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Bounded minimalisation and bounded counting in argument-bounded idc's

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We define and investigate a number of small inductively defined classes (idc's), *à la* Gregorczyk, that are based on *argument-bounded* initial functions and the *bounded minimalisation* and *bounded counting* schemata. We establish equivalences between these and other classes in the literature, with an emphasis on minimalism. We also obtain characterisations of the classes in terms of well-known fragments of first-order predicate logic.

1. Introduction

This paper is based on the talk given by the author at the *TAMC 2008 conference*, which was held between 25–29 April in Xi'an, China, and emerges from some investigations into very small sub-recursive classes, the so-called *inductively defined classes (idc's)*. At the talk I presented results from the pre-proceedings paper *A Characterisation of the Relations Definable in Presburger Arithmetic* (Barra 2008) in addition to various results which were either unfinished at deadline or omitted due to space limitations.

The original motivation for this work was to discard all *non-argument-bounded* (definitions follow) functions from the set of initial functions of idc's, and then compare the resulting classes to otherwise similar classes. This approach of *banning all growth* has proved successful in the past, and has repeatedly yielded surprising and enlightening results – see, for example, Jones (1999; 2001); Kristiansen and Voda (2003a) and Kristiansen and Voda (2003b) for work with functionals of higher types and imperative programming languages, respectively; Kristiansen and Barra (2005) for work with function algebras and the λ-calculus; and Kristiansen (2005; 2006).

Recently, argument-bounded idc's have found a new use in the context of *detour degrees à la* Kristiansen and Voda (see Kristiansen and Voda (2008)), and I expect that some of the results presented here will be very useful in the further development of that theory.

However, to find the source of inspiration for the specific work in this paper, we must look further back in time. A. Grzegorczyk's seminal paper *Some classes of recursive functions* (Grzegorczyk 1953) was the source of great inspiration to many researchers

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during the decades following its publication. A significant contribution was made in Harrow's Ph.D. dissertation (Harrow 1973), the findings of which were later summarised and enhanced in Harrow (1975). Harrow answered several questions, which had been originally posed by Grzegorczyk, with regard to the interchangeability of the *bounded primitive recursion* and *bounded minimalisation* schemata in the small Grzegorczyk-classes $\mathcal{E}^i$ ($i = 0, 1, 2$). Another result from Harrow (1973) is that $G^1_\star$ (this and other classes mentioned below are defined later in the paper) is identical to the set of predicates $\mathfrak{PrA}_\star$ – those subsets of $\mathbb{N}^k$ that are definable by a formula in the language of *Presburger Arithmetic*.

We will show that the classes G^i contain redundancies in the sense that the *increasing* functions 'S', '$+$' and '$\times$' can be substituted with their argument-bounded inverses *predecessor*, *(truncated) difference* and *integer division and remainder*, without affecting the induced relational classes $G^i_\star$. That is, the growth provided by, for example, addition, in the restricted framework of composition and bounded minimalisation, does not contribute to the number of computable predicates. In fact, we show that the *quantifier-free fragment* of Presburger Arithmetic may be captured in a much weaker system: essentially, only truncated difference and composition is necessary.

Next, we investigate the seemingly stronger schemata of *bounded counting* and *bounded n-ary counting*, and show that an analogous result holds. Indeed, with bounded counting, not only are the increasing functions substitutable for their argument-bounded inverses – in some cases they are completely redundant.

2. Notation and basic definitions

Unless otherwise specified, a *function* in this paper means a function $f : \mathbb{N}^k \to \mathbb{N}$, and the *arity* of f is then k.

A function is *argument-bounded*† (a.b.) if, for some $c_f \in \mathbb{N}$ we have‡ $f(\vec{x}) \leqslant \mathsf{max}(\vec{x}, c_f)$ for all $\vec{x} \in \mathbb{N}^k$.

We say that f has *top-index* i if $f(\vec{x}) \leqslant \mathsf{max}(x_i, c_f)$. If $c_f = 0$, we say that f is *strictly argument-bounded*, and that i is a *strict top-index*.

Whenever a symbol 'x' occurs under an arrow, for example, '$\vec{x}$', we will usually not mention the length of the list explicitly but adopt the convention that $\vec{x}$ has length k and $\vec{g}$ has length ℓ.

The *bounded* f, denoted $\hat{f}$, is the $(k+1)$-ary function $\hat{f}(\vec{x}, b) \stackrel{\text{def}}{=} \mathsf{min}(f(\vec{x}), b)$. These bounded versions, in particular the bounded versions of increasing functions like $\hat{\mathsf{S}}(x, b) = \mathsf{min}(x + 1, b)$ (*bounded successor*), will be of major importance for the ensuing developments. The *predecessor*, denoted P, is defined by $\mathsf{P}(x) \stackrel{\text{def}}{=} \mathsf{max}(x - 1, 0)$. The *case function*,

† In Barra (2008) we employed the term *non-increasing* rather than *argument-bounded*, but, following the advice of one of the referees, we have changed it here.

‡ For readers familiar with $\mathcal{E}^0$, the bound that holds for f in G^0 and $\mathcal{E}^0$ is $f(\vec{x}) \leqslant \mathsf{max}(\vec{x}) + c_f$ – note the distinction. The latter bound is sometimes referred to as 0-*boundedness*.

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denoted C, and the *(truncated) difference function*, denoted $\dot{-}$, are defined by

$$\mathsf{C}(x,y,z) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } z = 0 \\ y & \text{otherwise} \end{cases}$$

$$x \dot{-} y \stackrel{\text{def}}{=} \max(x-y,0) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } x \leqslant y \\ x-y & \text{if } x > y. \end{cases}$$

When we use the symbol '$-$' without the dot in an expression or formula, we mean the usual minus on $\mathbb{Z}$.

Let $\phi(x,y,n,r) \stackrel{\text{def}}{\Leftrightarrow} 0 \leqslant r < y \wedge x = ny + r$. The *remainder function* and *integer division function*, denoted rem and $\lfloor \frac{\cdot}{\cdot} \rfloor$, respectively, are defined by

$$\left\lfloor \frac{x}{y} \right\rfloor \stackrel{\text{def}}{=} \begin{cases} x & \text{if } y = 0 \\ n & \text{if } \phi(x,y,n,r) \end{cases}$$

$$\mathsf{rem}(x,y) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } y = 0 \\ r & \text{if } \phi(x,y,n,r). \end{cases}$$

The choice of $\lfloor \frac{x}{0} \rfloor \stackrel{\text{def}}{=} x$ makes the functions total on $\mathbb{N}^2$. We also have $\lfloor \frac{x}{y} \rfloor y + \mathsf{rem}(x,y) = x$ for all x and y.

$\mathscr{I}$ is the set of all *projections* $\mathsf{I}_i^k(\vec{x}) = x_i$, and $\mathscr{N}$ is the set of all *constant functions* $\mathsf{c}(x) = c$ for all $c \in \mathbb{N}$.

A *relation* is a subset R of $\mathbb{N}^k$ for some k. Relations are also, and interchangeably, called *predicates*. Sets of predicates are usually sub-scripted with a '$\star$'. For a set $\mathscr{F}_\star$ of relations, we say that *$\mathscr{F}_\star$ is Boolean*, when $\mathscr{F}_\star$ is closed under finite intersections and complements.

When $R = f^{-1}(0) \stackrel{\text{def}}{=} \{\vec{x} \in \mathbb{N}^k \mid f(\vec{x}) = 0\}$, the function f is referred to as *a characteristic function for R*, and is denoted χ_R. However, this function is not unique, and we use χ_R^c to denote the unique characteristic function for R satisfying $\chi_R^c(x) = c$ when $x \notin R$.

Let $\mathscr{F}$ be a set of functions. $\mathscr{F}_\star$ denotes the set of *relations of $\mathscr{F}$*, in other words, those subsets $R \subseteq \mathbb{N}^k$ with $\chi_R \in \mathscr{F}$: formally, $\mathscr{F}_\star \stackrel{\text{def}}{=} \{f^{-1}(0) \mid f \in \mathscr{F}\}$.

The *graph of f*, denoted Γ_f, is the relation $\{(\vec{x},y) \in \mathbb{N}^{k+1} \mid f(\vec{x}) = y\}$. We overload Γ_f by using it to denote its characteristic function as well.

When A is a set, we use $|A|$ to denote the *cardinality of A*, and $f{\restriction}_A$ to denote the *restriction of the function f to the set A*.

3. Schemata, idc's and overview

In this section we set the stage for the later development. We first introduce our most fundamental notions, *schemata*, or *operations*, for defining new functions from previously defined functions, and *inductively defined classes*, which form our computational model. Next we give a pointer to the kind of results to expect†. We will then prove a few useful lemmas.

† Since quite a lot of notation will be needed, we will not give an overview of the results until the final section since at this stage they would either be too cumbersome to state or incomprehensible.

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In the main sections following this one, we will consider each of our classes in turn by first introducing it and then presenting our main results with respect to it, together with the corresponding proofs.

3.1. *Schemata*

In this paper we are concerned with the following *schemata*:

Definition 1 (Composition and bounded minimalisation). We say that f is generated from h and $\vec{g}$ by *composition* when $f(\vec{x}) \stackrel{\text{def}}{=} h(g_1(\vec{x}), \dots, g_\ell(\vec{x}))$. The schema of composition will be denoted by COMP, and we also write $h \circ \vec{g}$ for the generated function.

We say that f is generated from g_1 and g_2 by *bounded minimalisation*[‡] when $f(\vec{x}, y)$ equals the least $z \leqslant y$ satisfying the equation $g_1(\vec{x}, z) = g_2(\vec{x}, z)$, if such exists, and y otherwise. The schema is denoted BMIN, and we write $\mu_{z \leqslant y}[g_1(\vec{x}, z) = g_2(\vec{x}, z)]$, or simply $\mu_{z \leqslant y}[g_1, g_2]$, for the generated function.

In Barra (2008) we considered a slightly different version of the bounded minimalisation schema. There, a failed search would return 0 rather than y. In most contexts the two versions are equivalent, but in some very restricted settings this may not be the case. We shall have more to say about this in Section 4.3.

The third and fourth schema we will study are actually families of schemata: for each $n \geqslant 1$ we will define the schemata of *n-ary bounded counting* and *argument-bounded n-ary bounded counting*.

Definition 2 (n-ary bounded counting). The function f is generated from $g_1(\vec{x}, \vec{z})$ and $g_2(\vec{x}, \vec{z})$, where $|\vec{z}| = n$, by *n-ary bounded counting* when

$$f(\vec{x}, y) \stackrel{\text{def}}{=} |\{\vec{z} \mid \max(\vec{z}) < y \wedge g_1(\vec{x}, \vec{z}) = g_1(\vec{x}, \vec{z})\}| .$$

The schema is denoted BCOUNT^n, and we write $\sharp_{\vec{z}<y}[g_1(\vec{x}, z) = g_2(\vec{x}, z)]$, or $\sharp_{\vec{z}<y}[g_1, g_2]$, for the generated function.

The function f is generated from $g_1(\vec{x}, \vec{z})$ and $g_2(\vec{x}, \vec{z})$, where $|\vec{z}| = n$, by *argument-bounded n-ary bounded counting* when

$$f(\vec{x}, y) \stackrel{\text{def}}{=} \max\left(y, |\{\vec{z} \mid \max(\vec{z}) < y \wedge g_1(\vec{x}, \vec{z}) = g_1(\vec{x}, \vec{z})\}|\right) .$$

The schema is denoted $\overline{\text{BCOUNT}}^n$, and we write $\overline{\sharp}_{\vec{z}<y}[g_1, g_2]$ for the generated function.

3.2. *Inductively defined classes*

A fundamental notion underlying the work presented here is that of an *inductively defined class of functions* (an *idc*). An idc is generated from a set $\mathscr{X}$, whose contents are called the *initial*, *primitive* or *basic* functions, as the least class containing $\mathscr{X}$ and closed under the *schemata*, *functionals* or *operations* of some set OP of functionals. We write $[\mathscr{X}\,;\,\text{OP}]$ for this set[†].

[‡] Bounded minimalisation is also known as *bounded search* or *limited minimum*.
[†] This notation is adopted from Clote (1996), where an idc is called a *function algebra*.

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We will *always* assume that our classes contain projections and constants, and that they are are closed under composition, and we will omit $\mathscr{I} \cup \mathscr{N}$ and COMP from our notation. Thus, for example, $[\{\mathsf{C}\}\ ;\ \text{BMIN}]$ abbreviates $[\mathscr{I} \cup \mathscr{N} \cup \{\mathsf{C}\}\ ;\ \text{COMP}, \text{BMIN}]$. This simply means that our idc's are closed under so-called *explicit definitions*.

