DERIVING ROBUST BAYESIAN PREMIUMS UNDER BANDS OF PRIOR DISTRIBUTIONS WITH APPLICATIONS

BY

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ABSTRACT

We study the propagation of uncertainty from a class of priors introduced by Arias-Nicolás *et al.* [(2016) *Bayesian Analysis*, **11**(4), 1107–1136] to the premiums (both the collective and the Bayesian), for a wide family of premium principles (specifically, those that preserve the likelihood ratio order). The class under study reflects the prior uncertainty using distortion functions and fulfills some desirable requirements: elicitation is easy, the prior uncertainty can be measured by different metrics, and the range of quantities of interest is easily obtained from the extremal members of the class. We illustrate the methodology with several examples based on different claim counts models.

KEYWORDS

Credibility, class of priors, distortion functions, Kolmogorov and Kantorovich metrics, premium calculation principle, robust Bayesian analysis, stochastic orders.

JEL codes: IM30.

1. INTRODUCTION AND MOTIVATION

Given a risk X, a premium principle is a functional $H[X]$ that maps X to a non-negative real number, which is the premium charged to the policyholder to compensate the insurer for bearing the risk X. From the simplest net premium (which is the expected claim amount) to other more sophisticated ones based on utility and economic theories, such as the Esscher premium principle (Bühlmann, 1980; Gerber, 1980) or the distortion premium principle (Denneberg, 1990; Wang, 1996), the actuarial literature offers a number of premium principles that differ from each other by the properties that they satisfy. For an overview on this topic, the reader is referred to Young (2004) and Chapter 2 in Denuit *et al.* (2005).

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Let X be a risk with density function $f(x|\theta)$, where θ is a risk parameter belonging to the parameter space Θ. Under the Bayesian approach, prior beliefs about parameters are combined with sample information to update the model and determine the future premium (see, e.g., Eichenauer *et al.*, 1988; Heilmann, 1989; Makov *et al.*, 1996; Klugman *et al.*, 1998). For example, third-party liability motor insurance claims (which are rare events that occur randomly) are often modeled as Poisson random variables. However, experience from data suggests that the expected claim frequency is not equal for all policies in the same cell. Consequently, the actuary incorporates heterogeneity into the model using a prior distribution on the parameter to determine the cell tariff. A similar procedure is often followed in other branches of insurance.

In this framework, we first define, over the set of states Θ, a prior belief or structure function π that incorporates our beliefs about the parameter θ. Then we consider the conditional random variable $[X|\Theta=\theta]$, denoted by X_θ. Finally, based on the experience from a sample $x=(x_1,\ldots,x_n)$, the marginal density, $m(x)$, and the likelihood function, $l(\theta|x)$, we obtain, via Bayes theorem, the posterior belief density function π_x, given by $\pi_x(\theta)=l(\theta|x)\pi(\theta)/m(x)$. At this point, we must distinguish the following three premiums. The first one is $H[X_\theta]$, which is known as the true individual premium or the risk premium based on H. We will denote $H[X_\theta]=P_{R,H}(\pi)$ to make explicit that the premium depends on the prior belief. Since, from the Bayesian perspective, $P_{R,H}(\pi)$ is again a random risk, given a premium principle H^* (not necessarily equal to H), we can consider the premium $H^*[P_{R,H}(\pi)]=P_{C,H,H^*}(\pi)$, which is called the collective premium. A similar argument, using the posterior belief π_x instead of π, produces the Bayes or individual premium, denoted by $H^*[P_{R,H}(\pi_x)]=P_{B,H,H^*}(\pi_x)$ (see Gómez-Déniz, 2009, for further information). We remark that H and H^* are not necessarily equal: the collective and the Bayes premiums can be computed using first, for example, the net premium, H, and then the Esscher premium, H^*, or any other combination, such as Esscher–Esscher, Esscher–net, exponential–net, etc.

A key issue in this approach is the elicitation of an appropriate prior distribution for the parameter θ when there is not enough information to identify it (see, e.g., Eichenauer *et al.*, 1988). One possibility to avoid an arbitrary choice is to use robust methods that involve an entire class or family of prior distributions rather than a single one. In the literature, these classes have been specified taking the form of parametric families, contamination classes, densities with a few determined percentiles or distribution bands, among others. A question of natural interest is to study the propagation of uncertainty from the class of prior distributions to the premium. References on this topic include Heilmann and Schroter (1987), Eichenauer *et al.* (1988), Makov (1995), Young (1999), Gómez-Déniz *et al.* (1999, 2000, 2002), Schnieper (2004), Calderín and Gómez-Déniz (2007), Chan *et al.* (2008), and Boratyńska (2017).

The aim of this paper is to study the propagation of uncertainty from a class of priors recently introduced by Arias-Nicolás *et al.* (2016), called the distorted band of priors, to the premiums (both the collective and the Bayesian). The distorted band of priors fulfills some desirable requirements: elicitation is easy, the prior uncertainty can be measured by different metrics, and the range of quantities of interest is easily obtained from the extremal members of the class. Moreover, this class possesses a characteristic that makes it particularly interesting for actuarial applications: it quantifies the prior uncertainty in terms of distortion functions and stochastic orders, tools often used to evaluate and compare risks. The research is conducted by considering the propagation of uncertainty on a wide family of combinations of premium principles (unlike other studies on the same topic[1] that only consider a single premium principle).

The rest of the paper is structured as follows. Section 2 contains a background about some stochastic orders and metrics, distortion functions, and the distorted band class of priors. Section 3 shows how the uncertainty of this class of priors propagates to the premiums. Section 4 contains some actuarial applications. Finally, Section 5 contains conclusions.

2. THE DISTORTED CLASS

We start by recalling the definition of the stochastic orders that appear in this paper.

Definition 1. *Let X and Y be two random variables with distribution functions F and G, densities [discrete densities] f_X and f_Y, and supports supp(f_X) and supp(f_Y), respectively.*

(a) X is said to be smaller than Y in the stochastic order, the increasing convex order, and the increasing concave order (denoted by $X \leq_{\mathrm{st}} Y$, $X \leq_{\mathrm{icx}} Y$ and $X \leq_{\mathrm{icv}} Y$, respectively), if $E[\phi(X)] \leq E[\phi(Y)]$, for all non-decreasing, non-decreasing convex, and non-decreasing concave functions $\phi : \mathbb{R} \to \mathbb{R}$, respectively, provided these expectations exist.

(b) X is said to be smaller than Y in the likelihood ratio order, denoted by $X \leq_{\mathrm{lr}} Y$, if the ratio $f_Y(t)/f_X(t)$ increases over the union of the supports of X and Y (here $a/0$ is taken to be equal to ∞ whenever $a > 0$).

(c) X is said to be smaller than Y in the uniform conditional variability order, denoted by $X \leq_{\mathrm{uv}} Y$, if supp$(f_X) \subseteq$ supp(f_Y) and the ratio $f_X(t)/f_Y(t)$, $t \in$ supp(f_Y), is unimodal (where the mode is a supremum) but f_X and f_Y are not stochastically ordered.

The following chains of implications are well known (see Whitt, 1985; Müller and Stoyan, 2002; Shaked and Shanthikumar, 2007):

$$
\begin{array}{c}
X \leq_{\mathrm{lr}} Y \Rightarrow X \leq_{\mathrm{st}} Y \Rightarrow X \leq_{\mathrm{icx}} Y \Rightarrow E[X] \leq E[Y] \\
\Downarrow \\
X \leq_{\mathrm{icv}} Y \Rightarrow E[X] \leq E[Y],
\end{array}
\tag{2.1}
$$

$$
\begin{array}{c}
X \leq_{\mathrm{uv}} Y \text{ and } E[X] \leq E[Y] \Rightarrow X \leq_{\mathrm{icx}} Y, \\
X \leq_{\mathrm{uv}} Y \text{ and } E[X] \geq E[Y] \Rightarrow X \geq_{\mathrm{icv}} Y.
\end{array}
\tag{2.2}
$$

The class of priors considered in this paper is based on the notion of distortion function. A distortion function h is a non-decreasing continuous function from [0, 1] to [0, 1] such that $h(0) = 0$ and $h(1) = 1$. Distortion functions were introduced in actuarial science by Denneberg (1990) and have been applied to a wide variety of insurance problems, in particular to construct premium principles and risk measures (see, e.g., Wang, 1996; Sordo *et al.*, 2016, 2018).

