SPACES $A \times B$ OF CONILPOTENCY ≤ 1

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Let A and B be spaces having the homotopy type of countable CW-complexes. Then we prove the following theorems.

THEOREM 1. *If* $\operatorname{conil}(A \times B) \leq 1$, *then for each integer* $n \geq 1$, *the inclusion* $j : \Sigma^n A \vee \Sigma^n B \to \Sigma^n A \times \Sigma^n B$ *is a homotopy equivalence.*

This result is obtained as a corollary of Theorem 2.

THEOREM 2. *If* $\operatorname{conil}(A \times B) \leq 1$, *then for all spaces* X *and all elements* α *of* $[\Sigma A, X]$ *and* β *of* $[\Sigma B, X]$, *the generalized Whitehead product* $[\alpha, \beta] = 0$.

In case $A = B$, we obtain a much stronger result.

THEOREM 3. *If* $\operatorname{conil}(A \times A) \leq 1$, *then* ΣA *is contractible. Further, if* $\pi_1(A)$ *is abelian, then* A *is contractible.*

We will work in the category of spaces with base point and having the homotopy type of countable CW-complexes. All maps and homotopies are to respect base points. For simplicity, we shall frequently use the same symbol for a map and its homotopy class. Given spaces X and Y, we denote the set of homotopy classes of maps from X to Y by $[X, Y]$. We have an isomorphism $\tau : [\Sigma X, Y] \to [X, \Omega Y]$ where Σ is the suspension functor and Ω is the loop functor.

1. We recall some constructions we need. Let A, B, X be spaces. In [1], Arkowitz defined a generalized Whitehead product $[\ ,\ ] : [\Sigma A, X] \times [\Sigma B, X] \to [\Sigma(A \wedge B), X]$ where $A \wedge B$ is the smash product. Let X be an H-space. Then in [2], he defined a generalized Samelson product $\langle\ ,\ \rangle : [A, X] \times [B, X] \to [A \wedge B, X]$. These homotopy operations are related in the following way. Let A, B, X be spaces and let α be an element of $[\Sigma A, X]$, β an element of $[\Sigma B, X]$. Then $\tau[\alpha, \beta] = \langle \tau(\alpha), \tau(\beta) \rangle$.

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We also need the Hopf construction (see [7], [8]). We briefly recall this. Let $f: A \times B \to X$ be a map. Then the Hopf construction gives a map $J(f): \Sigma(A \wedge B) \to \Sigma X$. This is the unique element satisfying the relation $\Sigma f = J(f)\,\Sigma q + \Sigma(f\,j\,p_1) + \Sigma(f\,j\,p_2)$ where $q: A \times B \to A \wedge B$, is the projection, $j: A \vee B \to A \times B$ is the inclusion and $p_1: A \times B \to A \vee B$, $p_2: A \times B \to A \vee B$ are defined by $p_1(a, b) = (a, *)$, $p_2(a, b) = (*, b)$. Then the Hopf construction gives a function $J: [A \times B,\ X] \to [\Sigma(A \wedge B),\ \Sigma X]$. Even if X is an H-space, J is in general not a homomorphism. We shall see below that it is if $\mathrm{conil}(A \times B) \leq 1$.

2. We now state our results.

LEMMA 1. _Suppose_ (X, ϕ) _is an_ H-space. _If_ $\mathrm{conil}(A \times B) \leq 1$ _then_ $J: [A \times B,\ X] \to [\Sigma(A \wedge B),\ \Sigma X]$ _is a homomorphism_.

Proof. Let $f, g: A \times B \to X$ be maps. We need to show that $J(f + g) = J(f) + J(g)$. By the definition of J we have $\Sigma f = J(f)\,\Sigma q + \Sigma(f\,j\,p_1) + \Sigma(f\,j\,p_2)$ and $\Sigma g = J(g)\,\Sigma q + \Sigma(g\,j\,p_1) + \Sigma(g\,j\,p_2)$. Since $\mathrm{conil}(A \times B) \leq 1$, by [8], we have that $\Sigma: [A \times B,\ X] \to [\Sigma(A \times B),\ \Sigma X]$ is a monomorphism of abelian groups. Hence we now have $\Sigma(f + g) = \{J(f) + J(g)\}\,\Sigma q + \Sigma\{(f + g)\,j\,p_1\} + \Sigma\{(f + g)\,j\,p_2\}$. But $J(f + g)$ is the unique element satisfying this relation. Hence $J(f + g) = J(f) + J(g)$.

Now let (X, ϕ, μ) be a G-space with multiplication ϕ and homotopy inverse μ. Let $c: X \times X \to X$ be the basic commutator (see [3] for definitions). Let $f: A \to X$, $g: B \to X$ be maps. Then we have their generalized Samelson product $\langle f, g \rangle : A \wedge B \to X$ and hence $\Sigma\langle f, g \rangle : \Sigma(A \wedge B) \to \Sigma X$. By the definition of the product, we have $\Sigma\langle f, g \rangle\,\Sigma q = \Sigma\{c(f \times g)\}$. On the other hand, we have the map $c(f \times g): A \times B \to X$ and hence the Hopf-construction $J\{c(f \times g)\}: \Sigma(A \wedge B) \to \Sigma X$.