Remark 3. The careful reader will notice that none of the proofs presented depend on the presence of any constants other than 0 and 1. Consequently, all results regarding the induced relational classes are valid under the assumption that $\mathscr{I} \cup \{0, 1\}$ is substituted for $\mathscr{I} \cup \mathscr{N}$, and all other results will also hold by minor modifications of some definitions. Informally, all results will hold *almost everywhere*. Whether we choose $\mathscr{N}$ or $\{0, 1\}$ is largely a matter of style, and my preference is for the straightforwardness afforded by having $\mathscr{N}$ available over the (unnecessary) exception-handling incurred by the more minimalist $\{0, 1\}$.

3.3. *Results*

The results of this paper fall roughly into one of three categories:

(1) Results on the strength of our schemata when working on their own: in other words, we investigate idc's of the type $[\ ;\ \text{OP}]$ where OP is bounded minimalisation or bounded counting. We also investigate the strength of the truncated difference function in the context $[\{\dot{-}\}\ ;\ ]$.
(2) Results concerned with the *redundancy of basic functions* when the (only) operator is counting, and the *equivalence of basic functions with argument-bounded inverses* when the operator is minimalisation.
(3) Descriptive-complexity-like characterisations of the induced relational classes. That is, we characterise them by well-known fragments of first-order logics. We will introduce these fragments and suitable notation on the fly.

3.4. *Preliminary lemmas*

Lemma 4. Let $\mathscr{G}$ be any idc closed under BMIN, and let $f(\vec{x})$ be any function satisfying *either* $f(\vec{x}) \leqslant x_i$ *or* $f(\vec{x}) \leqslant c$. Then:

$$\Gamma_f \in \mathscr{G} \;\Rightarrow\; f \in \mathscr{G}.$$

Proof. The fact that $\Gamma_f \in \mathscr{G}$ means that for some function, $\Gamma_f \in \mathscr{G}$ satisfies $\Gamma_f(\vec{x}, y) = 0 \;\Leftrightarrow\; f(\vec{x}) = y$. By hypothesis, we either have $f(\vec{x}) \leqslant x_i$ for fixed i, in which case $f(\vec{x}) = \mu_{z \leqslant x_i}[\Gamma_f(\vec{x}, z) = 0] \in \mathscr{G}$, or we have $f(\vec{x}) \leqslant c$ for fixed c, in which case $f(\vec{x}) = \mu_{z \leqslant c}[\Gamma_f(\vec{x}, z) = 0] \in \mathscr{G}$. □

Lemma 5. Let $\mathscr{G}$ and $\mathscr{G}'$ be arbitrary idc's. Assume $\chi_= \in \mathscr{G}$, and that for every $f \in \mathscr{G}'$, we have $\Gamma_f \in \mathscr{G}$. Then $\mathscr{G}'_\star \subseteq \mathscr{G}_\star$.

Proof. Let $f' \in \mathscr{G}'$ be arbitrary. We must show that the predicate $R \overset{\text{def}}{=} (f')^{-1}(0)$ is also the pre-image of some function $f \in \mathscr{G}$. By hypothesis, $\chi_=, \Gamma_{f'} \in \mathscr{G}$, and thus, so is the

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function $f \stackrel{\text{def}}{=} \chi_=(\Gamma_{f'}(\vec{x},0),0)$. The fact that

$$f(\vec{x}) = 0 \Leftrightarrow \chi_=(\Gamma_{f'}(\vec{x},0),0) = 0 \Leftrightarrow \Gamma_{f'}(\vec{x},0) = 0 \Leftrightarrow f'(\vec{x}) = 0$$

then allows us to conclude the proof. □

Lemma 6. Let $h, \vec{g}$ be argument-bounded. Then so are the functions $h \circ \vec{g}$, $\mu_{z \leqslant y}[g_1, g_2]$ and $\overline{\sharp}_{\vec{z}<y}[g_1, g_2]$. Hence, if all $f \in \mathscr{X}$ are a.b., then all $f \in [\mathscr{X}\ ;\ \text{OP}]$ are a.b., when OP is any of COMP, BMIN or $\overline{\text{BCOUNT}}^n$.

Proof. By definition, for some $c_h, c_1, \ldots, c_\ell$, we have

$$h \circ \vec{g}(\vec{x}) \leqslant \max(g_1(\vec{x}), \ldots, g_\ell(\vec{x}), c_h) \leqslant \max(\vec{x}, \max(\vec{c}, c_h)),$$

which proves the case of composition. The two remaining cases are trivial since

$$\mu_{z \leqslant y}[g_1, g_2],\ \overline{\sharp}_{\vec{z}<y}[g_1, g_2] \leqslant y$$

by definition. □

4. The class $\mathscr{F}^\mu$

The first class we will consider is the class $\mathscr{F}^\mu \stackrel{\text{def}}{=} [\ ;\ \text{BMIN}]$. Note that there are no initial functions except projections and constants.

4.1. *Bootstrapping with* BMIN

The following section represents a recurrent theme of this paper: *bootstrapping*, or, establishing the existence of various functions in our small idc's.

Proposition 7.

(i) $\chi_=,\ \chi_{=n} \in \mathscr{F}^\mu$.
(ii) $\min \in \mathscr{F}^\mu$.
(iii) $\mathscr{F}^\mu_\star$ is Boolean.
(iv) $\mathscr{F}^\mu_\star$ is closed under $\exists_{z \leqslant y}$-type quantifiers.
(v) $\chi_<, \chi_\leqslant, \chi_\neq \in \mathscr{F}^\mu$.

Proof. Recall that χ^1_R denotes the characteristic function of R, which is 1 on the complement.

(i) Set

$$f(x_1, x_2, y) = \mu_{z \leqslant y}\left[\mathsf{I}^3_1(x_1, x_2, z) = \mathsf{I}^3_2(x_1, x_2, z)\right].$$

Then

$$\chi^c_=(x_1, x_2) = f(x_1, x_2, c) = \begin{cases} 0 & \text{if } x_1 = x_2 \\ c & \text{otherwise,} \end{cases}$$

which, since $\chi_{=n}(x) = \chi^c_=(x, \mathsf{n})$, gives the result.

(ii) Clearly,

$$\min(x, y) = \mu_{z \leqslant y}[x = z].$$

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Hence, if $\chi_R \in \mathscr{F}$, then so is $\chi^1_R(\vec{x}) = \mathsf{min}(\chi_R(\vec{x}), 1)$, and, finally,

$$\chi^c_R = \mu_{z \leqslant c}[\chi_R(\vec{x}) = 0].$$

(iii) We have

$$\chi^c_{R \wedge S} = \chi^c_= \circ \left(\chi^1_R, \chi^2_S\right) \quad \text{and} \quad \chi^c_{\neg R} = \chi^c_= \circ \left(\chi^1_R, 1\right).$$

Hence $\chi^c_{\neq} \in \mathscr{F}$.

(iv) This part is especially easy to show since bounded minimalisation is tailor-made for the purpose:

$$\exists z \leqslant y \big(R(\vec{x}, z)\big) \Leftrightarrow \chi_R(\vec{x}, \mu_{z \leqslant y}[\chi_R(\vec{x}, z) = 0]) = 0.$$

(v) This part follows because

$$x < y \Leftrightarrow \neg\exists z < y\big(x = z\big)$$

and $\mathscr{F}^\mu$ is Boolean by (iii). □

We can also do some argument-bounded basic arithmetic, as in the following proposition.

Proposition 8. $\hat{\mathsf{S}}, \mathsf{P} \in \mathscr{F}^\mu$.

Proof. Observe that $\hat{\mathsf{S}}(x, b) \stackrel{\text{def}}{=} \mathsf{min}(x+1, b) = \mu_{z \leqslant b}[\chi_<(x, z) = 0]$. Next, set

$$\mathsf{P}'(x, y) = \mu_{z \leqslant y}[\hat{\mathsf{S}}(z+1, x) = x].$$

Then $\mathsf{P}(x) = \mathsf{P}'(x, x) = \mu_{z \leqslant x}[\hat{\mathsf{S}}(z+1, x) = x]$. □

Proposition 9. $\hat{\mathsf{C}} \in \mathscr{F}^\mu$.

Proof. Since $\hat{\mathsf{C}}(x, y, z, b)$ has strict top-index b, by Lemma 4 it is sufficient to show that the graph of $\hat{\mathsf{C}}$ belongs in $\mathscr{F}^\mu$. But

$$\hat{\mathsf{C}}(x, y, z, b) = u \Leftrightarrow \bigvee \begin{cases} z = 0 \ \wedge \ x \leqslant b \ \wedge \ u = x \\ z \neq 0 \ \wedge \ y \leqslant b \ \wedge \ u = y \\ u = b\,, \end{cases}$$

and the formula to the right is a $\mathscr{F}^\mu_\star$-formula by Proposition 7. □

4.2. *$\mathscr{F}^\mu$ versus the classes G^0 and $\mathscr{PL}$*

Let $G^0 \stackrel{\text{def}}{=} \big[\{\mathsf{P}, \mathsf{S}\}\ ;\ \text{BMIN}^0\big]$, where the superscript '0' to 'BMIN' indicates that the schema we have in mind is the variant of bounded minimalisation mentioned above, which is a class that was originally defined by Grzegorczyk in Grzegorczyk (1953). Hence G^0 is the smallest *Grzegorczyk-class* $\mathscr{E}^0$ with minimalisation substituted for *primitive recursion*. Grzegorczyk posed the problem of whether the inclusion $G^0 \subseteq \mathscr{E}^0$ was proper or not, and the question remained open for some twenty years until K. Harrow answered the question in the negative by proving the following theorem.

Theorem 10 (Harrow 1975). $G^0 = \mathscr{E}^0 \cap \mathscr{PL}$.

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In the title of this section, $\mathscr{PL}$ refers to the set of *piecewise linear functions* – see Definition 11.

Part of my motivation for this research was to find classes of argument-bounded functions that characterise previously studied classes. The reader will have noticed that $\mathscr{F}^{\mu}$ is 'G^0 without successor', and that all $f \in \mathscr{F}^{\mu}$ are argument-bounded (Lemma 6). Our question is thus: *how do G^0 and $\mathscr{F}^{\mu}$ compare?*

Definition 11 (Piecewise linear). A function is *piecewise linear* if it may be written in the form

$$f(\vec{x}) = y \;\Leftrightarrow\; \bigvee_{1\leqslant i\leqslant \ell} (y = L_{3i}(\vec{x}) \;\wedge\; L_{3i+1} \leqslant L_{3i+2})$$

for some $\ell \in \mathbb{N}$, where each sub-scripted 'L' is either a constant or of the form $x_j \dot{-} c$ or $x_j + c$.

Note that there are finitely many clauses†.

Lemma 12. $f \in \mathscr{PL} \;\Rightarrow\; \Gamma_f \in \mathscr{F}^{\mu}$.

Proof. We have already shown that $\chi_{\leqslant}$, $\mathsf{P} \in \mathscr{F}^{\mu}$. By nesting the predecessor function, we have that the function $x \dot{-} c$ is in $\mathscr{F}^{\mu}$ for all (fixed) $c \in \mathbb{N}$. Observe next that for $x, y \in \mathbb{N}$, we have $x \leqslant y + c \;\Leftrightarrow\; x \dot{-} c \leqslant y$, since $x < c$ implies both $x \dot{-} c = 0 \leqslant y$ and $x < c + y$, and because $c \leqslant x$ implies $x \dot{-} c = x - c$. We also have the function $\hat{\mathsf{S}} \in \mathscr{F}^{\mu}$, so, by nesting, we obtain $\mathsf{min}(x + c, y) \in \mathscr{F}^{\mu}$. Furthermore, $x + c \leqslant y \;\Leftrightarrow\; \mathsf{min}(x + (c-1), y) \neq \mathsf{min}(x + c, y)$.

The above means we can decide any predicate of the form $L_{3i+1} \leqslant L_{3i+2}$. Let f be represented by $L_{3i} \;\Leftrightarrow\; L_{3i+1} \leqslant L_{3i+2}$ for $i \leqslant \ell$. Clearly,

$$f(\vec{x}) = y \;\Leftrightarrow\; \bigvee_{1\leqslant i\leqslant \ell} (y = L_{3i} \;\wedge\; L_{3i+1} \leqslant L_{3i+2}),$$

which, by the above, is an $\mathscr{F}^{\mu}_{\star}$-predicate. □

Theorem 13. $\mathscr{F}^{\mu}_{\star} = G^0_{\star} = \mathscr{PL}_{\star}$ and $\mathscr{F}^{\mu} \subsetneq G^0$.

