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ADJOINT ORBITS OF SEMI-SIMPLE LIE GROUPS AND LAGRANGIAN SUBMANIFOLDS

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Abstract We give various realizations of the adjoint orbits of a semi-simple Lie group and describe their symplectic geometry. We then use these realizations to identify a family of Lagrangian submanifolds of the orbits.

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1. Introduction

Let G be a non-compact (real or complex) semi-simple Lie group with Lie algebra $\mathfrak{g}$. The purpose of this paper is to describe various realizations of the homogeneous spaces G/Z_H with Z_H the centralizer in G of an element H belonging to a Cartan subalgebra of $\mathfrak{g}$. We then use these descriptions to study the symplectic geometry of adjoint orbits and to identify a family of Lagrangian submanifolds.

Our motivation in studying these homogeneous spaces is the construction of Lefschetz fibrations in [**4**]. The full description of these fibrations requires a further understanding of the symplectic geometry (or, rather, geometries) of G/Z_H, and particularly of those properties related to the description of the Fukaya category of the Lagrangian vanishing cycles. In this paper we study concepts that are relevant to questions motivated by mirror symmetry (see [**5**] and [**10**]) and that are also of general interest in Lie theory.

To be more specific, let $\mathfrak{a}$ be a Cartan–Chevalley algebra of $\mathfrak{g}$, that is, the Lie algebra of the A component of an Iwasawa decomposition $G = KAN$. We select a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ and pick $H_0 \in \operatorname{cl} \mathfrak{a}^+$. The adjoint orbit $\operatorname{Ad}(G)H_0$ is diffeomorphic to the homogeneous space G/Z_{H_0}. Also the subadjoint orbit $\operatorname{Ad}(K)H_0$ is diffeomorphic to a flag manifold $\mathbb{F}_{H_0} = G/P_{H_0}$, where P_{H_0} is the parabolic subgroup defined by H_0, which contains Z_{H_0}.

In this paper we get other realizations of G/Z_{H_0}. First we prove that G/Z_{H_0} has the structure of a vector bundle over $\mathbb{F}_{H_0}$ isomorphic to the cotangent bundle $T^*\mathbb{F}_{H_0}$. This fact was proved before by Azad, van den Ban and Biswas [**2**] using a different approach. Here we exploit more decisively the associated vector bundle construction obtained by P_{H_0}-representations by viewing $G \to \mathbb{F}_{H_0} = G/P_{H_0}$ as a P_{H_0}-principal bundle (see § 2.1).

The isomorphism $\mathrm{Ad}(G)H_0 \approx T^*\mathbb{F}_{H_0}$ provides the adjoint orbit with two different actions: namely, the natural transitive action on $\mathrm{Ad}(G)H_0$ and the linear action on $T^*\mathbb{F}_{H_0}$ obtained by lifting the action of G on $\mathbb{F}_{H_0}$. The latter action is not transitive since the zero section is invariant. One is therefore asked to build a transitive action on the cotangent bundle $T^*\mathbb{F}_{H_0}$ that is different from the linear action. We do so by constructing a Lie algebra $\theta(\mathfrak{g})$ of Hamiltonian vector fields (with respect to the canonical symplectic form Ω of $T^*\mathbb{F}_{\Theta}$) that is isomorphic to $\mathfrak{g}$. The elements of $\theta(\mathfrak{g})$ are complete vector fields and hence the infinitesimal action given by $\theta(\mathfrak{g})$ integrates to an action of a Lie group, by a classical theorem of Palais [**7**]. This action is transitive and Hamiltonian by construction. The isotropy subgroup of the transitive action is Z_{H_0} and thus $T^*\mathbb{F}_{H_0}$ gets identified with G/Z_{H_0}. It turns out that the moment map $\mu\colon T^*\mathbb{F}_{H_0} \to \mathfrak{g}$ of the Hamiltonian action takes values in $\mathrm{Ad}(G)H_0$ and is the inverse of the previously defined map $\mathrm{Ad}(G)H_0 \to T^*\mathbb{F}_{H_0}$.

In another realization of G/Z_{H_0}, it is compactified to an algebraic projective variety, namely, the product $\mathbb{F}_{H_0} \times \mathbb{F}_{H_0^*}$, where $\mathbb{F}_{H_0^*}$ is the flag manifold dual to $\mathbb{F}_{H_0}$ (see § 3). This is obtained by the diagonal action $g(x, y) = (gx, gy)$ of G on $\mathbb{F}_{H_0} \times \mathbb{F}_{H_0^*}$, which has just one open and dense orbit whose isotropy group is $Z_{H_0} = Z_{H_0^*}$ and hence realizes G/Z_{H_0}. The embedding $G/Z_{H_0} \to \mathbb{F}_{H_0} \times \mathbb{F}_{H_0^*}$ induces several geometric structures on G/Z_{H_0} inherited from those of $\mathbb{F}_{H_0} \times \mathbb{F}_{H_0^*}$. The point is that $\mathbb{F}_{H_0} \times \mathbb{F}_{H_0^*}$ is a flag manifold of $G \times G$ and hence admits Riemannian metrics (Hermitian in the complex case) that are invariant by the compact group $K \times K$. These metrics on $\mathbb{F}_{H_0} \times \mathbb{F}_{H_0^*}$ induce new metrics on G/Z_{H_0}, as well as new symplectic structures in the complex case.

The embedding $G/Z_{H_0} \to \mathbb{F}_{H_0} \times \mathbb{F}_{H_0^*}$ combined with representations of $\mathfrak{g}$ yields realizations of G/Z_{H_0} as orbits on $V \otimes V^*$, where V is the space of an irreducible representation of $\mathfrak{g}$ with highest weight defined by H_0 (see § 4).

The last two realizations of G/Z_{H_0} are used in §§ 5 and 6 to build a class of Lagrangian submanifolds in G/Z_{H_0} with respect to the symplectic structures inherited from the embedding $G/Z_{H_0} \to \mathbb{F}_{H_0} \times \mathbb{F}_{H_0^*}$.

2. Adjoint orbits and cotangent bundles of flags

Let $\mathfrak{g}$ be a non-compact semi-simple Lie algebra (real or complex) and let G be a connected Lie group with finite centre and Lie algebra $\mathfrak{g}$ (e.g. G may be $\mathrm{Aut}_0(\mathfrak{g})$, the component of the identity of the group of automorphisms).

The usual notation is given below.

(1) The Cartan decomposition: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, with global decomposition $G = KS$.

(2) The Iwasawa decomposition: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, with global decomposition $G = KAN$.

(3) Π is a set of roots of $\mathfrak{a}$, with a choice of a set of positive roots Π^+ and simple roots $\Sigma \subset \Pi^+$ such that $\mathfrak{n}^+ = \sum_{\alpha>0} \mathfrak{g}_\alpha$ and $\mathfrak{g}_\alpha$ is the root space of the root α. The corresponding Weyl chamber is $\mathfrak{a}^+$.

(4) A subset $\Theta \subset \Sigma$ defines a parabolic subalgebra $\mathfrak{p}_\Theta$ with parabolic subgroup P_Θ and a flag $\mathbb{F}_\Theta = G/P_\Theta$. The flag is also $\mathbb{F}_\Theta = K/K_\Theta$, where $K_\Theta = K \cap P_\Theta$. The Lie algebra of K_Θ is denoted by $\mathfrak{k}_\Theta$.

(5) $H_\Theta \in \operatorname{cl} \mathfrak{a}^+$ is *characteristic* for $\Theta \subset \Sigma$ if $\Theta = \{\alpha \in \Sigma \colon \alpha(H_\Theta) = 0\}$. Then, $\mathfrak{p}_\Theta = \bigoplus_{\lambda \geqslant 0} \mathfrak{g}_\lambda$, where λ runs through the non-negative eigenvalues of $\operatorname{ad}(H_\Theta)$. Conversely, starting with $H_0 \in \operatorname{cl} \mathfrak{a}^+$ we define $\Theta_{H_0} = \{\alpha \in \Sigma \colon \alpha(H_0) = 0\}$, and for the several objects requiring a subscript Θ we use H_0 instead of Θ_{H_0}. For instance, $\mathbb{F}_{H_0} = \mathbb{F}_{\Theta_{H_0}}$, etc.

(6) $b_\Theta = 1 \cdot K_\Theta = 1 \cdot P_\Theta$ denotes the origin of the flag $\mathbb{F}_\Theta = K/K_\Theta = G/P_\Theta$.

(7) We write

$$\mathfrak{n}_\Theta^+ = \sum_{\alpha(H_\Theta)>0} \mathfrak{g}_\alpha, \qquad \mathfrak{n}_\Theta^- = \sum_{\alpha(H_\Theta)<0} \mathfrak{g}_\alpha,$$

so that $\mathfrak{g} = \mathfrak{n}_\Theta^- \oplus \mathfrak{z}_\Theta \oplus \mathfrak{n}_\Theta^+$, where $\mathfrak{z}_\Theta$ is the centralizer of H_Θ in $\mathfrak{g}$.

(8) $Z_\Theta = \{g \in G \colon \operatorname{Ad}(g) H_\Theta = H_\Theta\}$ is the centralizer in G of the characteristic element H_Θ. Its Lie algebra is $\mathfrak{z}_\Theta$. Moreover, K_Θ is the centralizer of H_Θ in K:

$$K_\Theta = Z_K(H_\Theta) = Z_\Theta \cap K = \{k \in K \colon \operatorname{Ad}(k) H_\Theta = H_\Theta\}.$$

Theorem 2.1. *The adjoint orbit $\mathcal{O}(H_\Theta) = \operatorname{Ad}(G) \cdot H_\Theta \approx G/Z_\Theta$ of the characteristic element H_Θ is a C^∞ vector bundle over $\mathbb{F}_\Theta$ that is isomorphic to the cotangent bundle $T^*\mathbb{F}_\Theta$. Moreover, we can write down a diffeomorphism $\iota \colon \operatorname{Ad}(G) \cdot H_\Theta \to T^*\mathbb{F}_\Theta$ such that*

(1) *ι is equivariant with respect to the actions of K, that is, for all $k \in K$,*

$$\iota \circ \operatorname{Ad}(k) = \tilde{k} \circ \iota,$$

where $\tilde{k}$ is the lifting to $T^*\mathbb{F}_\Theta$ (via the differential) of the action of k on $\mathbb{F}_\Theta$; and

(2) *the pullback of the canonical symplectic form on $T^*\mathbb{F}_\Theta$ by ι is the (real) Kirillov–Kostant–Souriaux form on the orbit.*

The diffeomorphism $\iota \colon \mathcal{O}(H_\Theta) \to T^*\mathbb{F}_\Theta$ (see (2.1)) will be defined in two steps: first $\mathcal{O}(H_\Theta)$ is proved to be diffeomorphic to a vector bundle $\mathcal{V} \to K/K_\Theta$ associated with the principal bundle $K \to K/K_\Theta$, built from a representation of K_Θ; then $\mathcal{V} \to K/K_\Theta$ is proved to be isomorphic to $T^*\mathbb{F}_\Theta$.

Remark 2.2. The equivariance of item (1) holds only for the action of K. However, there exists also an action of G on the vector bundle, obtained via the diffeomorphism with $\mathcal{O}(H_\Theta)$. Unlike the action of K, this action is nonlinear since the linear action is not transitive.

