J. Inst. Math. Jussieu (2013) **12**(3), 635–650
doi:10.1017/S1474748012000801

IMPROVED LOCAL ENERGY DECAY FOR THE WAVE EQUATION ON ASYMPTOTICALLY EUCLIDEAN ODD DIMENSIONAL MANIFOLDS IN THE SHORT RANGE CASE

JEAN-FRANÇOIS BONY[1] AND DIETRICH HÄFNER[2]

[1] *Institut de Mathématiques de Bordeaux, UMR 5251 du CNRS, Université de Bordeaux I, 351 cours de la Libération, 33405 Talence Cedex, France* (bony@math.u-bordeaux1.fr)

[2] *Université de Grenoble 1, Institut Fourier, UMR 5582 du CNRS, BP 74, 38402 St Martin d'Hères, France* (Dietrich.Hafner@ujf-grenoble.fr)

(Received 21 September 2011; accepted 15 June 2012; first published online 13 July 2012)

Abstract We show improved local energy decay for the wave equation on asymptotically Euclidean manifolds in odd dimensions in the short range case. The precise decay rate depends on the decay of the metric towards the Euclidean metric. We also give estimates of powers of the resolvent of the wave propagator between weighted spaces.

Keywords: local energy decay; resolvent smoothness; wave equation; odd dimensions; low frequencies; asymptotically Euclidean manifolds

AMS 2000 *Mathematics subject classification:* Primary 35L05; 35P25; 47A10; 58J45; 81U30

1. Introduction

The aim of this paper is to investigate the decay of the local energy for the wave equation associated with short range metric perturbations of the Euclidean Laplacian on $\mathbb{R}^d$, $d \geqslant 3$ and odd. More precisely, for any $\rho > 0$, we show that the local energy decays like $\langle t \rangle^{-\rho}$ if the metric converges like $\langle x \rangle^{-\rho-2-\varepsilon}$ toward the Euclidean metric. This result rests on the $C^{\rho+1}$ smoothness of the weighted resolvent of the wave generator.

The case of the wave equation in dimension $d \geqslant 3$ and odd is very specific. Indeed, in flat space, the strong Huygens principle guarantees that the local energy decays as fast as we want. For compactly supported perturbations, this no longer holds in general but one can use the theory of resonances (see [21] for a general presentation of this field) to prove dispersive estimates. In non-trapping situations, this theory gives a resonance expansion of the cut-off propagator which implies an exponential decay of the local energy with an optimal decay rate as in [17] for example. Such properties are related to the meromorphic extension to the whole complex plane (and, in particular, in a neighbourhood of 0) of the cut-off resolvent of the wave generator (see [22, 24]). The resonance theory can also be used in trapping situations, but there is necessarily a 'loss of derivatives' in the local energy estimate; see [18]. There is a large literature on this subject; we merely mention [7, 23].

One can also obtain exponential decay of the local energy using the theory of resonances for exponentially decaying perturbations. In this case, the weighted resolvent has a meromorphic extension only in a half-plane containing the real axis and the exponential decay rate of the local energy is controlled by the exponential decay of the perturbation at infinity. Such ideas were developed in [8, 15, 20]. It is therefore natural to ask what are the decay rate and the regularity properties of the resolvent for polynomially decaying perturbations. In such situations, it is unlikely that the resolvent is analytic near the real axis. However, we might hope that the weighted resolvent has some C^k regularity properties up to the real line, depending on the decay rate of the perturbation. In the same way, the exponential decay should be replaced by a polynomial one. In this paper, we show that this is indeed the case.

Note that the definition of the resonances by complex dilation or distortion (see [1, 13]) does not seem to be appropriate for showing local energy decay at low frequencies. Indeed, such methods do not give good estimates of the resolvent near the thresholds. As regards the resonances, we also mention the dynamical definition of [10] which describes the long time evolution of well-prepared initial data.

To prove the local energy decay, one can also apply other techniques like the vector field methods (there is a huge literature on this field; see e.g. [16] and the books [2, 12]), the Mourre theory (see [5, 6]), etc. However, in general, these methods do not distinguish the parity of the dimension and give the polynomial decay of the local energy that one expects in even dimensions mutatis mutandis. Eventually, the theory of perturbations can be used to get resolvent estimates at low energy and then decay of the local energy for 'small perturbations' (short range interactions, lower order terms, etc). This approach, close to the one developed in this paper, has been followed in numerous papers concerning the local energy decay for the Schrödinger equation perturbed by a potential (see [14, 19] for example).

In this paper, we consider the following operator on $\mathbb{R}^d$, with $d \geqslant 3$ and odd:

$$P = -b \operatorname{div}(A \nabla b) = -\sum_{i,j=1}^{d} b(x) \frac{\partial}{\partial x_i} A_{i,j}(x) \frac{\partial}{\partial x_j} b(x), \tag{1.1}$$

where $b(x) \in C^{\infty}(\mathbb{R}^d)$ and $A(x) \in C^{\infty}(\mathbb{R}^d; \mathbb{R}^{d\times d})$ is a real symmetric $d \times d$ matrix. The C^{∞} hypothesis is made mostly for convenience; much weaker regularity could actually be considered. We make an ellipticity assumption:

$$\exists \delta > 0, \quad \forall x \in \mathbb{R}^d \quad A(x) \geqslant \delta I_d \quad \text{and} \quad b(x) \geqslant \delta, \tag{H1}$$

I_d being the identity matrix on $\mathbb{R}^d$. We also assume that P is a perturbation of the Euclidean Laplacian: more precisely,

$$\forall \alpha \in \mathbb{N}^d \quad |\partial_x^{\alpha}(A(x) - I_d)| + |\partial_x^{\alpha}(b(x) - 1)| \lesssim \langle x \rangle^{-\rho-|\alpha|}, \tag{H2$_\rho$}$$

for some $\rho > 0$.

In particular, if $b = 1$, we are concerned with an elliptic operator in divergence form, $P = -\operatorname{div}(A\nabla)$. On the other hand, if $A = (g^2 g^{i,j}(x))_{i,j}$, $b = (\det g^{i,j})^{1/4}$, $g = \frac{1}{b}$, then the

https://doi.org/10.1017/S1474748012000801 Published online by Cambridge University Press

above operator is unitarily equivalent to the Laplace–Beltrami operator $-\Delta_{\mathfrak{g}}$ on $(\mathbb{R}^d, \mathfrak{g})$ with metric

$$\mathfrak{g} = \sum_{i,j=1}^{d} g_{i,j}(x)\, dx^i\, dx^j,$$

where $(g_{i,j})_{i,j}$ is inverse to $(g^{i,j})_{i,j}$ and the unitary transform is just multiplication by g. We are mainly interested in the low frequency behaviour, but our result is global in energy if we suppose in addition

$$P \text{ is non-trapping.} \tag{H3}$$

In the following, $\|\cdot\|$ will designate the norm on $L^2(\mathbb{R}^d)$ or $\mathcal{L}(L^2)$. Let H^s be the usual Sobolev space on $\mathbb{R}^d$. Then it is well known that $(P, D(P) = H^2)$ is self-adjoint on L^2. Let us first rewrite the wave equation associated with P as a first-order equation. The wave equation

$$\begin{cases} (\partial_t^2 + P)u = 0, \\ u(0) = u_0, \\ \partial_t u(0) = u_1, \end{cases} \tag{1.2}$$

is equivalent to the first-order equation

$$\begin{cases} i\partial_t \psi = G\psi, \\ \psi(0) = (u_0, u_1), \end{cases} \tag{1.3}$$

with $\psi = (u, \partial_t u)$ and

$$G = i \begin{pmatrix} 0 & 1 \\ -P & 0 \end{pmatrix}. \tag{1.4}$$

We also put $P_0 = -\Delta$ and

$$G_0 = i \begin{pmatrix} 0 & 1 \\ -P_0 & 0 \end{pmatrix}. \tag{1.5}$$

Let $\dot{H}^1_P$ (resp., $\dot{H}^2_P$) be the completion of $C_0^\infty(\mathbb{R}^d)$ in the norm $\|u\|^2_{\dot{H}^1_P} = \langle Pu, u\rangle$ (resp., $\|u\|^2_{\dot{H}^2_P} = \langle Pu, u\rangle + \|Pu\|^2$). Then it is well known that $(G, (\dot{H}^2_P \oplus \dot{H}^1_P))$ is self-adjoint on $\mathcal{E} = \dot{H}^1_P \oplus L^2$. We put $\mathcal{H}^s := H^{s+1} \oplus H^s$. Our main result is the following.

