ON A PROBLEM OF L. MOSER

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1. Introduction. In [1], Moser derives a recurrence relation and studies the limiting behaviour of the Expectations E_n of the following game. "A real number is drawn at random from [0, 1]. We may either keep the number selected, or reject it and draw again. We can then either keep the second number chosen or reject it, and draw again, and so on. Suppose we have at most n choices. What stopping rule gives the largest E_n and how can we estimate E_n?"

The solution turns out to be that, given at most $m = n + 1$ choices, we should stop after the first choice if and only if we have found at least E_n. Following a request made in the second last sentence of [1], we consider the effect of the same procedure when the choices are made at random from the normal distribution, mean zero, variance one, and then compare the E_n's of Moser's game with the E_n's of picking from a normal distribution, mean 1/2, variance 1/12.

2. Derivation of the recurrence relation for arbitrary distributions. Let $f(x)$ be a probability density defined for $a \leq x \leq b$ (a and b may be $-\infty$ and $+\infty$), and further let $f(x)$ be continuous in $[a, b]$. Let E_j denote the expectation when picking at random from $f(x)$, when we have at most j choices. Let $E_0 = 0$, $E_1 = \mu$, where μ is the mean value of $f(x)$, i.e., $\mu = \int_a^b xf(x)dx$. (We will use the symbol σ^2 to denote the variance, i.e., $\sigma^2 = \int_a^b (x-\mu)^2 f(x)dx$.)

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If we follow the Moser procedure, and if we are given at most $n + 1$ choices, we stop after the first choice if we have found at least E_n. This occurs with probability

$$\int_{E_n}^{b} f(x)dx.$$

The mean value obtainable in this case is

$$\int_{E_n}^{b} xf(x \mid E_n \leq x \leq b)dx$$

where the function f in the integrand is the conditional probability density of x, given that x lies in $[E_n, b]$. But this probability density function is

$$f(x \mid E_n \leq x \leq b) = f(x) \Big/ \int_{E_n}^{b} f(x)dx.$$

The probability that we draw again is now

$$\int_{a}^{E_n} f(x)dx$$

and since there are now n choices left, the expectation is E_n.

Hence, the total expectation is

$$E_{n+1} = \int_{E_n}^{b} f(x)dx \cdot \int_{E_n}^{b} xf(x \mid E_n \leq x \leq b)dx + E_n \cdot \int_{a}^{E_n} f(x)dx.$$

That is, we now have that

$$E_{n+1} = \int_{E_n}^{b} xf(x)dx + E_n \cdot \int_{a}^{E_n} f(x)dx, \quad n = 0, 1, 2, \ldots$$

with $E_0 = 0$, $E_1 = \mu$.

For example, if $f_1(x) = 1$, when $0 \leq x \leq 1$, the above becomes

$$E_{n+1} = \tfrac{1}{2}(1 - E_n{}^2) + E_n^2$$

i.e., $E_{n+1} = \frac{1}{2}(1 + E_n{}^2)$, which, of course, is the recurrence relation found by Moser. For the sequel, it is important to note that $\mu = 1/2$, $\sigma^2 = 1/12$ for the uniform distribution $f_1(x) = 1$, $(0 \leq x \leq 1)$.

If we now consider picking from

$$f_2(x) = (1/\sqrt{2\pi}\,) \exp(-\tfrac{1}{2}x^2), \qquad - \infty \leq x \leq \infty \quad ,$$

the normal density with $\mu = 0$, $\sigma^2 = 1$, then we have that

$$E_{n+1} = \int_{E_n}^{\infty} x(1/\sqrt{2\pi}\,) \exp(-\tfrac{1}{2}x^2)dx + E_n. \int_{-\infty}^{E_n} (1/\sqrt{2\pi}\,) \exp(-\tfrac{1}{2}x^2)dx.$$

The second term, using an approximating function found in [2], significant to 8 figures, was programmed for the IBM 650 by Miss Gillian Richardson of Statistical Techniques Research Group, Princeton University, and a set of E_{n+1}'s generated for the normal density $f_2(x)$. (We use the notation $E_{n+1}^{(2)}$ to denote these E's, and $E_{n+1}^{(1)}$ to denote the expectations when picking from $f_1(x)$.) $E_{n+1}^{(1)}$ and $E_{n+1}^{(2)}$ are tabulated in tables 1 and 2.