The projection $\pi\colon \mathcal{O}(H_\Theta) \to \mathbb{F}_\Theta$ is obtained via the action of G. Given the homogeneous spaces $\mathcal{O}(H_\Theta) = G/Z_\Theta$ and $\mathbb{F}_\Theta = G/P_\Theta$, the centralizer Z_Θ is contained in P_Θ. We obtain a canonical fibration $gZ_\Theta \mapsto gP_\Theta$ with fibre P_Θ/Z_Θ. On the one hand, in terms of the adjoint action the fibre is $\mathrm{Ad}(P_\Theta)\cdot H_\Theta$, whereas on the other hand, it is the affine subspace $H_\Theta + \mathfrak{n}_\Theta^+$, where $\mathfrak{n}_\Theta^+$ is the sum of the eigenspaces of $\mathrm{ad}(H_\Theta)$ associated with eigenvalues greater than 0: that is,

$$\mathfrak{n}_\Theta^+ = \sum \mathfrak{g}_\alpha,$$

with the sum running over the positive roots α outside $\langle\Theta\rangle$, i.e. with $\alpha(H_\Theta) > 0$. Indeed, if $g \in P_\Theta$, then $\mathrm{Ad}(g)H_\Theta = H_\Theta + X$, with $X \in \mathfrak{n}_\Theta^+$. Moreover, if $N_\Theta = \mathrm{e}^{\mathfrak{n}_\Theta}$, then the map $g \in N_\Theta \mapsto \mathrm{Ad}(g)H_\Theta - H_\Theta \in \mathfrak{n}_\Theta$ is a diffeomorphism.

Example 2.3. The example of $\mathfrak{sl}(n)$ ($\mathbb{R}$ or $\mathbb{C}$) is enlightening: P_Θ is the group of matrices that are block upper triangular. The diagonal part (in blocks) is Z_Θ, whereas $\mathfrak{n}_\Theta^+$ is the upper triangular part above the blocks. H_Θ is a diagonal matrix that has one scalar matrix in each block. Thus, conjugation $\mathrm{Ad}(g)H_\Theta = gH_\Theta g^{-1}$ keeps H_Θ inside the blocks and adds an upper triangular part above the blocks, that is, $gH_\Theta g^{-1} = H_\Theta + X$ for some $X \in \mathfrak{n}_\Theta^+$.

The fibre of $\pi\colon \mathcal{O}(H_\Theta) \to \mathbb{F}_\Theta$ is a vector space. This alone does not guarantee the structure of a vector bundle. Nevertheless, the structure of a vector bundle can be obtained as a bundle associated with the principal bundle $K \to K/K_\Theta$ with structure group K_Θ.

2.1. $\mathcal{O}(H_\Theta) \to \mathbb{F}_\Theta$ is a vector bundle

The adjoint representation of K_Θ on $\mathfrak{g}$ leaves invariant the subspace $\mathfrak{n}_\Theta^+$ and, consequently, $\mathrm{Ad}(k)$ takes eigenspaces of $\mathrm{ad}(H_\Theta)$ to eigenspaces. It follows that the restriction of Ad defines a representation ρ of K_Θ on $\mathfrak{n}_\Theta^+$. This allows us to define the vector bundle $K \times_\rho \mathfrak{n}_\Theta^+$ associated with the principal bundle $K \to K/K_\Theta$. To define a diffeomorphism between $\mathcal{O}(H_\Theta)$ and $K \times_\rho \mathfrak{n}_\Theta^+$ recall that $\mathcal{O}(H_\Theta) = \bigcup_{k\in K} \mathrm{Ad}(k)(H_\Theta + \mathfrak{n}_\Theta^+)$.

Proposition 2.4. *The map $\gamma\colon \mathcal{O}(H_\Theta) \to K \times_\rho \mathfrak{n}_\Theta^+$ defined by*

$$Y = \mathrm{Ad}(k)(H_\Theta + X) \in \mathcal{O}(H_\Theta) \mapsto k \cdot X \in K \times_\rho \mathfrak{n}_\Theta^+$$

is a diffeomorphism satisfying that

(1) *γ is equivariant with respect to the actions of K,*

(2) *γ maps fibres onto fibres, and*

(3) *γ maps the orbit $\mathrm{Ad}(K)H_\Theta$ onto the zero section of $K \times_\rho \mathfrak{n}_\Theta^+$.*

Proof. To see that γ is well defined: if $\mathrm{Ad}(k)(H_\Theta + X) = \mathrm{Ad}(k_1)(H_\Theta + X_1)$, then $\mathrm{Ad}(u)(H_\Theta + X) = H_\Theta + X_1$, where $u = k_1^{-1}k$. By equivariance, it then follows that

$$\begin{aligned} u \cdot b_\Theta &= u \cdot \pi(H_\Theta + X) \\ &= \pi(\mathrm{Ad}(u)(H_\Theta + X)) \\ &= \pi(H_\Theta + X_1) \\ &= b_\Theta. \end{aligned}$$

Consequently, $u \in K_\Theta$, and therefore $\mathrm{Ad}(u)(H_\Theta + X) = H_\Theta + \mathrm{Ad}(u)X = H_\Theta + X_1$, with $X_1 = \mathrm{Ad}(u)X$. Hence,

$$k_1 \cdot X_1 = ku^{-1} \cdot \rho(u)X = k \cdot X,$$

showing that γ is well defined. Surjectivity follows because of $k \cdot X = \gamma(\mathrm{Ad}(k)(H_\Theta + X))$. For injectivity, since $k_1 \cdot X_1 = k \cdot X$ implies $k_1 = ku$ and $X_1 = \mathrm{Ad}(u^{-1})X$, $u \in K_\Theta$. Hence,

$$\begin{aligned}\mathrm{Ad}(k_1)(H_\Theta + X_1) &= \mathrm{Ad}(k)(\mathrm{Ad}(u)H_\Theta + \mathrm{Ad}(u)X_1)\\ &= \mathrm{Ad}(k)(H_\Theta + X).\end{aligned}$$

Now, the fibre of $\mathcal{O}(H_\Theta)$ over $k \cdot b_\Theta$ is $\mathrm{Ad}(k)(H_\Theta + \mathfrak{n}_\Theta^+)$, which is taken by γ to elements of the type $k \cdot X$, which are in the fibre over $k \cdot b_\Theta$ of $K \times_\rho \mathfrak{n}_\Theta^+$. Also, $\gamma(\mathrm{Ad}(k)(H_\Theta)) = k \cdot 0$, which is in the zero section of $K \times_\rho \mathfrak{n}_\Theta^+$. Equivariance holds because

$$\gamma \circ \mathrm{Ad}(u)(\mathrm{Ad}(k)(H_\Theta + X)) = \gamma(\mathrm{Ad}(uk)(H_\Theta + X)) = uk \cdot X$$

and the last term is the left action of $u \in K$ on the vector bundle. Finally, diffeomorphism follows from the manifold constructions of $\mathcal{O}(H_\Theta)$ (as a homogeneous space) and $K \times_\rho \mathfrak{n}_\Theta^+$ (as an associated bundle). □

From the diffeomorphism γ we endow $\mathcal{O}(H_\Theta)$ with the structure of a vector bundle coming from $K \times_\rho \mathfrak{n}_\Theta^+$. Its fibres are the affine subspaces $\mathrm{Ad}(k)(H_\Theta + \mathfrak{n}_\Theta^+)$.

2.2. Isomorphism with $T^*\mathbb{F}_\Theta$

Let L be a Lie group, let M be a closed subgroup, and let $\iota\colon M \to \mathrm{Gl}(T_{x_0}(L/M))$ be the isotropy representation of M on the tangent space of L/M at x_0. The tangent bundle $T(L/M)$ is isomorphic to the vector bundle $L \times_\iota T_{x_0}(L/M)$, associated with the principal bundle $L \to L/M$ via the representation ι. Similarly, if ι^* is the dual representation, then $T^*(L/M)$ is isomorphic to the vector bundle $L \times_{\iota^*} (T_{x_0}(L/M))^*$. Observe that if $Q \times_{\rho_1} V$ and $Q \times_{\rho_2} W$ are vector bundles associated with the principal bundle $Q \to X$, via equivalent representations ρ_1 and ρ_2, then $Q \times_{\rho_1} V$ is isomorphic to $Q \times_{\rho_2} W$.

The tangent space $T_{b_\Theta}\mathbb{F}_\Theta$ can be identified with $\mathfrak{n}_\Theta^- = \sum_{\alpha(H_\Theta)<0} \mathfrak{g}_\alpha$, and the isotropy representation becomes the restriction of the adjoint representation. The subspace $\mathfrak{n}_\Theta^+$ is isomorphic to the dual $(\mathfrak{n}_\Theta^-)^*$ of $\mathfrak{n}_\Theta^-$ via the Cartan–Killing form $\langle\cdot,\cdot\rangle$ of $\mathfrak{g}$. Thus, the map

$$X \in \mathfrak{n}_\Theta^+ \mapsto \langle X, \cdot\rangle \in (\mathfrak{n}_\Theta^-)^*$$

is an isomorphism.

Therefore, $T^*\mathbb{F}_\Theta = T^*(K/K_\Theta)$ is isomorphic to $K \times_\rho \mathfrak{n}_\Theta^+$, which in turn is diffeomorphic to the adjoint orbit $\mathcal{O}(H_\Theta)$. Both diffeomorphisms permute the action of K. This finishes the proof of the first part of Theorem 2.1, as well as of item (1). Thus, the diffeomorphism $\iota\colon \mathcal{O}(H_\Theta) \to T^*\mathbb{F}_\Theta$ is obtained by composing $\gamma\colon \mathcal{O}(H_\Theta) \to K \times_\rho \mathfrak{n}_\Theta^+$ with the vector bundle isomorphism between $K \times_\rho \mathfrak{n}_\Theta^+$ and $T^*\mathbb{F}_\Theta$. It is explicitly given by

$$\iota\colon \mathrm{Ad}(k)(H_\Theta + X) \in \mathcal{O}(H_\Theta) \mapsto \langle \mathrm{Ad}(k)X, \cdot\rangle \in T^*_{kb_\Theta}\mathbb{F}_\Theta, \tag{2.1}$$

where $X \in \mathfrak{n}_\Theta^+$ and $T_{kb_\Theta}\mathbb{F}_\Theta$ is identified with $\mathrm{Ad}(k)\mathfrak{n}_\Theta^-$.

Item (2) of Theorem 2.1 is a consequence of Proposition 2.16 below.

2.3. The action of G on $T^*\mathbb{F}_\Theta$

The diffeomorphism $\iota\colon \mathcal{O}(H_\Theta) \to T^*\mathbb{F}_\Theta$ induces an action of G on $T^*\mathbb{F}_\Theta$ by $g\alpha = \iota \circ \mathrm{Ad}(g) \circ \iota^{-1}(\alpha)$, $g \in G$, $\alpha \in T^*\mathbb{F}_\Theta$. The action of K is linear since it is given by the lifting of the linear action on $\mathbb{F}_\Theta$. However, the action of G is not linear because the linear action on $T^*\mathbb{F}_\Theta$ is not transitive (the zero section is invariant). It is therefore natural to ask how the action of G behaves in terms of the geometry of $T^*\mathbb{F}_\Theta$. The description of this action will be made via an infinitesimal action of the Lie algebra $\mathfrak{g}$ of G, that is, through a homomorphism $\theta\colon \mathfrak{g} \to \Gamma(T^*\mathbb{F}_\Theta)$, which associates with each element of the Lie algebra $\mathfrak{g}$ a Hamiltonian vector field on $T^*\mathbb{F}_\Theta$.