Theorem 1. *Assume $d \geqslant 3$ and odd, (H1) and (H2)$_\rho$. Let $\mu \geqslant 0$ be such that $\rho > \mu + 2$ ($\rho > \mu + 1$ in dimension $d = 3$).*

(i) *For all $\chi \in C_0^\infty(\mathbb{R})$ and $\varepsilon > 0$, we have*

$$\left\| \langle x\rangle^{-\mu-1-\varepsilon} e^{-itG} \chi(G) \langle x\rangle^{-\mu-1-\varepsilon} \right\|_{\mathcal{L}(\mathcal{H}^s)} \lesssim \langle t\rangle^{-\mu}.$$

https://doi.org/10.1017/S1474748012000801 Published online by Cambridge University Press

(ii) *If we suppose in addition (H3), then the above estimate holds globally in energy:*

$$\left\| \langle x \rangle^{-\mu-1-\varepsilon} e^{-itG} \langle x \rangle^{-\mu-1-\varepsilon} \right\|_{\mathcal{L}(\mathcal{H}^s)} \lesssim \langle t \rangle^{-\mu}.$$

For $d = 3$, we can replace $\langle x \rangle^{-\mu-1-\varepsilon}$ by $\langle x \rangle^{-\mu-1/2-\varepsilon}$ in the above estimates.

Remark 2. Combining the previous theorem with [5] and an interpolation argument, we can replace $\langle x \rangle^{-\mu-1-\varepsilon}$ by $\langle x \rangle^{-\mu-\varepsilon}$ in Theorem 1 if $\rho = +\infty$ and $\mu > 1$.

Note that one can express the wave propagator at low frequencies in terms of P using the classical formula

$$e^{-itG} = \begin{pmatrix} \cos t\sqrt{P} & \dfrac{\sin t\sqrt{P}}{\sqrt{P}} \\ -\sqrt{P}\sin t\sqrt{P} & \cos t\sqrt{P} \end{pmatrix}, \tag{1.6}$$

and that $\chi(G) = \chi(\sqrt{P}) \oplus \chi(\sqrt{P})$ for χ even. The proof of Theorem 1 rests on the following smoothness property of the weighted resolvent at low frequencies (see also the Hölder regularity stated in Proposition 11).

Theorem 3. *Assume $d \geqslant 3$ and odd, (H1) and $(H2)_\rho$. Let $k \in \mathbb{N}^*$ be such that $\rho > k+1$ ($\rho > k$ for $d = 3$) and let $\kappa = k$ ($\kappa = k - 1/2$ for $d = 3$, $k \geqslant 2$). Then, for all $s \in \mathbb{R}$ and $C, \varepsilon > 0$, we have*

$$\sup_{z \in \mathbb{C}\setminus\mathbb{R}, |z| \leqslant C} \left\| \langle x \rangle^{-\kappa-\varepsilon} (G - z)^{-k} \langle x \rangle^{-\kappa-\varepsilon} \right\|_{\mathcal{L}(\mathcal{H}^s, \mathcal{H}^{s+k})} \lesssim 1.$$

The polynomial decay of the local energy (resp., the C^k smoothness of the weighted resolvent) for polynomially decaying perturbations is analogous to the exponential decay of the local energy (resp., the analytic extension of the resolvent) given by the resonance theory for compactly supported or exponentially decaying perturbations.

2. The free resolvent

The goal of this section is to show the following estimate on the free resolvent.

Proposition 4. *Let $d \geqslant 3$ be odd. For all $k \in \mathbb{N}$, $s \in \mathbb{R}$ and $C, \varepsilon > 0$, we have*

$$\sup_{z \in \mathbb{C}\setminus\mathbb{R}, |z| \leqslant C} \left\| \langle x \rangle^{-k-\varepsilon} (G_0 - z)^{-k} \langle x \rangle^{-k-\varepsilon} \right\|_{\mathcal{L}(\mathcal{H}^s, \mathcal{H}^{s+k})} \lesssim 1.$$

To prove this result, we will write the free resolvent as an integral in time over the evolution and then use the strong Huygens principle. Note that we estimate the powers of the resolvent in a scale of Sobolev spaces rather than in a scale of energy spaces. We therefore first need rough estimates for the evolution on $H^1 \oplus L^2$.

Lemma 5. *Uniformly for $t \in \mathbb{R}$, we have*

$$\left\| e^{-itG_0} \right\|_{\mathcal{L}(H^1 \oplus L^2)} \lesssim \langle t \rangle, \tag{2.1}$$

$$\left\| e^{-itG_0} \langle x \rangle^{-1} \right\|_{\mathcal{L}(H^1 \oplus L^2)} \lesssim 1. \tag{2.2}$$

Proof. Using the functional calculus, we obtain

$$\|\cos t\sqrt{P_0}\|_{\mathcal{L}(H^1)} \lesssim 1, \qquad \|\cos t\sqrt{P_0}\|_{\mathcal{L}(L^2)} \lesssim 1,$$

$$\|-\sqrt{P_0}\sin t\sqrt{P_0}\|_{\mathcal{L}(H^1,L^2)} = \left\|-\sqrt{P_0}\sin(t\sqrt{P_0})\langle P_0\rangle^{-1/2}\right\|_{\mathcal{L}(L^2)} \lesssim 1,$$

$$\begin{aligned}\left\|\frac{\sin t\sqrt{P_0}}{\sqrt{P_0}}\right\|_{\mathcal{L}(L^2,H^1)} &= \left\|\langle P_0\rangle^{1/2}\frac{\sin t\sqrt{P_0}}{\sqrt{P_0}}\right\|_{\mathcal{L}(L^2)} \\ &\lesssim \left\|\langle P_0\rangle^{1/2} t\frac{\sin t\sqrt{P_0}}{t\sqrt{P_0}}\chi(P_0\leqslant 1)\right\|_{\mathcal{L}(L^2)} \\ &\quad + \left\|\langle P_0\rangle^{1/2}\frac{\sin t\sqrt{P_0}}{\sqrt{P_0}}\chi(P_0\geqslant 1)\right\|_{\mathcal{L}(L^2)} \lesssim \langle t\rangle.\end{aligned}$$

Combined with (1.6), this implies the first estimate.