Proof. Since $\mathscr{F}^{\mu} \subseteq G^0 \subseteq \mathscr{PL}$, we need only show that $\mathscr{PL}_{\star} \subseteq \mathscr{F}^{\mu}_{\star}$. Since $\chi_{=} \in \mathscr{F}^{\mu}$ by Proposition 7, the theorem now follows from Lemmas 5 and 12. The fact that $\mathscr{F}^{\mu} \subsetneq G^0$ as function classes is clear since $\mathsf{S} \in G^0$, but $\mathsf{S}(x) > x$ is not a.b. □

4.3. *A remark on the failed-search-value of* BMIN

As we mentioned earlier, in Barra (2008) we employed a $\mu_{z\leqslant y}$-schema that returned the value '0' upon a failed search rather than the value 'y'. During this discussion, we will use $\mu^0_{\leqslant y}$ and $\mu^y_{\leqslant y}$, respectively, to denote these variants. We also let $\mathscr{F}^{\mu^y}$ and $\mathscr{F}^{\mu^0}$ have the obvious meaning, and define $\mathscr{C}^{\mu^0} \stackrel{\text{def}}{=} [\{\hat{\mathsf{C}}\}\,;\mu^0_{\leqslant y}]$.

† For example, $f(x_1, x_2) = \begin{cases} 5 & \text{if } x_1 \leqslant x_2 \wedge 3 \leqslant x_1 \\ x_2 + 4 & \text{otherwise} \end{cases}$ is a member of $\mathscr{PL}$, since the 'otherwise clause' can be split into the two clauses $x_2 + 1 \leqslant x_1$ and $x_1 \leqslant 2$.

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Now, it is quite easy to see that $\mathscr{F}^{\mu^0} \subseteq \mathscr{F}^{\mu^y}$ since

$$\mu^0_{z\leqslant y}[g_1, g_2] = \hat{\mathsf{C}}(\mu^y_{z\leqslant y}[g_1, g_2], 0, \chi_=(g_1(\vec{x}, \mu^y_{z\leqslant y}[g_1, g_2], g_2(\vec{x}, \mu^y_{z\leqslant y}[g_1, g_2]), y)$$

and that the function on the right-hand side belongs in $\mathscr{F}^{\mu^y}$, since it involves composing functions, which has already been shown to belong to $\mathscr{F}^{\mu^y}$.

By verifying that $\hat{\mathsf{C}}(1, 0, x, 1) = 1 \dot{-} x$ and that $\hat{\mathsf{C}}(0, \hat{\mathsf{C}}(0, 1, y, 1), x, 1)$ is zero exactly when either x or y is zero, we conclude that $\mathscr{C}^{\mu^0}$ *is Boolean*. Also, the fact that

$$\mu^0_{z\leqslant y}[z = x] = \begin{cases} x & \text{if } x \leqslant y \\ 0 & \text{otherwise} \end{cases}$$

implies

$$\chi_{\leqslant}(x, y) = \hat{\mathsf{C}}\left(0, 1 \dot{-} \mu^0_{z\leqslant y}[z = x], x, 1\right) \in \mathscr{C}^{\mu^0}.$$

Since $\mathscr{C}^{\mu^0}$ is Boolean, we also have characteristic functions for the other standard order-predicates. Since

$$\mu^y_{z\leqslant y}[g_1, g_2] = \hat{\mathsf{C}}(\mu^0_{z\leqslant y}[g_1, g_2], y, \chi_=(g_1(\vec{x}, \mu^0_{z\leqslant y}[g_1, g_2]), g_2(\vec{x}, \mu^0_{z\leqslant y}[g_1, g_2]), y),$$

we also see that $\mathscr{F}^{\mu^y} = \mathscr{C}^{\mu^0}$. This means that we have the following theorem.

Theorem 14. $\mathscr{F}^{\mu^0} \subseteq \mathscr{F}^{\mu^y} = \mathscr{C}^{\mu^0}$.

The status of the inclusion above remains open. I conjectured in Barra (2008) that it is proper, but a proof is lacking. The class $\mathscr{F}^{\mu^0}$ seems to be very ill-behaved in the sense that if $f(\vec{x}) \in \mathscr{F}^{\mu^y}$, then there is some function $f'(\vec{x}, \vec{y}) \in \mathscr{F}^{\mu^0}$ and c such that $f'(\vec{x}, \vec{y}) = f(\vec{x})$ when $\mathsf{max}(\vec{x}) + c < \mathsf{min}(\vec{y})$, but otherwise the function degenerates.

5. The class $\mathscr{D}$

In this section we study the class $\mathscr{D} \stackrel{\text{def}}{=} [\{\dot{-}\}\ ;\ ]$. Note that composition is the *sole* closure operation of $\mathscr{D}$.

5.1. *Bootstrapping with* $\dot{-}$

The lemma below is our starting point for showing that the class $\mathscr{D}$ is surprisingly powerful.

Lemma 15 (bounded addition). The function $\mathsf{min}(x + y, z)$ belongs to $\mathscr{D}$.

Proof. Set $f(x, y, z) \stackrel{\text{def}}{=} z \dot{-} ((z \dot{-} x) \dot{-} y)$. If $x + y \geqslant z$, then $(z \dot{-} x) \dot{-} y = 0$, which yields $z \dot{-} ((z \dot{-} x) \dot{-} y) = z - 0 = z$ On the other hand, if $z > x + y \geqslant x$, then $z \dot{-} x = z - x > y > 0$. Hence $(z - x) \dot{-} y = ((z - x) - y) > 0$. But now $z > (z - (x + y)) > 0$, so

$$z \dot{-} ((z \dot{-} x) \dot{-} y) = z - (z - (x + y)) = z - z + (x + y) = x + y. \qquad \square$$

This function is the key to proving several properties of $\mathscr{D}$ and $\mathscr{D}_\star$.

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Proposition 16.

(i) $\mathsf{min}(x,y) \in \mathscr{D}$.
(ii) $\mathscr{D}_\star$ is Boolean.
(iii) $\chi_=, \chi_< \in \mathscr{D}$.
(iv) $\Gamma_{\mathsf{max}}, \Gamma_{\mathsf{C}} \in \mathscr{D}$.
(v) If A or $\mathbb{N}^k \setminus A$ is finite, then $A \in \mathscr{D}_\star$.

Proof.

(i) Clearly, $\mathsf{min}(x,y) = \mathsf{min}(x+0,y)$, which gives part (i).
(ii) Given χ_{R_1} and χ_{R_2}, we have

$$\chi_{R_1 \cap R_2} = \mathsf{min}(\chi_{R_1} + \chi_{R_2}, 1) \text{ and } \chi_{\mathbb{N}^k \setminus R} = 1 \dot{-} \chi_{R_1},$$

which gives part (ii).
(iii) This part follows from $x \dot{-} y = 0 \Leftrightarrow x \leqslant y$ and $y \dot{-} x = 0 \Leftrightarrow y \leqslant x$ in conjunction with part (ii).
(iv) This part follows from

$$\mathsf{max}(x,y) = z \Leftrightarrow \big(y \leqslant x \wedge z = x\big) \vee \big(x < y \wedge z = y\big)$$

and

$$\mathsf{C}(x,y,z) = w \Leftrightarrow \big(z = 0 \wedge w = x\big) \vee \big(z < 0 \wedge w = y\big).$$

(v) This part follows from parts (ii) and (iii) and the observation that any singleton $\{n\}$ have characteristic function in $\mathscr{D}$ by $\chi_{\{n\}}(x) = \chi_=(x, \mathsf{n})$. □

Armed with Proposition 16, we can prove the following lemma.

Lemma 17. Let $\mathbf{r} \in \{<,=\}$, and let $1 \leqslant j < k \in \mathbb{N}$ be arbitrary. Then, the following relations belong to $\mathscr{D}_\star$:

$$\sum_{i=1}^{j} x_i \;\mathbf{r}\; \sum_{i=j+1}^{k} x_i.$$

Proof. Observe first that the function $f(x,\vec{y}) = x \dot{-} \big(\sum_{i=1}^{k} y_i\big) \in \mathscr{D}$, by k consecutive applications of composition. Note also that when $R(\vec{x}) \in \mathscr{D}_\star$ and $\vec{f} \in \mathscr{D}$, the relation $S(\vec{y})$ defined by $S(\vec{y}) \stackrel{\text{def}}{\Leftrightarrow} R(f_1(\vec{y}), \ldots, f_k(\vec{y}))$ belongs to $\mathscr{D}_\star$.

We prove the lemma by induction on k:

Induction start: This is given by Proposition 16.
Induction step: Note that

$$\sum_{i=1}^{j} x_i = \sum_{i=j+1}^{k} x_i \Leftrightarrow \neg\left(\sum_{i=1}^{j} x_i < \sum_{i=j+1}^{k} x_i\right) \wedge \neg\left(\sum_{i=j+1}^{k} x_i < \sum_{i=1}^{j} x_i\right).$$

Hence, it is sufficient to perform the induction step when $\mathbf{r}$ is '$<$'. Moreover, as $\mathscr{D}_\star$ is Boolean, we may invoke the induction hypothesis for $\mathbf{r} \in \{\geqslant, \leqslant, <, >, =\}$.

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We proceed to the case $k+1$, with $2 \leqslant k$, and first prove the special cases of $j = k$. Clearly,

$$\sum_{i=1}^{k} x_i < x_{k+1} \Leftrightarrow 0 < x_{k+1} \dot{-} \left(\sum_{i=1}^{k} x_i\right).$$

This shows that $\sum_{i=1}^{k} x_i < x_{k+1} \in \mathscr{D}_\star$ by the initial remarks.
Next consider the general case $1 \leqslant j < k$.

$$\sum_{i=1}^{j} x_i < \left(\sum_{i=j+1}^{k} x_i\right) + x_{k+1} \Leftrightarrow$$

$$\left(\sum_{i=1}^{j} x_i < x_{k+1}\right) \vee \left(\underbrace{\left(x_{k+1} \leqslant \sum_{i=1}^{j} x_i\right)}_{\psi} \wedge \underbrace{\left(\left(\left(\sum_{i=1}^{j} x_i\right) \dot{-} x_{k+1}\right) < \sum_{i=j+1}^{k} x_i\right)}_{\phi}\right).$$

The conjunct marked ϕ above requires special attention. Consider the function

$$g_j(x_1, \ldots, x_j, x_{k+1}) = \left(\sum_{i=1}^{j} x_i\right) \dot{-} x_{k+1}.$$

Now g_j is *not* in $\mathscr{D}$, but we see that $g_j(\vec{x}, y)$ is equal to

$$(x_1 \dot{-} y) + (x_2 \dot{-} (y \dot{-} x_1)) + \cdots + (x_j \dot{-} (y \dot{-} x_1 \dot{-} x_2 \dot{-} \cdots \dot{-} x_{j-1})) \cdots)). \qquad (\dagger)$$

Furthermore, when ψ is true, we have $g_j(\vec{x}, y) = \left(\sum_i x_i\right) - y$. Importantly for us, the expression ($\dagger$) is a sum of j summands, each summand being a $\mathscr{D}$-function of the variables involved. Hence,

$$\psi \wedge \phi \Leftrightarrow \psi \wedge \underbrace{\left(\sum_{i=1}^{j} \left(x_i \dot{-} \left(x_{k+1} \dot{-} \sum_{\ell=1}^{i-1} x_\ell\right)\right) < \sum_{i=j+1}^{k} x_i\right)}_{\phi'},$$

which, since the ϕ' is in $\mathscr{D}_\star$ by the induction hypothesis, concludes the proof when **r** is '<'. □

5.2. *Presburger Arithmetic and* $\mathscr{D}_\star$

Let $\mathfrak{PrA}$ be the first-order language $\{0, \mathsf{S}, +, <, =\}$ with the intended structure $\mathfrak{N} \stackrel{\text{def}}{=} (\mathbb{N}, 0, \mathsf{S}, +, <)$ – the natural numbers with the usual order, successor and addition. Terms, (atomic) formulae and the numerals $\underline{m}$ are defined in the standard way. As usual, we use abbreviations extensively: for example, we use $t \leqslant s$ for $t = s \vee t < s$.

Many readers will no doubt have recognised $\mathfrak{PrA}$ as the language of *Presburger Arithmetic*, see, for example, Enderton (1972, page 188). It is known that the theory of $\mathfrak{N}$ is decidable (we will return to this point in Section 6), and that it does *not* admit quantifier elimination.