Let Ω be the canonical symplectic form on $T^*\mathbb{F}_\Theta$. Given a vector field X on $\mathbb{F}_\Theta$ denote by $X^\#$ the lifting of X to $T^*\mathbb{F}_\Theta$. The flow of $X^\#$ is linear and is defined by $\alpha \in T^*_x\mathbb{F}_\Theta \mapsto \alpha \circ (\mathrm{d}\phi_{-t})_{\phi_t(x)}$, where ϕ_t is the flow of X. The lifting satisfies

(1) $\pi_* X^\# = X$, where $\pi\colon T^*\mathbb{F}_\Theta \mapsto \mathbb{F}_\Theta$ is the projection;

(2) $X^\#$ is the Hamiltonian vector field with respect to Ω for the function $h_X(\xi) = \xi(X(x))$, $\xi \in T^*_x\mathbb{F}_\Theta$; and

(3) if X and Y are vector fields, then $[X,Y]^\# = [X^\#, Y^\#]$, that is, $X \mapsto X^\#$ is a homomorphism of Lie algebras.

Now, for $Y \in \mathfrak{g}$ we denote the vector field on $\mathbb{F}_\Theta$ whose flow is e^{tY} by $\tilde{Y}$ or simply by Y if there is no confusion.

Since the action of K in $T^*\mathbb{F}_\Theta$ is linear, it follows that the vector field induced by $A \in \mathfrak{k}$ on $T^*\mathbb{F}_\Theta$ is $X^\#$, that is, $\theta(X) = X^\#$ if $X \in \mathfrak{k}$. Using the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, it remains to describe $\theta(X)$ when $X \in \mathfrak{s}$. This is done by modifying the vector field $X^\#$ by a vertical one so that the new vector field still projects on X.

The following lemma is well known. We include it here for the sake of completeness.

Lemma 2.5. *Let M be a manifold and let $f\colon M \to \mathbb{R}$. Define $F\colon T^*M \to \mathbb{R}$ by $F = f \circ \pi$ ($\pi\colon T^*M \to M$ is the projection). Let V_F be the Hamiltonian vector field of F with respect to Ω. Then, V_F is vertical ($\pi_* V_F = 0$), and V_F is the constant parallel vector field whose restriction to a fibre T^*_xM is $-\mathrm{d}f_x \in T^*_xM$.*

Proof. A straightforward way to see this is to use local coordinates q, p of M and the fibre, respectively. The Hamiltonian vector field is then

$$V_F = \sum_i \frac{\partial F}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial F}{\partial q_i}\frac{\partial}{\partial p_i}.$$

Since the function F does not depend on p, only the second term remains, showing that the vector field is vertical. If $x = (q_1, \dots, q_n) \in M$ is fixed, then the second term becomes

$$\sum_i -\frac{\partial F}{\partial q_i}\frac{\partial}{\partial p_i} = -\mathrm{d}f_x,$$

since $\partial F/\partial q_i = \partial f/\partial q_i$. $\square$

We return to $\mathbb{F}_\Theta$, which coincides with the adjoint orbit $\mathrm{Ad}(K)\cdot H_\Theta \subset \mathfrak{s}$. Given $X \in \mathfrak{s}$, we can define the height function

$$f_X(x) = \langle x, X\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the Cartan–Killing form, which is an inner product when restricted to $\mathfrak{s}$.

Now choose a K-invariant Riemannian metric $(\cdot,\cdot)_{\mathrm{B}}$ on $\mathbb{F}_\Theta$. The most convenient for our purposes is the so-called Borel metric, which has the property that for any $X \in \mathfrak{s}$ the gradient of f_X is exactly the vector field X induced by X (see [**3**]).

For $X \in \mathfrak{s}$ set $F_X = f_X \circ \pi$ and denote by V_X its Hamiltonian vector field on $T^*\mathbb{F}_\Theta$. By Lemma 2.5 V_X is vertical.

The following lemma will be used to evaluate the symplectic form on the several Hamiltonian vector fields defined above.

Lemma 2.6. *We have the following directional derivatives.*

(1) *If $A \in \mathfrak{k}$ and $X \in \mathfrak{s}$, then $A^\# F_X = F_{[A,X]}$.*

(2) *If $X, Y \in \mathfrak{s}$, then $X^\# F_Y = Y^\# F_X$.*

(3) *If $X, Y \in \mathfrak{s}$, then $V_X F_Y = 0$.*

Proof. Straightforward calculation. □

Remark 2.7. In the computation of the partial derivative of item (1) we used the fact that the Lie algebra of G is formed by *right* invariant fields. For the bracket $[\cdot,\cdot]$ in $\mathfrak{g}$ formed by the right invariant vector fields, the following equality holds: $\mathrm{Ad}(\mathrm{e}^A) = \mathrm{e}^{-\mathrm{ad}(A)}$. The reason to use right invariant vector fields is so that we can project onto homogeneous spaces.

We deduce the Lie brackets between the Hamiltonian vector fields.

Corollary 2.8. *We have the following Lie brackets.*

(1) *If $A \in \mathfrak{k}$ and $X \in \mathfrak{s}$, then $[A^\#, V_X] = V_{[A,X]}$.*

(2) *If $X, Y \in \mathfrak{s}$, then $[X^\#, V_Y] = [Y^\#, V_X]$.*

(3) *If $X, Y \in \mathfrak{s}$, then $[V_X, V_Y] = 0$.*

Corollary 2.9. *The map θ defined on $\mathfrak{g}$ and taking values on vector fields of $T^*\mathbb{F}_\Theta$ defined by $\theta(A) = A^\#$ if $A \in \mathfrak{k}$ and $\theta(X) = X^\# + V_X$ is a homomorphism of Lie algebras.*

Proof. This follows directly from the brackets in Corollary 2.8. □

In other words, θ is an infinitesimal action of $\mathfrak{g}$ on $T^*\mathbb{F}_\Theta$. By a classical result of Palais this action is integrated to an action of a connected Lie group G whose Lie algebra is $\mathfrak{g}$, provided the vector fields are complete.

Lemma 2.10. *The vector fields $\theta(Z)$, $Z \in \mathfrak{g}$ are complete.*

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Proof. Take $Z = A + X$ with $A \in \mathfrak{k}$ and $X \in \mathfrak{s}$ so that $\theta(Z) = A^{\#} + X^{\#} + V_X = (A+X)^{\#} + V_X$. Suppose by contradiction that there exists a maximal trajectory $z(t)$ of Z defined in a proper interval $(a,b) \subset \mathbb{R}$, with, for example, $b < \infty$. This implies that $\lim_{t\to b} z(t) = \infty$. Let $x(t)$ be the projection of $z(t)$ onto $\mathbb{F}_\Theta$. Then $x(t)$ is a trajectory of the vector field $\widetilde{A+X}$ on $\mathbb{F}_\Theta$ induced by $A+X$. Since $\widetilde{A+X}$ is complete (by compactness of $\mathbb{F}_\Theta$), there exists $\lim_{t\to b} x(t) = x(b)$.

In a local trivialization $T^*\mathbb{F}_\Theta \approx U \times \mathbb{R}^n$ around $x(b)$ we have $z(t) = (x(t), y(t))$. The second component $y(t)$ satisfies a linear equation

$$\dot{y} = A(t)y + c(t),$$

where $A(t)$ is the derivative at $x(t)$ of the vector field $\widetilde{A+X}$ and $c(t) = V_X(x(t))$. The solution of this linear equation is defined in a neighbourhood of b, contradicting the fact that $z(t) \to \infty$ as $t \to b$. □

As a consequence we obtain the following result.

Proposition 2.11. *The infinitesimal action θ integrates to an action $a\colon G \times T^*\mathbb{F}_\Theta \to T^*\mathbb{F}_\Theta$ of a connected Lie group G with Lie algebra $\mathfrak{g}$. This action $a(g,x) = g \cdot x$ satisfies*

(1) $\theta(Y)(x) = \frac{\mathrm{d}}{\mathrm{d}t} a(\mathrm{e}^{tY}, x)_{|t=0}$ *for all* $Y \in \mathfrak{g}$*;*

(2) *the action is Hamiltonian since the vector fields $\theta(Y)$, $Y \in \mathfrak{g}$ are Hamiltonian vector fields;*

(3) *the projection $\pi\colon T^*\mathbb{F}_\Theta \to \mathbb{F}_\Theta$ is equivariant with respect to this new action and the action of G on $\mathbb{F}_\Theta$; and*

(4) *the action a is transitive.*

Proof. The first two items are due to the construction of θ and a. As to equivariance, it holds because for any $Y \in \mathfrak{g}$ the projection $\pi_*\theta(Y)$ is the vector field $\tilde{Y}$ induced by Y via the action on $\mathbb{F}_\Theta$.

To prove transitivity we observe that the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ induces the Cartan decomposition $G = KS$. The group K acts on $T^*\mathbb{F}_\Theta$ by linear transformations among the fibres, since $\theta(A) = A^{\#}$ for $A \in \mathfrak{k}$. Since K acts transitively on $\mathbb{F}_\Theta$, it suffices to verify that G acts transitively on a single fibre.

Let $b_\Theta \in \mathbb{F}_\Theta$ be the origin of $\mathbb{F}_\Theta$, which is also seen as the null vector of $T^*_{b_\Theta}\mathbb{F}_\Theta$. The orbit $G \cdot b_\Theta$ on $T^*\mathbb{F}_\Theta$ is then open, because the tangent space to the orbit

$$\{\theta(Z)(b_\Theta)\colon Z \in \mathfrak{g}\}$$

coincides with the tangent space $T_{b_\Theta}(T^*\mathbb{F}_\Theta)$.

In fact, $T_{b_\Theta}(T^*\mathbb{F}_\Theta)$ is the sum of the (horizontal) tangent space $T\mathbb{F}_\Theta$ with the (vertical) fibre $T^*_{b_\Theta}\mathbb{F}_\Theta$. The transitive action of K on $\mathbb{F}_\Theta$ guarantees that $T\mathbb{F}_\Theta = \{\theta(A)(b_\Theta)\colon A \in \mathfrak{k}\}$. On the other hand, given $X \in \mathfrak{s}$ there exists $A \in \mathfrak{k}$ such that $\tilde{X}(b_\Theta) = \tilde{A}(b_\Theta)$. In such a case, $\widetilde{X-A}(b_\Theta) = 0$, which implies that $(X-A)^{\#}(b_\Theta) = V_X(b_\Theta)$. The vertical vector

$V_X(b_\Theta)$ is the linear functional of $T_{b_\Theta}\mathbb{F}_\Theta$ given by $v \mapsto (\tilde{X}(b_\Theta), v)_{\mathrm{B}} = (\mathrm{d}f_X)_{b_\Theta}(v)$. These linear functionals generate $T^*_{b_\Theta}\mathbb{F}_\Theta$, since $\tilde{X}(b_\Theta)$, $X \in \mathfrak{s}$, generates $T_{b_\Theta}\mathbb{F}_\Theta$. This shows that the vertical space is contained in the space tangent to the orbit, concluding the proof that the orbit is open.

Finally, take $H \in \mathfrak{a}^+$. Then, $V_H(b_\Theta) = 0$, since $\tilde{H}(b_\Theta) = 0$. Moreover, $H^\#$ is vertical in the fibre over b_Θ and restricts to the fibre as a linear vector field. Since H was chosen in the positive chamber $\mathfrak{a}^+$, such a linear vector field is given by a linear transformation whose eigenvalues are all negative. This implies that any trajectory of $H^\#$ in the fibre intercepts every neighbourhood of the origin. Since $G \cdot b_\Theta$ contains a neighbourhood of the origin, we conclude that G is transitive in the fibre $T^*_{b_\Theta}\mathbb{F}_\Theta$, showing that the action is transitive. □

The next step is to identify $T^*\mathbb{F}_\Theta$ as a homogeneous space of G, via the transitive action of the previous proposition. First of all we will find the isotropy algebra $\mathfrak{l}$ at b_Θ, that is,

$$\mathfrak{l} = \{Y \in \mathfrak{g} \colon \theta(Y)(b_\Theta) = 0\},$$

where the origin of the flag b_Θ is also seen as the null vector of $T^*_{b_\Theta}\mathbb{F}_\Theta$.