To our purposes, given a prior belief π with distribution function F_π and a distortion function h, the transformation of F_π, given by

$$F_{\pi_h}(x) = h \circ F_\pi(x) = h\,[F_\pi(x)], \tag{2.3}$$

represents a perturbation of the accumulated probability that is used to quantify the uncertainty about the specification of the prior belief (a similar idea was used in Furman and Landsman (2006) in the context of some tail-based risk measures). Note that $F_{\pi_h}(x)$ is again a distribution function for a particular distorted random variable, denoted by X_{π_h}, with density function π_h. The following lemma, given in Arias-Nicolás *et al.* (2016), formalizes the idea, in terms of the likelihood ratio order, that X_{π_h} gives more weight to higher (lower) risk events when h is convex (respectively, concave). The result is also a reformulation of Theorem 1 of Blazej (2008), which is a more general result stated in terms of weighted distributions for absolutely continuous distributions.

Lemma 2. *Let π be a specific prior belief with distribution function F_π (absolutely continuous or discrete) and let h be a convex (concave) distortion function in* [0, 1]. *Then $\pi \le_{\mathrm{lr}} (\ge_{\mathrm{lr}})\pi_h$.*

Now suppose that, instead of requiring a complete specification of the prior belief, the actuary assumes that any distribution close enough to π is a good representation of it. One possibility to perturbate π, giving more (or less) weight to extreme events, is to consider two distortion functions: one concave, h_1, and one convex, h_2. From Lemma 2, we have $\pi_{h_1} \le_{\mathrm{lr}} \pi \le_{\mathrm{lr}} \pi_{h_2}$. This led Arias-Nicolás *et al.* (2016) to define the following class of priors.

Definition 3. *Given a concave distortion function h_1 and a convex distortion function h_2, the distorted band associated with a specific prior π, denoted by $\Gamma_{h_1,h_2,\pi}$, is defined as*

$$\Gamma_{h_1,h_2,\pi} = \left\{\pi' : \pi_{h_1} \le_{\mathrm{lr}} \pi' \le_{\mathrm{lr}} \pi_{h_2}\right\}. \tag{2.4}$$

Since $\pi \in \Gamma_{h_1,h_2,\pi}$, the distorted band can be seen as a particular "neighborhood" band of π, where the lower and upper bounds are its distortions by h_1 and h_2, respectively. Examples of distortion functions that can be used to define the band include the power families:

$$h_1(x) = 1 - (1-x)^{\alpha_1} \text{ and } h_2(x) = x^{\alpha_2},\ \alpha_i > 1,\ i = 1, 2.$$

By making $\alpha_i = n \in \mathbf{N}, n > 1, i = 1, 2,$ then $F_{\pi_{h_1}}(\theta) = 1 - (1 - F_\pi(\theta))^n$ and $F_{\pi_{h_2}}(\theta) = (F_\pi(\theta))^n$ correspond to the distribution functions of the minimum and the maximum, respectively, of an i.i.d. random sample of size n from the baseline prior distribution π, which seem to be reasonable bounds for the confidence band. Other examples are given in Arias-Nicolás *et al.* (2016) where distortions plays different roles. The distorted band satisfies some nice properties (see Arias-Nicolás *et al.*, 2016). For example, $(1-\epsilon)\pi + \epsilon\pi' \in \Gamma_{h_1,h_2,\pi}$, for all $\pi' \in \Gamma_{h_1,h_2,\pi}$ and for all $0 \le \epsilon \le 1$ (which is related to the ϵ-contamination classes). Additionally, posterior distributions inherit the likelihood ratio order, that is, for all $\pi' \in \Gamma_{h_1,h_2,\pi}$ we obtain that

$$\pi_{h_1,x} \le_{\rm lr} \pi'_x \le_{\rm lr} \pi_{h_2,x}. \tag{2.5}$$

Another good property of the distorted band is that the prior uncertainty can be measured by the Kantorovich (or Wasserstein) metric. Given two random variables X and Y, this metric is defined by

$$KW(X, Y) = \int_{-\infty}^{\infty} |F_X(x) - F_Y(x)| dx. \tag{2.6}$$

The tractability of Kantorovich metric between a distribution function F and its distortion F_h has been used to study the variability of F (López-Díaz *et al.*, 2012). As pointed out in Arias-Nicolás *et al.* (2016), if $\pi_{h_1} \le_{\rm lr} \pi_{h_2}$, the Kantorovich metric between π_{h_1} and π_{h_2} is simply the difference of their expectations, that is,

$$\begin{aligned} KW(\pi_{h_1}, \pi_{h_2}) &= E^{\pi_{h_2}}(\theta) - E^{\pi_{h_1}}(\theta), \\ KW(\pi, \pi_{h_1}) &= E^{\pi}(\theta) - E^{\pi_{h_1}}(\theta), \\ KW(\pi, \pi_{h_2}) &= E^{\pi_{h_2}}(\theta) - E^{\pi}(\theta), \\ KW(\pi_x, \pi_{h_1,x}) &= E^{\pi_x}(\theta) - E^{\pi_{h_1,x}}(\theta), \\ KW(\pi_x, \pi_{h_2,x}) &= E^{\pi_{h_2,x}}(\theta) - E^{\pi_x}(\theta), \\ KW(\pi_{h_1,x}, \pi_{h_2,x}) &= E^{\pi_{h_2,x}}(\theta) - E^{\pi_{h_1,x}}(\theta). \end{aligned} \tag{2.7}$$

Given two distortions h_1 and h_2, since $KW(\pi_{h_1}, \pi_{h_2}) = KW(\pi, \pi_{h_2}) + KW(\pi, \pi_{h_1})$, we can study which one contributes more to the uncertainty measure.

3. The main contributions

Let X be a random variable such that the conditional random variable $X_\theta = [X|\Theta = \theta]$ represents a random risk depending on a parameter θ. Let π be a prior belief in the parameter space Θ. We are interested in situations where the risk is a non-decreasing function of the parameter θ. For example, when the number of claims is modeled by a Poisson distribution, the risk is an

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increasing function of the parameter, which is the expected number of claims. This motivates the following definition.

Definition 4. *Given a premium principle H, we say that X_θ is increasing in risk for H, in short* $\mathbf{IR_H}$*, if the risk premium $P_{R,H}(\theta)$ is non-decreasing in $\theta \in \Theta$.*

Premium principles are usually required to preserve some stochastic orderings, such as the usual stochastic order and the increasing convex order (see Young, 2004). Given X and Y two random risks, we denote by $\mathcal{H}_{\text{st}}$ and $\mathcal{H}_{\text{icx}}$ the classes of premium principles preserving these orders, respectively,

$$\mathcal{H}_{\text{st}} = \{H : \text{If } X \leq_{\text{st}} Y, \text{ then } H[X] \leq H[Y]\}$$

and

$$\mathcal{H}_{\text{icx}} = \{H : \text{If } X \leq_{\text{icx}} Y, \text{ then } H[X] \leq H[Y]\}.$$

As a direct consequence of the implications in Equation (2.1), a wider class of premium principles can be defined in terms of the likelihood ratio order:

$$\mathcal{H}_{\text{lr}} = \{H : \text{If } X \leq_{\text{lr}} Y, \text{ then } H[X] \leq H[Y]\}.$$

It is apparent that $\mathcal{H}_{\text{icx}} \subset \mathcal{H}_{\text{st}} \subset \mathcal{H}_{\text{lr}}$. A remarkable example of a class of premium principles that belongs to $\mathcal{H}_{\text{lr}}$ and possesses some members that do not belong to the other two classes is the family of weighted premium principles, which includes, among others, the Esscher premium, the modified variance premium, and the Kamp premium (see Bartoszewicz and Skolimowska, 2006; Furman and Zitikis, 2008, for the relation between weighted distributions and the likelihood ratio order). As pointed out in Young (2004), the Esscher premium does not belong to $\mathcal{H}_{\text{st}}$.

