VANISHING OF THE NEGATIVE HOMOTOPY K-THEORY OF QUOTIENT SINGULARITIES

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Abstract   Making use of Gruson–Raynaud's technique of 'platification par éclatement', Kerz and Strunk proved that the negative homotopy K-theory groups of a Noetherian scheme X of Krull dimension d vanish below $-d$. In this article, making use of noncommutative algebraic geometry, we improve this result in the case of quotient singularities by proving that the negative homotopy K-theory groups vanish below -1. Furthermore, in the case of cyclic quotient singularities, we provide an explicit 'upper bound' for the first negative homotopy K-theory group.

1. Introduction and statement of results

Given a Noetherian scheme X of Krull dimension d, Kerz and Strunk proved[1] in [11, Theorem 1] that $KH_n(X) = 0$ for every $n < -d$. The first goal of this article is to improve this vanishing result in the case where X is a quotient singularity.

Let k be a field of characteristic zero, $d \geqslant 2$ an integer, and G a finite subgroup of $\mathrm{SL}_d(k)$. The group G acts naturally on the polynomial ring $S := k[t_1, \ldots, t_d]$ and the associated invariant ring $R := S^G$ is Gorenstein and of Krull dimension d. Throughout the article we assume that the local ring R is an isolated singularity.

Example 1.1 (Cyclic quotient singularities). Let k be an algebraically closed field[2] of characteristic zero, $m \geqslant 2$ an integer, ζ a primitive mth root of unit, and $a_1, \ldots, a_d$

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[1]Recently, Kerz, Strunk and Tamme proved in [12, Theorem B] a stronger result: the negative K-theory groups of a Noetherian scheme X of Krull dimension d vanish below $-d$.

[2]More generally, it suffices that k contains all the mth roots of unit.

integers satisfying the following three conditions: $0 < a_j < m$, $\gcd(a_j, m) = 1$ for every $1 \leqslant j \leqslant d$, and $a_1 + \cdots + a_d = m$. When G is the cyclic subgroup of $\mathrm{SL}_d(k)$ generated by $\mathrm{diag}(\zeta^{a_1}, \ldots, \zeta^{a_d})$, the ring R is an isolated quotient singularity of Krull dimension d. For example, when $d = 2$, $a_1 = 1$, and $a_2 = m - 1$, the ring R identifies with the Kleinian singularity $k[u, v, w]/(u^m + vw)$ of type A_{m-1}.

Remark 1.2. As proved by Kurano and Nishi in [13], all Gorenstein isolated quotient singularities of odd prime (Krull) dimension are cyclic. Moreover, in all the other (Krull) dimensions there exist *noncyclic* Gorenstein isolated quotient singularities. For example, in dimension two all Gorenstein quotient singularities are isolated.

Let us write X for the affine (singular) k-scheme $\mathrm{Spec}(R)$.

Theorem 1.3. *We have* $KH_n(X) = 0$ *for every* $n < -1$.

Intuitively speaking, Theorem 1.3 shows that in the case of quotient singularities the vanishing of the negative homotopy K-theory groups is independent of the Krull dimension! To the best of the author's knowledge, this result is new in the literature. Such a vanishing result does not hold in general because, for every integer $d \geqslant 2$, there exist Gorenstein isolated singularities X of Krull dimension d with $KH_{-d}(X) \neq 0$; consult Reid [16] for details. Theorem 1.3 leads naturally to the following divisibility properties of nonconnective algebraic K-theory:

Proposition 1.4.   (i) *The abelian group* $\mathbb{K}_{-2}(X)$ *is divisible;*
   (ii) *The abelian groups* $\mathbb{K}_n(X), n < -2$, *are uniquely divisible.*

Remark 1.5. Since the base field k is of characteristic zero and the scheme X is of finite type over k, it follows from the work of Cortiñas *et al.* [4] that $\mathbb{K}_n(X) = 0$ for every $n < -d$. Moreover, $\mathbb{K}_{-d}(X) \simeq KH_{-d}(X)$. Consequently, making use of Theorem 1.3, we conclude that $\mathbb{K}_{-d}(X) = 0$.