Lemma 2.12. *The isotropy subalgebra $\mathfrak{l} = \{Y \in \mathfrak{g} \colon \theta(Y)(b_\Theta) = 0\}$ coincides with the isotropy subalgebra at H_Θ of the adjoint orbit, that is, $\mathfrak{l} = \mathfrak{z}_\Theta$.*

Proof. Let $Y \in \mathfrak{g}$ with $\theta(Y)(b_\Theta) = 0$ and $Y = A + X$, $A \in \mathfrak{k}$ and $X \in \mathfrak{s}$. Then, $\theta(Y) = A^\# + X^\# + V_X$, and since $A^\#(b_\Theta) = X^\#(b_\Theta) = 0$, it follows that $V_X(b_\Theta) = 0$. However, as in the previous proof, $V_X(b_\Theta)$ is the linear functional $v \mapsto (\tilde{X}(b_\Theta), v)_{\mathrm{B}}$. Therefore, $\tilde{X}(b_\Theta) = 0$. On the other hand, $\theta(Y)(b_\Theta) = 0$ implies that $\tilde{Y}(b_\Theta) = 0$, and consequently $\tilde{A}(b_\Theta) = -\tilde{X}(b_\Theta) = 0$. This shows that $A \in \mathfrak{p}_\Theta \cap \mathfrak{k} \subset \mathfrak{z}_\Theta$ and $B \in \mathfrak{p}_\Theta \cap \mathfrak{s} \subset \mathfrak{z}_\Theta$, thus $Y \in \mathfrak{z}_\Theta$. Therefore,

$$\{Y \in \mathfrak{g} \colon \theta(Y)(b_\Theta) = 0\} \subset \mathfrak{z}_\Theta.$$

Equality follows from the fact that these algebras have the same dimension, since they are isotropy algebras of spaces of equal dimension. □

The equality of isotropy Lie algebras $\mathfrak{l} = \mathfrak{z}_\Theta$ immediately shows the equality of isotropy subgroups if we know in advance that they are connected, as happens in the case of complex Lie algebras, for instance. The next statement shows that the isotropy groups do indeed coincide.

Proposition 2.13. *Let L be the isotropy group of the action $a \colon G \times T^*\mathbb{F}_\Theta \to T^*\mathbb{F}_\Theta$ at b_Θ. Then, $L = Z_\Theta$.*

Proof. By the previous lemma the Lie algebras of these groups coincide, and therefore their connected components of the identity $(Z_\Theta)_0$ and L_0 are equal. Since L normalizes its Lie algebra, it follows that L normalizes $\mathfrak{z}_\Theta$. Nevertheless, the normalizer of $\mathfrak{z}_\Theta$ is Z_Θ. Therefore, $L \subset Z_\Theta$.

To verify the opposite inclusion, consider the restriction of the action a to the subgroup K. For $A \in \mathfrak{k}$, $\theta(A) = A^\#$. Thus, the action of K on $T^*\mathbb{F}_\Theta$ is linear. Therefore, the isotropy group $K \cap L$ coincides with the isotropy group of the action on $\mathbb{F}_\Theta$ at b_Θ, that is, $K \cap L = K_\Theta$. Now, we know that K_Θ intercepts all connected components of Z_Θ: see [**11**, Lemma 1.2.4.5]. Therefore, the relations $K_\Theta \subset L$ and $(Z_\Theta)_0 = L_0$ imply that $Z_\Theta \subset L$. □

Remark 2.14. Lemma 2.12 and Proposition 2.13 are needed to verify that the homogeneous space obtained via the infinitesimal action indeed coincides with the homogeneous space of the adjoint orbit. Although the manifolds are isomorphic, it is not granted *a priori* that the group actions agree.

Remark 2.15. The group G that integrates the infinitesimal action θ is necessarily the adjoint group $\mathrm{Aut}_0(\mathfrak{g})$, whose centre is trivial. This happens because the action of G on $T^*\mathbb{F}_\Theta$ is effective, as G is a subgroup of diffeomorphisms of $T^*\mathbb{F}_\Theta$. An effective action on the adjoint orbit only happens for the adjoint group, since the centre $Z(G) \subset Z_\Theta$, and if $z \in Z(G)$, then $\mathrm{Ad}(z) = \mathrm{id}$.

2.4. Moment map on $T^*\mathbb{F}$

The action $a\colon G \times T^*\mathbb{F}_\Theta \to T^*\mathbb{F}_\Theta$ defined above is a Hamiltonian action, since $\theta(Y)$ is a Hamiltonian field for each $Y \in \mathfrak{g}$. We can then define a moment map $\mu\colon T^*\mathbb{F}_\Theta \to \mathfrak{g}^*$, by $\mu(\xi)(Y) = \mathrm{en}_Y(\xi)$, where $\mathrm{en}_Y : T^*\mathbb{F}_\Theta \to \mathbb{R}$ is the energy function of $\theta(Y)$ and $\xi \in T^*\mathbb{F}_\Theta$. That is,

- if $A \in \mathfrak{k}$, then $\mu(\xi)(A) = \xi(\tilde{A}(x))$, $x = \pi(\xi)$; and
- if $X \in \mathfrak{s}$, then $\mu(\xi)(X) = \xi(\tilde{X}(x)) + \langle X, x\rangle$, $x = \pi(\xi)$, where $\langle\cdot,\cdot\rangle$ is Cartan–Killing.

Associated with μ we define a cocycle $c\colon G \to \mathfrak{g}^*$ by

$$c(g) = \mu(g \cdot \xi) - \mathrm{Ad}^* \mu(\xi),$$

where $\xi \in T^*\mathbb{F}_\Theta$ is arbitrary since the right-hand side is constant as a function of ξ (see [**1**]). The map c is a cocycle in the sense that

$$c(gh) = \mathrm{Ad}^*(g)c(h) + c(g),$$

which means that c is a 1-cocycle of the cohomology of the coadjoint representation of G on $\mathfrak{g}^*$.

In the case when $\mathfrak{g}$ is semi-simple, the Cartan–Killing form $\langle\cdot,\cdot\rangle$ interchanges the representations: coadjoint Ad^* and adjoint Ad. With this we can define a moment map $\mu\colon T^*\mathbb{F}_\Theta \to \mathfrak{g}$ (same notation) by $\langle\mu(\xi),\cdot\rangle = \mathrm{en}_Y(\xi)$. So the cocycle becomes $c(g) = \mu(g \cdot \xi) - \mathrm{Ad}(g)\mu(\xi)$, which satisfies $c(gh) = \mathrm{Ad}(g)c(h) + c(g)$.

Theorem 2.16. *Let $\mu\colon T^*\mathbb{F}_\Theta \mapsto \mathfrak{g}$ be the moment map of the action $a\colon G \times T^*\mathbb{F}_\Theta \to T^*\mathbb{F}_\Theta$ constructed above, and let $c\colon G \to \mathfrak{g}$ be the corresponding cocycle. Then,*

(1) *c is identically zero, which means that $\mu\colon T^*\mathbb{F}_\Theta \to \mathfrak{g}$ is equivariant, that is, $\mu(g\cdot\xi) = \mathrm{Ad}\,\mu(\xi)$;*

(2) *μ is a diffeomorphism between $T^*\mathbb{F}_\Theta$ and the adjoint orbit $\mathrm{Ad}(G)H_\Theta$;*

(3) *$\mu^*\omega = \Omega$, where Ω is the canonical symplectic form of $T^*\mathbb{F}_\Theta$ and ω the (real) Kirillov–Kostant–Souriaux form on $\mathrm{Ad}(G)H_\Theta$; and*

(4) *$\mu\colon T^*\mathbb{F}_\Theta \to \mathrm{Ad}(G)H_\Theta$ is the inverse of the map $\iota\colon \mathrm{Ad}(G)H_\Theta \to T^*\mathbb{F}_\Theta$ of Theorem 2.1 given in (2.1).*

Proof. The result is a consequence of the following items.

(1) $\mu(b_\Theta) = H_\Theta$, where b_Θ is the origin of $\mathbb{F}_\Theta$, which is also regarded as the null vector in $T^*_{b_\Theta}\mathbb{F}_\Theta$. In fact, if $A \in \mathfrak{k}$, then $\mu(b_\Theta)(A) = b_\Theta(\tilde{A}(b_\Theta)) = 0$. Whereas if $X \in \mathfrak{s}$, then

$$\begin{aligned}\mu(b_\Theta)(X) &= b_\Theta(\tilde{X}(b_\Theta)) + \langle X, H_\Theta\rangle \\ &= \langle X, H_\Theta\rangle.\end{aligned}$$

Therefore, $H_\Theta \in \mathfrak{g}$ satisfies $\mu(b_\Theta)(Y) = \langle Y, H_\Theta\rangle$ for all $Y \in \mathfrak{g}$, which means that $\mu(b_\Theta) = H_\Theta$.

(2) If $x \in \mathbb{F}_\Theta$ with $x = \mathrm{Ad}(k)H_\Theta$, $k \in K$, then $\mu(x) = \mathrm{Ad}(k)H_\Theta$. This follows by the same argument as in the previous item, where we regard x as the zero vector in $T_x\mathbb{F}_\Theta$ and thus obtain $x(\tilde{X}(x)) = 0$ for any $X \in \mathfrak{g}$.

(3) $c(k) = 0$ if $k \in K$, as follows by definition $c(k) = \mu(k \cdot b_\Theta) - \mathrm{Ad}(k)\mu(b_\Theta)$ and the previous items.

(4) $c(h) = 0$ if $h \in A$. In fact, $\mathrm{Ad}(h)\mu(b_\Theta) = \mathrm{Ad}(h)H_\Theta = H_\Theta$. On the other hand, if $H \in \mathfrak{a}$, then $\theta(H)(b_\Theta) = 0$, since $H^\#(b_\Theta) = 0$ and $V_H(b_\Theta) = 0$, since $(\mathrm{d}f_H)_{b_\Theta}(\cdot) = (\tilde{H}(b_\Theta), \cdot) = 0$. This implies that b_Θ is a fixed point by the action of A on $T^*\mathbb{F}_\Theta$. Therefore, $\mu(h \cdot b_\Theta) = \mu(b_\Theta) = H_\Theta$, which tells us that $c(h) = \mu(h \cdot b_\Theta) - \mathrm{Ad}(h)\mu(b_\Theta) = 0$.

(5) $c \equiv 0$, that is, μ is equivariant: $\mu(g \cdot \xi) = \mathrm{Ad}\,\mu(\xi)$. This follows from the polar decomposition $G = K(\mathrm{cl}A^+)K$ and two applications of the cocycle property. In fact, if $g = uhv \in K(\mathrm{cl}A^+)K$, then

$$\begin{aligned}c(g) = c(uhv) &= \mathrm{Ad}(uh)c(v) + c(uh) \\ &= \mathrm{Ad}(u)c(h) + c(u) \\ &= 0.\end{aligned}$$

(6) Since μ is equivariant and $\mu(b_\Theta) = H_\Theta$, its image is contained in the adjoint orbit $\mathrm{Ad}(G)H_\Theta$. The diffeomorphism property is due to equivariance, transitivity of G on the spaces, and the fact that the isotropy subgroups on both spaces coincide. The pullback of item (3) is a standard fact about moment maps of Hamiltonian actions.