The following Lemma is immediate.

Lemma 5. *Given $\theta_1 < \theta_2$, if $X_{\theta_1} \leq_* X_{\theta_2}$ (where $*$ means icx, st or lr), then X_θ is* $\mathbf{IR_H}$ *for all $H \in \mathcal{H}_*$.*

Example 6. *Let suppose that the number of claims (risk) follows a binomial distribution with success probability parameter p and a fixed and known number of clients n, denoted by $X_p \sim B(n, p)$. From Table 2.5 in Belzunce et al. (2016), fixed n, the binomial distribution is ordered in the likelihood ratio order, that is, if $p_1 < p_2$ we obtain that $B(n, p_1) \leq_{lr} B(n, p_2)$. Then, using Lemma 5, the random risk X_p is* $\mathbf{IR_H}$ *for all $H \in \mathcal{H}_{\text{lr}}$.*

Now we present the main result. Theorem 7 allows us to quantify and interpret the uncertainty induced by the partial knowledge of the prior for a large number of premium principles. Note that the range of quantities of interest can be computed just looking for the extremal distributions generating the distorted class.

Theorem 7. *Let X_θ be a random risk depending on a parameter θ and let π be a prior belief in the parameter space Θ. Let $\Gamma_{h_1,h_2,\pi}$ be the distorted band associated with π based on the concave and convex distortions h_1 and h_2, respectively. Then*

(a) $P_{C,H,H^*}(\pi_{h_1}) \le P_{C,H,H^*}(\pi') \le P_{C,H,H^*}(\pi_{h_2})$,
(b) $P_{B,H,H^*}(\pi_{h_1,x}) \le P_{B,H,H^*}(\pi'_x) \le P_{B,H,H^*}(\pi_{h_2,x})$,

for all premium principle H such that X_θ is $\mathbf{IR_H}$, for all $H^* \in \mathcal{H}_{\mathrm{lr}}$ and for all $\pi' \in \Gamma_{h_1,h_2,\pi}$.

Proof. We only prove part (b) (part (a) follows a similar argument). By hypothesis, the risk premium $P_{R,H}(\theta)$ is a non-decreasing function of θ. From (2.5) and using that the likelihood ratio order is preserved by non-decreasing functions (see Belzunce *et al.*, 2016), we obtain that

$$P_{R,H}(\pi_{h_1,x}) \le_{\mathrm{lr}} P_{R,H}(\pi'_x) \le_{\mathrm{lr}} P_{R,H}(\pi_{h_2,x}),$$

for all $\pi' \in \Gamma_{h_1,h_2,\pi}$. The proof follows using that $H^* \in \mathcal{H}_{\mathrm{lr}}$. ■

Remark 8. *We know, from Remark 4 in Arias-Nicolás et al. (2016), that all priors of the form $\pi_\epsilon = (1-\epsilon)\pi_{h_{\alpha_1}} + \epsilon\pi_{h_{\alpha_2}}$ (obtained as a mixture of $\pi_{h_{\alpha_1}}$ and $\pi_{h_{\alpha_2}}$) belong to the class $\Gamma_{h_1,h_2,\pi}$, for all $0 \le \epsilon \le 1$. Since $\Gamma_{h_1,h_2,\pi}$ is a convex class of distributions and π_ϵ is continuous (see Lemma 3.1 in Ríos et al., 1995), it follows that any value in the interval $\left[P_{B,H,H^*}(\pi_{h_{\alpha_1},x}), P_{B,H,H^*}(\pi_{h_{\alpha_2},x})\right]$ can be expressed as $P_{B,H,H^*}(\pi_{\epsilon,x})$ for some ϵ. In particular, the posterior regret Bayesian premium (see Ríos et al., 1995; Gómez-Déniz, 2009) given by*

$$\frac{1}{2}\left[P_{B,H,H^*}(\pi_{h_{\alpha_1},x}) + P_{B,H,H^*}(\pi_{h_{\alpha_2},x})\right]$$

is also a Bayes action (premium).

To end this section, we provide a result that connects the prior and posterior distributions using the uniform conditional variability order given in Definition 1. Proposition 9 will help to interpret the premiums in a *bonus–malus* system.

Proposition 9. *Let X_θ be a random risk depending on a parameter θ and let π be a prior belief in the parameter space Θ. Let π_x be the corresponding posterior distribution. If the likelihood function $l(\theta|x)$, $\theta \in supp(\pi_x)$ is unimodal, where the mode is a supremum, then*

(a) If $E[\pi_x] \le E[\pi]$, then $\pi_x \le_{\mathrm{icx}} \pi$,
(b) If $E[\pi_x] \ge E[\pi]$, then $\pi_x \ge_{\mathrm{icv}} \pi$.

Proof. Since $\mathrm{supp}(\pi_x) \subseteq \mathrm{supp}(\pi)$, it is easy to see that $\pi_x(\theta)/\pi(\theta) = l(\theta|x)/m(x)$. Then, from the unimodality of $l(\theta|x)$, it follows $\pi_x \le_{\mathrm{uv}} \pi$. The rest of the proof follows directly from the chain of implications given in Equation (2.2). ■

Remark 10. *When $l(\theta|x)$ is strictly decreasing (respectively, increasing), then the supremum is reached at the minimum (or the maximum) of the union of the supports of π and π_x. In this case, $\pi_x \leq_{\mathrm{lr}} \pi$ (respectively, $\pi_x \geq_{\mathrm{lr}} \pi$) and the relation $\pi_x \leq_{\mathrm{icx}} \pi$ (respectively, $\pi_x \geq_{\mathrm{icv}} \pi$) follows directly from the chain of implications given in Equation (2.1).*

4. Applications

This section illustrates, with three examples, the methods described in this paper. In the three examples, uncertainty about the prior is incorporated by means of a distorted band class based on the power distortion functions $h_{\alpha_1}(x)$ (concave) and $h_{\alpha_2}(x)$ (convex), given by

$$h_{\alpha_1}(x) = 1 - (1-x)^{\alpha_1} \text{ and } h_{\alpha_2}(x) = x^{\alpha_2},\ \alpha_i > 1,\ i = 1, 2. \tag{4.1}$$

The aim is to study the propagation of the uncertainty to the Bayesian premiums. We focus on the case where the likelihood belongs to the exponential family of distributions, that is, it can be expressed as $l(\theta|x) = a(x)\exp(-\theta x)/c(\theta)$ for the continuous or discrete case and the natural conjugate prior density is given by $\pi(\theta) = [c(\theta)]^{-n_0}\exp(-x_0\theta)/d(n_0, x_0)$ (see Jewell, 1974, for details). From Equations (2.3) and (4.1), the prior distorted densities are given by

$$\pi_{h_{\alpha_1}}(\theta) = \frac{d}{d\theta}\left\{1 - [1 - F_\pi(\theta)]^{\alpha_1}\right\},$$

$$\pi_{h_{\alpha_2}}(\theta) = \frac{d}{d\theta}[F_\pi(\theta)]^{\alpha_2}.$$

In the first two examples, we consider a distorted class such that the collective premiums associated with the priors in the band are close among them according to the epsilon distance. In the third one, uncertainty is induced directly from the baseline prior.