Now, we have mentioned that the mean and variance of $f_1(x)$ are $1/2$ and $1/12$ respectively. If we take E's of the normal distribution and make the transformation

$$E_{n+1}^{(3)} = E_{n+1}^{(2)} \Big/ \sqrt{12} + \tfrac{1}{2}$$

we will have generated a set of E's when following Moser's procedure, with choices made from

$$f_3(x) = (\sqrt{12}/\sqrt{2\pi}) \exp(-6(x-\tfrac{1}{2})^2), \qquad - \infty \leq x \leq \infty$$

that is, from a normal distribution with mean $1/2$, variance $1/12$. Table 3 shows $E_{n+1}^{(3)} - E_{n+1}^{(1)}$. The table shows that if a benefactor offers a player m choices from either $f_1(x)$ or $f_3(x)$, then it is to the player's advantage if he picks from $f_1(x)$ for $m \leq 8$, but if $m > 8$ he should pick from $f_3(x)$.

3. Optimality of the procedure. Let g_{n+1} denote "a game in which a player is allowed at most $n + 1$ choices from an arbitrary probability density function $f(x)$" and let Moser's procedure be followed. Further, denote E_{n+1} by $\mathcal{E}(g_{n+1})$, that is, the expectation of the player's winnings (no entrance fee demanded) of the game g_{n+1}. It is clear that $E_i \leq E_j$ as long as $i < j$.

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Now from a basic definition of expectation we have

$$\mathcal{E}(g_{n+1}) = E\left\{\mathcal{E}(g_{n+1} \mid x_1)\right\}$$

where x_1 denotes the outcome of the first choice; that is, we have

$$\mathcal{E}(g_{n+1}) = \int_a^b \mathcal{E}(g_{n+1} \mid x_1)\, f(x_1)dx_1 .$$

Clearly then, to maximize $\mathcal{E}(g_{n+1})$ is to maximize $\mathcal{E}(g_{n+1} \mid x_1)$, the conditional expectation of the game, given the outcome of the first choice. It is clear that this is maximized if $x_1 > \mathcal{E}(g_n)$ and that if this be the case, the player should stop, for he now has n choices left, and his expectation is $E_n = \mathcal{E}(g_n)$, which is less than E_{n+1}.

REFERENCES

1. Leo Moser, On a problem of Cayley, Scripta Mathematica 22 (1956), 289-292.

2. Cecil Hastings, Jr., Approximations for Digital Computers, (Princeton, 1955).

TABLE 1.

$E_m^{(1)}$ of Moser's Procedure; $f_1(x) = 1$, $0 < x < 1$

m	E_m	m	E_m
1	.50000000	20	.91988745
2	.62500000	40	.95611755
3	.69531250	60	.96967375
4	.74172975	80	.97680340
5	.77508150	100	.98120855
6	.80037565	150	.98724655
7	.82030060	200	.99034290
8	.83644655	250	.99222765
9	.84982140	300	.99349595
10	.86109820	350	.99440815

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TABLE 2

$E_m^{(2)}$ of Moser's Procedure; $f_2(x) = (1/\sqrt{2\pi}) \exp(-\frac{1}{2}x^2)$,
$-\infty \leq x \leq \infty$.

m	E_m	m	E_m
1	.00000000	20	1.6120126
2	.39916388	40	1.9203302
3	.62976442	60	2.0887212
4	.79036107	80	2.2029409
5	.91266925	100	2.2885865
6	1.0108734	150	2.4378909
7	1.0925556	200	2.5392308
8	1.1622448	250	2.6154736
9	1.2228589	300	2.6763849
10	1.2763842	350	2.7270036

TABLE 3

$$D_m = E_m^{(3)} - E_m^{(1)},$$

where $f_3(x) = (\sqrt{12}/\sqrt{2\pi}) \exp(-6(x-\frac{1}{2})^2)$, $-\infty \leq x \leq \infty$

m	D_m	m	D_m
1	.00000000	20	.04546051
2	-.00977131	40	.09823400
3	-.01351517	60	.13328810
4	-.01357216	80	.15913090
5	-.01161658	100	.17944940
6	-.00856130	150	.21651190
7	-.00490696	200	.24266990
8	-.00093537	250	.26279450
9	+.00318756	300	.27910980
10	+.00736218	350	.29280990

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