Glasgow Mathematical Journal (2024), **66**, pp. 440–448
doi:10.1017/S0017089524000065

Glasgow Mathematical Journal

RESEARCH ARTICLE

Examples of hyperbolic spaces without the properties of quasi-ball or bounded eccentricity

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Received: 21 August 2023; **Revised:** 16 February 2024; **Accepted:** 19 February 2024; **First published online:** 11 March 2024

Keywords: Gromov's hyperbolic spaces; Non-(quasi)-geodesic; Quasi-ball property; Bounded eccentricity property

2020 Mathematics Subject Classification: *Primary* - 51M09; *Secondary* - 20F67

Abstract
In this note, we present examples of non-quasi-geodesic metric spaces which are hyperbolic (i.e., satisfying Gromov's 4-point condition) while the intersection of any two metric balls therein does not either 'look like' a ball or has uniformly bounded eccentricity. This answers an open question posed by Chatterji and Niblo.

1. Introduction

In the seminal work [4], Gromov introduced a notion of hyperbolicity for metric spaces which encodes the information of metric curvatures for the underlying spaces, with prototypes from classic hyperbolic geometry. Gromov's hyperbolic spaces have attracted a lot of interest since they are discovered and have fruitful applications in various aspects of mathematics (see, e.g., [1, 3]).

Recall that a geodesic metric space (X, d) is called *hyperbolic* (in the sense of Gromov [4]) if there exists $\delta > 0$ such that for any geodesic triangle in (X, d), the union of the δ-neighbourhoods of any two sides of the triangle contains the third. Gromov also provided a characterisation for his hyperbolicity using the so-called *Gromov product*:

$$(x|y)_p = \frac{1}{2}(d(x,p) + d(y,p) - d(x,y)) \quad \text{for} \quad x, y, p \in (X, d).$$

He proved in [4] that a geodesic metric space (X, d) is hyperbolic *if and only if* the following condition holds:

Gromov's 4-point condition:
There exists $\delta > 0$ such that

$$(x|y)_p \geq \min\{(x|z)_p, (y|z)_p\} - \delta \quad \text{for all} \quad x, y, z, p \in (X, d).$$

Note that the statement of Gromov's 4-point condition does not require that the underlying space (X, d) to be geodesic. Hence in the general context, we say that a (not necessarily geodesic) metric space is *hyperbolic* if Gromov's 4-point condition holds.

Later in [2], Chatterji and Niblo discovered new characterisations of Gromov's hyperbolicity for geodesic metric spaces using the geometry of intersections of balls. More precisely, for a geodesic metric space, they showed that it is hyperbolic in the sense of Gromov *if and only if* the following holds:

Quasi-ball property:
The intersection of any two metric balls is at a uniformly bounded Hausdorff distance from a ball.

†JZ was partly supported by National Key R&D Program of China 2022YFA100700.

They also considered the eccentricity of the intersection of balls. Recall that for a metric space (X, d) and $\delta > 0$, we say that the *eccentricity* of a subset S of X is less than δ if there exist $c, c' \in X$ and $R \geq 0$ such that

$$B(c, R) \subseteq S \subseteq B(c', R + \delta). \tag{1.1}$$

Here we use $B(x, r) := \{y \in X : d(x, y) \leq r\}$ to denote the closed metric ball. The *eccentricity* of S is the infimum of δ satisfying (1.1). By convention, the eccentricity of the empty set is 0. Chatterji and Niblo proved in [2] that a geodesic metric space is hyperbolic *if and only if* the following holds:

Bounded eccentricity property:

The intersection of any two metric balls has uniformly bounded eccentricity.