We now prove (2.2). Using the previous arguments, we only have to show

$$\left\|\langle P_0\rangle^{1/2}\frac{\sin t\sqrt{P_0}}{\sqrt{P_0}}\chi(P_0\leqslant 1)\langle x\rangle^{-1}\right\|_{\mathcal{L}(L^2)} \lesssim 1.$$

By the classical Hardy estimate, we have

$$\left\|\frac{1}{|x|}u\right\| \lesssim \||\nabla|u\|,$$

which gives by the Fourier transform

$$\left\|\frac{1}{\sqrt{P_0}}u\right\| \lesssim \||x|u\| \lesssim \|\langle x\rangle u\|.$$

We conclude that

$$\left\|\langle P_0\rangle^{1/2}\frac{\sin t\sqrt{P_0}}{\sqrt{P_0}}\chi(P_0\leqslant 1)\langle x\rangle^{-1}\right\| \lesssim \left\|\frac{1}{\sqrt{P_0}}\langle x\rangle^{-1}\right\| \lesssim 1,$$

and the second estimate of the lemma follows. □

The following estimate on the free evolution is fundamental for the proof of Proposition 4.

Lemma 6. *Let $d\geqslant 3$ be odd and $\alpha\geqslant 1$. Then, uniformly in $t\in\mathbb{R}$, we have*

$$\left\|\langle x\rangle^{-\alpha}e^{-itG_0}\langle x\rangle^{-\alpha}\right\|_{\mathcal{L}(H^1\oplus L^2)} \lesssim \langle t\rangle^{-\alpha}.$$

Proof. By (2.1), we can suppose $t\geqslant 1$. Let $\varphi\in C_0^\infty(]-\infty,\frac{1}{2}[)$ be such that $\varphi=1$ close to zero. In particular, this implies

$$\left\|\varphi\left(\frac{|x|}{t}\right)\right\|_{\mathcal{L}(H^1\oplus L^2)} \lesssim 1, \tag{2.3}$$

and

$$\begin{aligned}\left\|\langle x\rangle^{-\alpha}(1-\varphi)\left(\frac{|x|}{t}\right)\right\|_{\mathcal{L}(H^1\oplus L^2)} &\leqslant \left\|\langle x\rangle^{-\alpha}(1-\varphi)\left(\frac{|x|}{t}\right)\right\|_{\mathcal{L}(L^2)} \\ &\quad + \left\|\langle x\rangle^{-\alpha}(1-\varphi)\left(\frac{|x|}{t}\right)\right\|_{\mathcal{L}(H^1)}.\end{aligned}$$

https://doi.org/10.1017/S1474748012000801 Published online by Cambridge University Press

We obviously have

$$\left\| \langle x\rangle^{-\alpha}(1-\varphi)\left(\frac{|x|}{t}\right)\right\|_{\mathcal{L}(L^2)} \lesssim \langle t\rangle^{-\alpha}.$$

On the other hand,

$$\begin{aligned}\left\| \langle x\rangle^{-\alpha}(1-\varphi)\left(\frac{|x|}{t}\right)u\right\|_{H^1} &\lesssim \left\| \langle x\rangle^{-\alpha}(1-\varphi)\left(\frac{|x|}{t}\right)u\right\|_{L^2} + \left\| \langle x\rangle^{-\alpha-1}(1-\varphi)\left(\frac{|x|}{t}\right)u\right\|_{L^2}\\ &\quad + \left\| \langle x\rangle^{-\alpha}\varphi'\left(\frac{|x|}{t}\right)\frac{x}{t|x|}u\right\|_{L^2} + \left\| \langle x\rangle^{-\alpha}(1-\varphi)\left(\frac{|x|}{t}\right)\nabla u\right\|_{L^2}\\ &\lesssim \langle t\rangle^{-\alpha}\|u\|_{H^1}.\end{aligned}$$

Combining the three previous estimates, this yields

$$\left\| \langle x\rangle^{-\alpha}(1-\varphi)\left(\frac{|x|}{t}\right)\right\|_{\mathcal{L}(H^1\oplus L^2)} \lesssim \langle t\rangle^{-\alpha}. \tag{2.4}$$

Using the previous estimates, we can now finish the proof of the lemma. We write

$$\begin{aligned}\left\| \langle x\rangle^{-\alpha}e^{-itG_0}\langle x\rangle^{-\alpha}\right\|_{\mathcal{L}(H^1\oplus L^2)} &\leqslant \left\| \langle x\rangle^{-\alpha}(1-\varphi)\left(\frac{|x|}{t}\right)e^{-itG_0}\langle x\rangle^{-\alpha}\right\|_{\mathcal{L}(H^1\oplus L^2)}\\ &\quad + \left\| \langle x\rangle^{-\alpha}\varphi\left(\frac{|x|}{t}\right)e^{-itG_0}(1-\varphi)\left(\frac{|x|}{t}\right)\langle x\rangle^{-\alpha}\right\|_{\mathcal{L}(H^1\oplus L^2)}\\ &\quad + \left\| \langle x\rangle^{-\alpha}\varphi\left(\frac{|x|}{t}\right)e^{-itG_0}\varphi\left(\frac{|x|}{t}\right)\langle x\rangle^{-\alpha}\right\|_{\mathcal{L}(H^1\oplus L^2)}\\ &=: I_1+I_2+I_3.\end{aligned} \tag{2.5}$$

Using (2.2), (2.4) and $\alpha \geqslant 1$, we get

$$I_1 \lesssim \left\| \langle x\rangle^{-\alpha}(1-\varphi)\left(\frac{|x|}{t}\right)\right\|_{\mathcal{L}(H^1\oplus L^2)} \left\| e^{-itG_0}\langle x\rangle^{-\alpha}\right\|_{\mathcal{L}(H^1\oplus L^2)} \lesssim \langle t\rangle^{-\alpha}. \tag{2.6}$$

In the same way, (2.2)–(2.4) and $\alpha \geqslant 1$ imply

$$\begin{aligned}I_2 &\lesssim \left\| \varphi\left(\frac{|x|}{t}\right)\right\|_{\mathcal{L}(H^1\oplus L^2)} \left\| \langle x\rangle^{-\alpha}e^{-itG_0}\right\|_{\mathcal{L}(H^1\oplus L^2)} \left\| (1-\varphi)\left(\frac{|x|}{t}\right)\langle x\rangle^{-\alpha}\right\|_{\mathcal{L}(H^1\oplus L^2)}\\ &\lesssim \langle t\rangle^{-\alpha}.\end{aligned} \tag{2.7}$$

Eventually, the strong Huygens principle and the assumptions on the support of φ give

$$I_3 = 0. \tag{2.8}$$

Thus, the lemma follows from (2.5) and the estimates (2.6)–(2.8). □

Proof of Proposition 4. We have, for $\operatorname{Im} z > 0$,

$$(G_0 - z)^{-1} = i\int_0^{+\infty} e^{-it(G_0-z)}dt,$$

https://doi.org/10.1017/S1474748012000801 Published online by Cambridge University Press

and thus

$$(G_0 - z)^{-k} = i \int_0^{+\infty} \frac{(it)^{k-1}}{(k-1)!} e^{-it(G_0 - z)} dt.$$

We then estimate, for $\operatorname{Im} z > 0$,

$$\begin{aligned}\left\| \langle x \rangle^{-k-\varepsilon} (G_0 - z)^{-k} \langle x \rangle^{-k-\varepsilon} \right\|_{\mathcal{L}(\mathcal{H}^0)} &\leqslant \int_0^{+\infty} |t|^{k-1} \left\| \langle x \rangle^{-k-\varepsilon} e^{-itG_0} \langle x \rangle^{-k-\varepsilon} \right\|_{\mathcal{L}(\mathcal{H}^0)} dt \\ &\lesssim \int_0^{+\infty} \langle t \rangle^{-1-\varepsilon} dt \lesssim 1,\end{aligned}$$

where we have used Lemma 6. To obtain the higher order estimates, we observe that

$$\|v\|_{\mathcal{H}^\beta} \asymp \left\| (G_0 + i)^\beta v \right\|_{\mathcal{H}^0},$$

and use

$$\begin{aligned}(G_0 + i)^\beta \langle x \rangle^{-k-\varepsilon} &= \mathcal{O}(1) \langle x \rangle^{-k-\varepsilon} (G_0 + i)^\beta, \\ (G_0 + i)(G_0 - z)^{-1} &= 1 + (z + i)(G_0 - z)^{-1},\end{aligned}$$

with $|z| \leqslant C$. The proof for $\operatorname{Im} z < 0$ is analogous. □

3. Improved estimates for the free resolvent in dimension 3

In this section, we show improved resolvent estimates in dimension 3 using the explicit form of the kernel.