The second goal of this article is to provide some information about $KH_{-1}(X)$ in the case of cyclic quotient singularities. Let k be an algebraically closed field of characteristic zero and (Q, ρ) the quiver with relations defined as follows:

(s1) consider the quiver with vertices $\mathbb{Z}/m$ and with arrows $x_j^i \colon i \to i + a_j$, where $i \in \mathbb{Z}/m$ and $1 \leqslant j \leqslant d$. The relations ρ are given by $x_{j'}^{i+a_j} x_j^i = x_j^{i+a_{j'}} x_{j'}^i$ for every $i \in \mathbb{Z}/m$ and $1 \leqslant j, j' \leqslant d$;

(s2) remove from (s1) all arrows $x_j^i \colon i \to i'$ with $i > i'$;

(s3) remove from (s2) the vertex 0.

Consider the $(m - 1) \times (m - 1)$ matrix C such that C_{ij} equals the number of arrows in Q from j to i (counted modulo the relations). Let $M := (-1)^{d-1} C(C^{-1})^T - \mathrm{id}$ and $\mathbf{M} \colon \oplus_{r=1}^{m-1} \mathbb{Z} \to \oplus_{r=1}^{m-1} \mathbb{Z}$ be the associated (matrix) homomorphism.

Theorem 1.6. *The abelian group $KH_{-1}(X)$ is a quotient of the cokernel of* **M**.

Intuitively speaking, Theorem 1.6 provides an explicit 'upper bound' for the first negative homotopy K-theory group of cyclic quotient singularities.

Example 1.7 (Kleinian singularities of type A). When $d = 2$, $a_1 = 1$, and $a_2 = m - 1$, the three steps (s1)–(s3) lead to the quiver $Q \colon 1 \to 2 \to 3 \to \cdots \to m - 2 \to m - 1$ (without relations). Consequently, we obtain the following matrix:

$$M = \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ -1 & -1 & \ddots & \ddots & \vdots \\ -1 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ -1 & 0 & \cdots & 0 & -1 \end{bmatrix}_{(m-1)\times(m-1)} .$$

The cokernel of **M** is isomorphic to $\mathbb{Z}/m$; a generator is given by the image of the vector $(0, \ldots, 0, -1) \in \oplus_{r=1}^{m-1} \mathbb{Z}$. Thanks to Theorem 1.6, we hence conclude that $KH_{-1}(\mathrm{Spec}(k[u, v, w]/(u^m + vw)))$ is a quotient of $\mathbb{Z}/m$.

Example 1.8 (A three-dimensional singularity). When $d = m = 3$ and $a_1 = a_2 = a_3 = 1$, the three steps (s1)–(s3) lead to the quiver $Q \colon 1 \Rrightarrow 2$ (without relations). Consequently, we obtain the matrix $M = \left(\begin{smallmatrix} 0 & -3 \\ 3 & -9 \end{smallmatrix} \right)$. The cokernel of **M** is isomorphic to $\mathbb{Z}/3 \times \mathbb{Z}/3$; generators are given by the images of $(1, 0)$ and $(-1, -3)$. Thanks to Theorem 1.6, we hence conclude that $KH_{-1}(X)$ is a quotient of $\mathbb{Z}/3 \times \mathbb{Z}/3$.

Preliminaries

We assume the reader is familiar with the language of differential graded (=dg) categories; consult Keller's ICM survey [9]. Let $\mathbf{dgcat}(k)$ be the category of dg categories and dg functors. Given a (dg) k-algebra A, we still write A for the associated dg category with a single object. Consider the simplicial k-algebra $\Delta_n := k[t_0, \ldots, t_n]/(\sum_i t_i - 1), n \geqslant 0$, with faces and degeneracies:

$$\partial_j(t_i) := \begin{cases} t_i & \text{if } i < j \\ 0 & \text{if } i = j \\ t_{i-1} & \text{if } i > j \end{cases} \qquad \delta_j(t_i) := \begin{cases} t_i & \text{if } i < j \\ t_i + t_{i+1} & \text{if } i = j \\ t_{i+1} & \text{if } i > j. \end{cases}$$

Following [20, §2.2.5], the *homotopy K-theory functor* is defined by the formula

$$KH \colon \mathbf{dgcat}(k) \longrightarrow \mathbf{Spt} \quad \mathcal{A} \mapsto \mathrm{hocolim}_{n \geqslant 0} \mathbb{K}(\mathcal{A} \otimes \Delta_n),$$

where **Spt** stands for the homotopy category of spectra.