Remark 11. *In the exponential family, a reparametrization often leads to obtain $P_{R,H}(\theta) = \theta$, for H the net premium. If H^* is also the net premium, in the continuous case we have $P_{C,H,H^*}(\pi_{h_{\alpha_1}}) = \int [1 - F_\pi(\theta)]^{1/p}\, d\theta$. This is simply the premium based on the risk-adjusted premium, where $p = 1/\alpha_1 < 1$ is the risk index (see Drozdenko, 2008, for details about the risk-adjusted premium). This transformation gives more weight to large claims (sizes) and reduces the probability of obtaining small claims (sizes). Similar arguments apply when the prior is $\pi_{h_{\alpha_2}}(\theta)$, which gives more weight to small claims (sizes) and reduces, therefore, the probability of obtaining large claims (sizes). Therefore, the prior distribution in the band acts as a mechanism to balance the collective and Bayes premiums based on the initial prior distribution, giving more prominence to small or large claims.*

TABLE 1

FITTED DATA TO A PORTFOLIO OF AUTOMOBILE INSURANCE IN GERMANY (1969).

No. of claims	Observed	Geometric fitted
0	20,592	20,615.80
1	2651	2598.46
2	297	327.51
3	41	41.28
4	7	5.20
5	0	0.65
6	1	0.08
Total	23,589	23,589

4.1. Example 1 [real data set]

We consider a portfolio of automobile insurance policies from Germany (1960) (see Table 1 and Willmot, 1987, for details)). The number of claims is supposed to follow a Poisson distribution with parameter $\theta > 0$, denoted by $X_\theta \sim P(\theta)$, and π is supposed to be an exponential distribution with rate parameter $b > 0$, that is, the baseline prior density is given by $\pi(\theta) = b\exp(-b\theta)$. The corresponding posterior distribution is a gamma distribution with shape parameter equal to $n\bar{x} + 1$ and rate parameter equal to $b + n$, denoted by $\pi_x \sim G(n\bar{x} + 1, b + n)$.

We compute H and H^* using the net premiums. It is easy to see that the individual, collective, and Bayesian premiums are given by

$$P_{R,H}(\theta) = \theta, \quad P_{C,H,H^*}(\pi) = \frac{1}{b}, \quad \text{and} \quad P_{B,H,H^*}(\pi_x) = \frac{n\bar{x}+1}{b+n}. \tag{4.2}$$

The marginal (unconditional) distribution of the risk X is a geometric distribution with parameter $b/(b+1)$. Using this distribution, the maximum likelihood (ml) estimate of b is $\widehat{b} = 6.934$ with a standard error of 0.127.

Now, we introduce a perturbation scheme on the prior distribution by considering the distorted band $\Gamma_{h_{\alpha_1},h_{\alpha_2},\pi}$, where h_{α_1} and h_{α_2} are defined by Equation (4.1). Then,

$$\begin{aligned} \pi_{h_{\alpha_1}}(\theta) &= \alpha_1 b\exp(-\alpha_1 b\theta), \\ \pi_{h_{\alpha_2}}(\theta) &= \alpha_2 b\exp(-b\theta)(1-\exp(-b\theta))^{\alpha_2-1}. \end{aligned} \tag{4.3}$$

It is easy to see that Poisson distributions are ordered in the likelihood ratio order in terms of their parameters. Specifically, $\theta_1 < \theta_2$ implies $P(\theta_1) \leq_{\mathrm{lr}} P(\theta_2)$. Hence, using Lemma 5, $X_\theta = P(\theta)$ is $\mathbf{IR_H}$ for all $H \in \mathcal{H}_{\mathrm{lr}}$. In particular, $X_\theta = P(\theta)$ is $\mathbf{IR_H}$ when H is the net premium.

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After some computations we get

$$P_{C,H,H^*}(\pi_{h_{\alpha_1}}) = (\alpha_1 b)^{-1},$$
$$P_{C,H,H^*}(\pi_{h_{\alpha_2}}) = \frac{\mathcal{H}_{\alpha_2}}{b}, \tag{4.4}$$

where $\mathcal{H}_z$ represents the zth harmonic number. From Theorem 7 (a) it follows that

$$P_{C,H,H^*}(\pi_{h_{\alpha_1}}) \leq P_{C,H,H^*}(\pi) \leq P_{C,H,H^*}(\pi_{h_{\alpha_2}}). \tag{4.5}$$

A natural question is how to choose the distortion parameters α_1 and α_2. One possibility is to require that the resulting collective premiums are close enough to the premium associated to the prior distribution π. This can be done taking α_1 and α_2 such that

$$P_{C,H,H^*}(\pi_{h_{\alpha_1}}) + \epsilon = P_{C,H,H^*}(\pi) = P_{C,H,H^*}(\pi_{h_{\alpha_2}}) - \epsilon \tag{4.6}$$

for some $\epsilon > 0$ small enough (a similar argument has been used in Eichenauer *et al.* (1988) and Gómez-Déniz *et al.* (2002)). Combining Equations (4.2), (4.4), and (4.6) and replacing b by $\hat{b}$, we get

$$(\alpha_1 \widehat{b})^{-1} + \varepsilon = \frac{1}{\widehat{b}},$$
$$\frac{\mathcal{H}_{\alpha_2}}{\widehat{b}} - \varepsilon = \frac{1}{\widehat{b}}. \tag{4.7}$$

The equations system (4.7) has been solved numerically using Wolfram Mathematica software for $\varepsilon = 0.05$, 0.1, and 0.14. The solutions for α_1 and α_2 are 1.53067, 3.26143, 34.1772, and 1.63976, 2.53965, and 3.51876, respectively.

From Theorem 7 (b), the Bayes premiums satisfy

$$P_{B,H,H^*}(\pi_{h_{\alpha_1},x}) \leq P_{B,H,H^*}(\pi'_x) \leq P_{B,H,H^*}(\pi_{h_{\alpha_2},x}),\ \forall \pi' \in \Gamma_{h_{\alpha_1},h_{\alpha_2},\pi}.$$

Since the posterior distorted distributions do not have closed-form expressions, the bounds in these inequalities have been computed numerically by using Wolfram Mathematica software. Figure 1 shows the effect of the distortion functions on the Bayesian premiums combining some values of the sample mean, $\bar{x}$ (with sample sizes $n = 1$, $n = 5$, and $n = 10$). At first glance, as usual, uncertainty decreases when the sample size increases.

As expected, the range of Bayesian premiums is larger when the uncertainty about the baseline prior π increases, that is, when α_1 and α_2 increase. Moreover, the range decreases when the sample size increases and/or the sample mean of the number of claims is close to $1/\hat{b} = 0.1442$. It is also worth mentioning that the contribution to uncertainty of concave (respectively, convex) distortions is bigger when the sample mean of the number of claims is