Proposition 7. *Let $d = 3$, $k \in \mathbb{N}^*$, $s \in \mathbb{R}$ and*

$$\kappa = \begin{cases} 1 & \text{for } k = 1, \\ k - 1/2 & \text{for } k \geqslant 2. \end{cases}$$

Then, for all $C, \varepsilon > 0$, we have

$$\sup_{z \in \mathbb{C} \setminus \mathbb{R}, |z| \leqslant C} \left\| \langle x \rangle^{-\kappa-\varepsilon} (G_0 - z)^{-k} \langle x \rangle^{-\kappa-\varepsilon} \right\|_{\mathcal{L}(\mathcal{H}^s, \mathcal{H}^{s+k})} \lesssim 1.$$

In order to prove this proposition, we will need the following lemma valid in all dimensions.

Lemma 8. *Let $0 < \alpha, \gamma < \frac{d}{2}$ and $0 < \beta < d$ be such that $\frac{d}{2} < \alpha + \beta < d$ and $\alpha + \beta + \gamma > d$. Then the operator with integral kernel $k(x, y) = \langle x \rangle^{-\alpha} |x - y|^{-\beta} \langle y \rangle^{-\gamma}$ is a bounded operator on $L^2(\mathbb{R}^d)$.*

Proof. Let $u \in L^2(\mathbb{R}^d)$ and

$$v(x) = \int \langle x \rangle^{-\alpha} |x - y|^{-\beta} \langle y \rangle^{-\gamma} u(y)\, dy.$$

Then, by the Hölder inequality, we have

$$\|v\| \lesssim \left\| \langle x \rangle^{-\alpha+\varepsilon} \right\|_{p_1} \left\| \int |x - y|^{-\beta} \langle x - y \rangle^{-\varepsilon} \langle y \rangle^{-\gamma+\varepsilon} |u|(y)\, dy \right\|_{p_2},$$

https://doi.org/10.1017/S1474748012000801 Published online by Cambridge University Press

with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}$. For $0 < \alpha < \frac{d}{2}$, we can take $p_1 = \frac{d+\varepsilon}{\alpha-\varepsilon}$ and $p_2 = \frac{2d+2\varepsilon}{d-2\alpha+3\varepsilon}$ with $\varepsilon > 0$ small enough. Now, by [11, Corollary 4.5.2], we have

$$\big\| |x|^{-\beta} \langle x \rangle^{-\varepsilon} * \langle y \rangle^{-\gamma+\varepsilon} |u|(y) \big\|_{p_2} \lesssim \big\| |x|^{-\beta} \langle x \rangle^{-\varepsilon} \big\|_{q_1} \big\| \langle y \rangle^{-\gamma+\varepsilon} |u| \big\|_{q_2},$$

with $1 \leqslant q_1, q_2 \leqslant +\infty$ and

$$\frac{1}{q_1} + \frac{1}{q_2} = 2 - 1 + \frac{1}{p_2}.$$

As $0 < \beta < d$ and $\alpha + \beta > d/2$, we can take $q_1 = \frac{d+\varepsilon}{\beta+\varepsilon}$ and $q_2 = \frac{2d+2\varepsilon}{3d-2(\alpha+\beta)+3\varepsilon}$. We now estimate again by the Hölder inequality

$$\big\| \langle y \rangle^{-\gamma+\varepsilon} |u| \big\|_{q_2} \lesssim \big\| \langle y \rangle^{-\gamma+\varepsilon} \big\|_{r_1} \|u\|_{r_2},$$

with

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{q_2}.$$

As $\alpha + \beta < d$, we can take $r_2 = 2$ and $\frac{1}{r_1} = 1 - \frac{\alpha+\beta}{d+\varepsilon}$. We need $(\gamma - \varepsilon) r_1 > d$ or equivalently

$$\gamma - \varepsilon > \frac{d}{r_1} = d - \frac{d(\alpha+\beta)}{d+\varepsilon} \Longleftrightarrow d(\alpha+\beta+\gamma) > d^2 + 2\varepsilon d - \varepsilon\gamma + \varepsilon^2,$$

which is fulfilled for $\varepsilon > 0$ small enough since $\alpha + \beta + \gamma > d$. □

Proof of Proposition 7. Using Proposition 4, it is sufficient to consider the case $k \geqslant 2$. Let us first recall that the kernel of the free resolvent in dimension 3 is given by

$$(P_0 - z^2)^{-1}\delta = \frac{e^{iz|x|}}{4\pi |x|}.$$

Using (1.5), we write

$$\begin{aligned} (G_0 - z)^{-1} &= (P_0 - z^2)^{-1} \begin{pmatrix} z & i \\ -iP_0 & z \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix} + (P_0 - z^2)^{-1} \begin{pmatrix} z & i \\ -iz^2 & z \end{pmatrix}. \end{aligned}$$

The kernel of the second operator in the above line is given by

$$\begin{pmatrix} z & i \\ -iz^2 & z \end{pmatrix} \frac{1}{4\pi |x-y|} e^{iz|x-y|}.$$

Note also that, for $k \geqslant 2$,

$$(G_0 - z)^{-k} = \frac{1}{(k-1)!} \partial_z^{k-1} (G_0 - z)^{-1}.$$

Thus, the kernel of this operator decomposes into a sum of terms of the form

$$z^\beta |x-y|^{\gamma-1} e^{iz|x-y|}, \tag{3.1}$$

with $0 \leqslant \beta \leqslant 2$, $0 \leqslant \gamma \leqslant k-1$. We therefore have to bound kernels of the form

$$\langle x\rangle^{-\kappa-\varepsilon}|x-y|^{\gamma-1}e^{iz|x-y|}\langle y\rangle^{-\kappa-\varepsilon}.$$

If $\gamma-1=-1$, then Lemma 8 tells us that the corresponding operator is bounded on L^2 for $\kappa \geqslant 1$. If $\gamma-1 \geqslant 0$, then for $\kappa=\gamma-1+3/2$ the above kernel can be estimated by

$$\langle x\rangle^{-3/2-\varepsilon}\langle y\rangle^{-3/2-\varepsilon},$$

which clearly defines a bounded operator on L^2. The worst case is $\gamma=k-1$ and thus $\kappa=k-1/2$. This proves the estimate in $\mathcal{L}(\mathcal{H}^0)$ and then in $\mathcal{L}(\mathcal{H}^s,\mathcal{H}^{s+k})$ by the same argument as at the end of the proof of Proposition 4. □

4. Resolvent estimates for the perturbed operator

Using the results obtained in the previous sections, we now prove the estimates for the weighted resolvent of G stated in Theorem 3. To lighten the exposition, we will use the notation $R(z)=(G-z)^{-1}$ and $R_0(z)=(G_0-z)^{-1}$ in the sequel. Let $\widetilde{\partial}_j=\partial_j b$ and $\widetilde{\partial}_j^*=b\partial_j$. In the following, r_j will stand for an error term fulfilling

$$\partial_x^\alpha r_j(x)=\mathcal{O}(\langle x\rangle^{-|\alpha|-\rho-j}). \tag{4.1}$$

Let us now introduce

$$V:=G_0-G=i\begin{pmatrix}0 & 0\\ P-P_0 & 0\end{pmatrix}:\mathcal{H}^s\longrightarrow\mathcal{H}^{s-1}, \tag{4.2}$$

which is continuous. Note that

$$P-P_0=\widetilde{\partial}^* r_0\widetilde{\partial}+\widetilde{\partial}^* r_1+r_1\widetilde{\partial}+r_2, \tag{4.3}$$

where we have not written the sum over the indexes on the right-hand side. In dimension 3, we will need the following lemma. Note that this result also applies to G replaced by G_0.