2. Proof of Theorem 1.3

Let us write $\mathcal{D}^b(\mathrm{Mod}(X))$ for the bounded derived category of $\mathcal{O}_X$-modules, $\mathcal{D}^b(\mathrm{coh}(X))$ for the bounded derived category of coherent $\mathcal{O}_X$-modules, and $\mathbf{perf}(X)$ for the category

of perfect complexes of $\mathcal{O}_X$-modules. These categories admit canonical dg enhancements $\mathcal{D}^b_{\mathrm{dg}}(\mathrm{Mod}(X))$, $\mathcal{D}^b_{\mathrm{dg}}(\mathrm{coh}(X))$, and $\mathrm{perf}_{\mathrm{dg}}(X)$, respectively; see [9, §§ 4.4–4.6], [14]. Recall from Orlov [15] that the dg category of singularities $\mathcal{D}^{\mathrm{sg}}_{\mathrm{dg}}(X)$ is defined as the Drinfeld's dg quotient $\mathcal{D}^b_{\mathrm{dg}}(\mathrm{coh}(X))/\mathrm{perf}_{\mathrm{dg}}(X)$. Consequently, we have the following short exact sequence of dg categories (see [9, § 4.6]):

$$0 \longrightarrow \mathrm{perf}_{\mathrm{dg}}(X) \longrightarrow \mathcal{D}^b_{\mathrm{dg}}(\mathrm{coh}(X)) \longrightarrow \mathcal{D}^{\mathrm{sg}}_{\mathrm{dg}}(X) \longrightarrow 0.$$

Since homotopy K-theory KH is a localizing invariant of dg categories (see [20, § 8.2.2]) and $KH(\mathrm{perf}_{\mathrm{dg}}(X))$ is isomorphic to $KH(X)$ (see [20, § 2.2.5]), we obtain an induced distinguished triangle of spectra:

$$KH(X) \longrightarrow KH(\mathcal{D}^b_{\mathrm{dg}}(\mathrm{coh}(X))) \longrightarrow KH(\mathcal{D}^{\mathrm{sg}}_{\mathrm{dg}}(X)) \longrightarrow \Sigma KH(X). \qquad (2.1)$$

Proposition 2.2. *We have* $KH_n(\mathcal{D}^b_{\mathrm{dg}}(\mathrm{coh}(X))) = 0$ *for every* $n < 0$.

Proof. Given a dg category $\mathcal{A}$ and an integer $p \geqslant 1$, let us write $\mathcal{A}[t_1, \ldots, t_p]$ for the tensor product $\mathcal{A} \otimes k[t_1, \ldots, t_p]$. Since homotopy K-theory is defined as the 'realization' of a simplicial spectrum, we have the following standard convergent right half-plane spectral sequence :

$$E^1_{pq} = N^p \mathbb{K}_q(\mathcal{A}) \Rightarrow KH_{p+q}(\mathcal{A}). \qquad (2.3)$$

Here, $N^0 \mathbb{K}_q(\mathcal{A}) := \mathbb{K}_q(\mathcal{A})$ and $N^p \mathbb{K}_q(\mathcal{A})$, $p \geqslant 1$, is defined as the intersection

$$\bigcap_{i=1}^{p} \mathrm{Kernel}\left(\mathbb{K}_q(\mathcal{A}[t_1, \ldots, t_p]) \overset{\mathrm{id} \otimes (t_i = 0)}{\longrightarrow} \mathbb{K}_q(\mathcal{A}[t_1, \ldots, \widehat{t_i}, \ldots, t_p]) \right).$$