In [2, Section 4], Chatterji and Niblo also discussed the situation of non-geodesic metric spaces. They recorded an example due to Viktor Schroeder (see [2, Example 18]) that there exists a non-geodesic metric space with the quasi-ball property but not hyperbolic (*i.e.*, does not satisfy Gromov's 4-point condition). However, the other direction is unclear and hence, they asked the following:

Question 1.1. *Does there exist a non-geodesic hyperbolic metric space which does not satisfy the quasi-ball property or the bounded eccentricity property?*

In this short note, we provide an affirmative answer to Question 1.1 by constructing concrete examples. The main result is the following:

Theorem 1.2. *There exists a non-quasi-geodesic hyperbolic (i.e., satisfying Gromov's 4-point condition) space which does not satisfy either the quasi-ball property or the bounded eccentricity property.*

Our construction is motivated by Gromov's observation in [4, Section 1.2] (also suggested in [2, Section 4]) that for a metric space (X, d), we can endow another metric d' on X defined by

$$d'(x, y) = \ln(1 + d(x, y)) \quad \text{for} \quad x, y \in X \tag{1.2}$$

such that (X, d') satisfies Gromov's 4-point condition. We show that if (X, d) is geodesic and unbounded, then (X, d') cannot be quasi-geodesic (see Corollary 3.3). Then, we study the relation of the quasi-ball property and the bounded eccentricity property between (X, d) and (X, d') (see Lemmas 3.5, 3.7 and 3.9). Finally, we show in Example 3.10 that for the Euclidean plane $\mathbb{R}^2$ with the Euclidean metric d_E, the construction $(\mathbb{R}^2, d'_E)$ in (1.2) provides an example to conclude Theorem 1.2.

2. Preliminaries

Here we collect some necessary notions and notation for this note.

Let (X, d) be a metric space. For $x \in X$ and $R \geq 0$, denote the closed (metric) ball by $B(x, R) = \{y \in X : d(x, y) \leq R\}$. We say that (X, d) is *bounded* if there exist $x \in X$ and $R \geq 0$ such that $X = B(x, R)$, and *unbounded* if it is not bounded. For a subset $A \subset X$ and $R \geq 0$, denote $\mathcal{N}_R(A) = \{x \in X : d(x, A) \leq R\}$ the closed R-*neighbourhood* of A in X. For subsets $A, B \subset X$, the *Hausdorff distance* between A and B is

$$d_H(A, B) = \inf\{R \geq 0 : A \subseteq \mathcal{N}_R(B) \text{ and } B \subseteq \mathcal{N}_R(A)\}.$$

Recall that a *path* in a metric space (X, d) is a continuous map $\gamma : [a, b] \to X$. A path γ is called *rectifiable* if its *length*

$$\ell(\gamma) := \sup\left\{\sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) : a = t_0 < t_1 < \cdots < t_n = b, n \in \mathbb{N}\right\}$$

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is finite. Usually, it is convenient to change the parameter $t \in [a, b]$ to the *standard arc parameter* $s \in [0, \ell(\gamma)]$ as follows. Define a map $\varphi : [a, b] \to [0, \ell(\gamma)]$ by $t \mapsto s = \ell(\gamma|_{[a,t]})$, and let $\tilde{\gamma} : [0, \ell(\gamma)] \to X$ to be the unique path satisfying $\tilde{\gamma} \circ \varphi = \gamma$. Then, we have $\ell(\tilde{\gamma}|_{[s_1,s_2]}) = |s_1 - s_2|$ for any $0 \le s_1 \le s_2 \le \ell(\gamma)$.

Now we recall the notions of quasi-isometry and (quasi-)geodesic.

Definition 2.1. *Let* $(X, d_X), (Y, d_Y)$ *be metric spaces and* $L \ge 1, C > 0$ *be constants. An* (L, C)-quasi-isometric embedding *from* (X, d_X) *to* (Y, d_Y) *is a map* $f : X \to Y$ *such that for any* $x, x' \in X$*, we have*

$$L^{-1}d_X(x, x') - C \le d_Y(f(x), f(x')) \le Ld_X(x, x') + C.$$

If additionally, we have $\mathcal{N}_C(f(X)) = Y$*, then* f *is called an* (L, C)-quasi-isometry. *In this case, we say that* (X, d_X) *and* (Y, d_Y) *are* (L, C)-quasi-isometric.