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We overload the symbol $\mathfrak{PrA}$ and use it to denote the set of $\mathfrak{PrA}$-formulae as well. For $\phi(\vec{x}) \in \mathfrak{PrA}$, we define $R_\phi \subseteq \mathbb{N}^k$ by

$$R_\phi \stackrel{\text{def}}{=} \left\{\vec{m} \in \mathbb{N}^k \mid \mathfrak{N} \models \phi(\underline{\vec{m}})\right\}$$

and set

$$\mathfrak{PrA}_\star \stackrel{\text{def}}{=} \{R_\phi \mid \phi \in \mathfrak{PrA}\}.$$

We use $\mathfrak{PrA}^{\text{qf}}$ to denote the set of *quantifier free* $\mathfrak{PrA}$-formulae and $\mathfrak{PrA}^{\Delta_0}$ for the Δ_0-formulae. $\mathfrak{PrA}_\star^{\text{qf}}$ and $\mathfrak{PrA}_\star^{\Delta_0}$ are defined as expected.

Any $\mathfrak{PrA}$-term t is clearly equivalent to some term $\left(\sum_{i\leqslant k} a_i x_i\right) + \underline{m}$, which is shorthand for

$$\underbrace{(x_1 + \cdots + x_1)}_{a_1-\text{times}} + \cdots + \underbrace{(x_k + \cdots + x_k)}_{a_k-\text{times}} + \underline{m}.$$

Thus, any atomic formula $\phi(\vec{x}) \in \mathfrak{PrA}$ is equivalent to a formula of the form†

$$\sum a_i x_i + \underline{a}\ \mathbf{r}\ \sum b_i y_i + \underline{b}. \qquad (\ddagger)$$

The main result of this section is given by the following theorem.

Theorem 18. $\mathfrak{PrA}_\star^{\text{qf}} = \mathscr{D}_\star$.

Lemma 17 provides us with most of the proof of $\mathfrak{PrA}_\star^{\text{qf}} \subseteq \mathscr{D}_\star$.

Proof of $\mathfrak{PrA}_\star^{\text{qf}} \subseteq \mathscr{D}_\star$. Let $R \in \mathfrak{PrA}_\star^{\text{qf}}$. Then, by definition, we have $R = R_\phi$ for some $\phi(\vec{x}) \in \mathfrak{PrA}^{\text{qf}}$. Since $\mathscr{D}_\star$ is Boolean, it is sufficient to prove the lemma for ϕ atomic.

Let $\phi(\vec{x})$ be atomic. Hence, it has the form specified in (‡) above. If we let R be the relation

$$R(\vec{x}, \vec{y}) \stackrel{\text{def}}{=} \sum a_i x_i + z\ \mathbf{r}\ \sum b_i y_i + w\,,$$

we have, by Lemma 17, that $\chi_R \in \mathscr{D}$. But then

$$\chi_R(\vec{n}, \vec{m}, \mathsf{a}, \mathsf{b}) = 0 \Leftrightarrow \sum a_i n_i + a\ \mathbf{r}\ \sum b_i m_i + b \Leftrightarrow \mathfrak{N} \models \phi(\underline{\vec{n}}, \underline{\vec{m}}),$$

so $\chi_R(\vec{x}, \vec{y}, \mathsf{a}, \mathsf{b}) = \chi_{R_\phi}$. Hence $\chi_{R_\phi} \in \mathscr{D}$. □

To facilitate the proof of the opposite inclusion, we first consider the language $\mathfrak{PrA}^{\dot{-}} \stackrel{\text{def}}{=} \mathfrak{PrA} \cup \{\dot{-}, -\}$, in other words, $\mathfrak{PrA}$ augmented with new function symbols '$\dot{-}$' and '$-$', and with intended model $\mathfrak{Z} \stackrel{\text{def}}{=} (\mathbb{Z}, 0, \mathsf{S}, +, -, \dot{-}, <)$. Note that $\dot{-}$ is well defined on all of $\mathbb{Z}^2$ by its original definition $\max(x - y, 0)$. We also note the fact that for $\phi \in \mathfrak{PrA}$ variable free, we have $\mathfrak{Z} \models \phi \Leftrightarrow \mathfrak{N} \models \phi$.

Also, to every function $f \in \mathscr{D}^k$, there is a $\mathfrak{PrA}^{\dot{-}}$-term $t_f(\vec{x})$ such that for all $\vec{n}, m \in \mathbb{N}$ we have $\mathfrak{Z} \models t_f(\underline{\vec{n}}) = \underline{m} \Leftrightarrow f(\vec{n}) = m$.

Lemma 19. Let $\phi(\vec{x}) \in \mathfrak{PrA}^{\dot{-}}$ be atomic. Then there is a $\phi'(\vec{x}) \in \mathfrak{PrA}^{\text{qf}}$ such that for all $\vec{n} \in \mathbb{N}$ we have $\mathfrak{N} \models \phi'(\underline{\vec{n}}) \Leftrightarrow \mathfrak{Z} \models \phi(\underline{\vec{n}})$.

† We continue to use $\mathbf{r}$ as a meta variable ranging over $\{<, =\}$, unless we specify otherwise.

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Proof. For a $\mathfrak{PrA}^{\dot{-}}$-term t, we define $\mathrm{rk}_{\dot{-}}(t)$ to be the number of occurrences of the symbol $\dot{-}$ in t. Note that when $\phi \in \mathfrak{PrA}^{\dot{-}}$ is atomic, it has the form $t_1 \mathbf{\ r\ } t_2$, and that $\mathrm{rk}_{\dot{-}}(t_1) + \mathrm{rk}_{\dot{-}}(t_2) = 0$ implies that either $\phi \in \mathfrak{PrA}^{\mathrm{qf}}$ or it is equivalent to $\phi' \in \mathfrak{PrA}^{\mathrm{qf}}$ by basic arithmetical considerations.

The proof is by induction on $\mathrm{rk}_{\dot{-}}(t_1) + \mathrm{rk}_{\dot{-}}(t_2)$:

Induction start: This is given by the comments above.

Induction step: Let $\mathrm{rk}_{\dot{-}}(t_1)+\mathrm{rk}_{\dot{-}}(t_2) = \ell+1$. At least one of the t_i must contain a sub-term s of the form $s_1 \dot{-} s_2$, and with $\mathrm{rk}_{\dot{-}}(s) = 1$, that is, s may be chosen to satisfy $\mathrm{rk}_{\dot{-}}(s_1) = \mathrm{rk}_{\dot{-}}(s_2) = 0$. Now consider the terms defined by $t_i^- \stackrel{\mathrm{def}}{=} t_i[s := s_1 - s_2]$ and $t_i^0 \stackrel{\mathrm{def}}{=} t_i[s := \underline{0}]$, where $t_i[s := s']$ denotes the result of substituting s' for all occurrences of the sub-term s. Note that $\mathrm{rk}_{\dot{-}}(t_i^-) = \mathrm{rk}_{\dot{-}}(t_i^0) < \mathrm{rk}_{\dot{-}}(t_i)$ for at least one i since we have removed *at least* one occurrence of $\dot{-}$ from one of the t_i in each construction. So we have $\mathrm{rk}_{\dot{-}}(t_1^-) + \mathrm{rk}_{\dot{-}}(t_2) = \mathrm{rk}_{\dot{-}}(t_1^0) + \mathrm{rk}_{\dot{-}}(t_2) \leqslant \ell$, and, moreover,

$$t_1(\vec{x}) \mathbf{\ r\ } t_2(\vec{x}) \Leftrightarrow \bigvee \begin{cases} s_1(\vec{x}) < s_2(\vec{x}) \wedge t_1^0(\vec{x}) \mathbf{\ r\ } t_2^0(\vec{x}) \\ s_2(\vec{x}) \leqslant s_1(\vec{x}) \wedge t_1^-(\vec{x}) \mathbf{\ r\ } t_2^-(\vec{x}). \end{cases}$$

By the induction hypothesis, the relation $t_1(\vec{x}) \mathbf{\ r\ } t_2(\vec{x})$ is in $\mathfrak{PrA}^{\mathrm{qf}}_\star$ since each disjunct above is a conjunction of atomic $\mathfrak{PrA}^{\dot{-}}$-terms of rank strictly less than $\ell + 1$. □

We are now ready to finish the proof of Theorem 18.

Proof of $\mathscr{D}_\star \subseteq \mathfrak{PrA}^{\mathrm{qf}}_\star$. Let $R \in \mathscr{D}^k_\star$. Then $R = f^{-1}(0)$ for some $f \in \mathscr{D}^k$. We now fix a $\mathfrak{PrA}^{\dot{-}}$-term t_f such that $\mathfrak{Z} \models t(\vec{x}) = y$ if and only if $f(\vec{x}) = y$ and apply Lemma 19 to obtain $\phi \in \mathfrak{PrA}^{\mathrm{qf}}$ satisfying

$$\mathfrak{N} \models \phi(\vec{x}, y) \Leftrightarrow \mathfrak{Z} \models t(\vec{x}) = y.$$

But then we also have

$$\vec{x} \in f^{-1}(0) \Leftrightarrow \mathfrak{N} \models \phi(\vec{x}, \underline{0}),$$

so $f^{-1}(0) = R_\phi \in \mathfrak{PrA}^{\mathrm{qf}}_\star$. □

Bearing in mind that no increasing functions are available in $\mathscr{D}$, and, perhaps even more strikingly, that composition is the *only* schema, the class $\mathscr{D}$ really delivers rather more than one would at first glance expect†.

It is also of interest that when only composition is available, none of the standard linear initial functions add anything to $\mathscr{D}_\star$. More precisely, we have the following theorem.

Theorem 20. $[\{\mathsf{max}, \mathsf{min}, \mathsf{C}, \mathsf{S}, \mathsf{P}, +, \dot{-}\}\ ;\ ]_\star = \mathscr{D}_\star$.

Proof. It is sufficient to show that

$$[\mathscr{I} \cup \mathscr{N} \cup \{\mathsf{C}, \dot{-}, +\}\ ;\ \mathrm{COMP}]_\star \subseteq \mathfrak{PrA}^{\mathrm{qf}}_\star,$$

† In contrast, for example, $[\{\mathsf{C}\}\ ;\ ]_\star$ consists essentially of $\varnothing$, $\{0\}$, $\mathbb{N} \setminus \{0\}$ and $\mathbb{N}$ (and their products), and most other familiar functions, like S, P or $+$, even fail to produce a Boolean set of relations.

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since $\mathsf{max}, \mathsf{min}, \mathsf{S}$ and P are clearly definable in this idc To this end, we augment the language $\mathfrak{PrA}$ with function symbols '$\dot{-}$', 'C' and '$-$', and define a $(\dot{-}, \mathsf{C})$-rank analogously to the '$\dot{-}$'-rank defined above.

The proof is now almost identical to the one given for Lemma 19, except that in the induction step we must also allow for the case where the minimal positive-rank-sub-term is $\mathsf{C}(s_0, s_1, s_2)$ rather than $s_1 \dot{-} s_2$:

Induction step:

— Case $s \equiv \mathsf{C}(s_0, s_1, s_2)$):
We have

$$t_1(\vec{x}) \,\mathbf{r}\, t_2(\vec{x}) \;\Leftrightarrow\; \bigvee \begin{cases} s_2(\vec{x}) = 0 \;\wedge\; t_1[s := s_0](\vec{x}) \,\mathbf{r}\, t_2[s := s_0](\vec{x}) \\ s_2(\vec{x}) > 0 \;\wedge\; t_1[s := s_1](\vec{x}) \,\mathbf{r}\, t_2[s := s_1](\vec{x}) \end{cases}$$

As before, the formula on the right-hand side has strictly smaller rank than the formula on the left-hand side, and we are done. We do not need to worry about '$+$' since it is already accommodated for in the language $\mathfrak{PrA}$. □

6. The class $\mathscr{D}^\mu$

In this section we turn our attention to the class $\mathscr{D}^\mu \stackrel{\text{def}}{=} [\{\dot{-}\} \,;\, \text{BMIN}]$: that is, the class produced by merging $\mathscr{D}$ and $\mathscr{F}^\mu$. Obviously, $\mathscr{D}, \mathscr{F}^\mu \subseteq \mathscr{D}^\mu$, so all of the required *bootstrapping* has already been carried out in the previous sections. Indeed, the first thing we will do is to prove a proposition that *limits* the possible candidates for membership in $\mathscr{D}^\mu$. We will prove a slightly stronger version than we actually need for this section by including $\overline{\text{BCOUNT}}^n$, because this stronger result will be useful later in the paper.

Proposition 21 (top-index). Let $f \in [\{\dot{-}\} \,;\, \text{BMIN}, \overline{\text{BCOUNT}}^n]$. Then f has a top-index. Furthermore, if $f(\mathbb{N}^k)$ is infinite, the top-index is strict.