Let us treat now the case where $\mathcal{A}$ is the dg category $\mathcal{D}^b_{\mathrm{dg}}(\mathrm{coh}(X))$. Using the fact that the dg category $k[t_1, \ldots, t_p]$ is Morita equivalent to $\mathrm{perf}(\mathrm{Spec}(k[t_1, \ldots, t_p])) \simeq \mathrm{perf}(\mathrm{Spec}(k[t_1])) \otimes \cdots \otimes \mathrm{perf}(\mathrm{Spec}(k[t_p]))$, Lemma 2.4 below (applied inductively) implies that $\mathcal{D}^b_{\mathrm{dg}}(\mathrm{coh}(X))[t_1, \ldots, t_p]$ is Morita equivalent to $\mathcal{D}^b_{\mathrm{dg}}(\mathrm{coh}(X[t_1, \ldots, t_p]))$. Consequently, since nonconnective algebraic K-theory is invariant under Morita equivalences, we obtain the following identifications

$$\mathbb{K}(\mathcal{D}^b_{\mathrm{dg}}(\mathrm{coh}(X))[t_1, \ldots, t_p]) \simeq \mathbb{K}(\mathcal{D}^b_{\mathrm{dg}}(\mathrm{coh}(X[t_1, \ldots, t_p]))) = G(X[t_1, \ldots, t_p]),$$

where G stands for G-theory; see [9, Theorem 5.1]. As proved by Schlichting in [19, Theorem 7], the abelian groups $G_q(X[t_1, \ldots, t_p])$, $q < 0$, are zero; this uses the fact that X, and hence $X[t_1, \ldots, t_p]$, is Noetherian. Therefore, we conclude that $N^p \mathbb{K}_q(\mathcal{D}^b_{\mathrm{dg}}(\mathrm{coh}(X))) = 0$ for every $p \geqslant 0$ and $q < 0$. This implies that the spectral sequence (2.3) degenerates below the line $q = -1$ and consequently that

$$KH_n(\mathcal{D}^b_{\mathrm{dg}}(\mathrm{coh}(X))) \simeq \mathbb{K}_n(\mathcal{D}^b_{\mathrm{dg}}(\mathrm{coh}(X))) = G_n(X) \quad n < 0.$$

The proof follows now from the fact that $G_n(X) = 0$ for every $n < 0$. □

Lemma 2.4 (Rouquier[3]). *Given an affine k-scheme of finite type $X = \mathrm{Spec}(R)$, we have the following induced Morita equivalence:*

$$\mathcal{D}^b_{\mathrm{dg}}(\mathrm{coh}(X)) \otimes \mathrm{perf}_{\mathrm{dg}}(\mathrm{Spec}(k[t])) \longrightarrow \mathcal{D}^b_{\mathrm{dg}}(\mathrm{coh}(X[t])) \quad (\mathcal{F}, \mathcal{F}') \mapsto \mathcal{F} \boxtimes \mathcal{F}'.$$

Proof. Recall first that $\mathcal{O}_X$ and $\mathcal{O}_{\mathrm{Spec}(k[t])}$ are generators of the triangulated categories $\mathrm{perf}(X)$ and $\mathrm{perf}(\mathrm{Spec}(k[t]))$, respectively. As proved by Rouquier in [17, Theorem 7.38], the triangulated category $\mathcal{D}^b(\mathrm{coh}(X))$ admits a generator $\mathcal{G}$. We claim that $\mathcal{G}[t] := \mathcal{G} \boxtimes \mathcal{O}_{\mathrm{Spec}(k[t])}$ is a generator of the triangulated category $\mathcal{D}^b(\mathrm{coh}(X[t]))$. In order to prove this claim, we proceed by induction on the dimension of X. The case $\dim(X) = 0$ is clear. Assume then that the claim holds for every affine k-scheme of finite type of dimension $< \dim(X)$. Let us write $\mathcal{T}$ for the smallest thick triangulated subcategory of $\mathcal{D}^b(\mathrm{coh}(X[t]))$ containing the object $\mathcal{G}[t]$. Given any object $\mathcal{H} \in \mathcal{D}^b(\mathrm{coh}(X[t]))$, we need to show that $\mathcal{H} \in \mathcal{T}$. Let us denote by $i \colon X^s \hookrightarrow X$ the singular locus of X and by $\mathcal{D}^b_{X^s[t]}(\mathrm{coh}(X[t]))$ the full triangulated subcategory of $\mathcal{D}^b(\mathrm{coh}(X[t]))$ consisting of those bounded complexes of coherent $\mathcal{O}_{X[t]}$-modules that are supported on the closed subscheme $X^s[t]$. Similar to the proof of [17, Theorem 7.38], we have a distinguished triangle