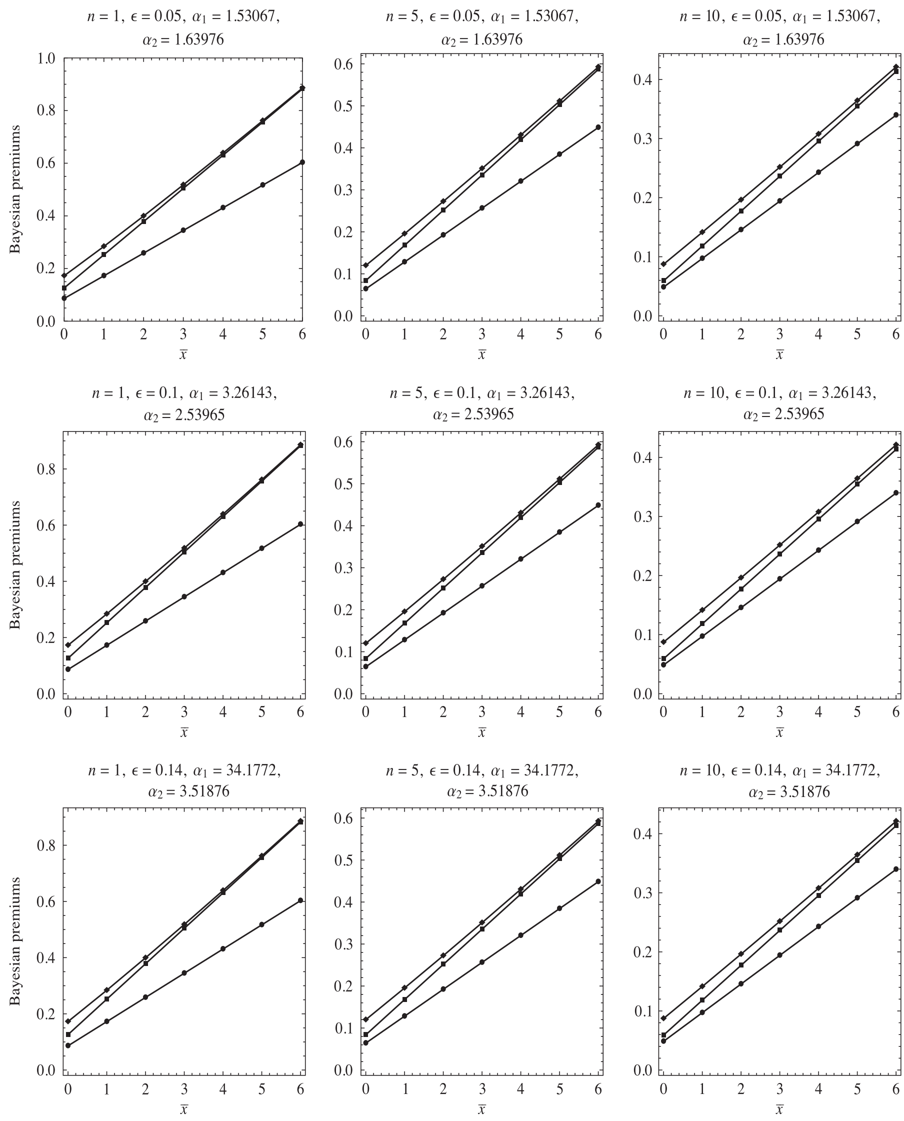

FIGURE 1: Range of the Bayesian premiums based on the net premium against $\bar{x}$, for $\varepsilon = 0.05$, 0.1, and 0.14, $\alpha_1 = 1.53067$, 3.26143, and 34.1772, $\alpha_2 = 1.63976$, 2.53965, and 3.51876, and $n = 1$, 5, and 10, for the Poisson–exponential model.

smaller (respectively, larger) than $1/\hat{b} = 0.1442$. This is coherent with the fact that the likelihood, given by

$$l(\theta|x) = \frac{e^{n\theta}\theta^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i},\ \theta \in (0, \infty),$$

is unimodal and the supremum is achieved at the maximum likelihood estimator (mls) of θ, given by the sample mean $\hat{\theta}_{\text{mls}} = \bar{x}$. From Proposition 9, we see that $\pi_x \leq_{\text{icx}} (\geq_{\text{icv}})\pi$ if and only if $E[\pi_x] \leq (\geq) E[\pi]$ or equivalently if and only if $\bar{x} \leq (\geq) 1/b$. Therefore, the Bayesian premiums $P_{B,H,H^*}(\pi_{h_{\alpha_1},x})$ and $P_{B,H,H^*}(\pi_{h_{\alpha_2},x})$ can be seen as a competitive value of the premium and a prudent one, respectively, in a *bonus–malus* system.

Finally, note that since the Kantorovich metric between the lower and upper distorted priors is given by $KW(\pi_{h_{\alpha_1}}, \pi_{h_{\alpha_2}}) = 2\varepsilon$, the uncertainty induced in the collective premium increases with the "size" of the distorted band.

4.1.1. *Connections with credibility theory.*

From Equation (4.3), it is apparent that the Bayesian premium associated with the lower bound of the distorted band can be rewritten as

$$P_{B,H,H^*}(\pi_{h_{\alpha_1},x}) = \frac{n\bar{x}+1}{\alpha_1 b + n} = Z_{h_1}^{\alpha_1}(n)\bar{x} + \left(1 - Z_{h_1}^{\alpha_1}(n)\right) P_{C,H,H^*}(\pi_{h_{\alpha_1}}),$$

that is, as a credibility expression, where

$$Z_{h_1}^{\alpha_1}(n) = \frac{n}{\alpha_1 b + n} \tag{4.8}$$

is the credibility factor varying between 0 and 1. Straightforward computations provide that this credibility factor obeys the expression of the classical Bühlmann credibility factor. That is, $Z = n/(n+K)$, where $K = E_{\pi_{\alpha_1}}[\text{Var}[X_\theta]]/\text{Var}_{\pi_{\alpha_1}}[E[X_\theta]]$ (see Bühlmann, 1967; Bühlmann and Gisler, 2005, for further details).

On the other hand, given α_2 a positive integer and making use of the Newton binomial, the density of the upper bound of the distorted band can be rewritten as

$$\pi_{h_{\alpha_2}}(\theta) = \alpha_2 b \exp(-b\theta) \sum_{j=0}^{\alpha_2-1} (-1)^{\alpha_2-1-j} \binom{\alpha_2-1}{j} \exp[-b\theta(\alpha_2-1-j)].$$

Therefore, the posterior distribution can be expressed as a convex sum of α terms of gamma random variables:

$$\pi_{h_{\alpha_2},x} =_d \frac{1}{\sum_{j=0}^{\alpha-1} \kappa(j)} \sum_{j=0}^{\alpha-1} \kappa(j) \mathcal{G}(n\bar{x}+1, n+b(\alpha_2-j)),$$

where

$$\kappa(j) = (-1)^{\alpha_2-1-j} \binom{\alpha_2-1}{j} \frac{1}{[n+b(\alpha_2-j)]^{n\bar{x}+1}}.$$

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Consequently,

$$
\begin{aligned}
P_{B,H,H^*}(\pi_{h_{\alpha_2},x}) &= \frac{1}{\sum_{j=0}^{\alpha_2-1}\kappa(j)} \sum_{j=0}^{\alpha_2-1} \kappa(j) \frac{n\bar{x}+1}{n+b(\alpha_2-j)} \\
&= \frac{1}{\sum_{j=0}^{\alpha_2-1}\kappa(j)} \sum_{j=0}^{\alpha_2-1} \kappa(j) \left[Z_{h_2}^{\alpha_2}(n)\bar{x} + \left(1 - Z_{h_2}^{\alpha_2}(n)\right) \frac{1}{b(\alpha_2-j)} \right],
\end{aligned}
$$

where

$$
Z_{h_2}^{\alpha_2}(n) = \frac{n}{n+b(\alpha_2-j)}. \tag{4.9}
$$

Therefore, the premium is a sum of α_2 terms, where each term presents a factor of credibility given by Equation (4.9). Observe that the higher α_1 and α_2 are, the smaller the credibility factors in Equations (4.8) and (4.9), respectively, are. In other words, higher α_1 and α_2 give more weight to the collective compared to the sample data through the upper and lower bounds of the premium.