LEMMA 2. $\Sigma\langle f, g \rangle = J\{c(f \times g)\}$ _for all maps_ $f: A \to X$, $g: B \to X$ _where_ (X, ϕ, μ) _is a_ G-space _and_ $c: X \times X \to X$ _is the basic commutator_.

Proof. The definition of J gives the relation

$$\Sigma\{c(f \times g)\} = J\{c(f \times g)\}\,\Sigma q + \Sigma\{c(f \times g)\,j\,p_1\} + \Sigma\{c(f \times g)\,j\,p_2\}.$$

Now $c(f \times g)\,j = cj(f \vee g) \simeq *$ since c is the commutator. Hence $J\{c(f \times g)\}\,\Sigma q = \Sigma\{c(f \times g)\}$. Since we also have $\Sigma\langle f, g \rangle\,\Sigma q = \Sigma\{c(f \times g)\}$ and $(\Sigma q)^{\#}$ is a monomorphism, the result follows.

LEMMA 3. _Let_ (X, ϕ, μ) _be a_ G-space _and let_ $f: A \to X$, $g: B \to X$ _be maps_. _If_ $\mathrm{conil}(A \times B) \leq 1$, _then_ $\langle f, g \rangle = 0$.

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Proof. We have $\Sigma\langle f, g\rangle = J\{c(f\times g)\} = J(f\pi_1 + g\pi_2 - f\pi_1 - g\pi_2)$ where $\pi_1 : A\times B \to A$, $\pi_2 : A\times B\to B$ are the projections. Since $\mathrm{conil}(A\times B) \leq 1$, J is a homomorphism. Since $\Sigma q : \Sigma(A\times B)\to\Sigma(A\wedge B)$ is an H'-map and $(\Sigma q)^{\#}$ is a monomorphism, it is easily checked that $\mathrm{conil}(A\wedge B) \leq \mathrm{conil}(A\times B)$. Thus the image of J is an abelian group and hence $\Sigma\langle f, g\rangle = 0$. We now have $\langle f, g\rangle = 0$ since Σ is a monomorphism. This proves the lemma.

We now prove the theorems announced above. We begin with Theorem 2.

Proof of Theorem 2. We assume $\mathrm{conil}(A\times B) \leq 1$. Let X be any space and consider the generalized Whitehead product $[\alpha, \beta]$ where $\alpha\in[\Sigma A, X]$, $\beta\in[\Sigma B, X]$,. The relation between the Whitehead product and the Samelson product gives $\tau[\alpha,\beta] = \langle\tau(\alpha), \tau(\beta)\rangle$. Now $\tau(\alpha)\in[A, \Omega X]$, $\tau(\beta)\in[B, \Omega X]$ and ΩX is a G-space. Hence by Lemma 3 we have $\langle\tau(\alpha), \tau(\beta)\rangle = 0$. Since τ is an isomorphism, we have $[\alpha, \beta] = 0$.

Proof of Theorem 1. We proceed by induction on n. Consider the case $n = 1$. Let $i_1 : \Sigma A\to\Sigma A\vee\Sigma B$, $i_2 : \Sigma B\to\Sigma A\vee\Sigma B$ be the inclusion maps. Applying Theorem 2, the hypothesis of Theorem 1 implies that $[i_1, i_2] = 0$. It follows then from Proposition 5.2 of [1] that the inclusion $j : \Sigma A\vee\Sigma B\to\Sigma A\times\Sigma B$ is a homotopy-equivalence. Clearly $\Sigma A\vee\Sigma B$ has an obvious suspension structure. Hence $\mathrm{conil}(\Sigma A\times\Sigma B) \leq 1$. The proof is completed by induction on n.

Proof of Theorem 3. In case $A = B$, we have that if $\mathrm{conil}(A\times A) \leq 1$, then $\Sigma A\times\Sigma A$ has the homotopy type of a suspension. Hence $\mathrm{cat}(\Sigma A\times\Sigma A) \leq 1$ where we have normalised category. We now apply Theorem 1.1 of [5] and conclude that ΣA is contractible. If $\pi_1(A)$ is abelian this now implies that A itself is contractible. This proves Theorem 3.

Remark 1. We can apply the above results to various examples. Thus, let π be a non-trivial abelian group. Then while it is possible for $\mathrm{conil}\, K(\pi, n) \leq 1$, we always have $\mathrm{conil}\, K(\pi + \pi, n) \geq 2$. We can also show that for spheres S^m, S^n, we have $\mathrm{conil}\,(S^m\times S^n) = 2$.

Remark 2. Since the generalised Whitehead product, the generalised Samelson product and the Hopf construction can all be dualised, many of the above results can be dualised. We leave it to the reader to check which results can be dualised.

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