$$\mathcal{H}_1 \longrightarrow \mathcal{H} \oplus \Sigma\mathcal{H} \longrightarrow \mathcal{H}_2 \longrightarrow \Sigma\mathcal{H}_1$$

with $\mathcal{H}_1 \in \mathrm{perf}(X[t])$ and $\mathcal{H}_2 \in \mathcal{D}^b_{X^s[t]}(\mathrm{coh}(X[t]))$. Therefore, in order to prove our claim, it suffices to show that $\mathcal{H}_1$ and $\mathcal{H}_2$ belong to $\mathcal{T}$. Since $\mathcal{G}$ is a generator of $\mathcal{D}^b(\mathrm{coh}(X))$, the object $\mathcal{O}_{X[t]} = \mathcal{O}_X \boxtimes \mathcal{O}_{\mathrm{Spec}(k[t])}$ belongs to $\mathcal{T}$. Consequently, since $\mathcal{O}_{X[t]}$ is a generator of $\mathrm{perf}(X[t])$, the object $\mathcal{H}_1$ also belongs to $\mathcal{T}$. In what concerns the object $\mathcal{H}_2$, let us start by choosing a generator $\mathcal{G}'$ of the triangulated category $\mathcal{D}^b(\mathrm{coh}(X^s))$. Since $\dim(X^s) < \dim(X)$, the induction assumption implies that $\mathcal{G}'[t] := \mathcal{G}' \boxtimes \mathcal{O}_{\mathrm{Spec}(k[t])}$ is a generator of the triangulated category $\mathcal{D}^b(\mathrm{coh}(X^s[t]))$. Now, a proof similar to the one of [22, Theorem 6.8(i)], with [6, Proposition (19.1.1)] replaced by the finite type assumption of X and with [2, Proposition 6.1] replaced by [18, Proposition 6.6], shows that $i_*(\mathcal{O}_{X^s}) \boxtimes \mathcal{O}_{\mathrm{Spec}(k[t])}$ is a generator of the triangulated category $\mathcal{D}^b_{X^s[t]}(\mathrm{coh}(X[t]))$. This implies that $i_*(\mathcal{G}')[t] := i_*(\mathcal{G}') \boxtimes \mathcal{O}_{\mathrm{Spec}(k[t])}$ is also a generator of $\mathcal{D}^b_{X^s[t]}(\mathrm{coh}(X[t]))$. Since $i_*(\mathcal{G}')$ belongs to the category $\mathcal{D}^b(\mathrm{coh}(X))$ and $\mathcal{G}$ is a generator of $\mathcal{D}^b(\mathrm{coh}(X))$, the object $i_*(\mathcal{G}')[t]$ belongs to $\mathcal{T}$. Consequently, $\mathcal{H}_2$ also belongs to $\mathcal{T}$. This concludes the proof of our claim.

Since $\mathcal{G}$ is a generator of $\mathcal{D}^b(\mathrm{coh}(X))$ and $\mathcal{G}[t]$ a generator of $\mathcal{D}^b(\mathrm{coh}(X[t]))$, we have induced Morita equivalences between dg categories

$$\mathcal{D}^b_{\mathrm{dg}}(\mathrm{coh}(X)) \simeq \mathrm{perf}_{\mathrm{dg}}(\mathbf{R}\mathrm{End}(\mathcal{G})) \quad \mathcal{D}^b_{\mathrm{dg}}(\mathrm{coh}(X[t])) \simeq \mathrm{perf}_{\mathrm{dg}}(\mathbf{R}\mathrm{End}(\mathcal{G}[t])),$$

where $\mathbf{R}\mathrm{End}(\mathcal{G})$ and $\mathbf{R}\mathrm{End}(\mathcal{G}[t])$ stand for the (derived) dg k-algebra of endomorphisms of $\mathcal{G}$ and $\mathcal{G}[t]$, respectively. Consider the following classical adjunction

$$
\begin{array}{c}
\mathcal{D}^b(\mathrm{Mod}(X[t])) \\[4pt]
\mathcal{F} \mapsto \mathcal{F}[t] \Big\uparrow \Big\downarrow \mathrm{Res} \\[4pt]
\mathcal{D}^b(\mathrm{Mod}(X)).
\end{array}
\tag{2.5}
$$