4.2. Example 2 [real data set]

This example is taken from Lau *et al.* (2006). The prior distribution of the risk parameter θ is supposed to be uniform on (0, 10), denoted by $\pi \sim U(0, 10)$. The distribution of claims size is a Pareto distribution with shape parameter $b > 0$ and mode parameter $\theta > 0$, denoted by $X_\theta \sim Pa(b, \theta)$, with density function $f(x|\theta) = b\theta^b/x^{b+1}$, $x \geq \theta$. From Bayes theorem, the posterior distribution is given by

$$
\pi_x(\theta) = \frac{\theta^{nb}(nb+1)}{\min[x_{(1)}, 10]^{nb+1}} = \frac{f_{B(nb+1,1)}(\theta/10)}{10F_{B(nb+1,1)}(\min[x_{(1)}, 10]/10)},
$$

where $\theta \in (0, \min[x_{(1)}, 10])$ and $f_{B(a_1,a_2)}(x)$ and $F_{B(a_1,a_2)}(x)$ represent the density and the distribution functions, respectively, of a classical beta distribution with shape parameters a_1 and a_2 in the interval (0, 1). It is remarkable that the posterior distribution results from a change of scale, equal to 10, of a right-truncated beta distribution, truncated at $\min[x_{(1)}, 10]$. By considering the net premium principle for H and H^*, a straightforward computation provides the individual, the collective, and the Bayesian premiums as

$$
P_{R,H}(\theta) = \frac{b\theta}{b-1},\quad P_{C,H,H^*}(\pi) = \frac{5b}{b-1},\quad P_{B,H,H^*}(\pi_x) = \frac{b(nb+1)\min[x_{(1)}, 10]}{(b-1)(nb+2)}, \tag{4.10}
$$

where $x_{(1)}$ is the sample minimum. Lau *et al.* (2006) suggest to take $b = 3$.

We consider again a perturbation scheme on the prior distribution by using the distorted band $\Gamma_{h_{\alpha_1},h_{\alpha_2},\pi}$, where h_{α_1} and h_{α_2} are defined by Equation (4.1). In this case, the bounds are given by

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$$\pi_{h_{\alpha_1}}(\theta) = \frac{\alpha_1}{10}\left(1-\frac{\theta}{10}\right)^{\alpha_1-1} = \frac{f_{B(1,\alpha_1)}(\theta/10)}{10},\ \ \theta \in (0,10),$$

$$\pi_{h_{\alpha_2}}(\theta) = \frac{\alpha_2}{10}\left(\frac{\theta}{10}\right)^{\alpha_2-1} = \frac{f_{B(\alpha_2,1)}(\theta/10)}{10},\ \ \theta \in (0,10). \tag{4.11}$$

It is well known (see, e.g., Table 2.1 in Belzunce *et al.*, 2016) that Pareto distributions are ordered in the likelihood ratio order according to their location parameters. Specifically, $\theta_1 < \theta_2$ implies $Pa(b,\theta_1) \leq_{\mathrm{lr}} Pa(b,\theta_2)$. It follows from Lemma 5 that the random risk $X_\theta = Pa(b,\theta)$ is $\mathbf{IR_H}$ for all $H \in \mathcal{H}_{\mathrm{lr}}$ (in particular, X_θ is $\mathbf{IR_H}$ for the net premium).

Some computation yields to

$$P_{C,H,H^*}(\pi_{h_{\alpha_1}}) = \frac{10b}{(b-1)(1+\alpha_1)},$$

$$P_{C,H,H^*}(\pi_{h_{\alpha_2}}) = \frac{10\alpha_2 b}{(b-1)(1+\alpha_2)}. \tag{4.12}$$

As in Section 4.1 (Example 1), α_1 and α_2 must verify Equation (4.6) for a fixed $\epsilon > 0$. Combining Equations (4.6), (4.10), and (4.12), we need to solve the following equation system with $b = 3$:

$$\frac{10b}{(b-1)(1+\alpha_1)} + \varepsilon = \frac{5b}{b-1},$$

$$\frac{10\alpha_2 b}{(b-1)(1+\alpha_2)} - \varepsilon = \frac{5b}{b-1}.$$

The solution satisfies $\alpha = \alpha_1 = \alpha_2$. Of course, this is coherent with the fact that both distortions produce a symmetric effect in the uniform prior distribution. For $\varepsilon = 3, 5$, and 6 we obtain $\alpha = 2.33, 5$, and 9, respectively. The distorted posterior distributions are given by

$$\pi_{h_{\alpha_1},x}(\theta) = \frac{f_{B(nb+1,\alpha_1)}(\theta/10)}{10F_{B(nb+1,\alpha_1)}(\min[x_{(1)},10]/10)},$$

$$\pi_{h_{\alpha_2},x}(\theta) = \frac{\theta^{nb+\alpha_2-1}(nb+\alpha_2)}{\min(x_{(1)},10)^{nb+\alpha_2}} = \frac{f_{B(nb+\alpha_2,1)}(\theta/10)}{10F_{B(nb+\alpha_2,1)}(\min[x_{(1)},10]/10)}, \tag{4.13}$$

where $\theta \in (0, \min[x_{(1)}, 10])$. From Equation (4.13), it is easy to compute a closed-form expression for the distorted Bayesian premiums:

$$P_{B,H,H^*}(\pi_{h_{\alpha_1},x}) = 10\frac{nb+1}{nb+\alpha_1+1}\,\frac{F_{Beta(nb+2,\alpha_1)}\left(\frac{\min(x_{(1)},10)}{10}\right)}{F_{Beta(nb+1,\alpha_1)}\left(\frac{\min(x_{(1)},10)}{10}\right)},$$

$$P_{B,H,H^*}(\pi_{h_{\alpha_2},x}) = \frac{nb+\alpha_2}{nb+\alpha_2+1}\min(x_{(1)},10). \tag{4.14}$$

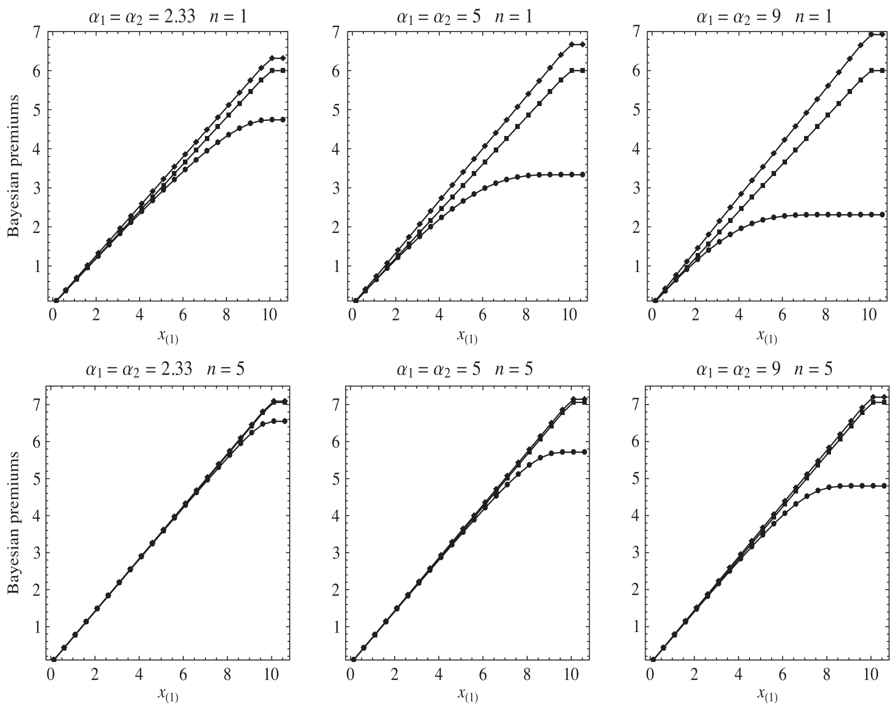

FIGURE 2: Range of the Bayesian premiums based on the net premium against $x_{(1)}$, for $\varepsilon = 0.2$, 0.6, and 1, $\alpha_1 = \alpha_2 = 1.05479$, 1.17391, and 1.30769, and $n = 1$ and 5 for the pareto–uniform model.