Lemma 9. *For all $s\in\mathbb{R}$ and $C,\varepsilon>0$, we have*

$$\left\|\langle x\rangle^{-1/2-\varepsilon}\begin{pmatrix}0 & 0\\ \widetilde{\partial}_j & 0\end{pmatrix}R(z)\langle x\rangle^{-1/2-\varepsilon}\right\|_{\mathcal{L}(\mathcal{H}^s,\mathcal{H}^{s+1})}\lesssim 1, \tag{4.4}$$

$$\left\|\langle x\rangle^{-1/2-\varepsilon}R(z)\begin{pmatrix}0 & 0\\ \widetilde{\partial}_j^* & 0\end{pmatrix}\langle x\rangle^{-1/2-\varepsilon}\right\|_{\mathcal{L}(\mathcal{H}^s,\mathcal{H}^{s+1})}\lesssim 1, \tag{4.5}$$

uniformly in $z\in\mathbb{C}\setminus\mathbb{R}$, $|z|\leqslant C$.

Proof. We only show (4.4), the proof for (4.5) being analogous. Let us first recall that

$$G=U^{-1}LU \quad \text{with} \quad L=\begin{pmatrix}P^{1/2} & 0\\ 0 & -P^{1/2}\end{pmatrix}, \tag{4.6}$$

$$U=\frac{1}{\sqrt{2}}\begin{pmatrix}P^{1/2} & i\\ P^{1/2} & -i\end{pmatrix},\quad U^{-1}=\frac{1}{\sqrt{2}}\begin{pmatrix}P^{-1/2} & P^{-1/2}\\ -i & i\end{pmatrix}. \tag{4.7}$$

https://doi.org/10.1017/S1474748012000801 Published online by Cambridge University Press

Therefore

$$R(z) = U^{-1}(L - z)^{-1}U. \tag{4.8}$$

Note that $U : \dot{H}^1_P \oplus L^2 \to L^2 \oplus L^2$ is a unitary transform and that L is self-adjoint on $L^2 \oplus L^2$ with domain $D(L) = H^1 \oplus H^1$. Using (4.6)–(4.8), we compute

$$\begin{pmatrix} 0 & 0 \\ \widetilde{\partial}_j & 0 \end{pmatrix} R(z) = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ A & B \end{pmatrix},$$

with

$$A = \widetilde{\partial}_j (P^{1/2} - z)^{-1} + \widetilde{\partial}_j (-P^{1/2} - z)^{-1},$$
$$B = i(\widetilde{\partial}_j (P^{1/2} - z)^{-1} - \widetilde{\partial}_j (-P^{1/2} - z)^{-1})P^{-1/2}.$$

In order to prove a bound on $\mathcal{L}(\mathcal{H}^0, \mathcal{H}^1)$, it is therefore sufficient to show that

$$\langle x\rangle^{-1/2-\varepsilon}\widetilde{\partial}_j (P^{1/2} - z)^{-1}\langle x\rangle^{-1/2-\varepsilon} : H^1 \longrightarrow H^1,$$
$$\langle x\rangle^{-1/2-\varepsilon}\widetilde{\partial}_j P^{-1/2}(P^{1/2} - z)^{-1}\langle x\rangle^{-1/2-\varepsilon} : L^2 \longrightarrow H^1,$$

are bounded uniformly in $z \in \mathbb{C} \setminus \mathbb{R}$, $|z| \leqslant C$. This follows from [3, Lemmas 4.1, 4.7 and 4.8] and [4, Theorem 1]. In order to prove the estimates for $s \in \mathbb{N}$, we commute the two previous operators with the partial derivatives $\widetilde{\partial}_k$ and use [3, Lemmas 4.1, 4.7 and 4.8]. For $s \in \mathbb{Z} \setminus \mathbb{N}$, it is enough to consider the adjoint of these two operators and proceed as before. Eventually, the case $s \in \mathbb{R}$ follows from an interpolation argument. □

To prove Theorem 3, it will be useful to have an explicit form of the powers of the perturbed resolvent $R^k(z)$ in terms of the powers of the free resolvent $R_0^j(z)$, $1 \leqslant j \leqslant k$, and of the perturbed resolvent $R(z)$.

Lemma 10. *For all $k \in \mathbb{N}^*$ and $z \in \mathbb{C} \setminus \mathbb{R}$, we can write*

$$R^k(z) = \sum_{\text{finite}} M_0 V \cdots V M_n,$$

where $\sum_{\text{finite}}$ means a finite sum of terms of the type $M_0 V \cdots V M_n$ with $n \in \mathbb{N}$, $M_0 = R_0^{\alpha_0}(z)$, $M_n = R_0^{\alpha_n}(z)$ and $M_j = R(z)$ (in which case we put $\alpha_j = 1$) or $M_j = R_0^{\alpha_j}(z)$ for $1 \leqslant j \leqslant n-1$. Moreover, the α_j satisfy

$$\forall j \in \{0, \dots, n\} \quad 0 < \alpha_j \leqslant k, \quad \alpha_j + \alpha_{j+1} \leqslant k+1 \quad \text{and} \quad \sum_{j=0}^{n} \alpha_j = n + k.$$

Proof. We prove the lemma by induction over k. In the case $k = 1$, we use twice the resolvent identity:

$$\begin{aligned} R(z) &= R_0(z) + R_0(z)VR(z) \\ &= R_0(z) + R_0(z)VR_0(z) + R_0(z)VR(z)VR_0(z). \end{aligned}$$

https://doi.org/10.1017/S1474748012000801 Published online by Cambridge University Press

Let us now suppose the lemma for $k \geqslant 1$. We write

$$\begin{aligned} R^{k+1}(z) &= R(z)R^k(z) \\ &= (R_0(z) + R_0(z)VR_0(z) + R_0(z)VR(z)VR_0(z)) \sum_{\text{finite}} M_0 V \cdots VM_n \\ &= \sum_{\text{finite}} M_0 V \cdots VM_m, \end{aligned}$$

where the last sum has the required properties. □

Proof of Theorem 3. Let us first consider the case $k = 1$. From (1.4), we have

$$R(z) = (P - z^2)^{-1} \begin{pmatrix} z & i \\ -iP & z \end{pmatrix}. \tag{4.9}$$

Using [4, Theorem 1] and a simple calculation, we get

$$\left\| \langle x \rangle^{-1-\varepsilon} (P - z^2)^{-1} \langle x \rangle^{-1-\varepsilon} \right\|_{\mathcal{L}(H^s, H^{s+2})} \lesssim 1,$$

uniformly in $z \in \mathbb{C} \setminus \mathbb{R}$, $|z| \leqslant C$. It then follows by (4.9) that

$$\left\| \langle x \rangle^{-1-\varepsilon} R(z) \langle x \rangle^{-1-\varepsilon} \right\|_{\mathcal{L}(\mathcal{H}^s, \mathcal{H}^{s+1})} \lesssim 1, \tag{4.10}$$

and the case $k = 1$ follows.