Proof. We use induction on f:

Induction start: This is $f \in \mathscr{I} \cup \mathscr{N} \cup \{\dot{-}\}$, and the result is obvious since constants are bounded by themselves, $\mathsf{I}_i^k(\vec{x}) \leqslant x_i$ and $x \dot{-} y \leqslant x$.

Induction step: We need to analyse three cases:

— Case $f = h \circ \vec{g}$:
We have that f is bounded by the x_i such that i is the top-index of g_j where j is the top-index of h. The c_f may be fixed to $\mathsf{max}(c_h, \vec{c})$. A prerequisite for $f(\mathbb{N}^k)$ to be infinite is that $h(\mathbb{N}^\ell)$ is infinite. Hence, the j is a strict top-index for h by the induction hypothesis. Also, since $h(\vec{y}) \leqslant y_j$, again if f is to have infinite image, g_j must have infinite image, so a second appeal to the induction hypothesis implies that i is a strict top-index for g_j. But then i is a strict top-index for f.

— Case $f = \mu_{z \leqslant y}[g_1, g_2]$:
f has strict top-index y by definition.

— Case $f = \overline{\sharp}_{\vec{z} < y}[g_1, g_2]$:
As in the previous case, f has strict top-index y by definition. □

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6.1. *Presburger Arithmetic and* $\mathscr{D}^{\mu}_{\star}$

We define $\mathfrak{PrA}^{\Delta_V}$ as the set of $\mathfrak{PrA}$-formulae, where all quantifiers occur in the context $\exists_z(z \leqslant y \ \wedge \ \phi)$, which we abbreviate by $\exists_{z \leqslant y}\phi$. The '$V$' in '$\Delta_V$' is meant to reflect the requirement that the quantified variable must be bounded by a *variable*, and not a general *term*, which is also known as *finite* or *linear* quantification.

Lemma 22. If $f \in \mathscr{D}^{\mu}$, then $\Gamma_f \in \mathfrak{PrA}^{\Delta_V}_{\star}$.

Proof. We use induction on f:

Induction start: This is given by $\mathscr{D}_{\star} = \mathfrak{PrA}^{\mathrm{qf}}_{\star}$.
Induction step:

— Case $f = h \circ \vec{g}$:
Let i_j (for $1 \leqslant j \leqslant m$) be the strict top-index of g_j if $g_j(\mathbb{N}^k)$ is infinite, and $\max g_j(\mathbb{N}^k)$ otherwise. We fix $\mathfrak{PrA}^{\Delta_V}$-formulae $\phi_h(\vec{z}, y)$ and $\phi_j(\vec{x}, z_j)$ representing the graphs of h and the g_j's, respectively. Then

$$(h \circ \vec{g})(\vec{x}) = y \ \Leftrightarrow \ \exists_{z_1 \leqslant t_1} \cdots \exists_{z_m \leqslant t_m} \left(\left(\bigwedge_{1 \leqslant i \leqslant m} \phi(\vec{x}, z_i) \right) \wedge \phi_h(\vec{z}, y) \right),$$

where $t_j = x_{i_j}$ for unbounded g_j, and i_j otherwise. Clearly, quantification bounded by a constant is merely a finite disjunction, and, as such, even $\mathfrak{PrA}^{\mathrm{qf}}_{\star}$ is closed under $\exists_{x \leqslant \underline{c}}$ type quantifiers.

— Case $f = \mu_{z \leqslant y}[g_1, g_2]$:
Let ϕ_i represent $g_i(\vec{x}) = y$, and define

$$\phi(\vec{x}, y, w) \ \Leftrightarrow \ \bigvee \left\{ \begin{array}{l} w = y \ \wedge \ \neg\exists_{z \leqslant y}(\phi_1(\vec{x}, z) \wedge \phi_2(\vec{x}, z)) \\ \exists_{z \leqslant y}\left(\bigwedge \left\{ \begin{array}{l} w = z \wedge \phi_1(\vec{x}, z) \wedge \phi_2(\vec{x}, z) \\ \forall_{u \leqslant z}(\neg\phi_1(\vec{x}, u) \vee \neg\phi_2(\vec{x}, u) \vee u \neq z) \end{array} \right. \right) \end{array} \right.$$

Then $\phi \in \mathfrak{PrA}^{\Delta_V}$, and $\mathfrak{N} \models \phi(\vec{x}, y, w) \ \Leftrightarrow \ f(\vec{x}, y) = w$ as required. □

Corollary 23. $\mathscr{D}^{\mu}_{\star} = \mathfrak{PrA}^{\Delta_V}_{\star}$.

Proof. The $\mathscr{D}^{\mu}_{\star} \subseteq \mathfrak{PrA}^{\Delta_V}_{\star}$-direction follows immediately from Lemma 22. The opposite direction follows from the definition of $\mathfrak{PrA}^{\Delta_V}_{\star}$ since $\mathfrak{PrA}^{\mathrm{qf}}_{\star} \subseteq \mathscr{D}_{\star} \subseteq \mathscr{D}^{\mu}_{\star}$ and the fact that just like $\mathscr{F}^{\mu}_{\star}$, the class $\mathscr{D}^{\mu}_{\star}$ is closed under linear quantification. □

Theorem 24. $\mathscr{D}^{\mu}_{\star} = \mathfrak{PrA}_{\star}$.

Theorem 24 follows from the original proof that the theory of $\mathfrak{PrA}$ is decidable. In 1930, Mojżes Presburger demonstrated this now well-known fact in Presburger (1930)[†] by proving that the theory of the intended structure $\mathfrak{N}^{\equiv} = (\mathbb{N}, 0, \mathsf{S}, +, <, \equiv_2, \equiv_3, \ldots)$ for the

[†] I consulted D. Jaquette's excellent translation – see Presburger and Jaquette (1991).

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language $\mathfrak{PrA}^{\equiv} \stackrel{\text{def}}{=} \mathfrak{PrA} \cup \{\equiv_2, \equiv_3, \ldots\}$ does admit quantifier elimination. In particular, for any $\mathfrak{PrA}$-formula ϕ, there is a $\phi' \in \mathfrak{PrA}^{\equiv,\text{qf}}$ such that

$$\mathfrak{N}^{\equiv} \models \phi' \;\Leftrightarrow\; \mathfrak{N}^{\equiv} \models \phi \;\Leftrightarrow\; \mathfrak{N} \models \phi.$$

The relation 'congruence modulo λ' is definable by

$$x \equiv_\lambda y \;\Leftrightarrow\; \exists_{u \leqslant x}\big(x = y + \underbrace{u + \cdots + u}_{\lambda\text{-terms}}\big) \vee \exists_{u \leqslant y}\big(y = x + \underbrace{u + \cdots + u}_{\lambda\text{-terms}}\big).$$

Since the right-hand side is clearly $\mathfrak{PrA}_\star^{\Delta_V}$, by Corollary 23, these predicates belong to $\mathscr{D}_\star^\mu$ for all $\lambda \in \mathbb{N}$. But we can do even better.

Lemma 25. Let $p, q \in \mathbb{N}[\vec{x}]$ be linear polynomials, and let $\lambda \in \mathbb{N}$. Then, the relation $p(\vec{x}) \equiv_\lambda q(\vec{x})$ is in $\mathscr{D}_\star^\mu$.

Proof. First note that the unary function $\mathsf{rem}_\lambda(x) \stackrel{\text{def}}{=} \mathsf{rem}(x, \lambda) \in \mathscr{D}^\mu$ for each fixed $\lambda \in \mathbb{N}$ since $\mathsf{rem}(x, \lambda) = \mu_{z \leqslant \lambda}[\chi_{\equiv_\lambda}(x, z) = 0]$.

Write $p(\vec{x}) = \sum_{i=1}^{k_p} a_i^p x_i + m_p$. Set $A_p = \sum_{i=1}^{k_p} a_i$, and note that A_p is independent of $\vec{x}$. Also, since $\mathsf{rem}_\lambda(x) < \lambda$, we have

$$s^p(\vec{x}) = \sum_{i=1}^{k_p} a_i \,\mathsf{rem}_\lambda(x_i) < A_p \lambda.$$

We can argue similarly for $q(\vec{x})$. So $p(\vec{x}) \equiv_\lambda q(\vec{x}) \;\Leftrightarrow\; s^p(\vec{x}) \equiv_\lambda s^q(\vec{x})$.

We still need to show that the relation $s^p(\vec{x}) \equiv_\lambda s^q(\vec{x})$ is in $\mathscr{D}_\star^\mu$. Since bounded addition is in $\mathscr{D}$, we also have $\hat{s}^p(\vec{x}, z) \stackrel{\text{def}}{=} \mathsf{min}(s^p(\vec{x}), z)$ in $\mathscr{D}^\mu$. But then, for $A = \mathsf{max}(\lambda A_p, \lambda A_q)$, we have $p(\vec{x}) \equiv_\lambda q(\vec{x}) \;\Leftrightarrow\; \hat{s}^p(\vec{x}, A) \equiv_\lambda \hat{s}^q(\vec{x}, A)$. □

Because Lemma 25 yields a decision procedure for all atomic $\mathfrak{PrA}^{\equiv}$-formulae within $\mathscr{D}_\star^\mu$, and since $\mathscr{D}_\star^\mu$ is Boolean, we have $\mathfrak{PrA}_\star^{\equiv,\text{qf}} \subseteq \mathscr{D}_\star^\mu$. So, using Presburger's original results, we have

$$\mathscr{D}_\star^\mu \supseteq \mathfrak{PrA}_\star^{\equiv,\text{qf}} = \mathfrak{PrA}_\star \supseteq \mathfrak{PrA}_\star^{\Delta_V} = \mathscr{D}_\star^\mu,$$

which constitutes a proof of Theorem 24. Note that this also proves the following corollary.

Corollary 26. $\mathfrak{PrA}_\star^{\Delta_V} = \mathfrak{PrA}_\star$.

6.2. *$\mathscr{D}^\mu$ versus the class G^1*

The class G^1 is defined by $G^1 \stackrel{\text{def}}{=} \big[\{0, \mathsf{S}, +\}\ ;\ \textsc{bmin}^0\big]$. K. Harrow proved the following theorem.

Theorem 27 (Harrow 1975). $\mathfrak{PrA}_\star = G_\star^1 \subsetneq \mathscr{E}^1$.

Here $\mathscr{E}^1$ is the familiar Grzegorczyk-class – see the discussion in Section 4.2. As with G^0 and $\mathscr{F}^\mu$, we obtain the following theorem on the induced relational classes.

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Theorem 28. $\mathscr{D}^{\mu}_{\star} = G^1_{\star}$ and $\mathscr{D}^{\mu} \subsetneq G^1$.

Proof. The only part we have not already proved is $\mathscr{D}^{\mu} \subsetneq G^1$, and, again, $\mathsf{S} \in G^1 \setminus D^{\mu}$ suffices as proof for this: S is not argument-bounded, while all functions of $\mathscr{D}^{\mu}$ are. □

7. The class $\mathscr{D}\mathscr{D}^{\mu}$

We let $\mathfrak{PA} \stackrel{\text{def}}{=} \mathfrak{PrA} \cup \{\times\}$ be the language of *Peano Arithmetic*, and use the notational conventions of the previous sections.

The relational class $\mathfrak{PA}^{\Delta_0}_{\star}$ can thus be described as those predicates that are definable by a Δ_0-formula in the language $\mathfrak{PA}$ of Peano Arithmetic. This class has been the focus of intense and varied studies over the years, and is known by many names. Perhaps the most widely used are $\Delta_0^{\mathbb{N}}$ or $\mathfrak{R}$. The class is known to equal the set of *Rudimentary relations*, and the class of *Constructive Arithmetic* (both defined by Smullyan – see Smullyan (1961)), and has many other characterisations.

The definition of the class G^2, using our notation, is

$$G^2 \stackrel{\text{def}}{=} \left[\{0, \mathsf{S}, +, \times\}\ ;\ \text{BMIN}^0\right].$$

Theorem 29 (Harrow). $\mathfrak{PA}^{\Delta_0}_{\star} = G^2_{\star}$ and $\mathfrak{PA}^{\Delta_0}_{\star} = \mathfrak{PA}^{\Delta_V}_{\star}$.

Of course, the fact that $\mathfrak{PA}^{\Delta_0}_{\star} \subsetneq \mathfrak{PA}_{\star}$ is well known.