From Theorem 7 (b), the Bayesian premiums in Equations (4.10) and (4.14) satisfy

$$P_{B,H,H^*}(\pi_{h_{\alpha_1},x}) \leq P_{B,H,H^*}(\pi'_x) \leq P_{B,H,H^*}(\pi_{h_{\alpha_1},x}),\ \forall \pi' \in \Gamma_{h_{\alpha_1},h_{\alpha_2},\pi}.$$

We show in Figure 2 the effect of the distortion functions on the Bayesian premiums combining several values of the minimum sample $x_{(1)}$ with two sample sizes, $n = 1$ and $n = 5$. At first sight, uncertainty decreases when the sample size increases, as expected.

As in Section 4.1 (Example 1), the range of Bayesian premiums is larger when α increases. Likewise, the range decreases when the sample size increases and/or the sample minimum decreases. Recall that the sample minimum is a biased estimator of θ with a positive bias. Observe that the convex distortion contributes more to the uncertainty when the sample minimum increases and the concave distortion contributes more when the sample minimum decreases. This property is again coherent with the behavior of the likelihood, given by

$$l(\theta|x) = \frac{b^n \theta^{nb}}{\prod_{i=1}^{n} x_i^{b+1}},\ \theta \in (0, x_{(1)}),$$

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TABLE 2

VALUES FOR THE PREMIUMS DEPENDING ON THE PREMIUM PRINCIPLES.

H–H^*	Net–Net	Esscher–Net	Esscher–Esscher	Exponential utility–Net
Collective premium	$\frac{a}{b}$	$e^{\beta}\frac{a}{b}$	$\frac{e^{\beta}a}{b-\beta e^{\beta}}$	$(e^{\beta}-1)\frac{a}{b^2}$
Bayesian premium	$\frac{a+n\bar{x}}{b+n}$	$e^{\beta}\frac{a+n\bar{x}}{b+n}$	$e^{\beta}\frac{a+n\bar{x}}{(b+n)-\beta e^{\beta}}$	$(e^{\beta}-1)\frac{a+n\bar{x}}{b(b+n)}$

which is strictly increasing and unimodal, with the supremum achieved at the mls, given by $\hat{\theta}_{\text{mls}} = x_{(1)}$. Then, from Proposition 9, $\pi_x \leq_{\text{icv}} (\geq_{\text{icv}})\pi$ holds if and only if $E[\pi_x] \leq (\geq) E[\pi]$ or, equivalently, if and only if $\min(x_{(1)}, 10) \leq (\geq) 5(nb+2)/(nb+1)$. If $x_{(1)} \geq 10$, it follows from Remark 10 that $\pi \leq_{\text{lr}} \pi_x$.

The Kantorovich distance between the lower and upper distorted priors is given by

$$KW(\pi_{h_{\alpha_2}}, \pi_{h_{\alpha_1}}) = \frac{b}{(b-1)} \frac{10(\alpha-1)}{(1+\alpha)} = \frac{b}{(b-1)} 2\varepsilon. \tag{4.15}$$

As in Section 4.1 (Example 1), the Kantorovich distance is proportional to ϵ; therefore, it can be used to control the effect of the distortions in the collective premium.

4.3. Example 3

In Gómez-Déniz *et al.* (1999), the uncertainty with regard to the prior distribution is represented by the assumption that π belongs to the classical contamination class of priors. Starting from this class, the authors make a Bayesian robustness analysis to measure the sensitivity with respect to the prior of the Bayesian premium for the Esscher principle in the Poisson-gamma model. Now we extend the study by considering different premium principles and the distorted band class.

Let suppose that the number of claims follows a Poisson distribution with parameter $\theta > 0$, $X_\theta \sim P(\theta)$, and let π be a gamma distribution with shape parameter $a > 0$ and scale parameter, $b > 0$, denoted by $\pi \sim G(a, b)$, with density function

$$\pi(\theta) = \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta}.$$

The posterior distribution is also a gamma distribution with shape parameter $n\bar{x} + a$ and scale parameter $b + n$, denoted by $\pi_x \sim G(n\bar{x} + a, b + n)$.

Table 2 shows the collective and Bayesian premiums for different combinations of H and H^*.

We consider again a perturbation scheme on the prior distribution by using the distorted band $\Gamma_{h_{\alpha_1}, h_{\alpha_2}, \pi}$, where h_{α_1} and h_{α_2} are given by Equation (4.1).

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TABLE 3

KW METRIC DEPENDING ON THE DISTORTION PARAMETERS.

KW metric	$\alpha_1 = \alpha_2 = 1.05$	$\alpha_1 = \alpha_2 = 1.11$	$\alpha_1 = \alpha_2 = 1.15$	$\alpha_1 = \alpha_2 = 2$
$KW(\pi_{h_{\alpha_2}}, \pi_{h_{\alpha_1}})$	0.03406	0.06697	0.09875	0.12945
$KW(\pi_{h_{\alpha_2},x}, \pi_{h_{\alpha_1},x})$	0.01577	0.02494	0.04352	0.05654

In this case, there are no closed-form expressions for the bounds, neither for the prior bounds $\pi_{h_{\alpha_1}}(\theta)$ and $\pi_{h_{\alpha_2}}(\theta)$ nor for the posterior ones $\pi_{h_{\alpha_1},x}(\theta)$ and $\pi_{h_{\alpha_2},x}(\theta)$. As in Section 4.1 (Example 1), $X_\theta = P(\theta)$ is $\mathbf{IR_H}$ for all $H \in \mathcal{H}_{\mathrm{lr}}$ (in particular, for the net, the Esscher and the exponential utility premium principles). Therefore, it follows from Theorem 7 (b) that

$$P_{B,H,H^*}(\pi_{h_{\alpha_1},x}) \leq P_{B,H,H^*}(\pi'_x) \leq P_{B,H,H^*}(\pi_{h_{\alpha_1},x}),\ \forall \pi' \in \Gamma_{h_{\alpha_1},h_{\alpha_2},\pi},$$

for any combination of the principles H and H^* considered in Table 2. This band is illustrated in Figures 3 and 4 for different scenarios. As in Gómez-Déniz *et al.* (1999), we have assumed a fixed expected amount of claims, $c = 100$ monetary units, and a prior gamma distribution with shape and scale parameters equal to 5 and 2, respectively, $G(5, 2)$. We have fixed the sample size $n = 10$ under two scenarios: the first one with sample mean $\overline{x} = 2$ and the second one with sample mean $\overline{x} = 5$. We have considered different distortion parameters (namely $\alpha_1 = \alpha_2 = 1.05, 1.11, 1.15$, and 1.2).

To obtain the risk aversion constant β in the Esscher premium, we have supposed that the Esscher premium differs from the net premium in a $\sigma\%$, that is, $\theta e^\beta = (1 + \sigma\%)\theta$. Taking $\sigma = 10$ we obtain $\beta = 0.0953$. The same risk aversion constant has been considered for the exponential utility principle. The Bayesian premiums $P_{B,H,H^*}(\pi_{h_{\alpha_1},x})$ and $P_{B,H,H^*}(\pi_{h_{\alpha_2},x})$ have been estimated by simulation using the algorithms described in Arias-Nicolás *et al.* (2016).

On one hand, observe that the range of the Bayesian premiums is larger when the uncertainty about the baseline prior π increases, that is, when α increases. On the other hand, the range decreases when the sample mean of the number of claims is close to $a/b = 2.5$. Concave distortions contribute more to the uncertainty when the sample mean of the number of claim is smaller than $a/b = 2.5$, while convex distortions contribute more when it is larger. As in Section 4.1 (Example 1), this is coherent with the fact that the likelihood is unimodal and the supremum is achieved at the mls of θ, given by the sample mean $\hat{\theta}_{\mathrm{mls}} = \bar{x}$. Then, from Proposition 9, $\pi_x \leq_{\mathrm{icx}} (\geq_{\mathrm{icv}})\pi$ if and only if $E[\pi_x] \leq (\geq) E[\pi]$ or, equivalently, if and only if $\bar{x} \leq (\geq) a/b$.