We now treat the case $k \geqslant 2$, $d \geqslant 3$ and odd. Using Lemma 10, we can write

$$\begin{aligned} \langle x \rangle^{-k-\varepsilon} R^k(z) \langle x \rangle^{-k-\varepsilon} &= \sum_{\text{finite}} \langle x \rangle^{-k-\varepsilon} M_0 V \cdots M_j V M_{j+1} \cdots V M_n \langle x \rangle^{-k-\varepsilon} \\ &= \sum_{\text{finite}} \langle x \rangle^{\alpha_0 - k} \langle x \rangle^{-\alpha_0-\varepsilon} M_0 \langle x \rangle^{-\alpha_0-\varepsilon} \langle x \rangle^{\alpha_0+\varepsilon} V \cdots \\ &\quad \cdots M_j \langle x \rangle^{-\alpha_j-\varepsilon} \langle x \rangle^{\alpha_j+\varepsilon} V \langle x \rangle^{\alpha_{j+1}+\varepsilon} \langle x \rangle^{-\alpha_{j+1}-\varepsilon} M_{j+1} \cdots \\ &\quad \cdots V \langle x \rangle^{\alpha_n+\varepsilon} \langle x \rangle^{-\alpha_n-\varepsilon} M_n \langle x \rangle^{-\alpha_n-\varepsilon} \langle x \rangle^{\alpha_n-k}. \end{aligned} \tag{4.11}$$

Since $\alpha_j + \alpha_{j+1} \leqslant k + 1 < \rho$, (4.2) and (4.3) imply that

$$\langle x \rangle^{\alpha_j+\varepsilon} V \langle x \rangle^{\alpha_{j+1}+\varepsilon} : \mathcal{H}^s \longrightarrow \mathcal{H}^{s-1},$$

is a bounded operator. Moreover, from Proposition 4 and (4.10), we have

$$\left\| \langle x \rangle^{-\alpha_j-\varepsilon} M_j \langle x \rangle^{-\alpha_j-\varepsilon} \right\|_{\mathcal{L}(\mathcal{H}^s, \mathcal{H}^{s+\alpha_j})} \lesssim 1,$$

uniformly in $z \in \mathbb{C} \setminus \mathbb{R}$, $|z| \leqslant C$. Combining (4.11) with the previous estimates, $\alpha_0 \leqslant k$, $\alpha_n \leqslant k$ and $\sum \alpha_j = k + n$, we get that $\langle x \rangle^{-k-\varepsilon} R^k(z) \langle x \rangle^{-k-\varepsilon}$ is bounded uniformly in $z \in \mathbb{C} \setminus \mathbb{R}$, $|z| \leqslant C$, as an operator from $\mathcal{H}^s$ to $\mathcal{H}^{s+k}$.

It remains to study the case $k \geqslant 2$ and $d = 3$. As before, Lemma 10 gives

$$\begin{aligned} &\langle x \rangle^{-k+1/2-\varepsilon} R^k(z) \langle x \rangle^{-k+1/2-\varepsilon} \\ &\quad = \sum_{\text{finite}} \langle x \rangle^{-k+1/2-\varepsilon} M_0 V \cdots M_j V M_{j+1} \cdots V M_n \langle x \rangle^{-k+1/2-\varepsilon}. \end{aligned} \tag{4.12}$$

https://doi.org/10.1017/S1474748012000801 Published online by Cambridge University Press

From (4.2) and (4.3), we have

$$V = \begin{pmatrix} 0 & 0 \\ \widetilde{\partial}^* & 0 \end{pmatrix} \begin{pmatrix} 0 & ir_0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \widetilde{\partial} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \widetilde{\partial}^* & 0 \end{pmatrix} \begin{pmatrix} ir_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & ir_1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \widetilde{\partial} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ ir_2 & 0 \end{pmatrix}$$

where the sum over the indexes does not appear. In particular, since $\alpha_j + \alpha_{j+1} \leqslant k+1 < \rho + 1$, V can be written as

$$V = \sum_{\text{finite}} A_j^* B A_{j+1}, \tag{4.13}$$

where

$$A_j = \langle x \rangle^{-\alpha_j + 1/2 - \varepsilon} \begin{pmatrix} 0 & 0 \\ \widetilde{\partial} & 0 \end{pmatrix} \quad \text{or} \quad A_j = \langle x \rangle^{-\alpha_j - \varepsilon},$$
$$A_j^* = \begin{pmatrix} 0 & 0 \\ \widetilde{\partial}^* & 0 \end{pmatrix} \langle x \rangle^{-\alpha_j + 1/2 - \varepsilon} \quad \text{or} \quad A_j^* = \langle x \rangle^{-\alpha_j - \varepsilon},$$

and B is a bounded operator from $\mathcal{H}^s$ to $\mathcal{H}^{s-1}$. From Proposition 7, Lemma 9 and (4.10), we have, for $j \in \{1, \dots n-1\}$,

$$\left\| A_j M_j A_j^* \right\|_{\mathcal{L}(\mathcal{H}^s, \mathcal{H}^{s+\alpha_j})} \lesssim 1, \tag{4.14}$$

uniformly in $z \in \mathbb{C} \setminus \mathbb{R}$, $|z| \leqslant C$. Moreover, since $k \geqslant 2$, we have $k - 1/2 \geqslant \max(1, \alpha_0 - 1/2)$ and $k - 1/2 \geqslant \max(1, \alpha_n - 1/2)$. Then, Proposition 7 and Lemma 9 give

$$\left\| \langle x \rangle^{-k+1/2-\varepsilon} M_0 A_0^* \right\|_{\mathcal{L}(\mathcal{H}^s, \mathcal{H}^{s+\alpha_0})} \lesssim 1 \quad \text{and} \quad \left\| A_n M_n \langle x \rangle^{-k+1/2-\varepsilon} \right\|_{\mathcal{L}(\mathcal{H}^s, \mathcal{H}^{s+\alpha_n})} \lesssim 1. \tag{4.15}$$

Putting together (4.12)–(4.15), we get that $\langle x \rangle^{-k+1/2-\varepsilon} R^k(z) \langle x \rangle^{-k+1/2-\varepsilon}$ is bounded uniformly in $z \in \mathbb{C} \setminus \mathbb{R}$, $|z| \leqslant C$, as an operator from $\mathcal{H}^s$ to $\mathcal{H}^{s+k}$. □

It turns out that the weighted resolvent of G not only is bounded, but also has some Hölder regularity which will be used in the proof of Theorem 1.

Proposition 11. *Assume $d \geqslant 3$ and odd, $k \in \mathbb{N}$ with $k \geqslant 2$, $\alpha \in]0, 1[$ and $\rho > k + \alpha + 1$ ($\rho > k + \alpha$ for $d = 3$). Let $\kappa = k$ ($\kappa = k - 1/2$ for $d = 3$). Then, for all $s \in \mathbb{R}$ and $C, \varepsilon > 0$, we have*

$$\left\| \langle x \rangle^{-\kappa-\alpha-\varepsilon} (R^k(z) - R^k(z')) \langle x \rangle^{-\kappa-\alpha-\varepsilon} \right\|_{\mathcal{L}(\mathcal{H}^s, \mathcal{H}^{s+k})} \lesssim |z - z'|^{\alpha},$$

uniformly in $|z|, |z'| \leqslant C$ with $\operatorname{Im} z \cdot \operatorname{Im} z' > 0$.