(7) To see the inverse of μ take $\xi = \iota(H_\Theta + Z) \in T^*_{b_\Theta}\mathbb{F}_\Theta$. If $A \in \mathfrak{k}$ and $x \in \mathfrak{s}$, then

$$\langle \mu(\xi), A\rangle = \xi(\tilde{A}(b_\Theta)), \qquad \langle \mu(\xi), X\rangle = \xi(\tilde{X}(b_\Theta)) + f_X(b_\Theta).$$

Write $A = A^- + A^0 + A^+ \in \mathfrak{g} = \mathfrak{n}^-_\Theta \oplus \mathfrak{z}_\Theta \oplus \mathfrak{n}^+_\Theta$. Then $\tilde{A}(b_\Theta) = \widetilde{A^-}(b_\Theta)$, so $\xi(\tilde{A}(b_\Theta)) = \langle Z, A^-\rangle$. Since $\mathfrak{n}^+_\Theta$ is Cartan–Killing orthogonal to $\mathfrak{z}_\Theta \oplus \mathfrak{n}^+_\Theta$, we have $\xi(\tilde{A}(b_\Theta)) = \langle Z, A^-\rangle = \langle Z, A\rangle$, that is,

$$\langle \mu(\xi), A\rangle = \langle Z, A\rangle = \langle H_\Theta + Z, A\rangle$$

because $\langle H_\Theta, A\rangle = 0$. Similarly, $\xi(\tilde{X}(b_\Theta)) = \langle Z, X\rangle$ and, since $f_X(b_\Theta) = \langle H_\Theta, X\rangle$, we have $\langle \mu(\xi), X\rangle = \langle H_\Theta + Z, X\rangle$. Hence $\mu(\iota(H_\Theta + Z)) = H_\Theta + Z$, showing that μ and ι are inverse to each other.

□

Remark 2.17 (other actions). Besides the action defined above, there are other infinitesimal actions $\mathfrak{g}$ on $T^*\mathbb{F}_\Theta$ that play the same role.

(1) Take $\theta^-(A) = A^\#$ if $A \in \mathfrak{k}$ and $\theta^-(X) = X^\# - V_X$ if $X \in \mathfrak{s}$. Then, θ^- is still an infinitesimal representation, which gives rise to another action of G.

(2) If $(\cdot,\cdot)$ is a K-invariant Riemannian metric on $\mathbb{F}_\Theta$ such that each $\tilde{X}$, $X \in \mathfrak{s}$, is a gradient of the function $\hat{f}_X$, then the same game can be played with the Hamiltonian vector field of $\hat{F}_X = \hat{f} \circ \pi$ in place of V_X.

3. Embedding of adjoint orbits into products

We recall here a known realization of the homogeneous space G/Z_Θ as an orbit in a product of flag manifolds (see [**9**, § 3] for details).

Let w_0 be the principal involution of the Weyl group $\mathcal{W}$, that is, the element of highest length as a product of simple roots. Then $-w_0\mathfrak{a}^+ = \mathfrak{a}^+$ and $-w_0\Sigma = \Sigma$, so that $-w_0$ is a symmetry of the Dynkin diagram of Σ. For a subset $\Theta \subset \Sigma$ we put $\Theta^* = -w_0\Theta$ and refer to $\mathbb{F}_{\Theta^*}$ as the flag manifold dual to $\mathbb{F}_\Theta$. Clearly, if H_Θ is a characteristic element for Θ, then $-w_0H_\Theta$ is characteristic for Θ^*. (Except for the simple Lie algebras with diagrams A_l, D_l and E_6, all the flag manifolds are self-dual. In $A_l = \mathfrak{sl}(n)$, $n = l + 1$, we have, for instance, that the dual to the Grassmannian $\mathrm{Gr}_k(n)$ is $\mathrm{Gr}_{n-k}(n)$.)

Next we check that the diagonal action of G on the product $\mathbb{F}_\Theta \times \mathbb{F}_{\Theta^*}$ as $(g,(x,y)) \mapsto (gx, gy)$, $g \in G$, $x, y \in \mathbb{F}$, has just one open and dense orbit, which is G/Z_Θ.

Let x_0 be the origin of $\mathbb{F}_\Theta$. Since G acts transitively on $\mathbb{F}_H$, all the G-orbits of the diagonal action have the form $G\cdot(x_0, y)$, with $y \in \mathbb{F}_{\Theta^*}$. Thus, the G-orbits are in bijection with the orbits of the action of P_{Θ^*} on $\mathbb{F}_{\Theta^*}$, which is known to be the orbits through wy_0, $w \in \mathcal{W}$, where y_0 is the origin of $\mathbb{F}_{\Theta^*}$. Hence the G-orbits are $G\cdot(x_0, wy_0)$, $w \in \mathcal{W}$.

Now let w_0 be the principal involution of $\mathcal{W}$.

Proposition 3.1. *The orbit $G\cdot(x_0, \tilde{w}_0y_0)$ is open and dense in $\mathbb{F}_\Theta \times \mathbb{F}_{\Theta^*}$ and identifies to G/Z_H. (Here and elsewhere $\tilde{w}$ stands for a representative in K of $w \in \mathcal{W}$.)*

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Proof. The isotropy subgroup at the point $(x_0, \tilde{w}_0 y_0)$ is the intersection of the isotropy subgroups at x_0 and $w_0 y_0$. The first one is the parabolic subgroup P_{-H} associated with $\tilde{w}_0 H^* = -H$ and the second one is P_H, where H is a characteristic element of Θ. Since $Z_H = P_H \cap P_{-H}$, the identification follows. Now the Lie algebra $\mathfrak{z}_H = \mathfrak{p}_H \cap \mathfrak{p}_{-H}$ of $P_H \cap P_{-H}$ is complemented in $\mathfrak{g}$ by $\mathfrak{n}_H^+ \cap \mathfrak{n}_{-H}^+$, with $\mathfrak{n}_{-H}^+ = \sum_{\alpha(H)<0} \mathfrak{g}_\alpha$. Hence, the dimension of $G \cdot (x_0, \tilde{w}_0 y_0)$ is the same as the dimension of $\mathbb{F}_\Theta \times \mathbb{F}_{\Theta^*}$, so the orbit is open. Analogous reasoning shows that this is the only open, and hence dense, orbit. □

In terms of this realization of G/Z_Θ as an open orbit, the map $G/Z_\Theta \to \mathbb{F}_\Theta$ is just the projection onto the first factor. Also, if $\Theta \subset \Theta_1$, the projection $G/Z_\Theta \to G/Z_{\Theta_1}$ is inherited from the projections $\mathbb{F}_\Theta \to \mathbb{F}_{\Theta_1}$ and $\mathbb{F}_{\Theta^*} \to \mathbb{F}_{\Theta_1^*}$.

A flag manifold $\mathbb{F}_\Theta = G/P_\Theta$ is in bijection with the set of parabolic subalgebras conjugate to $\mathfrak{p}_\Theta$, since P_Θ is the normalizer of $\mathfrak{p}_\Theta$. From this point of view the open orbit $G \cdot (x_0, \tilde{w}_0 y_0) \subset \mathbb{F}_\Theta \times \mathbb{F}_{\Theta^*}$ is characterized by transversality: two parabolic subalgebras $\mathfrak{p}_1 \in \mathbb{F}_\Theta$ and $\mathfrak{p}_2 \in \mathbb{F}_{\Theta^*}$ are transversal if $\mathfrak{g} = \mathfrak{p}_1 + \mathfrak{p}_2$ or, equivalently, if $\mathfrak{n}(\mathfrak{p}_1) \cap \mathfrak{p}_2 = \mathfrak{p}_1 \cap \mathfrak{n}(\mathfrak{p}_2) = \{0\}$, where $\mathfrak{n}(\cdot)$ stands for the nilradical (see [**8**]). The open orbit $G \cdot (x_0, \tilde{w}_0 y_0)$ is then the set of pairs of transversal subalgebras. In particular, the set of subalgebras transversal to the origin $x_0 \in \mathbb{F}_\Theta$ is the open cell $N^+ \tilde{w}_0 y_0$ with y_0 the origin of $\mathbb{F}_{\Theta^*}$. More generally, the set of subalgebras transversal to gx_0, $g \in G$, is the open cell $gN^+ \tilde{w}_0 x_0$.

The fixed points of a split-regular element $h \in A^+ = \mathrm{e}^{\mathfrak{a}^+}$ in a flag manifold $\mathbb{F}_\Theta$ are isolated. The set of fixed points is the orbit through the origin of $\mathrm{Norm}_K(\mathfrak{a})$ and factors down to the Weyl group $\mathcal{W} = \mathrm{Norm}_K(\mathfrak{a})/\mathrm{Cent}_K(\mathfrak{a})$.

4. Adjoint orbits and representations of $\mathfrak{g}$

In this section we give realizations of the coset spaces G/Z_Θ based on representations of $\mathfrak{g}$. It will be convenient to assume that $\mathfrak{g}$ is a complex algebra, even though the theory works, with some adaptations, for real algebras. This new description helps to establish a bridge between the adjoint orbit and the open orbit in the product.

4.1. The adjoint action of G on End(V)

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and denote by $\mathfrak{h}_\mathbb{R}$ the real subspace of $\mathfrak{h}$ spanned by H_α, $\alpha \in \Pi$, where $\alpha(\cdot) = \langle H_\alpha, \cdot \rangle$. Fix a Weyl chamber $\mathfrak{h}_\mathbb{R}^+$ and let $\Sigma = \{\alpha_1, \dots, \alpha_l\}$ be the corresponding system of simple roots, with fundamental weights $\{\mu_1, \dots, \mu_l\}$. If $\mu = a_1 \mu_1 + \dots + a_l \mu_l$ with $a_i \in \mathbb{N}$, then there exists a unique irreducible representation ρ_μ of $\mathfrak{g}$ with highest weight μ. If $H \in \mathfrak{h}_\mathbb{R}^+$, then $\mu(H)$ is the largest eigenvalue of $\rho_\mu(H)$. If w_0 is the main involution, then $w_0 \mu$ is a lowest weight: that is, $(w_0 \mu)(H) = \mu(w_0 H)$ is the smallest eigenvalue of $\rho_\mu(H)$ if $H \in \mathfrak{h}_\mathbb{R}^+$.

If $K \subset G$ is the maximal compact subgroup, then V can be endowed with a K-invariant Hermitian form $(\cdot, \cdot)^\mu$ such that the weight spaces are pairwise orthogonal. Such a Hermitian form is unique up to scale, because the representation of K on V is irreducible.

In § 5 we study Lagrangian submanifolds of the adjoint orbits $\mathrm{Ad}(G) H_0$ with $H_0 \in \mathrm{cl}(\mathfrak{h}_\mathbb{R}^+)$ embedded into products $\mathbb{F}_{H_0} \times \mathbb{F}_{H_0^*}$. There is freedom of choice to pick the element H_0, producing the same flag $\mathbb{F}_{H_0}$. In what follows we will choose a convenient H_0.