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[3]Lemma 2.4 is a particular case of a general result of Rouquier [18]. The author is very grateful to Raphaël for kindly explaining him his ideas.

By assumption, the affine k-scheme X is of finite type and hence Noetherian. Therefore, since $\mathcal{G}$ belongs to the triangulated subcategory $\mathcal{D}^b(\mathrm{coh}(X)) \subset \mathcal{D}^b(\mathrm{mod}(X))$ and $\mathrm{Res}(\mathcal{G}[t]) \simeq \oplus_{i=1}^{\infty} \mathcal{G}$, the combination of [17, Corollary 6.17] with (2.5) implies that the dg k-algebra $\mathbf{R}\mathrm{End}(\mathcal{G}[t])$ is quasi-isomorphic to $\mathbf{R}\mathrm{End}(\mathcal{G})[t]$. This concludes the proof of Lemma 2.4 because $\mathrm{perf}_{\mathrm{dg}}(\mathrm{Spec}(k[t]))$ is Morita equivalent to $k[t]$. □

By combining Proposition 2.2 with the long exact sequence of (stable) homotopy groups associated to the distinguished triangle of spectra (2.1), we conclude that $KH_n(\mathcal{D}_{\mathrm{dg}}^{\mathrm{sg}}(X)) \simeq KH_{n-1}(X)$ for every $n < 0$. Consequently, the proof of Theorem 1.3 follows from the following result:

Proposition 2.6. *We have* $KH_n(\mathcal{D}_{\mathrm{dg}}^{\mathrm{sg}}(X)) = 0$ *for every* $n < 0$.

Proof. Consider the polynomial algebra $S := k[t_1, \ldots, t_d]$ as a $\mathbb{Z}$-graded k-algebra with $\deg(t_i) = 1$. Note that since the G-action on S preserves the $\mathbb{Z}$-grading, $R := S^G$ is a $\mathbb{Z}$-graded k-subalgebra of S. Following Orlov [15], in addition to the dg category of singularities $\mathcal{D}_{\mathrm{dg}}^{\mathrm{sg}}(X)$, we can also consider the dg category of graded singularities $\mathcal{D}_{\mathrm{dg}}^{\mathrm{sg,gr}}(X)$. By construction, this latter dg category comes equipped with a degree shift dg functor $(1) \colon \mathcal{D}_{\mathrm{dg}}^{\mathrm{sg,gr}}(X) \to \mathcal{D}_{\mathrm{dg}}^{\mathrm{sg,gr}}(X)$. Following Keller [8, §7.2], let us write $\mathcal{D}_{\mathrm{dg}}^{\mathrm{sg,gr}}(X)/(1)^{\mathbb{Z}}$ for the associated dg orbit category. As proved by Keller *et al.* in [10, Propositions A.2 and A.8], the forgetful dg functor $\mathcal{D}_{\mathrm{dg}}^{\mathrm{sg,gr}}(X) \to \mathcal{D}_{\mathrm{dg}}^{\mathrm{sg}}(X)$ induces a Morita equivalence $\mathcal{D}_{\mathrm{dg}}^{\mathrm{sg,gr}}(X)/(1)^{\mathbb{Z}} \simeq \mathcal{D}_{\mathrm{dg}}^{\mathrm{sg}}(X)$. Consequently, since homotopy K-theory is an $\mathbb{A}^1$-homotopy invariant of dg categories (see [20, §8.5]), [21, Theorem 1.5] yields a distinguished triangle of spectra:

$$KH(\mathcal{D}_{\mathrm{dg}}^{\mathrm{sg,gr}}(X)) \xrightarrow{KH((1))-\mathrm{id}} KH(\mathcal{D}_{\mathrm{dg}}^{\mathrm{sg,gr}}(X)) \to KH(\mathcal{D}_{\mathrm{dg}}^{\mathrm{sg}}(X)) \to \Sigma KH(\mathcal{D}_{\mathrm{dg}}^{\mathrm{sg,gr}}(X)).$$