Corollary 12. *Proposition 11 and a classical argument imply that the powers of the weighted resolvent have a limit on the real axis. More precisely, under the assumptions of*

Proposition 11, the limits

$$\langle x\rangle^{-\kappa-\varepsilon} R^j(\lambda \pm i0)\langle x\rangle^{-\kappa-\varepsilon} = \lim_{\delta\downarrow 0}\langle x\rangle^{-\kappa-\varepsilon} R^j(\lambda \pm i\delta)\langle x\rangle^{-\kappa-\varepsilon},$$

exist for $\lambda \in]-C, C[$ *and* $j \in \{1, \dots, k\}$. *Moreover, for* $j \in \{1, \dots, k-1\}$,

$$\langle x\rangle^{-\kappa-\varepsilon} R^{j+1}(\lambda \pm i0)\langle x\rangle^{-\kappa-\varepsilon} = j^{-1}\partial_\lambda \langle x\rangle^{-\kappa-\varepsilon} R^j(\lambda \pm i0)\langle x\rangle^{-\kappa-\varepsilon},$$

and

$$\left\| \langle x\rangle^{-\kappa-\alpha-\varepsilon}(R^k(\lambda \pm i0) - R^k(\lambda' \pm i0))\langle x\rangle^{-\kappa-\alpha-\varepsilon}\right\|_{\mathcal{L}(\mathcal{H}^s,\mathcal{H}^{s+k})} \lesssim |\lambda-\lambda'|^\alpha,$$

uniformly in $\lambda, \lambda' \in]-C, C[$.

Proof of Proposition 11. For $j \geqslant 2$, Proposition 4 (for $d \geqslant 5$ and odd) and Proposition 7 (for $d = 3$) yield

$$\left\| \langle x\rangle^{-\kappa-\varepsilon} R_0^j(z)\langle x\rangle^{-\kappa-\varepsilon}\right\|_{\mathcal{L}(\mathcal{H}^s,\mathcal{H}^{s+j})} \lesssim 1, \tag{4.16}$$

and

$$\left\| \langle x\rangle^{-\kappa-1-\varepsilon} R_0^{j+1}(z)\langle x\rangle^{-\kappa-1-\varepsilon}\right\|_{\mathcal{L}(\mathcal{H}^s,\mathcal{H}^{s+j})} \lesssim 1,$$

with

$$\kappa = \begin{cases} j & \text{for } d \geqslant 5, \\ j - 1/2 & \text{for } d = 3. \end{cases}$$

Since $\partial_z R_0^j(z) = jR_0^{j+1}(z)$, the last estimate gives

$$\left\| \langle x\rangle^{-\kappa-1-\varepsilon}(R_0^j(z) - R_0^j(z'))\langle x\rangle^{-\kappa-1-\varepsilon}\right\|_{\mathcal{L}(\mathcal{H}^s,\mathcal{H}^{s+j})} \lesssim |z-z'|. \tag{4.17}$$

Thus, an argument of interpolation between (4.16) and (4.17) implies

$$\left\| \langle x\rangle^{-\kappa-\alpha-\varepsilon}(R_0^j(z) - R_0^j(z'))\langle x\rangle^{-\kappa-\alpha-\varepsilon}\right\|_{\mathcal{L}(\mathcal{H}^s,\mathcal{H}^{s+j})} \lesssim |z-z'|^\alpha. \tag{4.18}$$

Since $k \geqslant 2$, Theorem 3 yields

$$\left\| \langle x\rangle^{-1-\varepsilon} R(z)\langle x\rangle^{-1-\varepsilon}\right\|_{\mathcal{L}(\mathcal{H}^s,\mathcal{H}^{s+1})} \lesssim 1, \quad \left\| \langle x\rangle^{-2-\varepsilon} R^2(z)\langle x\rangle^{-2-\varepsilon}\right\|_{\mathcal{L}(\mathcal{H}^s,\mathcal{H}^{s+2})} \lesssim 1,$$

for all $d \geqslant 3$ and odd. This gives

$$\left\| \langle x\rangle^{-1-\alpha-\varepsilon}(R(z) - R(z'))\langle x\rangle^{-1-\alpha-\varepsilon}\right\|_{\mathcal{L}(\mathcal{H}^s,\mathcal{H}^{s+1})} \lesssim |z-z'|^\alpha. \tag{4.19}$$

For the improvement in dimension $d = 3$, we need estimates in the spirit of Lemma 9. Since $k \geqslant 2$, Theorem 3 yields

$$\left\| \langle x\rangle^{-3/2-\varepsilon} R^2(z)\langle x\rangle^{-3/2-\varepsilon}\right\|_{\mathcal{L}(\mathcal{H}^s,\mathcal{H}^{s+2})} \lesssim 1,$$

and then

$$\left\| \langle x\rangle^{-3/2-\varepsilon} \begin{pmatrix} 0 & 0 \\ \tilde{\partial} & 0 \end{pmatrix} R^2(z)\langle x\rangle^{-3/2-\varepsilon}\right\|_{\mathcal{L}(\mathcal{H}^s,\mathcal{H}^{s+2})} \lesssim 1.$$

https://doi.org/10.1017/S1474748012000801 Published online by Cambridge University Press

Interpolating with Lemma 9, we get

$$\left\| \langle x\rangle^{-1/2-\alpha-\varepsilon} \begin{pmatrix} 0 & 0 \\ \widetilde{\partial} & 0 \end{pmatrix} (R(z)-R(z'))\langle x\rangle^{-1/2-\alpha-\varepsilon} \right\|_{\mathcal{L}(\mathcal{H}^s,\mathcal{H}^{s+1})} \lesssim |z-z'|^{\alpha}. \tag{4.20}$$

In the same way,

$$\left\| \langle x\rangle^{-1/2-\alpha-\varepsilon} (R(z)-R(z')) \begin{pmatrix} 0 & 0 \\ \widetilde{\partial}^* & 0 \end{pmatrix} \langle x\rangle^{-1/2-\alpha-\varepsilon} \right\|_{\mathcal{L}(\mathcal{H}^s,\mathcal{H}^{s+1})} \lesssim |z-z'|^{\alpha}. \tag{4.21}$$

By Lemma 10, we can write

$$R^k(z)-R^k(z') = \sum_{\text{finite}} \sum_{j=0}^{n} M_0(z)V\cdots V(M_j(z)-M_j(z'))V\cdots M_n(z'). \tag{4.22}$$

Now, the rest of the proof is similar to that of Theorem 3 and we omit the details. The difference is that we add an additional $\langle x\rangle^{-\alpha}$ on the left and on the right of $(M_j(z)-M_j(z'))$ and that we use (4.18)–(4.21) instead of Propositions 4, 7, Lemma 9 and (4.10) to estimate this term. □

5. Proof of the main theorem

In this part, we deduce Theorem 1 from the smoothness of the weighted resolvent obtained in § 4. First note that for $\mu < 2$, this theorem follows from [5]. Indeed, under the assumption $\rho > 0$, it is proved in [5, Theorem 1 i)] that

$$\left\| \langle x\rangle^{1-d} e^{-itG}\chi(G)\langle x\rangle^{1-d} \right\| \lesssim \langle t\rangle^{1-d+\varepsilon}.$$

On the other hand, [3, Lemma 4.2] gives $\|\langle x\rangle^{-1/2-\varepsilon}u\| \lesssim \|P^{1/4}u\|$, and then

$$\left\| \langle x\rangle^{-1/2-\varepsilon} e^{-itG}\chi(G)\langle x\rangle^{-1/2-\varepsilon} \right\| \lesssim 1.$$

Interpolating the two previous estimates yields Theorem 1 for $\mu < 2$.