Definition 2.2. *Let* (X, d) *be a metric space.*

1. *Given* $x, y \in X$*, a* geodesic *between* x *and* y *is an isometric embedding* $\gamma : [0, d(x, y)] \to X$ *with* $\gamma(0) = x$ *and* $\gamma(d(x, y)) = y$*. The space* (X, d) *is called geodesic if for any* x, y *in* X*, there exists a* geodesic *between* x *and* y*.*
2. *Given* $x, y \in X, L \ge 1$ *and* $C \ge 0$*, an* (L, C)-quasi-geodesic *between* x *and* y *is an* (L, C)*-quasi-isometric embedding* $\gamma : [0, T] \to X$ *such that* $\gamma(0) = x$ *and* $\gamma(T) = y$*. The space* (X, d) *is called* (L, C)-quasi-geodesic *if for any* x, y *in* X*, there exists an* (L, C)*-quasi-geodesic between* x *and* y*. We also say that* (X, d) *is* quasi-geodesic *if it is* (L, C)*-quasi-geodesic for some* L *and* C*.*

We also need the notion of ultrametric space. Recall that a metric space (X, d) is called *ultrametric* if there exists $\delta > 0$ such that for any points $x, y, z \in X$, we have

$$d(x, y) \le \max\{d(x, z), d(y, z)\} + \delta.$$

The following is due to Gromov:

Lemma 2.3 ([4, Section 1.2]). *Every ultrametric space satisfies Gromov's* 4*-point condition.*

Recall from Section 1 that for a metric space (X, d), Gromov considered another metric d' on X defined in (1.2) and noticed that

$$\begin{aligned} d'(x, y) &\le \ln(1 + d(x, z) + d(z, y)) \\ &\le \ln(2 + 2\max\{d(x, z), d(y, z)\}) \\ &= \max\{d'(x, z), d'(y, z)\} + \ln 2. \end{aligned}$$

Combining with Lemma 2.3, we obtain the following:

Lemma 2.4 ([4, Section 1.2]). *For a metric space* (X, d)*, the new metric* d' *on* X *defined in* (1.2) *satisfies Gromov's* 4*-point condition.*

3. Proof of Theorem 1.2

This whole section is devoted to the proof of Theorem 1.2, which is divided into several parts.

Firstly, we would like to study the property of (quasi-)geodesics for the new metric d' defined in (1.2). To simplify notation, for a path γ in X, we denote $\ell(\gamma)$ and $\ell'(\gamma)$ its length with respect to the metric d and d', respectively.

We need the following lemma:

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Lemma 3.1. *Let* (X, d) *be a metric space and* d' *be the metric on* X *defined in* (1.2) *. A path* $\gamma : [a, b] \to X$ *in* X *is rectifiable with respect to* d if and only if *it is rectifiable with respect to* d'*. In this case, we have* $\ell(\gamma) = \ell'(\gamma)$.

Proof. Firstly, we assume that γ is rectifiable with respect to d, *that is*, $\ell(\gamma) < \infty$. Note that $\ln(1 + x) \leq x$ holds for all $x \geq 0$. Hence for any partition $a = t_0 < t_1 < \cdots < t_n = b$, we have

$$\sum_{i=1}^{n} d'(\gamma(t_{i-1}), \gamma(t_i)) \leq \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) \leq \ell(\gamma),$$

which implies that $\ell'(\gamma) \leq \ell(\gamma)$. In particular, γ is rectifiable with respect to d'.

