence solution $(u_\varepsilon, v_\varepsilon)$ with $r_0 \leqslant \|u_\varepsilon\|_{L^\infty(Q_\omega)}, \|v_\varepsilon\|_{L^\infty(Q_\omega)} \leqslant R$, where positive constants r_0, R are independent of ε ($\leqslant \varepsilon_0$). Therefore, due to the periodicity of u_ε and v_ε, applying a rather standard argument similar to that in [**11**, **19**], we obtain

$$\iint_{Q_\omega} |\nabla u_\varepsilon^m|^p \,\mathrm{d}x\,\mathrm{d}t \leqslant C, \qquad \iint_{Q_\omega} \left|\frac{\partial u_\varepsilon^m}{\partial t}\right|^2 \mathrm{d}x\,\mathrm{d}t \leqslant C,$$
$$\iint_{Q_\omega} |\nabla v_\varepsilon^n|^q \,\mathrm{d}x\,\mathrm{d}t \leqslant C, \qquad \iint_{Q_\omega} \left|\frac{\partial v_\varepsilon^n}{\partial t}\right|^2 \mathrm{d}x\,\mathrm{d}t \leqslant C,$$

where C is a constant independent of ε. Moreover, by using the regularity results in [**14**, **17**], it follows that there exists a continuous and non-decreasing map $g\colon [0, \infty) \to [0, \infty)$, $g(0) = 0$, such that

$$|u_\varepsilon(t_2, x_2) - u_\varepsilon(t_1, x_1)| \leqslant g(|x_2 - x_1| + |t_2 - t_1|^{1/p}),$$

where the map g can be determined in terms of R and is independent of ε. The same inequality holds for v_ε. Therefore, using an argument similar to that in [**11**,**12**,**19**], we can obtain the coexistence solution $(u, v) \in C_\omega(\bar{Q}_\omega)\times \in C_\omega(\bar{Q}_\omega)$ for the problem (1.1)–(1.3) as a limit function of $(u_\varepsilon, v_\varepsilon)$. □

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