Consider the class $\mathscr{D}\mathscr{D}^{\mu} \stackrel{\text{def}}{=} \left[\{\dot{-}, \lfloor \frac{\cdot}{\cdot} \rfloor\}\ ;\ \text{BMIN}\right]$: informally, $\mathscr{D}\mathscr{D}^{\mu}$ is obtained by adding *integer division* to $\mathscr{D}^{\mu}$. Now, since we can think of division being in some way to multiplication what difference is to addition, a natural question to ask is whether the inclusion of this new function makes $\mathscr{D}\mathscr{D}^{\mu}$ to G^2 what $\mathscr{D}^{\mu}$ is to G^1. The answer is yes.

We need the following proposition.

Proposition 30. $\mathsf{rem} \in \mathscr{D}\mathscr{D}^{\mu}$.

Proof. We have already seen that $\hat{\mathsf{C}}$ and $\hat{\mathsf{S}}$ belong to sub-classes of $\mathscr{D}\mathscr{D}^{\mu}$. Now

$$\mathsf{rem}(x, y) = \hat{\mathsf{C}}\left(0, \min\left(\mu_{z \leqslant x}\left[\left\lfloor \frac{x \dot{-} z}{y} \right\rfloor \neq \left\lfloor \frac{x}{y} \right\rfloor\right] + 1, x\right), y, x\right).$$

Then, because $\mathsf{rem}(x, y) \leqslant y$, we have $\mathsf{rem} \in \mathscr{D}\mathscr{D}^{\mu}$ by Lemma 4. □

Theorem 31. $\mathscr{D}\mathscr{D}^{\mu}_{\star} = \mathfrak{PA}^{\Delta_0}_{\star} = G^2_{\star}$.

Proof. Note that

$$xy = z \Leftrightarrow \left(y > 0 \wedge x = \frac{z}{y}\right) \vee \phi \Leftrightarrow \left(y > 0 \wedge \mathsf{rem}(z, y) = 0 \wedge \left\lfloor \frac{z}{y} \right\rfloor = x\right) \vee \phi,$$

where $\phi(x, y, z)$ is, for example, $y = 0 \wedge z = 0$, so the graph of multiplication is computable in $\mathscr{D}\mathscr{D}^{\mu}_{\star}$.

Next, the class $\mathscr{C}\mathscr{A}$ of *Constructive Arithmetic predicates*, as introduced by Smullyan (see Smullyan (1961)), is defined as the closure of the graphs of addition and multiplication under explicit transformations, Boolean operations and quantification bounded by a

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variable. Hence $\mathscr{CA} \subseteq \mathscr{DD}^{\mu}_{\star}$. Next, since Harrow proved in Harrow (1975) that $G^2_{\star} = \mathfrak{PA}^{\Delta_0}_{\star}$, and since $\mathscr{DD}^{\mu} \subseteq G^2$ trivially, we clearly have $\mathscr{DD}^{\mu}_{\star} \subseteq \mathfrak{PA}^{\Delta_0}_{\star}$.

This is sufficient, since results in Bennett (1962), Wrathall (1978) and Lipton (1979) imply the non-trivial identities $\mathscr{CA} \overset{\text{Ben}}{=} \mathscr{RUD} \overset{\text{Wra}}{=} \mathsf{LH} \overset{\text{Lip}}{=} \mathfrak{PA}^{\Delta_0}_{\star}$, which completes the proof of Theorem 31. (Here $\mathscr{RUD}$ are Smullyan's *Rudimentary relations* and LH is the *Linear Hierarchy*.) □

8. The classes $\mathscr{F}^{n,\sharp}$ and $\overline{\mathscr{F}}^{n,\sharp}$

In this section we study classes

$$\mathscr{F}^{n,\sharp} \overset{\text{def}}{=} \left[\ ;\ \text{BCOUNT}^{n+1}\right] \quad \text{and} \quad \overline{\mathscr{F}}^{n,\sharp} \overset{\text{def}}{=} \left[\ ;\ \overline{\text{BCOUNT}}^{n+1}\right].$$

First note that for the case $n = 0$ above, the two schemata BCOUNT^1 and $\overline{\text{BCOUNT}}^1$ are the same. When we count solutions $\sharp_{z<y}[g_1(\vec{x}, z) = g_2(\vec{x}, z)]$, the answer is already bounded by $y = |\{0, \ldots, y-1\}|$. Accordingly, we will identify the two, and simply write $\mathscr{F}^{\sharp}$ for $\overline{\mathscr{F}}^{0,\sharp} = \mathscr{F}^{0,\sharp}$.

Now, observe that the schema BCOUNT^{n+1} for $n \geqslant 1$ does not necessarily generate argument-bounded functions from argument-bounded functions, and as such, the resulting classes $\mathscr{F}^{n,\mu}$ do not really fit the profile of the other classes introduced in this paper. However, as we shall see in Sections 8.3 and 8.4, we actually have $\overline{\mathscr{F}}^{1,\sharp} = \mathscr{F}^{1,\sharp} = \overline{\mathscr{F}}^{2,\sharp} = \mathscr{F}^{2,\sharp} = \cdots$.

Also, BCOUNT will prove to be the strongest schema considered so far. So-called *counting-quantifiers* and *counting-operations* have been subjected to extensive studies in the literature, see, for example, Paris and Wilkie (1985), Schweikardt (2005), Esbelin (1994) and Esbelin and More (1998).

We will first take a closer look at $\mathscr{F}^{\sharp}$.

8.1. *Bootstrapping with* BCOUNT^1

Let f be any function, and let $R = f^{-1}(0)$, so f is a characteristic function for R. Consider the function

$$F(\vec{x}, y) \overset{\text{def}}{=} \sharp_{z<y}[f(\vec{x}) = 0] = \begin{cases} y & \text{if } \vec{x} \in R \\ 0 & \text{if } \vec{x} \notin R. \end{cases}$$

In particular, $F(\vec{x}, \mathsf{c})$ is the specific characteristic function for $\neg R$, which we denoted using $\chi^c_{\neg R}$ earlier. Thus, by applying the above construction, twice if necessary, when a relation R belongs to $\mathscr{F}^{\sharp}_{\star}$, we may form χ^c_R and $\chi^c_{\neg R}$ for all $c > 0$. Hence:

(i) *$\mathscr{F}^{\sharp}_{\star}$ is closed under negation.*

Moreover, and not surprisingly, when $R \in \mathscr{F}^{\sharp}_{\star}$ we may count the number of $z < y$ for which $R(\vec{x}, z)$. We will use $\chi^{\sharp}_R$ to denote the functions that satisfies

$$\chi^{\sharp}_R(\vec{x}, y) \overset{\text{def}}{=} |\{z < y \mid R(\vec{x}, z)\}|.$$

Clearly, $\chi^{\sharp}_R(\vec{x}, y) = \sharp_{z<y}[\chi_R(\vec{x}, z) = 0]$, so:

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(ii) $R \in \mathscr{F}^{\sharp}_{\star} \Rightarrow \chi^{\sharp}_{R} \in \mathscr{F}^{\sharp}$.

Next, we have that

$$\sharp_{z<y}[x = z] = \begin{cases} 0 & \text{if } x \geqslant y \\ 1 & \text{if } x < y \end{cases}$$

$$\sharp_{z<1}[x_1 = x_2] = \begin{cases} 0 & \text{if } x_1 \neq x_2 \\ 1 & \text{if } x_1 = x_2 \end{cases}$$

belong to $\mathscr{F}^{\sharp}$ by applying BCOUNT to suitable projections, so we conclude that:

(iii) $\chi_{\leqslant}, \chi_{\neq} \in \mathscr{F}^{\sharp}$.

Closure under negation yields $\chi_{<}, \chi_{=} \in \mathscr{F}^{\sharp}$. Since $\chi_{R \wedge S} = \chi_{=} \circ (\chi^1_R, \chi^2_S)$ we also have closure under logical 'and', so we have:

(iv) $\mathscr{F}^{\sharp}_{\star}$ *is Boolean.*

Consider the function $\chi^{\sharp}_{\leqslant}$. By (i)–(iii), we have $\chi^{\sharp}_{\leqslant} \in \mathscr{F}^{\sharp}$. This function counts the set D, defined by

$$D \stackrel{\text{def}}{=} \{z < y \mid x \leqslant z\} = \{z \mid x \leqslant z < y\}.$$

Since $y \leqslant x \leqslant z < y$ is contradictory, when $y \leqslant x$, we have $|D| = |\varnothing| = 0 = y \dot{-} x$. On the other hand, if $y > x$, say $x + c$, then

$$|D| = |\{x, x+1, x+2, \ldots, x+(c-1)\}| = c = y \dot{-} x.$$

We conclude that:

(v) $\chi^{\sharp}_{\leqslant} = \dot{-} \in \mathscr{F}^{\sharp}$, and thus $\mathscr{D} \subseteq \mathscr{F}^{\sharp}$.

We can now prove closure of $\mathscr{F}^{\sharp}$ under BMIN. Recall that the schema we consider returns y upon a failed search. Set $f = 1 \dot{-} \left(1 \dot{-} \sharp_{v<z}[g_1, g_2]\right)$. We need to verify that f is 0–1-valued and satisfies

$$f(\vec{x}, z) = 1 \Leftrightarrow \exists_{v<z}\,(g_1(\vec{x}, v) = g_2(\vec{x}, v)).$$

Set

$$h(\vec{x}, y) \stackrel{\text{def}}{=} \sharp_{z<y}[f(\vec{x}, z) = 1].$$

Given $\vec{x}$, we assume v_0 is the least element of

$$\{v \mid g_1(\vec{x}, v) = g_2(\vec{x}, v)\}.$$

Now, because of the way we defined f, the function h counts the set

$$M_y \stackrel{\text{def}}{=} \{z < y \mid v_0 < z\} = \{z \mid v_0 < z < y\}.$$

We have

$$M_y = \begin{cases} \varnothing & \text{if } y \leqslant v_0 + 1 \\ \{v_0 + 1, \ldots, v_0 + (n+1)\} & \text{if } y = v_0 + (n+2) \end{cases}$$

$$\Rightarrow$$

$$|M_y| = \begin{cases} 0 & \text{if } y \leqslant v_0 + 1 \\ n + 1 & \text{if } y = v_0 + (n+2). \end{cases}$$

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The above implies that

$$h'(\vec{x},y) \stackrel{\text{def}}{=} y \dot{-} h(\vec{x},y) = \left\{ \begin{array}{ll} y & \text{if } y \leqslant v_0 + 1 \\ v_0 + 1 & \text{if } y = v_0 + (n+2) \end{array} \right\}$$
$$= \left\{ \begin{array}{ll} y & \text{if } y \leqslant v_0 \\ v_0 + 1 & \text{if } v_0 < y. \end{array} \right.$$

Finally, for

$$h''(\vec{x},y) \stackrel{\text{def}}{=} h'(\vec{x},y) \dot{-} \chi_{\neq} \left(g_1(\vec{x}, h'(\vec{x},y) \dot{-} 1), g_2(\vec{x}, h'(\vec{x},y) \dot{-} 1) \right),$$

we get

$$h'' = \left\{ \begin{array}{ll} y \dot{-} \chi_{\neq}(g_1(\vec{x}, y \dot{-} 1), g_2(\vec{x}, y \dot{-} 1)) & \text{if } y \leqslant v_0 \\ (v_0 + 1) \dot{-} \chi_{\neq}(g_1(\vec{x}, v_0), g_2(\vec{x}, v_0)) & \text{if } v_0 < y. \end{array} \right.$$

By the definition of v_0, we see that

$$\chi_{\neq}(g_1(\vec{x}, y \dot{-} 1), g_2(\vec{x}, y \dot{-} 1)) = 0.$$

Therefore

$$\chi_{\neq}(g_1(\vec{x}, v_0), g_2(\vec{x}, v_0)) = 1.$$

We can then conclude that

$$h'' = \left\{ \begin{array}{ll} y \dot{-} 0 = y & \text{if } y \leqslant v_0 \\ (v_0 + 1) \dot{-} 1 = v_0 & \text{if } v_0 < y \end{array} \right\}$$
$$= \left\{ \begin{array}{ll} y & \text{if } y < v_0 \\ v_0 & \text{if } v_0 \leqslant y \end{array} \right\}$$
$$= \mu_{z \leqslant y}[g_1, g_2].$$

If there is no solution v_0, the same function works since we are conceptually always in the case $y < v_0$.

Hence, we have:

(vi) *$\mathscr{F}^{\sharp}$ is closed under* BMIN.

Clearly, (i)–(vi) provide a complete proof of the following proposition.

Proposition 32. $\mathscr{D}^{\mu} \subseteq \mathscr{F}^{\sharp}$. Hence $\mathfrak{PrA}_{\star} \subseteq \mathscr{F}^{\mu}_{\star}$.