Conversely, we assume that γ is rectifiable with respect to d', *that is*, $\ell'(\gamma) < \infty$. Without loss of generality, we can assume that γ is parametrised by the standard arc parameter with $a = 0$ and $b = \ell'(\gamma)$. Note that for any $\alpha \in (0, 1)$, there exists $\delta > 0$ such that $\alpha x \leq \ln(1 + x)$ holds for all $x \in [0, \delta]$. Given a partition $a = s_0 < s_1 < \cdots < s_m = b$, we choose a refinement $a = t_0 < t_1 < \cdots < t_n = b$ such that $|t_{i-1} - t_i| < \ln(1 + \delta)$ holds for all i. Hence, we have

$$d'(\gamma(t_{i-1}), \gamma(t_i)) \leq l'(\gamma|_{[t_{i-1}, t_i]}) = |t_{i-1} - t_i| < \ln(1 + \delta),$$

which implies that $d(\gamma(t_{i-1}), \gamma(t_i)) < \delta$ due to (1.2). Therefore, we obtain

$$\sum_{i=1}^{m} d(\gamma(s_{i-1}), \gamma(s_i)) \leq \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) \leq \frac{1}{\alpha} \cdot \sum_{i=1}^{n} d'(\gamma(t_{i-1}), \gamma(t_i)) \leq \frac{1}{\alpha} \cdot \ell'(\gamma)$$

for all $\alpha \in (0, 1)$. Letting $\alpha \to 1$ and taking the supremum of the left hand side, we obtain that $\ell(\gamma) \leq \ell'(\gamma)$, which concludes the proof. □

As a direct corollary, we obtain the following:

Corollary 3.2. *Let* (X, d) *be a geodesic metric space which contains at least two elements, and* d' *be the metric defined in* (1.2)*. Then, the metric space* (X, d') *is* not *geodesic.*

Proof. By assumption, we take two distinct points $x, y \in X$. If (X, d') is geodesic, we choose a geodesic γ between x and y. In particular, γ is rectifiable with respect to d'. Hence by Lemma 3.1, we know that γ is also rectifiable with respect to d and we have

$$\ell'(\gamma) = \ell(\gamma) \geq d(x, y) > d'(x, y),$$

where the last inequality follows from the assumption that $x \neq y$. This is a contradiction to the assumption that γ is a geodesic between x and y with respect to the metric d'. Hence, we conclude the proof. □

Moreover, with an extra hypothesis, the new metric d' cannot even be quasi-geodesic.

Corollary 3.3. *Let* (X, d) *be an unbounded geodesic metric space and* d' *be the metric defined in* (1.2)*. Then for any* $L \geq 1$ *and* $C \geq 0$*, the metric space* (X, d') *cannot be* (L, C)*-quasi-geodesic.*

To prove Corollary 3.3, we need the following lemma to tame quasi-geodesics. The idea is similar to [1, Lemma III.H.1.11], but the setting is slightly different.

Lemma 3.4. *Let* (X, d) *be a geodesic metric space and* d' *be the metric defined in* (1.2)*. Given an* (L, C)*-quasi-geodesic* $\gamma : [a, b] \to X$ *with respect to* d'*, there exists a continuous and rectifiable* (L, C')*-quasi-geodesic* $\gamma' : [a, b] \to X$ *with respect to* d' *satisfying the following:*

1. $\gamma'(a) = \gamma(a)$ *and* $\gamma'(b) = \gamma(b)$*;*
2. $C' = 3(L + C)$*;*

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3. $\ell'(\gamma'|_{[t,t']}) \leq k_1 d'(\gamma'(t), \gamma'(t')) + k_2$ *for all* $t, t' \in [a, b]$, *where* $k_1 = L(L + C)$ *and* $k_2 = (LC' + 4)(L + C)$*;*
4. $d'_H(\mathrm{Im}(\gamma), \mathrm{Im}(\gamma')) \leq L + C$.

Careful readers might already notice that in the situation of [1, Lemma III.H.1.11], we need to assume that the *new* metric d' on X is geodesic instead of the current setting that the *original* metric d is geodesic. Although the proof is similar, here we also provide one for convenience to readers.