In the sequel, we assume that $\mu \geqslant 2$. Thus, we can apply Corollary 12 with $k = \lfloor \mu \rfloor + 1 \geqslant 3$. Using Stone's formula and integrating by parts, we get

$$\begin{aligned} &\langle x\rangle^{-\mu-1-\varepsilon} e^{-itG}\chi(G)\langle x\rangle^{-\mu-1-\varepsilon} \\ &= \frac{1}{2\pi i}\int \chi(\lambda)e^{-it\lambda}\langle x\rangle^{-\mu-1-\varepsilon}(R(\lambda+i0)-R(\lambda-i0))\langle x\rangle^{-\mu-1-\varepsilon}d\lambda \\ &= \frac{1}{2\pi i}\frac{1}{(it)^{\lfloor\mu\rfloor}}\sum_{\pm}\sum_{j=1}^{\lfloor\mu\rfloor+1} \pm C_{\lfloor\mu\rfloor}^{j-1}\int \chi_j(\lambda)e^{-it\lambda}\langle x\rangle^{-\mu-1-\varepsilon}R^j(\lambda\pm i0)\langle x\rangle^{-\mu-1-\varepsilon}d\lambda, \end{aligned} \tag{5.1}$$

with $\chi_j = \partial^{\lfloor\mu\rfloor+1-j}\chi \in C_0^{\infty}(\mathbb{R})$. Moreover, mimicking the proof of [9, Theorem 25], we obtain, for all $1 \leqslant j \leqslant \lfloor\mu\rfloor+1$,

$$A := \int \chi_j(\lambda)e^{-it\lambda}\langle x\rangle^{-\mu-1-\varepsilon}R^j(\lambda\pm i0)\langle x\rangle^{-\mu-1-\varepsilon}d\lambda$$

https://doi.org/10.1017/S1474748012000801 Published online by Cambridge University Press

$$
\begin{aligned}
&= \int \chi_j(\lambda + \pi/t) e^{-it(\lambda+\pi/t)} \langle x\rangle^{-\mu-1-\varepsilon} R^j(\lambda + \pi/t \pm i0) \langle x\rangle^{-\mu-1-\varepsilon} d\lambda \\
&= -\int \chi_j(\lambda + \pi/t) e^{-it\lambda} \langle x\rangle^{-\mu-1-\varepsilon} R^j(\lambda + \pi/t \pm i0) \langle x\rangle^{-\mu-1-\varepsilon} d\lambda \\
&= -A + \int (\chi_j(\lambda) - \chi_j(\lambda + \pi/t)) e^{-it\lambda} \langle x\rangle^{-\mu-1-\varepsilon} R^j(\lambda \pm i0) \langle x\rangle^{-\mu-1-\varepsilon} d\lambda \\
&\quad + \int \chi_j(\lambda + \pi/t) e^{-it\lambda} \langle x\rangle^{-\mu-1-\varepsilon} (R^j(\lambda \pm i0) - R^j(\lambda + \pi/t \pm i0)) \langle x\rangle^{-\mu-1-\varepsilon} d\lambda \\
&= \mathcal{O}(t^{\lfloor \mu \rfloor - \mu}),
\end{aligned} \tag{5.2}
$$

since $\lambda \mapsto \langle x\rangle^{-\mu-1-\varepsilon} R^j(\lambda \pm i0) \langle x\rangle^{-\mu-1-\varepsilon}$ (and of course $\lambda \mapsto \chi_j(\lambda)$) is Hölder continuous of order $\mu - \lfloor \mu \rfloor$ thanks to Corollary 12. Then, (5.1) and (5.2) imply part (i) of Theorem 1 in the case $d \geqslant 3$ and odd. This argument gives also the improvement in dimension $d = 3$. In order to prove part (ii) of the theorem, it is sufficient to use the high energy estimates of [5, Theorem 5 ii)] as well as the formula (1.6).

References

1. J. Aguilar and J. M. Combes, A class of analytic perturbations for one-body Schrödinger Hamiltonians, *Comm. Math. Phys.* **22** (1971), 269–279.
2. S. Alinhac, *Geometric analysis of hyperbolic differential equations: an introduction*, London Mathematical Society Lecture Note Series, Volume 374 (Cambridge University Press, 2010).
3. J.-F. Bony and D. Häfner, The semilinear wave equation on asymptotically Euclidean manifolds, *Comm. Partial Differential Equations* **35** (2010), 23–67.
4. J.-F. Bony and D. Häfner, Low frequency resolvent estimates for long range perturbations of the Euclidean Laplacian, *Math. Res. Lett.* **17**(2) (2010), 301–306.
5. J.-F. Bony and D. Häfner, Local energy decay for several evolution equations on asymptotically Euclidean manifolds, *Ann. Sci. Éc. Norm. Supér.* **45**(2) (2012), 311–335.
6. J.-M. Bouclet, Low frequency estimates and local energy decay for asymptotically euclidean Laplacians, *Comm. Partial Differential Equations* **36** (2011), 1239–1286.
7. N. Burq, Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel, *Acta Math.* **180**(1) (1998), 1–29.
8. C. L. Dolph, J. B. McLeod and D. Thoe, The analytic continuation of the resolvent kernel and scattering operator associated with the Schroedinger operator, *J. Math. Anal. Appl.* **16** (1966), 311–332.
9. J. Fröhlich, M. Griesemer and I. M. Sigal, Spectral theory for the standard model of non-relativistic QED, *Comm. Math. Phys.* **283**(3) (2008), 613–646.
10. C. Gérard and I. M. Sigal, Space-time picture of semiclassical resonances, *Comm. Math. Phys.* **145**(2) (1992), 281–328.
11. L. Hörmander, *The analysis of linear partial differential operators. I*, Grundlehren der Mathematischen Wissenschaften, Volume 256 (Springer, 1983).
12. L. Hörmander, *Lectures on nonlinear hyperbolic differential equations*, Mathématiques & Applications, Volume 26 (Springer, 1997).
13. W. Hunziker, Distortion analyticity and molecular resonance curves, *Ann. Inst. H. Poincaré Phys. Théor.* **45**(4) (1986), 339–358.
14. A. Jensen and T. Kato, Spectral properties of Schrödinger operators and time-decay of the wave functions, *Duke Math. J.* **46**(3) (1979), 583–611.

https://doi.org/10.1017/S1474748012000801 Published online by Cambridge University Press

15. Y. Kian, On the meromorphic continuation of the resolvent for the wave equation with time-periodic perturbations and applications, preprint, 2011 (arXiv:1103.2530).

16. S. Klainerman, A commuting vectorfields approach to Strichartz-type inequalities and applications to quasi-linear wave equations, *Int. Math. Res. Not.*(5) (2001), 221–274.

17. P. Lax and R. Phillips, *Scattering theory*, second ed., Pure and Applied Mathematics, Volume 26 (Academic Press Inc., 1989), With appendices by C. Morawetz and G. Schmidt.

18. J. Ralston, Solutions of the wave equation with localized energy, *Comm. Pure Appl. Math.* **22** (1969), 807–823.

19. J. Rauch, Local decay of scattering solutions to Schrödinger's equation, *Comm. Math. Phys.* **61**(2) (1978), 149–168.

20. A. Sá Barreto and M. Zworski, Existence of resonances in three dimensions, *Comm. Math. Phys.* **173**(2) (1995), 401–415.

21. J. Sjöstrand, Lectures on resonances, preprint on http://math.u-bourgogne.fr/IMB/sjostrand/index.html, 2007, 1–169.

22. J. Sjöstrand and M. Zworski, Complex scaling and the distribution of scattering poles, *J. Amer. Math. Soc.* **4**(4) (1991), 729–769.

23. S.-H. Tang and M. Zworski, Resonance expansions of scattered waves, *Comm. Pure Appl. Math.* **53**(10) (2000), 1305–1334.

24. B. Vaĭnberg, *Asymptotic methods in equations of mathematical physics.* (Gordon & Breach Science Publishers, 1989).