8.2. *Counting quantifiers and counting operations*

Counting in $\mathfrak{PA}^{\Delta_0}$ has been extensively studied in the literature. Most notably, while we know that when $R(\vec{x},y) \in \mathscr{E}^0_{\star}$, then so is the predicate

$$S(x,z,y) \stackrel{\text{def}}{\Leftrightarrow} z = |\{u < y \mid R(\vec{x},u)\}|,$$

the analogous statement with respect to $\mathfrak{PA}^{\Delta_0}$ is an open problem. The predicate S above can be viewed as having been generated from R by a so-called *counting operation*. This terminology is found in, for example, Esbelin and More (1998).

The fact that $z = |\{u \leqslant y \mid R(\vec{x},u)\}| \Leftrightarrow \sharp_{u<y}[\chi_R(\vec{x},u) = 0] = z$, yields the following proposition.

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Proposition 33. $\mathscr{F}^{\sharp}_{\star}$ is closed under the counting operation

Schweikardt (2005) contains a study and survey of the concept of *counting-quantifiers*. The approach taken in Schweikardt (2005) was to extend first-order logic with counting quantifiers '$\exists_y^{=z}$', with the intended interpretation that $\exists_u^{=z} R(x, u)$ holds if and only if $z = |\{u \mid R(\vec{x}, u)\}|$. These counting quantifiers are unbounded *a priori*, but can obviously be made bounded, in the sense that, for example,

$$S(\vec{x}, z, y) \stackrel{\text{def}}{=} \exists_u^{=z} (u < y \wedge R(\vec{x}, u)),$$

takes on the meaning:
There are $u_0 < u_1 < \cdots < u_z < y$ *such that:*

(i) *for each* $i < z$ *we have* $R(\vec{x}, u_i)$*; and*
(ii) *if* $v < y$ *satisfies* $R(\vec{x}, v)$*, then for some* $i < z$ *we have* $v = u_i$.

We abbreviate this construction by $\exists_{u<y}^{=z} R(\vec{x}, u)$.

Also, since

$$\forall_{u<y} R(\vec{x}, u) \Leftrightarrow \exists_{u<y}^{=y} R(\vec{x}, u)$$

and

$$\exists_{u \leqslant y} R(\vec{x}, u) \Leftrightarrow R(\vec{x}, y) \vee \exists_{u<y} R(\vec{x}, u),$$

we see that the closure of a set of relations $\mathfrak{R}_\star$ under the bounded counting operation, which we denote by $\mathfrak{R}_\sharp$, is also closed under variable-bounded ($\exists_{z \leqslant y}$-type) quantifiers.

Incidentally, we have the following theorem.

Theorem 34 (Schweikardt). $\mathfrak{PrA}_\star = \mathfrak{PrA}_\sharp$.

The theorem in Schweikardt (2005) actually asserts that one can extend the underlying first-order logic of Presburger Arithmetic with full unbounded counting quantifiers, and still retain equality. The important thing for us is that we can now easily prove the following theorem.

Theorem 35. $\mathscr{F}^{\sharp}_{\star} = \mathfrak{PrA}_\star$.

Proof. The proof is by induction on f:

Induction start: The base-cases are obvious since we only need to consider $\mathscr{I} \cup \mathscr{N}$-functions.
Induction step:

— Case COMP:
This is exactly as in the proof of Lemma 22.

— Case $f = \sharp_{u<y}[g_1, g_2]$:
For $\sharp_{u<y}[g_1(\vec{x}, u) = g_2(\vec{x}, u)]$, let $\phi_i(\vec{x}, u, w)$ represent the graphs of the g_i's, and consider the formula[†]:

$$\psi(\vec{x}, y, z) \stackrel{\text{def}}{=} \exists_{u<y}^{=z} \exists_{w \leqslant \max(\vec{x}, u)} (\phi_1(\vec{x}, u, w) \wedge \phi_2(\vec{x}, u, w)).$$

[†] In the expression, the '$\exists_{w \leqslant \max(\vec{x}, y)}$'-quantifier is formal shorthand for the finite disjunction $\exists_{w \leqslant x_1} \phi \vee \exists_{w \leqslant x_k} \phi \vee \cdots \vee \exists_{w \leqslant y} \phi$, which is a $\mathfrak{PrA}$-formula when ϕ is.

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Clearly, $f(\vec{x}, y) = z \Leftrightarrow \mathfrak{N} \models \psi(\vec{x}, y, z)$. Since Schweikardt's proof implies that one may find an equivalent $\mathfrak{PrA}$-formula ψ' when the $\phi_i \in \mathfrak{PrA}$, we are done. □

That is, in a quite precise sense:

Presburger Arithmetic is exactly counting!

8.3. *The classes $\mathscr{F}^{n,\sharp}$ for $n \geqslant 2$*

The first thing to note is that n-ary bounded counting, as defined above, is *not* an argument-bounded schema. The bound is actually polynomial in the sense that the number of n-tuples $\vec{z}$ that may be counted by $\sharp_{\vec{z}<y}[g_1, g_2]$ is bounded only by y^n.

Still, a top-index-like phenomenon arises for functions in $\mathscr{F}^{n,\sharp}$. We generalise the top-index-notion by saying that $f(\vec{x})$ has *polynomial top-index* i, if, for some $c, n \in \mathbb{N}$, we have $f(\vec{x}) \leqslant \max(x_i^n, c)$; and i is a *strict (polynomial) top-index* if $c = 0$.

Lemma 36 (polynomial top-index). Let $f \in \mathscr{F}^{n,\sharp}$. Then f has a polynomial top-index. Furthermore, if $f(\mathbb{N}^k)$ is infinite, then the top-index is strict.

Proof. The assertion has already been proved for $f \in \mathscr{I} \cup \mathscr{N}$.

Induction step:

— Case $f = h \circ \vec{g}$:
Let $h(\vec{y}) \leqslant \max(y_j^{n_h}, c_h)$ and $g_j(\vec{x}) \leqslant \max(x_i^{n_j}, c_j)$. Then

$$f(\vec{x}) \leqslant \max\left(\left(g_j(\vec{x})\right)^{n_h}, c_j\right) \leqslant$$

$$\max\left((\max(x_i^{n_j}, c_j))^{n_h}, c_h\right) \leqslant \max\left(x_i^{n_h n_j}, \max\left(c_j^{n_h}, c_h\right)\right).$$

As before, if f is to have infinite image, then h must have infinite image, so $c_h = 0$. Thus $f(\vec{x}) \leqslant (g(\vec{x}))^{n_h}$. Unless g has infinite image, this function is bounded, so $c_j = 0$ as well.

— Case $f = \sharp_{\vec{z}<y}[g_1, g_2]$:
$|\vec{z}| = n$ trivially implies strict top-index y^n. □

Hence, functions like $\max$, C, or $x \cdot y$ are excluded from $\mathscr{F}^{n,\sharp}$. Also, any function in $\mathscr{F}^{n,\sharp}$ is bounded by a polynomial, which can be expressed as $\mathscr{F}^{n,\sharp} \subseteq \mathscr{E}^2$ for all $n \in \mathbb{N}$.
Next, observe that $g(x) \stackrel{\text{def}}{=} \sharp_{z_1,z_2<x}[0 = 0]$ satisfies

$$g \in \mathscr{F}^{1,\sharp} \quad \text{and} \quad g(x) = x^2.$$

Hence, by composing g with itself k times, we have $g^{(k)}(x) = x^{2k} \in \mathscr{F}^{1,\sharp}$. In particular, any polynomial $p \in \mathbb{N}[x]$ with constant term equal to zero† is in $\mathscr{F}^{1,\sharp}$.

Let $f(x, y, b) \stackrel{\text{def}}{=} \sharp_{z_1,z_2<b}[z_1 < x \wedge z_2 < y]$. Then $f(x, y, b) = \min(x \cdot y, b^2) \in \mathscr{F}^{1,\sharp}$. Equipped with this function from the 'bounded multiplication family', we can easily define enough

† For example, $p(x) = x^3 + 5$ *cannot* be in $\mathscr{F}^{n,\sharp}$ since it has infinite image, yet $p(1) = 6 > 1^m$ for any m.

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to get $\mathscr{F}^{1,\sharp}$ off the ground. We have

$$\left\lfloor \frac{x}{y} \right\rfloor = v \Leftrightarrow \bigvee \begin{cases} (y = 0 \vee y > x) \wedge v = 0 \\ \bigwedge \begin{cases} 0 < v, y \leqslant x \\ \mathsf{min}(y(v \dot{-} 1), x^2) < x \\ \exists_{u \leqslant y} \left(\mathsf{min}(vy, x^2) + u = x\right). \end{cases} \end{cases}$$

Since this predicate is obtainable by substituting the $\mathscr{F}^{1,\sharp}$-function $\mathsf{min}(xy, b^2)$ into a $\mathfrak{PrA}_\star$-predicate, we conclude that $\lfloor \frac{\cdot}{\cdot} \rfloor \in \mathscr{F}^{1,\sharp}$. So $\mathscr{DD}^\mu \subseteq \mathscr{F}^{1,\sharp}$. Since $\mathscr{F}^{n,\sharp}$ is trivially closed under the counting-operation we obtain the following proposition.

Proposition 37. $\mathfrak{PA}_\sharp^{\Delta_0} \subseteq \mathscr{F}_\star^{1,\sharp}$.

Definition 38. Let $B_y^k \overset{\text{def}}{=} \{\vec{x} \in \mathbb{N}^k \mid \mathsf{max}(\vec{x}) < y\}$ and define $e_y^k : N^k \to \mathbb{N}$ by

$$e_y^k(\vec{x}) = \sum_{i=1}^{k} x_i \cdot y^{(i-1)}.$$

It is well known that $e_y^k \restriction_{B_y^k}$ is a bijection between B_y^k and $|B_y^k| = y^k$. We use $\pi_i^{k,y}(z)$ to denote any function such that $\pi_i^{k,y}(e_y^k(\vec{x})) = x_i$ for $\vec{x} \in B_y^k$. Exactly what $\pi_i^{k,y}$ does for $z = e_y^k(\vec{x})$ when $\vec{x} \notin B_y^k$ is insignificant; we may simply let $\pi_i^{k,y}(z) = z$ for such $z \geqslant |B_y^k|$. Also, for each fixed $k > 0$, the functions

$$E^k(\vec{x}, y) : \mathbb{N}^{k+1} \to \mathbb{N};\ E^k(\vec{x}, y) = e_y^k(\vec{x})$$
$$\Pi_i^k : \mathbb{N}^2 \to \mathbb{N};\ \Pi_i^k(z, y) = \pi_i^{k,y}(z),$$

all have $\mathfrak{PA}^{\Delta_0}$-graphs. Because of the way we defined $\pi_i^{k,y}$, we have $\Pi_i^k(x, y) \leqslant x$. This means Π_i^k has a top-index, so $\Pi_i^k \in \mathscr{F}^{1,\sharp}$ directly. Also, since $E^k(\vec{x}, y) < y^k$ and is a polynomial in $\vec{x}$ and y, we have $E^k \in \mathscr{F}^{1,\sharp}$ since it is equal to $\mathsf{min}(E^k(x, y), y^k)$.

Esbelin and More have shown (Esbelin and More 1998) that $\mathfrak{PA}_\sharp^{\Delta_0}$ is closed under polynomially bounded quantification. We therefore easily obtain $\mathscr{F}_\star^{n,\sharp} \subseteq \mathfrak{PA}_\sharp^{\Delta_0}$, or, equivalently, the following lemma.

Lemma 39. $f \in \mathscr{F}^{n,\sharp} \Rightarrow \Gamma_f \in \mathfrak{PA}_\sharp^{\Delta_0}$.

Proof. The proof is by induction on f, for all n simultaneously:

Induction start: This is just as in previous proofs.

Induction step:

— Case $f = h \circ \vec{g}$:
Let $\phi_h(\vec{y}, w)$ and $\phi_j(\vec{x}, z_j)$ be the representing formulae of h and the g_j's, respectively. Consider

$$\psi_m(\vec{x}, w) \overset{\text{def}}{\Leftrightarrow} \exists_{\vec{z} \leqslant \mathsf{max}(x_1^m, \ldots, x_k^m)} \left(\bigwedge_{1 \leqslant j \leqslant \ell} \phi_j(\vec{x}, z_j) \wedge \phi_h(\vec{z}, w) \right).$$

As usual, the bound in the quantifier poses no problem with respect to taking a max. Since $\mathfrak{PA}_\sharp^{\Delta_0}$ is closed under polynomially bounded quantification, ψ_m is in

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$\mathfrak{PA}^{\Delta_0}_{\sharp}$ for all m. By choosing m sufficiently large, *viz.* so that m is larger than the maximal degree of the polynomial top-indices of the g_j's, we see that ψ_m represents the graph of $h \circ \vec{g}$.