Proof of Lemma 3.4. Define γ' to agree with γ on $\Sigma := \{a, b\} \cup (\mathbb{Z} \cap (a, b))$, then choose geodesic segments with respect to d joining the images of successive points in Σ and define γ' by concatenating linear reparameterisations of these geodesic segments.

Let $[t]$ denote the point of Σ closest to $t \in [a, b]$. Note that the d'-distance of the images of successive points in Σ is at most $L + C$, and hence, we have

$$d'(\gamma'(t), \gamma'([t])) = \ln(1 + d(\gamma'(t), \gamma'([t]))) \leq \ln(1 + \exp(L + C) - 1) \leq L + C,$$

which implies that $d'_H(\mathrm{Im}(\gamma), \mathrm{Im}(\gamma')) \leq L + C$. Since γ is an (L, C)-quasi-geodesic with respect to d', and $\gamma([t]) = \gamma'([t])$ for all $t \in [a, b]$, we have

$$d'(\gamma'(t), \gamma'(t')) \leq d'(\gamma'([t]), \gamma'([t'])) + 2(L + C) \leq L|[t] - [t']| + C + 2(L + C) \leq L|t - t'| + 3(L + C),$$

and similarly, we have

$$d'(\gamma'(t), \gamma'(t')) \geq \frac{1}{L}|t - t'| - 3(L + C) \tag{3.1}$$

for all $t, t' \in [a, b]$. Hence, γ' is an (L, C')-quasi-geodesic with respect to d'.

For any integers $s, s' \in \Sigma$ with $s \leq s'$, Lemma 3.1 tells us that

$$\ell'(\gamma'|_{[s,s']}) = \ell(\gamma'|_{[s,s']}) = \sum_{k=s}^{s'-1} \ell(\gamma'|_{[k,k+1]}) \leq |s - s'|(L + C).$$

Similarly for any $s, s' \in \Sigma$, we have $\ell'(\gamma'|_{[s,s']}) \leq (|s - s'| + 2)(L + C)$. Hence for any $t, t' \in [a, b]$, we have

$$\ell'(\gamma'|_{[t,t']}) \leq (|[t] - [t']| + 2)(L + C) + (L + C) \leq (|t - t'| + 4)(L + C).$$

Combining with inequality (3.1), we obtain that $\ell(\gamma'|_{[t,t']}) \leq k_1 d'(\gamma'(t), \gamma'(t')) + k_2$ for k_1, k_2 defined in (3). Hence, we conclude the proof. □

Proof of Corollary 3.3. Assume that (X, d') is (L, C)-quasi-geodesic for some $L \geq 1$ and $C \geq 0$. For any $x, y \in X$, choose an (L, C)-quasi-geodesic $\gamma : [0, T] \to X$ (with respect to d') connecting them. Lemma 3.4 implies that there is a rectifiable path $\gamma' : [0, T] \to X$ which is an $(L, 3L + 3C)$-quasi-geodesic (with respect to d') connecting x and y. Moreover, we have $\ell'(\gamma') \leq k_1 d'(x, y) + k_2$ for $k_1 = L(L + C)$ and $k_2 = (3L(L + C) + 4)(L + C)$. Hence, Lemma 3.1 implies that

$$k_1 d'(x, y) + k_2 \geq \ell'(\gamma') = \ell(\gamma') \geq d(x, y) = \exp(d'(x, y)) - 1.$$

Note that (X, d) is unbounded, which implies that (X, d') is also unbounded. Hence taking $d'(x, y) \to \infty$, we conclude a contradiction and finish the proof. □

Next, we move to study the relation of the quasi-ball property and the bounded eccentricity property between the metric spaces (X, d) and (X, d'). Again to save the notation, for $x \in X$ and $r \geq 0$, we denote $B(x, r)$ and $B'(x, r)$ the closed balls with respect to the metrics d and d', respectively. For $A \subset X$ and $\delta \geq 0$, we denote $\mathcal{N}_\delta(A)$ and $\mathcal{N}'_\delta(A)$ the closed δ-neighbourhood of A with respect to the metrics d and d', respectively.