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Let $\Theta_0 = \Theta(H_0) = \{\alpha \in \Sigma \colon \alpha(H_0) = 0\}$; that is, H_0 is characteristic for Θ_0. Let μ be a highest weight such that, for $\alpha \in \Sigma$, $\langle \alpha^\vee, \mu \rangle = 0$ if and only if $\alpha \in \Theta_0$. (For example, $\mu = \mu_{i_1} + \cdots + \mu_{i_s}$ if $\Sigma \setminus \Theta_0 = \{\alpha_{i_1}, \ldots, \alpha_{i_s}\}$.) Define $H_\mu \in \mathfrak{h}_{\mathbb{R}}$ by $\mu(\cdot) = \langle H_\mu \cdot \rangle$. Then, the centralizers of H_μ and H_0 coincide, since Θ_0 is the set of simple roots that vanish on H_0 as well as on H_μ. Hence the adjoint orbits $\mathrm{Ad}(G)H_\mu$ and $\mathrm{Ad}(G)H_0$ give rise to the same homogeneous space $G/Z_{H_0} = G/Z_{H_\mu}$ and the flags $\mathbb{F}_{H_0}$ and $\mathbb{F}_{H_\mu}$ coincide. From now on we take $H_0 = H_\mu$, with μ a highest weight, $\mu = \mu_{i_1} + \cdots + \mu_{i_s}$.

Let G be the linear connected group with Lie algebra $\rho_\mu(\mathfrak{g}) \approx \mathfrak{g}$, and consider its action on the projective space $\mathbb{P}(V)$ of the representation space $V = V(\mu)$. It is well known that this choice of μ guarantees that the projective orbit of G by the subspace of highest weight $V_\mu \in \mathbb{P}(V)$ is the flag $\mathbb{F}_{H_\mu} = \mathbb{F}_{\Theta_0}$.

Consider the dual representations ρ_μ^* of $\mathfrak{g}$ and G on V^* as $\rho_\mu^*(X)(\varepsilon) = -\varepsilon \circ \rho_\mu(X)$ and $\rho_\mu^*(g)(\varepsilon) = \varepsilon \circ \rho_\mu(g^{-1})$ if $\varepsilon \in V^*$, $X \in \mathfrak{g}$ and $g \in G$. Choose a basis $\{v_0, \ldots, v_N\}$ of V adapted to the decomposition in weight spaces with $v_0 \in V_\mu$. Denote by $\{\varepsilon_0, \ldots, \varepsilon_N\}$ the dual basis $\varepsilon_i(v_j) = \delta_{ij}$. Then ε_0 generates a subspace of 'lowest' weight of V^*, in the sense that

(1) $\rho_\mu^*(H)(\varepsilon_0) = -\mu(H)\varepsilon_0$ if $H \in \mathfrak{h}$; and

(2) $\rho_\mu^*(X)(\varepsilon_0) = 0$ if $X \in \sum_{\alpha<0} \mathfrak{g}_\alpha$, and $\rho_\mu(X)$ takes a weight space V_ν to a sum of spaces of weights smaller than ν.

Therefore, $-\mu$ is the lowest weight of V^*. So the highest weight is $\mu^* = -w_0\mu$. This means that the projective orbit of the highest weight (and of ε_0) on V^* is the dual flag $\mathbb{F}_{H_\mu^*}$.

Example 4.1. If $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, then the fundamental weights are λ_1, $\lambda_1 + \lambda_2$, $\ldots$, $\lambda_1 + \cdots + \lambda_{n-1}$, where λ_i is the functional that is associated with the ith eigenvalue of the diagonal matrix $H \in \mathfrak{h}$. If μ is a fundamental weight $\mu = \lambda_1 + \cdots + \lambda_k$, then the irreducible representation with highest weight μ is the representation of $\mathfrak{g}$ on the kth exterior power $\Lambda^k\mathbb{C}^n$ of $\mathbb{C}^n$. The highest weight space is generated by $e_1 \wedge \cdots \wedge e_k$ (e_i are the basis vectors of $\mathbb{C}^n$). The G-orbit of $e_1 \wedge \cdots \wedge e_k$ is the set of decomposable elements of $\Lambda^k\mathbb{C}^n$, so the projective G-orbit is identified with the Grassmannian $\mathrm{Gr}_k(n)$. The dual flag of $\mathrm{Gr}_k(n)$ is $\mathrm{Gr}_{n-k}(n)$, which is the projective orbit on $\Lambda^{n-k}\mathbb{C}^n$, identified with the dual $\Lambda^k\mathbb{C}^n$ by a choice of volume form on $\mathbb{C}^n$. The lowest weight space on $\Lambda^{n-k}\mathbb{C}^n$ is generated by $e_{k+1} \wedge \cdots \wedge e_n$.

Keeping the same highest weight μ, consider the tensor product $V \otimes V^*$. G gets represented on $V \otimes V^*$ by $g \cdot (v \otimes \varepsilon) = \rho_\mu(g)v \otimes \rho_\alpha^*(g)\varepsilon$, which is isomorphic to the adjoint representation of G on $\mathrm{End}(V)$.

Once again, let v_0 and ε_0 be generators of the spaces of highest weight of V and lowest of V^*, respectively. Our fourth model of the adjoint orbit is the G-orbit of $v_0 \otimes \varepsilon_0$. To prove that this orbit is indeed G/Z_{H_0} we shall consider the moment map of the representation. Namely, the map $\bar{M} \colon V \otimes V^* \to \mathfrak{g}^*$ defined by

$$\bar{M}(v \otimes \varepsilon)(Z) = \varepsilon(\rho_\mu(Z)v), \quad v \in V,\ \varepsilon \in V^*,\ Z \in \mathfrak{g}.$$

Since $\mathfrak{g}$ is semi-simple and $\mathfrak{g} \approx \mathfrak{g}^*$ via the Cartan–Killing form $\langle \cdot, \cdot \rangle$, we can take the moment map $M\colon V \otimes V^* \to \mathfrak{g}$ given by

$$\langle M(v \otimes \varepsilon), Z \rangle = \varepsilon(\rho_\mu(Z)v), \quad v \in V,\ \varepsilon \in V^*,\ Z \in \mathfrak{g}.$$

It is well known and easy to prove that M is equivariant with respect to the representations on $V \otimes V^*$ and $\mathfrak{g}$. In fact, since $\rho_\mu(\mathrm{Ad}(g)Z) = \rho_\mu(g)\rho_\mu(Z)\rho_\mu(g^{-1})$ we have

$$\begin{aligned}\langle \mathrm{Ad}(g)M(v \otimes \varepsilon), Z \rangle &= \langle \mathrm{Ad}(g)M(v \otimes \varepsilon), \mathrm{Ad}(g^{-1})Z \rangle \\ &= \varepsilon(\rho_\mu(g^{-1})\rho_\mu(Z)\rho_\mu(g)v) \\ &= \rho_\mu(g)v \otimes \rho_\mu^*(g)\varepsilon = g \cdot (v \otimes \varepsilon).\end{aligned}$$

The same calculation shows that $\bar{M}$ is equivariant with respect to the coadjoint representation.

In the semi-simple case the moment map has the following geometric interpretation: ρ_μ is a faithful representation, thus $\mathfrak{g} \approx \rho_\mu(\mathfrak{g}) \subset \mathrm{End}(V)$. The trace form $\mathrm{tr}(AB)$ on $\mathrm{End}(V)$ is non-degenerate. Thus, the moment map is just the orthogonal projection with respect to the trace form of $\mathrm{End}(V) \approx V \otimes V^*$ onto $\rho_\mu(\mathfrak{g}) \approx \mathfrak{g}$.

As a consequence of equivariance, it follows that the image of a G-orbit on $V \otimes V^*$ by M is an adjoint orbit.

Lemma 4.2. *The image of the G-orbit $G \cdot (v_0 \otimes \varepsilon_0)$ by M is the adjoint orbit of H_μ defined by $\mu(\cdot) = \langle H_\mu, \cdot \rangle$.*

Proof. If α is a root and $X \in \mathfrak{g}_\alpha$, then

$$\varepsilon_0(\rho_\mu(X)v_0) = (\rho_\mu(X)v_0) \otimes \varepsilon_0 = -v_0 \otimes (\rho_\mu^*(X)\varepsilon_0).$$

The second term vanishes if $\alpha > 0$, whereas if $\alpha < 0$, the third term vanishes. Hence $\langle M(\varepsilon_0 \otimes v_0), X \rangle = 0$. But, if $H \in \mathfrak{h}$, then

$$\varepsilon_0(\rho_\mu(H)v_0) = \mu(H)\varepsilon_0(v_0) = \mu(H),$$

that is, $\langle M(\varepsilon_0 \otimes v_0), H \rangle = \mu(H)$, which shows that $M(\varepsilon_0 \otimes v_0) = H_\mu$. Consequently, $M(G \cdot (v_0 \otimes \varepsilon_0)) = \mathrm{Ad}(G)H_\mu$. □

Proposition 4.3. *The G-orbit $G \cdot (v_0 \otimes \varepsilon_0)$ is the homogeneous space G/Z_{H_μ}.*

Proof. Set $G \cdot (v_0 \otimes \varepsilon_0) = G/L$. We want to show that $L = Z_{H_\mu}$. The equivariance of M together with the equality $M(G \cdot (v_0 \otimes \varepsilon_0)) = \mathrm{Ad}(G)H_\mu$ imply that the isotropy subgroup at $v_0 \otimes \varepsilon_0$ is contained in the isotropy subgroup at H_μ, that is, $L \subset Z_{H_\mu}$. Since Z_{H_μ} is connected, to show the opposite inclusion it suffices to show that the Lie algebra $\mathfrak{z}_{H_\mu}$ of Z_{H_μ} is contained in the isotropy algebra of $v_0 \otimes \varepsilon_0$.

To verify this, we observe that the isotropy algebra of v_0 is

$$\ker \mu + \sum_{\alpha > 0} \mathfrak{g}_\alpha + \sum_{\alpha \in \langle \Theta_0 \rangle^-} \mathfrak{g}_\alpha,$$

where $\langle\Theta_0\rangle^-$ is the set of negative roots generated by Θ_0, which in turn is the set of simple roots that vanish on H_0 (or H_μ). In this sum, the first term is given by elements $H \in \mathfrak{h}$ such that $\rho_\mu(H)v_0 = 0$. The second term appears in the isotropy algebra because v_0 is a highest weight vector. Finally, the last term comes from the fact that if α is a negative root and $X \in \mathfrak{g}_\alpha$, then $\rho_\mu(X)v_0 = 0$ if and only if $\langle\alpha^\vee, \mu\rangle = 0$. The roots that satisfy this equality are precisely the roots in $\langle\Theta_0\rangle^-$. Analogously, the isotropy algebra at ε_0 is given by

$$\ker\mu + \sum_{\alpha<0}\mathfrak{g}_\alpha + \sum_{\alpha\in\langle\Theta_0\rangle^+}\mathfrak{g}_\alpha,$$

where $\langle\Theta_0\rangle^+$ is the set of positive roots generated by Θ_0.

Now, set

$$X \in \mathfrak{z}_{H_\mu} = \mathfrak{h} \oplus \sum_{\alpha\in\langle\Theta_0\rangle^\pm}\mathfrak{g}_\alpha.$$

If $X \in \sum_{\alpha\in\langle\Theta_0\rangle^\pm}\mathfrak{g}_\alpha$, then $\rho_\mu(X)v_0\otimes\varepsilon_0 + v_0\otimes\rho_\mu^*(X)\varepsilon_0 = 0$ since X belongs to the isotropy algebras of v_0 and ε_0. Whereas if $H \in \mathfrak{h}$, then

$$\rho_\mu(H)v_0\otimes\varepsilon_0 + v_0\otimes\rho_\mu^*(H)\varepsilon_0 = \mu(H)v_0\otimes\varepsilon_0 - \mu(H)v_0\otimes\varepsilon_0 = 0.$$

Therefore, $\mathfrak{z}_{H_\mu}$ is contained in the isotropy subalgebra of $v_0\otimes\varepsilon_0$. □

Corollary 4.4. *The restriction of the moment map defines a diffeomorphism* $M\colon G\cdot(v_0\otimes\varepsilon_0)\to \mathrm{Ad}(G)H_\mu$.