— Case $f = \sharp_{\vec{z}<y}[g_1 = g_2]$:
Let $\phi(\vec{x}, \vec{z})$ be the $\mathfrak{PA}^{\Delta_0}_{\sharp}$ formula asserting that $g_1(\vec{x}, \vec{z}) = g_2(\vec{x}, \vec{z})$, we have

$$f(\vec{x}, y) = v \Leftrightarrow v = \left| \left\{ u < y^n \;\middle|\; \exists \vec{z} < y \left(\left(\bigwedge_{i=1}^{n} \left(z_i = \Pi_i^n(u, y) \right) \right) \wedge \phi(\vec{x}, \vec{z}) \right) \right\} \right|,$$

which is again clearly a $\mathfrak{PA}^{\Delta_0}_{\sharp}$-formula. □

Define $\mathscr{F}^{\omega,\sharp} \stackrel{\text{def}}{=} \bigcup_{n\in\mathbb{N}} \mathscr{F}^{n\sharp}$. Then the above means that we have the following theorem.

Theorem 40. $\mathscr{F}^{\sharp} \subsetneq \mathscr{F}^{1,\sharp} = \mathscr{F}^{\omega,\sharp}$. Furthermore, $\mathscr{F}^{\omega,\sharp}_{\star} = \mathfrak{PA}^{\Delta_0}_{\sharp}$.

Proof. By Proposition 37, we have $\mathfrak{PA}^{\Delta_0}_{\sharp} \subseteq \mathscr{F}^{1,\sharp}_{\star}$. Combined with Lemma 39 this implies that the graph of any $f \in \mathscr{F}^{\omega,\sharp}$ belongs to $\mathscr{F}^{1,\sharp}$, and thus also $f^{-1}(0)$ for any such function. □

This result is not very surprising, and is only included for completeness. There is no way of achieving exponential growth in $\mathscr{F}^{n,\sharp}$ for any n, and, growth-wise, $\mathscr{F}^{1,\sharp}$ dominates the whole 'hierarchy'. In fact, it is easy to see that we have the following corollary.

Corollary 41. $[\times\,;\,\text{BCOUNT}]_{\star} = \mathfrak{PA}^{\Delta_0}_{\sharp}$.

This follows because we can still capture the graph of a $[\times\,;\,\text{BCOUNT}]$-function in $\mathfrak{PA}^{\Delta_0}_{\sharp}$ by essentially the same formula as above: since $\mathfrak{PA}^{\Delta_0}_{\sharp}$ is closed under polynomial substitutions, we do not depend upon a top-index.

8.4. *The classes $\overline{\mathscr{F}}^{n,\sharp}$ for $n \geqslant 2$*

Recall that for these classes, the situation returns to 'normal' – we have already proved the relevant top-index result as Lemma 21.

Is the n-ary schema any stronger than the unary? That is, do we have, for example, $\overline{\mathscr{F}}^{0,\sharp} \subsetneq \overline{\mathscr{F}}^{1,\sharp}$, when all functions are argument-bounded? The answer is yes.

We may still count *pairs*, and pairs have to do with multiplication. The point is that we can do most of what we have done in the previous section by observing that a bounded multiplication function is still available:

$$\begin{aligned} f(x, y, b) &= \min(b, \sharp_{z_1,z_2<b}[z_1 < x \wedge z_2 < y]) \\ &= \min(b, \min(x \cdot y, b^2)) \\ &= \min(x \cdot y, b). \end{aligned}$$

This means that the analogue of Proposition 37 still holds for $\overline{\mathscr{F}}^{1,\sharp}$ since we can substitute x for the two occurrences of x^2 without destroying the equivalence. This means we have the following proposition.

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Proposition 42. $\mathfrak{P}\mathfrak{A}_{\sharp}^{\Delta_0} \subseteq \overline{\mathscr{F}}_{\star}^{1,\sharp}$.

Proof. The result follows simply because $\mathfrak{P}\mathfrak{A}_{\star}^{\Delta_0} = \mathscr{D}\mathscr{D}_{\star}^{\mu} \subseteq \overline{\mathscr{F}}_{\star}^{1,\sharp}$, and because we inherit closure under the counting operation from the fact that $\overline{\mathscr{F}}_{\star}^{1,\sharp}$ is closed under bounded counting. □

By combining the obvious fact that $\overline{\mathscr{F}}^{n,\sharp} \subseteq \mathscr{F}^{n,\sharp}$ and Lemma 39 we have the following lemma.

Lemma 43. $f \in \overline{\mathscr{F}}^{n,\sharp} \Rightarrow \Gamma_f \in \mathfrak{P}\mathfrak{A}_{\sharp}^{\Delta_0}$.

Defining $\overline{\mathscr{F}}^{\omega,\sharp} \stackrel{\text{def}}{=} \bigcup_{n\in\mathbb{N}} \overline{\mathscr{F}}^{n\sharp}$, we again get the following result.

Theorem 44. $\mathscr{F}^{\sharp} \subsetneq \overline{\mathscr{F}}^{1,\sharp} = \overline{\mathscr{F}}^{\omega,\sharp}$. Furthermore, $\overline{\mathscr{F}}_{\star}^{\omega,\sharp} = \mathfrak{P}\mathfrak{A}_{\sharp}^{\Delta_0}$.

We see that the only essential use of the polynomial growth possible in $\mathscr{F}^{1,\sharp}$, is in defining $\lfloor\frac{\cdot}{\cdot}\rfloor$: *the computational difference between* $\mathfrak{P}\mathfrak{r}\mathfrak{A}_{\star}^{\sharp}$ *and* $\mathfrak{P}\mathfrak{A}_{\sharp}^{\Delta_0}$ *can be thought of as being the ability to divide.*

Theorem 45.

$$\mathscr{F}^{1,\sharp} = [\{\lfloor\tfrac{\cdot}{\cdot}\rfloor\}\ ;\ \text{BCOUNT}] =_{\star} \left[\{\times\}\ ;\ \text{BCOUNT}\right],$$

where '$=_{\star}$' indicates that the equality only holds for the relational classes.

9. Summary of results and concluding discussion

The following diagram summarises the results of this paper:

$$\begin{array}{ccccccccc}
\mathscr{F}_{\star}^{\mu} & = & G_{\star}^{0} & \stackrel{\text{HAR}}{=} & \mathscr{P}\mathscr{L}_{\star} & & & & \\
\cap\!\!\!\!\not\,| & & & & & & & & \\
\mathscr{D}_{\star} & = & \mathfrak{P}\mathfrak{r}\mathfrak{A}_{\star}^{\text{qf}} & & & & & & \\
\cap\!\!\!\!\not\,| & & & & & & & & \\
\mathscr{D}_{\star}^{\mu} & = & \mathscr{F}_{\star}^{\sharp} & = & G_{\star}^{1} & \stackrel{\text{HAR}}{=} & \mathfrak{P}\mathfrak{r}\mathfrak{A}_{\star} & & \\
\cap\!\!\!\!\not\,| & & & & & & & & \\
\mathscr{D}\mathscr{D}_{\star}^{\mu} & = & G_{\star}^{2} & \stackrel{\text{HAR}}{=} & \mathfrak{P}\mathfrak{A}_{\star}^{\Delta_0} & & & & \\
\cap| & & & & & & & & \\
\overline{\mathscr{F}}_{\star}^{1,\sharp} & = & \cdots & = & \overline{\mathscr{F}}_{\star}^{\omega,\sharp} & = & \mathfrak{P}\mathfrak{A}_{\sharp}^{\Delta_0} & & \\
\| & & & & & & & & \\
\mathscr{F}_{\star}^{1,\sharp} & = & \cdots & = & \mathscr{F}_{\star}^{\omega,\sharp} & = & \mathfrak{P}\mathfrak{A}_{\sharp}^{\Delta_0} & &
\end{array}$$

$$\mathfrak{P}\mathfrak{r}\mathfrak{A}_{\star} \stackrel{\text{SCH}}{=} \mathfrak{P}\mathfrak{r}\mathfrak{A}_{\sharp} = \mathscr{F}_{\star}^{\sharp} \subsetneq \mathfrak{P}\mathfrak{A}_{\star}^{\Delta_0} \stackrel{?}{\subseteq} \mathfrak{P}\mathfrak{A}_{\sharp}^{\Delta_0} = \mathscr{F}_{\star}^{1,\sharp} = \mathscr{F}_{\star}^{\omega,\sharp}$$

The first feature to note in these results are the three equalities $\mathscr{F}_{\star}^{\mu} = G_{\star}^{0}$, $\mathscr{D}_{\star}^{\mu} = G_{\star}^{1}$ and $\mathscr{D}\mathscr{D}_{\star}^{\mu} = G_{\star}^{2}$, which formed the original motivation for this research. In each equality we have:

(1) On the functional level there is strict inclusion to the right.

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(2) The class on the left results from the class on the right by substituting an *argument-bounded almost-everywhere inverse* for the non-argument-bounded functions of that class.

By this we mean that with, for example, G^1, the non-argument-bounded function is $+$, and $\dot{-}$ is an a.e. inverse in the sense that for n fixed $(n+m)\dot{-}m = n \overset{\text{a.e.}}{=} (n\dot{-}m)+m$. A similar equation holds for multiplication and the pair $\lfloor \dot{\cdot} \rfloor$ and rem. It is sensible to view an idc as a structure that has some connection with the notion of an *algorithm*; functions are inductively built up using more basic functions. Hence, our findings say that with the non-iterative schemata BMIN and BCOUNT^n it is their ability or inability to define and compute with P, $\dot{-}$, $\lfloor \dot{\cdot} \rfloor$ and rem that decides the induced relational class.

A second note-worthy detail is the way that the class $\mathscr{D}_\star$ provides a fresh view at what happens in Presburger Arithmetic with respect to quantification. There is a $\mathscr{D}$–$\mathscr{D}^\mu$-dichotomy, which induces a $\mathfrak{PrA}_\star^{\text{qf}}$–$\mathfrak{PrA}_\star$-dichotomy. Intuitively, this result states that in $\mathscr{D}^\mu$, bounded minimalisation plays exactly the role of a quantifier: if R is the matrix of some prenex normal form $\mathfrak{PrA}$-formula, then R is a $\mathscr{D}_\star$-predicate.

In contrast, in the case of $\mathscr{DD}$ versus $\mathscr{DD}^\mu$ (where $\mathscr{DD}$ is '$\mathscr{DD}^\mu$ without BMIN') there is no such dichotomy. Our proof that $\mathscr{DD}^\mu = \mathfrak{PA}_\star^{\Delta_0}$ only appeared to be easy. What we have proved in this paper was simply that $\mathscr{CA} \subseteq \mathscr{DD}_\star^\mu \subseteq \mathfrak{PA}_\star^{\Delta_0}$. The proof of the missing inclusion, $\mathfrak{PA}_\star^{\Delta_0} \subseteq \mathscr{CA}$, is highly non-trivial.

A third point is that the pair of equalities $\mathscr{F}_\star^\sharp = \mathfrak{PrA}_\star$ and $\mathscr{F}^{1,\sharp} = \mathfrak{PA}_\sharp^{\Delta_0}$ are rather striking:

$\mathfrak{PrA}_\sharp$ *is simply unary counting and* $\mathfrak{PA}_\sharp^{\Delta_0}$ *is simply binary counting.*

Finally, in this paper we have studied *non-iterative* schemata, where the term non-iterative is not very precise, but simply refers to the contrast with schemata like *primitive recursion* or *iteration*. Whether BCOUNT should be counted as a non-iterative schema is not totally clear. Let $I^- \overset{\text{def}}{=} [\mathsf{P}\ ;\ \text{IT}]$, where IT is the schema of *pure iteration*. Esbelin (1994) proved the second inclusion in the chain

$$\mathfrak{PA}_\star^{\Delta_0} \subseteq \mathfrak{PA}_\sharp^{\Delta_0} \subseteq I_\star^-,$$

but whether the inclusions are proper or not is still unknown. Moreover, Esbelin and More (1998) showed that $\mathfrak{PA}_\sharp^{\Delta_0} = I_\star^-$ implies $\mathfrak{PA}_\sharp^{\Delta_0} = \mathscr{E}_\star^2$. Thus, informally, BCOUNT is 'quasi-iterative' since its idc hovers between $\mathfrak{PA}_\star^{\Delta_0}$, where many believe one *cannot* count, and the idc $I_\star^-$, where one *can* count – precisely because one can iterate.

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