Lemma 3.5. *Let* (X, d) *be a metric space and* d' *be the metric on* X *defined in* (1.2)*. Then,* (X, d) *has the quasi-ball property* if and only if (X, d') *has the quasi-ball property.*

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Proof. Firstly, we assume that (X, d') has the quasi-ball property, *that is*, there exists $\delta \geq 0$ such that the intersection of any two balls in (X, d') is δ-close to another ball (*i.e.*, their Hausdorff distance is bounded by δ). Note that there exists $\alpha = \alpha(\delta) > 1$ such that $x \leq \alpha \ln(1 + x)$ for all $x \in [0, \delta]$. This implies that for $A, B \subseteq X$ with $A \subseteq \mathcal{N}'_\delta(B)$, we have $A \subseteq \mathcal{N}_{\alpha\delta}(B)$.

Given two balls $B(x, s)$ and $B(y, t)$ in (X, d), it is clear that $B(x, s) = B'(x, \ln(1 + s))$ and $B(y, t) = B'(y, \ln(1 + t))$. Hence by the assumption, there exists another ball $B'(c, r)$ in (X, d') such that

$$d'_H(B'(x, \ln(1 + s)) \cap B'(y, \ln(1 + t)), B'(c, r)) \leq \delta.$$

It follows from the previous paragraph that in this case, we have

$$d_H(B(x, s) \cap B(y, t), B(c, \exp(r) - 1)) = d_H(B'(x, \ln(1 + s)) \cap B'(y, \ln(1 + t)), B'(c, r)) \leq \alpha\delta.$$

Hence, (X, d) has the quasi-ball property.

The converse holds similarly, using the fact that $\ln(1 + x) \leq x$ holds for all $x \geq 0$. □

Combining Lemma 2.4, Corollary 3.3, Lemma 3.5 and [2, Theorem 1], we obtain the following, which concludes one part of Theorem 1.2.

Corollary 3.6. *Let (X, d) be a geodesic metric space which is not hyperbolic (e.g., the Euclidean plane $\mathbb{R}^2$ with the Euclidean metric) and d' be the metric on X defined in (1.2). Then, (X, d') is a non-quasi-geodesic metric space which satisfies Gromov's 4-point condition but* not *the quasi-ball property.*

Concerning the bounded eccentricity property, we have the following:

Lemma 3.7. *Let (X, d) be a metric space and d' be the metric on X defined in (1.2). If (X, d) satisfies the bounded eccentricity property with eccentricity δ_0, then so does (X, d').*

Proof. Given two balls $B'(x, \ln(1 + r))$ and $B'(y, \ln(1 + s))$ in (X, d'), we assume that their intersection Y is non-empty. Note that

$$Y = B'(x, \ln(1 + r)) \cap B'(y, \ln(1 + s)) = B(x, r) \cap B(y, s)$$

is again an intersection of balls in (X, d). Hence by the assumption, there exist $c, c' \in X$ and $R \geq 0$ such that

$$B(c, R) \subseteq Y \subseteq B(c', R + \delta_0).$$

Therefore in (X, d'), we have

$$B'(c, \ln(1 + R)) \subseteq Y \subseteq B'(c', \ln(1 + R + \delta_0)).$$

Then, the eccentricity of Y in (X, d') is bounded above by

$$\ln(1 + R + \delta_0) - \ln(1 + R) = \ln\left(1 + \frac{\delta_0}{1 + R}\right) \leq \ln(1 + \delta_0) \leq \delta_0,$$

which concludes the proof. □

Remark 3.8. *Readers might wonder whether the converse to Lemma 3.7 holds. Unfortunately, in general this is false. Note that if we have*

$$B'(c, \ln(1 + R)) \subseteq Y \subseteq B'(c', \ln(1 + R) + \delta_0)$$

in (X, d'), then it implies that

$$B(c, R) \subseteq Y \subseteq B(c', \exp(\delta_0) \cdot (1 + R) - 1)$$

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in (X, d). Hence, the eccentricity of Y in (X, d) is bounded by $(\exp(\delta_0) - 1)(1 + R)$. However, if R (i.e., the radius of the ball contained in Y) does not have a uniform upper bound, then neither does the eccentricity of Y.