Via this diffeomorphism, the height function $f_H\colon \mathrm{Ad}(G)H_\mu \to \mathbb{C}$ defines a function, also denoted by f_H, on the orbit $G\cdot(v_0\otimes\varepsilon_0)$. This function has a simple expression.

Proposition 4.5. *Let* $v\otimes\varepsilon \in G\cdot(v_0\otimes\varepsilon_0)$. *Then*

$$f_H(v\otimes\varepsilon) = \varepsilon(\rho_\mu(H)v) = \mathrm{tr}((v\otimes\varepsilon)\rho_\mu(H)).$$

Proof. For a moment, denote by $\tilde{f}_H$ the function f_H defined on $G\cdot(v_0\otimes\varepsilon_0)$. Then

$$\tilde{f}_H(v\otimes\varepsilon) = f_H(M(v\otimes\varepsilon)) = \langle M(v\otimes\varepsilon), H\rangle,$$

which is $\varepsilon(\rho_\mu(H)v)$ by the definition of M. In the expression involving the trace, $v\otimes\varepsilon$ is regarded as an element of $\mathrm{End}(V)$ and the second equality follows from $\varepsilon(Sv) = \mathrm{tr}((v\otimes\varepsilon)S)$, which holds for any $S\in\mathrm{End}(V)$. □

4.2. Isomorphism with the open orbit in $\mathbb{F}_{H_\mu}\times\mathbb{F}_{H_\mu^*}$

As mentioned earlier, the flags $\mathbb{F}_{H_\mu}$ and $\mathbb{F}_{H_\mu^*}$ are obtained as projective orbits in $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$, respectively. The origin of $\mathbb{F}_{H_\mu}$ is identified with the highest weight space $[v_0]$, and in the identification with the adjoint orbit of the compact group K, this origin is precisely H_μ.

On the other hand, $[\varepsilon_0]$ is the lowest weight space in V^*. The isotropy algebra at $[\varepsilon_0]$ contains $\sum_{\alpha<0}\mathfrak{g}_\alpha$. Thus, $[\varepsilon_0]\in\mathbb{P}(V^*)$ is identified with $w_0b^*\in\mathbb{F}_{H_\mu^*}$, where b^* is the origin of $\mathbb{F}_{H_\mu^*}$. Under the identification of $\mathbb{F}_{H_\mu^*}$ with the adjoint orbit of the compact group K, the origin is $-H_\mu = w_0 H_\mu^*$.

We use these identifications to see $\mathbb{F}_{H_0} \times \mathbb{F}_{H_0^*}$ as the product of the projective orbits $G\cdot[v_0]\times G\cdot[\varepsilon_0] \subset \mathbb{P}(V)\times\mathbb{P}(V^*)$. The open orbit in $\mathbb{F}_{H_0}\times\mathbb{F}_{H_0^*}$ then becomes the diagonal G-orbit of $([v_0],[\varepsilon_0]) \in \mathbb{P}(V)\times\mathbb{P}(V^*)$. Denote this orbit by $G\cdot([v_0],[\varepsilon_0])$, that is,

$$G\cdot([v_0],[\varepsilon_0]) = \{(\rho_\mu(g)[v_0], \rho_\mu^*(g)[\varepsilon_0]) \in \mathbb{P}(V)\times\mathbb{P}(V^*)\colon g\in G\}.$$

Now we describe the diffeomorphism between the orbit $G\cdot(v_0\otimes\varepsilon_0)\subset V\otimes V^*$ and the orbit $G\cdot([v_0],[\varepsilon_0]) \subset \mathbb{F}_{H_\mu}\times\mathbb{F}_{H_\mu^*} \subset \mathbb{P}(V)\times\mathbb{P}(V^*)$. In fact, the diffeomorphism associates $g\cdot([v_0],[\varepsilon_0]) = (\rho_\mu(g)[v_0], \rho_\mu^*(g)[\varepsilon_0])$ with $g\cdot(v_0\otimes\varepsilon_0) = \rho_\mu(g)v_0\otimes\rho_\mu^*(g)\varepsilon_0$. We obtain the following proposition.

Proposition 4.6. *Let $\Phi\colon G\cdot(v_0\otimes\varepsilon_0)\to G\cdot([v_0],[\varepsilon_0])$ be the diffeomorphism obtained by identification of both orbits with G/Z_{H_μ}. If $v\otimes\varepsilon\in G\cdot(v_0\otimes\varepsilon_0)$, then $\Phi(v\otimes\varepsilon) = ([v],[\varepsilon])$ with inverse $\Phi^{-1}([v],[\varepsilon]) = (v\otimes\varepsilon)$.*

Proof. Our previous argument already proved this. Nevertheless, it is worth observing that the maps $v\otimes\varepsilon\mapsto([v],[\varepsilon])$ and $([v],[\varepsilon])\mapsto v\otimes\varepsilon$ are well defined, since $v_1\otimes\varepsilon_1 = v\otimes\varepsilon$ is equivalent to $v_1 = av$ and $\varepsilon_1 = a^{-1}\varepsilon$, which is also equivalent to $([v_1],[\varepsilon_1]) = ([v],[\varepsilon])$. □

4.3. Isomorphism with $T^*\mathbb{F}_{H_\mu}$

First of all, we recall the isomorphism between the adjoint orbit $\mathcal{O}(H_\mu)$ and the cotangent bundle $T^*\mathbb{F}_\Theta$. Here, H_μ remains fixed and is characteristic for Θ, that is, $\Theta = \{\alpha\in\Sigma\colon \alpha(H_\mu)=0\}$.

By the Iwasawa decomposition $G = KAN$ we can write $G = KP_\Theta$, and the adjoint action of P_Θ on H_μ is given by $\mathrm{Ad}(P_\theta)\cdot H_\mu = H_\mu + \mathfrak{n}_\Theta^+$, where $\mathfrak{n}_\Theta^+ = \sum_{\Pi^+\setminus\langle\Theta\rangle}\mathfrak{g}_\alpha$. Thus,

$$\mathcal{O}(H_\mu) = \mathrm{Ad}(G)H_\mu = \mathrm{Ad}(K)(H_\mu+\mathfrak{n}_\Theta^+) = \bigcup_{k\in K}\mathrm{Ad}(k)(H_\mu+\mathfrak{n}_\Theta^+).$$

The identification of the adjoint orbit with the cotangent bundle is given by the map that associates with each element of the adjoint orbit $\mathrm{Ad}(k)(H_\mu+X)$, $X\in\mathfrak{n}_\Theta^+$, the linear functional $f\in(T_{kb_0}\mathbb{F}_\Theta)^*$ given by $f(Y) = \langle\mathrm{Ad}(k)X, Y\rangle$, $Y\in T_{kb_0}\mathbb{F}_\Theta$.

Proposition 4.7. *Let μ be a highest weight and let v_0 and ε_0 be the generators of the highest weight space on V and the lowest weight space on V^*, respectively. The diffeomorphism between $G\cdot(v_0\otimes\varepsilon_0)$ and $T^*\mathbb{F}_\Theta$ is given by*

$$g\cdot(v_0\otimes\varepsilon_0)\mapsto(Y\mapsto\langle\mathrm{Ad}(k)X,Y\rangle, Y\in T_{kb_0}\mathbb{F}_\Theta), \tag{4.1}$$

where $g = kp$ is the Iwasawa decomposition, $\mathrm{Ad}(p)H_\mu = H_\mu + X$, and the flag $\mathbb{F}_\Theta$ is determined by H_μ.

5. Compactified adjoint orbits

We compactify adjoint orbits $\mathcal{O}(H_0)$ to products of flags $\mathbb{F}_{H_0} \times \mathbb{F}_{H_0^*}$ as an auxiliary tool to identify Lagrangian submanifolds of the orbits. We choose canonical complex structures on $\mathbb{F}_{H_0}$ and $\mathbb{F}_{H_0^*}$ so that, for an element w_0 of the Weyl group $\mathcal{W}$, the right action $R_{w_0}\colon \mathbb{F}_{H_0} \to \mathbb{F}_{H_0^*}$ is anti-holomorphic (Proposition 5.7). Consequently, the map $R_{w_0}\colon \mathbb{F}_{H_0} \to \mathbb{F}_{H_0^*}$ is anti-symplectic with respect to the Kähler forms on $\mathbb{F}_{H_0}$ and $\mathbb{F}_{H_0^*}$ given by the Borel metric and canonical complex structures (Corollary 5.9). We then obtain further examples of Lagrangian graphs by composites (either on the left or on the right) of R_{w_0} with symplectic maps.

5.1. Lagrangian graphs in adjoint orbits

On the one hand, $\mathcal{O}(H_0)$ can be embedded as an open dense submanifold in a product of two flags (§ 3); on the other hand, graphs of symplectic maps are Lagrangian submanifolds inside the product, due to the following general fact.

Let (M, ω) and (N, ω_1) be symplectic manifolds. The Cartesian product $M \times N$ can be endowed with the symplectic form $\omega \times \omega_1$. If $\phi\colon M \to N$ is anti-symplectic, that is, $\phi^*\omega_1 = -\omega$, then $\mathrm{graph}(\phi) \subset M \times N$ is a Lagrangian submanifold with respect to $\omega \times \omega_1$. Similarly, we could use a symplectic map (symplectomorphism) $\phi\colon M \to M$ taking $\omega_1 = -\omega$, which is also a symplectic form.

With this in mind, to construct an assortment of Lagrangian submanifolds in $\mathcal{O}(H_0)$ we use an embedding $\mathcal{O}(H_0) \hookrightarrow \mathbb{F}_1 \times \mathbb{F}_2$ into a product of flags. Taking symplectic forms ω_1 and ω_2 on $\mathbb{F}_1$ and $\mathbb{F}_2$ we obtain a symplectic form $\omega_1 \times \omega_2$ on $\mathbb{F}_1 \times \mathbb{F}_2$ and consequently on $\mathcal{O}(H_0)$ by restriction. If $\phi\colon \mathbb{F}_1 \to \mathbb{F}_2$ is anti-symplectic, then $\mathrm{graph}(\phi)$ and $\mathrm{graph}(\phi) \cap \mathcal{O}(H_0)$ are Lagrangian submanifolds of $\mathbb{F}_1 \times \mathbb{F}_2$ and $\mathcal{O}(H_0)$, respectively. The intended construction involves, first of all, a discussion about the right action of the Weyl group.

5.2. The right action of the Weyl group

Let $\mathfrak{g}$ be a non-compact semi-simple Lie algebra (real or complex), let G be the adjoint group of $\mathfrak{g}$, and let $K \subset G$ be the maximal compact subgroup. The maximal flag of $\mathfrak{g}$ is given by $\mathbb{F} = G/P = K/M$, where $P = MAN$ is the minimal parabolic subgroup. The adjoint orbit of a regular element $H \in \mathfrak{a} = \log A$ is given by $\mathcal{O}(H) = G/MA$. The flag $\mathbb{F}$ is contained in $\mathcal{O}(H)$ since $\mathbb{F}$ is a K-orbit of H.