Let us denote by $\underline{\mathrm{MCM}}^{\mathrm{gr}}(R)$ the stable category of $\mathbb{Z}$-graded maximal Cohen–Macaulay R-modules. As proved by Iyama and Takahashi in [7, Corollary 2.12], the triangulated category $\underline{\mathrm{MCM}}^{\mathrm{gr}}(R)$ admits a full exceptional collection $(\mathcal{E}_1, \ldots, \mathcal{E}_v)$ with $D_r := \mathrm{End}(\mathcal{E}_r), 1 \leqslant r \leqslant v$, finite-dimensional division k-algebras. Making use of the classical equivalence of categories $\mathrm{H}^0(\mathcal{D}_{\mathrm{dg}}^{\mathrm{sg,gr}}(X)) = \mathcal{D}^{\mathrm{sg,gr}}(X) \simeq \underline{\mathrm{MCM}}^{\mathrm{gr}}(R)$ (see Buchweitz [3]), and of the fact that homotopy K-theory is an additive invariant of dg categories (see [20, §2]), we hence conclude that $KH(\mathcal{D}_{\mathrm{dg}}^{\mathrm{sg,gr}}(X)) \simeq \oplus_{r=1}^{v} KH(D_r)$. Since D_r is a (right) regular Noetherian ring, we have $KH(D_r) \simeq \mathbb{K}(D_r)$ (see Gersten [5, Proposition 3.14]) and $\mathbb{K}_n(D_r) = 0$ for every $n < 0$ (see Schlichting [19, Theorem 7]). Therefore, the proof follows now from the long exact sequence of (stable) homotopy groups associated to the above distinguished triangle of spectra. □

3. Proof of Proposition 1.4

Given any prime power l^v, we have the universal coefficient sequences:

$$0 \to KH_n(X) \otimes_{\mathbb{Z}} \mathbb{Z}/l^v \to KH_n(X; \mathbb{Z}/l^v) \to \{l^v\text{-torsion in } KH_{n-1}(X)\} \to 0 \quad (3.1)$$

$$0 \to \mathbb{K}_n(X) \otimes_{\mathbb{Z}} \mathbb{Z}/l^v \to \mathbb{K}_n(X; \mathbb{Z}/l^v) \to \{l^v\text{-torsion in } \mathbb{K}_{n-1}(X)\} \to 0. \quad (3.2)$$

Theorem 1.3, combined with (3.1), implies that $KH_n(X; \mathbb{Z}/l^v) = 0$ for every $n < -1$. Since the base field k is of characteristic zero, $KH_n(X; \mathbb{Z}/l^v)$ is isomorphic to $\mathbb{K}_n(X; \mathbb{Z}/l^v)$; see

Weibel [23, p. 391]. Therefore, it follows from (3.2) that $\mathbb{K}_{-2}(X)$ is l^ν-divisible and that $\mathbb{K}_n(X), n < -2$, is uniquely l^ν-divisible. The proof follows now from the fact that the prime power l^ν is arbitrary.

4. Proof of Theorem 1.6

The triangle of spectra (2.1) yields a long exact sequence of abelian groups

$$\cdots \longrightarrow KH_0(\mathcal{D}_{\mathrm{dg}}^{\mathrm{sg}}(X)) \longrightarrow KH_{-1}(X) \longrightarrow KH_{-1}(\mathcal{D}_{\mathrm{dg}}^b(\mathrm{coh}(X))) \longrightarrow \cdots$$