Remark 3.8 suggests the following partial converse to Lemma 3.7:

Lemma 3.9. *Let (X, d) be a metric space and d' be the metric on X defined in (1.2). Assume that there exists a sequence of subsets $\{Y_n\}_{n\in\mathbb{N}}$ of X satisfying the following:*

1. *each Y_n is the intersection of two balls in (X, d);*
2. *there exists $M > 0$ such that for all $n \in \mathbb{N}$, the radius (with respect to d) of any ball contained in Y_n is bounded above by M;*
3. *the eccentricity of Y_n in (X, d) is not uniformly bounded.*

Then, (X, d') does not satisfy the bounded eccentricity property.

Proof. Assume the opposite, *that is*, there exists $\delta_0 > 0$ such that the eccentricity of the intersection of any two balls in (X, d') is uniformly bounded by δ_0. Hence by condition (1), we know that for each $n \in \mathbb{N}$, there exist $c_n, c'_n \in X$ and $r_n \geq 0$ such that

$$B'(c_n, \ln(1 + r_n)) \subseteq Y_n \subseteq B'(c'_n, \ln(1 + r_n) + \delta_0).$$

Hence in (X, d), we have

$$B(c_n, r_n) \subseteq Y_n \subseteq B(c_n, \exp(\delta_0) \cdot (1 + r_n) - 1).$$

By condition (2), we know that $r_n \leq M$ for all $n \in \mathbb{N}$. Therefore, we have

$$\exp(\delta_0) \cdot (1 + r_n) - 1 - r_n = (\exp(\delta_0) - 1)(1 + r_n) \leq (\exp(\delta_0) - 1)(1 + M),$$

which is a contradiction to condition (3) in the assumption. □

Example 3.10. *Now we show that there exists a non-quasi-geodesic metric space which satisfies Gromov's 4-point condition but not the bounded eccentricity property. For example, consider the Euclidean plane $\mathbb{R}^2$ equipped with the Euclidean metric d_E, and let d'_E be the metric on $\mathbb{R}^2$ defined in (1.2). Lemma 2.4 and Corollary 3.3 imply that $(\mathbb{R}^2, d'_E)$ is not quasi-geodesic but satisfies Gromov's 4-point condition.*

For each $n \in \mathbb{N}$, take $x_n = (0, 0)$ and $y_n = (2n, 0)$ and set

$$Y_n = B(x_n, n + 1) \cap B(y_n, n + 1).$$

It is easy to see that in $(\mathbb{R}^2, d_E)$, the biggest ball contained in Y_n is $B((n, 0), 1)$. Moreover, the diameter of Y_n is

$$d\Big((n, \sqrt{2n+1}), (n, -\sqrt{2n+1})\Big) = 2\sqrt{2n+1},$$

which implies that the eccentricity of Y_n cannot be uniformly bounded. Therefore applying Lemma 3.9, we conclude the result.

Proof of Theorem 1.2. Combining Corollary 3.6 and Example 3.10, we conclude the proof for Theorem 1.2. □

Recall that in [5] the following weaker form of the bounded eccentricity property was considered: For a metric space (X, d), there exist $\lambda > 0$ and $\delta > 0$ such that for any two balls $B_1, B_2 \subseteq X$ with non-empty intersection there exist $z, z' \in X$ and $r \geq 0$ such that

$$B(z, r) \subseteq B_1 \cap B_2 \subseteq B(z', \lambda r + \delta).$$

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Wenger showed in [5] that this condition also implies Gromov's hyperbolicity for geodesic metric spaces.