Table 3 provides the Kantorovich metrics for the different α's used in this study.

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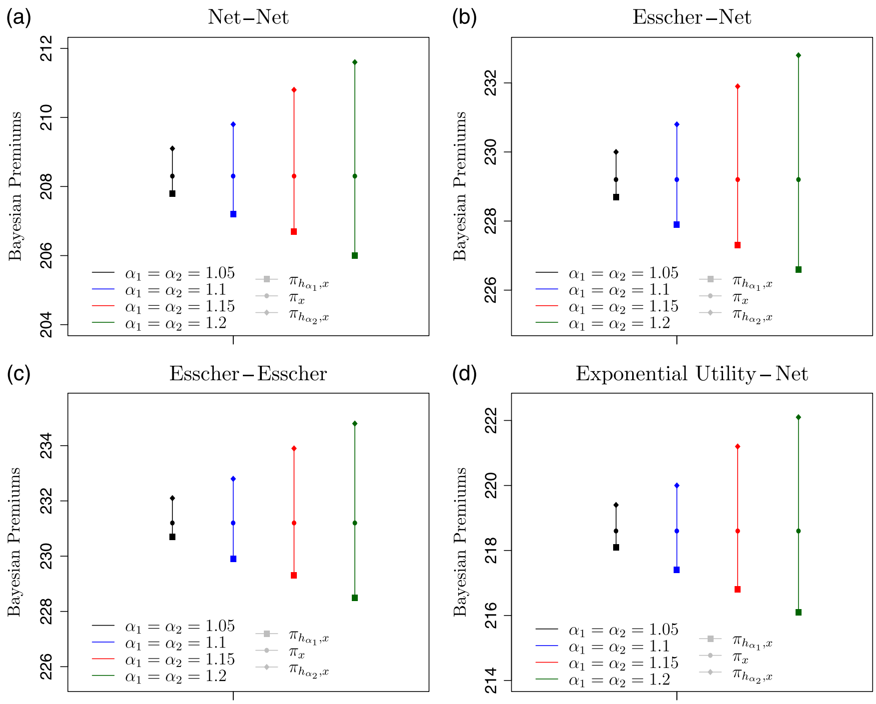

FIGURE 3: Range of the Bayesian premiums based on the different premiums in Table 2 with $\overline{x} = 2$, for $\alpha_1 = \alpha_2 = 1.05$, 1.1, 1.15, and 1.2 and $n = 10$ for the gamma–gamma model.

5. CONCLUDING REMARKS

Given a random risk that depends on a parameter, we have addressed the problem of computing collective and Bayesian premiums from a robust approach. We have focused on a class of priors, recently introduced in the literature, that fulfills the requirements described in Berger (1994) and reflects accurately the prior uncertainty using distortion functions. We have illustrated how the uncertainty propagates from this class of priors to collective and Bayesian premiums for a wide family of premium principles, specifically those that preserve the likelihood ratio order. One strength of this approach is that the sensitivity measures based on ranges of the premiums are easy to compute from the extremal distributions of the class.

An anonymous reviewer pointed out, in the light of Theorem 7, that weighted distributions also provide a natural framework for the ideas developed in this paper. In fact, if we restrict to absolutely continuous random variables, weighted distributions are more general objects than distorted distributions. For a non-negative random variable X with density function f and

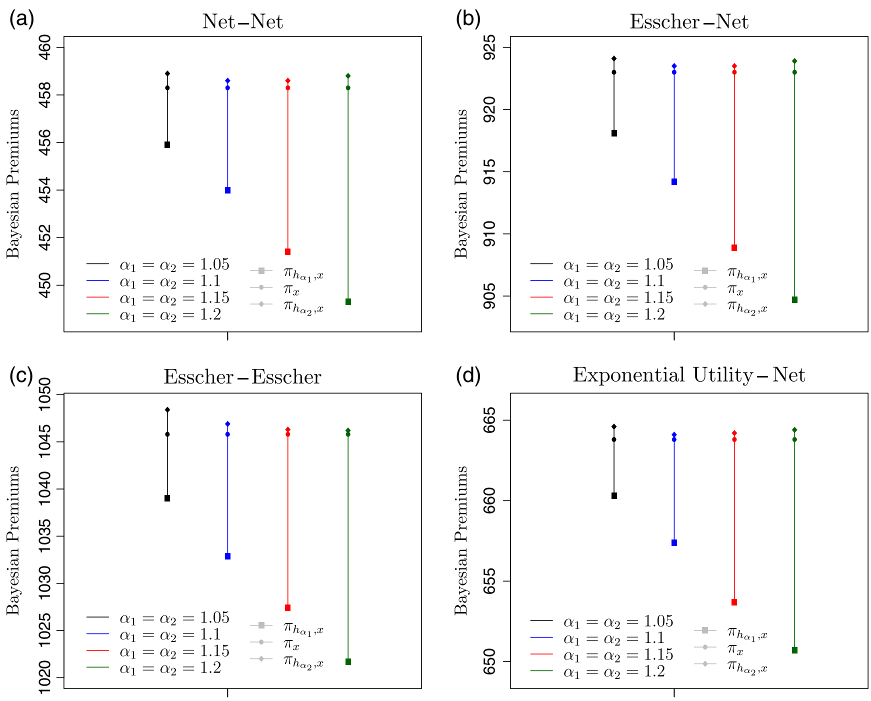

FIGURE 4: Range of the Bayesian premiums based on the different premiums in Table 2 with $\overline{x} = 5$, for $\alpha_1 = \alpha_2 = 1.05, 1.1, 1.15$, and 1.2 and $n = 10$ for the gamma–gamma model.

for a non-negative function ω such that $E[\omega(X)]$ is strictly positive and finite, a weighted random variable X^ω is a random variable with density function

$$f^\omega(x) = \frac{\omega(x)}{E[\omega(X)]} f(x), \quad x > 0. \tag{5.1}$$

A distorted distribution $h(F(x))$ is a particular case of weighted distribution by taking the weight function $w(x) = h'(F(x))$ (this is noted, e.g., in Furman and Zitikis, 2008). Moreover, the distortion h is convex (resp. concave) if and only if the weight function ω is increasing (resp. decreasing). In this new framework, we can perturbate the prior belief π by considering two weight functions: ω_1 (decreasing) and ω_2 (increasing). Then we have $\pi^{\omega_1} \leq_{\mathrm{lr}} \pi \leq_{\mathrm{lr}} \pi^{\omega_2}$ and we can define a class of priors based on weighted distributions. In this paper, we have adopted the distortion approach for several reasons. First, this work was motivated by the paper of Arias-Nicolás *et al.* (2016), which perturbated the prior belief π by using distortions. The second reason is that the distorted distribution approach enables to consider, at least from a theoretical point of view, more general random variables (not necessarily absolutely continuous). Finally, the literature provides some useful preservation results for distorted

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distributions that cannot be stated, in general, in terms of weighted distributions. For example, consider two prior beliefs π and $\overline{\pi}$ and two distortion functions h_1, concave, and h_2, convex. It follows from Theorem 7(a) in Sordo (2008) that if π is less disperse than $\overline{\pi}$ in the sense of Bickel and Lehmann (1979), then $KW\left(\pi_{h_1}, \pi_{h_2}\right) \leq KW\left(\overline{\pi}_{h_1}, \overline{\pi}_{h_2}\right)$, where KW is the Kantorovich metric. This is a very reasonable result: the more disperse prior belief, the wider uncertainty band. Unfortunately, we do not have a similar result for general weighted distributions.

In this paper, we have considered three classical claim counts models: exponential–Poisson, uniform–Pareto, and gamma–Poisson. Our future work will be addressed to the multivariate case, when the risk depends on more than one parameter.

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Notes

1. For example, Gómez-Déniz *et al.* (1999) study the propagation of uncertainty from certain class of priors to the Bayesian premium, which is computed using twice the Esscher premium.

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