Thanks to Proposition 2.2, we have $KH_{-1}(\mathcal{D}_{\mathrm{dg}}^b(\mathrm{coh}(X))) = 0$. Therefore, it follows that $KH_{-1}(X)$ is a quotient of $KH_0(\mathcal{D}_{\mathrm{dg}}^{\mathrm{sg}}(X))$. Let us now compute this latter group. As explained in the proof of Proposition 2.6, $KH(\mathcal{D}_{\mathrm{dg}}^{\mathrm{sg,gr}}(X))$ is isomorphic to the direct sum $\oplus_{r=1}^\nu KH(D_r)$. Since by assumption k is algebraically closed, all the finite-dimensional division k-algebras D_r are isomorphic to k. Therefore, since $KH_0(k) \simeq \mathbb{K}_0(k) \simeq \mathbb{Z}$ and $KH_{-1}(k) \simeq \mathbb{K}_{-1}(k) = 0$, we conclude from the long exact sequence of (stable) homotopy groups associated to the distinguished triangle of spectra constructed in the proof of Proposition 2.6 and from the natural identification $K_0(\mathcal{D}_{\mathrm{dg}}^{\mathrm{sg,gr}}(X)) = K_0(\mathcal{D}^{\mathrm{sg,gr}}(X))$, that the group $KH_0(\mathcal{D}_{\mathrm{dg}}^{\mathrm{sg}}(X))$ is isomorphic to the cokernel of the induced homomorphism:

$$K_0((1)) - \mathrm{id} \colon K_0(\mathcal{D}^{\mathrm{sg,gr}}(X)) \longrightarrow K_0(\mathcal{D}_{\mathrm{dg}}^{\mathrm{sg,gr}}(X)). \tag{4.1}$$

Via the equivalence of categories $\mathcal{D}^{\mathrm{sg,gr}}(X) \simeq \underline{\mathrm{MCM}}^{\mathrm{gr}}(R)$, (4.1) reduces to

$$K_0((1)) - \mathrm{id} \colon K_0(\underline{\mathrm{MCM}}^{\mathrm{gr}}(R)) \longrightarrow K_0(\underline{\mathrm{MCM}}^{\mathrm{gr}}(R)). \tag{4.2}$$

As proved by Amiot *et al.* in [1, Theorem 4.1], the triangulated category $\underline{\mathrm{MCM}}^{\mathrm{gr}}(R)$ admits a tilting object T. Moreover, the associated k-algebra $A := \mathrm{End}(T)$ is finite dimensional and of finite global dimension. Let us denote by $\mathcal{D}^b(\mathrm{mod}(A))$ the bounded derived category of finitely generated (right) A-modules. By construction, this latter category comes equipped with the Serre functor $\mathbf{S}$ and with the Auslander–Reiten translation functor τ. As proved in [1, Theorem 4.1], via the equivalence of categories $\underline{\mathrm{MCM}}^{\mathrm{gr}}(R) \simeq \mathcal{D}^b(\mathrm{mod}(A))$ induced by the tilting object T, the degree shift functor (1) corresponds to the functor $\mathbf{S}^{-1}\Sigma^{d-1}$. Since $\mathbf{S}^{-1}\Sigma = \tau$, we hence conclude that the above homomorphism (4.2) reduces to

$$(-1)^{d-2}\Phi_A - \mathrm{id} \colon K_0(\mathcal{D}^b(\mathrm{mod}(A))) \longrightarrow K_0(\mathcal{D}^b(\mathrm{mod}(A))), \tag{4.3}$$

where Φ_A stands for the inverse of the Coxeter matrix of A. As proved in [1, §5], the k-algebra A is isomorphic to the k-algebra $kQ/\langle\rho\rangle$ associated to the quiver with relations (Q, ρ) introduced at §1. On the one hand, this implies that the number of simple (right) A-modules agrees with the number of vertices of Q, i.e. $v = m - 1$. On the other hand, this implies that the matrix Φ_A can be written as $-C(C^{-1})^T$, where C_{ij} equals the number of arrows in Q from j to i (counted modulo the relations). Consequently, the above homomorphism (4.3) reduces to the (matrix) homomorphism $\mathrm{M} \colon \oplus_{r=1}^{m-1} \mathbb{Z} \to \oplus_{r=1}^{m-1} \mathbb{Z}$ introduced in §1. This concludes the proof.

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