We remark that this weaker form of the bounded eccentricity property does not hold for the space $(\mathbb{R}^2, d'_E)$ constructed in Example 3.10 either.

4. Miscellaneous comments

This section derives from comments by Indira Chatterji, who reminds us that our example is quasi-isometric to a horosphere in the hyperbolic space $\mathbb{H}^3$.

More precisely, we consider the half-space model $\mathbb{H}^3 = \{(x, t) : x \in \mathbb{R}^2, t > 0\}$ for the 3-dimensional hyperbolic space. Note that horospheres centred at ∞ are precisely the horizontal planes $H_k = \{(x, t) \in \mathbb{H}^3 : t = k\}$ with $k > 0$. Endow each H_k with the subspace metric d_k induced by the hyperbolic metric on $\mathbb{H}^3$. Direct calculations show that

$$d_k((x, k), (y, k)) = 2\ln\left(\sqrt{1 + \frac{d_E(x, y)^2}{4k^2}} + \frac{d_E(x, y)}{2k}\right),$$

where $x, y \in \mathbb{R}^2$ and d_E is the Euclidean metric on $\mathbb{R}^2$.

The following lemma shows that the space constructed in Example 3.10 is quasi-isometric to the horospheres H_k.

Lemma 4.1. *Consider the Euclidean plane $\mathbb{R}^2$ equipped with the Euclidean metric d_E, and let d'_E be the metric on $\mathbb{R}^2$ defined in (1.2). Then for all $k \geq 1$, the horosphere (H_k, d_k) is $(1, 2\ln(2k))$-quasi-isometric to $(\mathbb{R}^2, 2d'_E)$.*

Proof. Consider the map $f : \mathbb{R}^2 \to S_k$ by $f(x) = (x, k)$. Note that

$$1 + \frac{1}{2k}d_E(x, y) \leq \sqrt{1 + \frac{d_E(x, y)^2}{4k^2}} + \frac{d_E(x, y)}{2k} \leq 1 + \frac{1}{k}d_E(x, y).$$

Since $k \geq 1$, we have

$$d_k((x, k), (y, k)) \leq 2\ln\left(1 + \frac{1}{k}d_E(x, y)\right) \leq 2\ln(1 + d_E(x, y)) = 2d'_E(x, y),$$

and

$$d_k((x, k), (y, k)) \geq 2\ln\left(1 + \frac{1}{2k}d_E(x, y)\right) \geq 2d'_E(x, y) - 2\ln(2k),$$

which concludes the proof. □

Recall that $\mathbb{H}^3$ is a hyperbolic geodesic metric space, and hence, it satisfies the quasi-ball property and the bounded eccentricity property according to [2]. However as shown in Corollary 3.6 and Example 3.10, the space $(\mathbb{R}^2, 2d_E{}')$ does not satisfy either of these two properties for $d_E{}'$ defined in (1.2). This implies that the quasi-ball property *cannot* be preserved either by taking subspaces or quasi-isometries (even for $(1, C)$-quasi-isometries), and the same holds for the bounded eccentricity property. We guess that both of these properties should be preserved under $(1, C)$-quasi-isometries, which leads to the failure of the permanence by taking subspaces. Unfortunately, meanwhile we cannot exclude the suspicion of taking quasi-isometries. Therefore, we pose the following question to end this note.

Question 4.2. *Can the quasi-ball property and the bounded eccentricity property be preserved by taking subspace or quasi-isometries?*

https://doi.org/10.1017/S0017089524000065 Published online by Cambridge University Press

Acknowledgement. We would like to thank Indira Chatterji for her comments presented in Section 4 after reading an early draft of this paper, and Graham Niblo for helpful discussions and continuous support. We would also like to thank the anonymous referee for his/her suggestions to improve the paper.

Competing interests. The authors declare none.

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