The Weyl group $\mathcal{W}$ is isomorphic to $\mathrm{Norm}_G(A)/MA = \mathrm{Norm}_K(A)/M$. We obtain right actions of $\mathcal{W}$ on $\mathbb{F} = K/M$ (with $\mathcal{W} = \mathrm{Norm}_K(A)/M$) and on $\mathcal{O}(H) = G/MA$ (with $\mathcal{W} = \mathrm{Norm}_G(A)/MA$). Moreover, the fibrations $G/MA \to G/\mathrm{Norm}_G(A)$ and $\mathbb{F} = K/M \to K/\mathrm{Norm}_K(A)$ are principal bundles with structural group $\mathcal{W}$.

Example 5.1. Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ or $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. Hence, a regular element H is a diagonal matrix $H = \mathrm{diag}\{a_1, \ldots, a_n\}$ with $a_1 > \cdots > a_n$. $\mathcal{O}(H) = \{gHg^{-1}\colon g \in \mathrm{Sl}(n, \mathbb{R})\}$ (or $\mathbb{C}$), that is, the orbit is the set of diagonalizable matrices with the same eigenvalues as H. The Weyl group $\mathcal{W}$ is the permutation group of n elements, whereas $\mathrm{Norm}_K(A)$

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is the set of signed permutation matrices (matrices such that each row or column has exactly one non-zero entry ± 1). The right action of a permutation $w \in \mathcal{W}$ is given by

$$R_w \colon gHg^{-1} \mapsto g\bar{w}H(g\bar{w})^{-1} = g(\bar{w}H\bar{w}^{-1})g^{-1},$$

where $\bar{w} \in \mathrm{Norm}_K(A)$ is the permutation matrix that represents $w \in \mathcal{W}$. In this expression for R_w the term $\bar{w}H\bar{w}^{-1}$ is the matrix whose diagonal entries are the same as the ones of H permuted by w in the permutation group $\mathcal{W}$.

The right action R_w of $w \in \mathcal{W}$ is in general completely different from the left action of any of its representatives $\bar{w} \in \mathrm{Norm}_K(A)$. For example, in the case of $\mathfrak{sl}(2, \mathbb{C})$, the Weyl group is $\{1, (12)\}$ and the right action of $w = (12)$ in the flag $S^2 = \mathbb{C}P^1$ is the antipodal map. On the other hand,

$$\bar{w} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{Norm}_K(A)$$

is a representative of (12). The left action of $\bar{w}$ has two fixed points.

The right action of $\mathcal{W}$ leaves invariant the induced vector fields.

Proposition 5.2. *Given an element A in the Lie algebra, denote by $\tilde{A}$ the induced vector field on the homogeneous space (G/MA or K/M). Then, $(R_w)_*\tilde{A} = \tilde{A}$ for all $w \in \mathcal{W}$.*

Proof. Indeed, R_w commutes with the flow of $\tilde{A}$, which is the left action of e^{tA}. □

5.3. The K-orbit and graphs

In § 3, we defined an embedding of the adjoint orbit into the product $\mathbb{F}_{H_0} \times \mathbb{F}_{H_0^*}$, where $\mathbb{F}_{H_0^*}$ is the dual flag of $\mathbb{F}_{H_0}$.

Consider first of all the case when $\mathbb{F}_H = \mathbb{F}$ is the maximal flag, which is self-dual. In this flag, the right action of $\mathcal{W}$ is well defined. Denote by b_0 the origin of $\mathbb{F}$ and set $b_w = R_w b_0$, $w \in \mathcal{W}$.

Let $w_0 \in \mathcal{W}$ be the principal involution (the element of largest length as a product of simple reflections). The embedding of $\mathcal{O}(H_0)$ is given by the G-orbit of (b_0, b_{w_0}) under the diagonal action $g(x, y) = (gx, gy)$. This G-orbit is identified with the adjoint orbit $\mathcal{O}(H_0) = G/MA$ for any H_0 regular and real. Let K be the maximal compact subgroup of G (the real compact form in the case of complex G).

Proposition 5.3. *For $w \in \mathcal{W}$, the K-orbit of (b_0, b_w) by the diagonal action coincides with the graph of R_w.*

Proof. Take $x = k \cdot b_0 \in \mathbb{F}$, $k \in K$. Then, $R_w(x) = R_w(k \cdot b_0) = k \cdot R_w(b_0)$ since the left and right actions commute. Thus, $(x, R_w(x)) = (k \cdot b_0, k \cdot R_w(b_0)) = k \cdot (b_0, b_w)$ belongs to the K-orbit of (b_0, b_w). Conversely, an element of the orbit $k \cdot (b_0, b_w) = (x, R_w(x))$, $x = k \cdot b_0$, belongs to the graph of R_w. □

Remark 5.4. In the case when $w = w_0$ is the principal involution, the K-orbit of Proposition 5.3 corresponds to the zero section of $T^*\mathbb{F}$ when $\mathcal{O}(H_0) = G/MA$ is identified with the cotangent bundle. This happens because the origin G/MA gets mapped to $H_0 \in \mathcal{O}(H_0)$ and the K-orbit of H_0 is identified to the zero section. On the other hand, the origin of the open orbit $G \cdot (b_0, b_{w_0}) \in \mathbb{F} \times \mathbb{F}$ is (b_0, b_{w_0}), so that its K-orbit gets identified to the K-orbit of H_0.

Remark 5.5. It follows directly from Proposition 5.3 that the graphs of right translations R_w, $w \in \mathcal{W}$, are contained in the diagonal G-orbits and consequently are compact inside these orbits. This does not happen with left translations, not even by elements of $\mathrm{Norm}_K(A)$, which represent elements of the Weyl group.

5.4. Example

For $\mathfrak{sl}(2, \mathbb{C})$ the flag is $\mathbb{C}P^1 = S^2$ and $\mathcal{W} = \{1, (12)\}$. The right action of $R_{(12)}$ on S^2 is the antipodal map. Another way to see this right action is to identify $\mathbb{C}P^1$ with the set of Hermitian matrices with eigenvalues ± 1 (the adjoint orbit of the compact group $\mathrm{SU}(2)$). This identification associates with a Hermitian matrix the eigenspace associated with the eigenvalue $+1$. In this case, if $\xi = \langle (x, y) \rangle \in \mathbb{C}P^1$, then $R_{(12)}(\xi)$ is the eigenspace of the Hermitian matrix associated with the eigenvalue -1. That is, $R_{(12)}(\xi)$ is the Hermitian orthogonal $\xi^\perp$ of ξ, which is generated by $(-\bar{y}, \bar{x})$ if $\xi = \langle (x, y) \rangle$.

Consider now the Cartesian product $S^2 \times S^2$ with the diagonal action of $G = \mathrm{Sl}(2, \mathbb{C})$: $g(\xi, \eta) = (g\xi, g\eta)$. There are two orbits, as follows.

(1) The diagonal $\Delta = \{(\xi, \xi) \colon \xi \in S^2\}$.

(2) An open and dense orbit $\{(\xi, \eta) \colon \xi, \eta \in S^2,\ \xi \neq \eta\}$. As a homogenous space of G this open orbit is given by G/MA, where MA is the Cartan subgroup of diagonal matrices. Thus, it can be identified with the adjoint orbit of

$$H_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The right action $R_{(12)}$ on G/MA admits good descriptions in terms of the identifications with the adjoint orbit $\mathrm{Ad}(G)H_0$ and with the open orbit $G \cdot o$ in $S^2 \times S^2$. They are as follows.

(1) If $A \in \mathrm{Ad}(G)H_0$, then $R_{(12)}(A)$ is the unique 2×2 matrix with eigenvalues ± 1 that has the same eigenspaces as those of A, but with the order of the eigenvalues switched.

(2) If $(\xi, \eta) \in G \cdot o$, then $R_{(12)}(\xi, \eta) = (\eta, \xi)$, since in the first case the order of the eigenspaces is switched.

5.5. Hermitian structures and symplectic forms

Suppose here that $\mathfrak{g}$ is a complex algebra and take a Weyl basis $X_\alpha \in \mathfrak{g}_\alpha$. The real compact form $\mathfrak{u}$ is generated by $A_\alpha = X_\alpha - X_{-\alpha}$ and $Z_\alpha = \mathrm{i}S_\alpha = \mathrm{i}(X_\alpha + X_{-\alpha})$ with $\alpha > 0$. If $\mathfrak{u}_\alpha = \operatorname{span}\{A_\alpha, Z_\alpha\}$, then the tangent space at the origin b_{H_0} of $\mathbb{F}_{H_0}$ is isomorphic to

$$T_{H_0} = \sum_{\alpha(H_0)>0} \mathfrak{u}_\alpha$$

via the isomorphism

$$Y \in T_{H_0} \mapsto \tilde{Y}(b_{H_0}) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{tY} \cdot b_{H_0})_{|t=0} \in T_{b_{H_0}} \mathbb{F}_{H_0}.$$

The canonical complex structure J on $\mathbb{F}_{H_0}$ is invariant by the compact group $K = \mathrm{e}^{\mathfrak{u}}$, and at the origin of the subspaces $\mathfrak{u}_\alpha$, $\alpha > 0$, it is given by

$$JA_\alpha = Z_\alpha, \qquad JZ_\alpha = -A_\alpha.$$

If $\tilde{w}$ is a representative of $w \in \mathcal{W}$. Then the tangent space to $\tilde{w}H_0$ is identified with

$$T_{\tilde{w}H_0} = \sum_{\alpha(\tilde{w}H_0)>0} \mathfrak{u}_\alpha,$$

and the canonical complex structure J_w on $T_{\tilde{w}H_0}$ is given by

$$JA_\alpha = Z_\alpha, \qquad JZ_\alpha = -A_\alpha$$

with A_α and Z_α, with the caveat that we take roots α such that $\alpha(\tilde{w}H_0) > 0$ (which are not in general positive roots).

Every K-invariant Riemannian metric on $\mathbb{F}_{H_0}$ is almost Hermitian with respect to J (see [**6**]). In general, the corresponding Kähler form Ω is not closed and, consequently, is not symplectic. However, the Kähler form is symplectic for the case of the Borel metric $(\cdot,\cdot)^{\mathrm{B}}$, which is the K-invariant metric defined at the origin by $(\mathfrak{u}_\alpha, \mathfrak{u}_\alpha)^{\mathrm{B}} = 0$ if $\alpha \neq \beta$ and satisfying

$$\begin{aligned}
&(\tilde{A}_\alpha(H_0), \tilde{A}_\alpha(H_0))^{\mathrm{B}}_{H_0} = (\tilde{Z}_\alpha(H_0), \tilde{Z}_\alpha(H_0))^{\mathrm{B}}_{H_0} = \alpha(H_0),\\
&(\tilde{A}_\alpha(H_0), \tilde{Z}_\alpha(H_0))^{\mathrm{B}}_{H_0} = 0
\end{aligned}$$

if $\alpha(H_0) > 0$.

This description of the Borel metric also holds at other points of $\mathbb{F}_{H_0} = \mathrm{Ad}(U) \cdot H_0$. For example, the tangent space at $\mathrm{Ad}(\tilde{w}) \cdot H_0$ is $\sum_{\alpha(w \cdot H_0)>0} \mathfrak{u}_\alpha$ and the metric at $\mathfrak{u}_\alpha$ is given by the same expression provided $\alpha(w \cdot H_0) > 0$.

Proposition 5.6. *The map* $R_{w_0} \colon \mathbb{F}_{H_0} \to \mathbb{F}_{H_0^*}$ *is an isometry of Borel metrics.*

Proof. Since R_{w_0} is equivariant by the left actions on $\mathbb{F}_{H_0}$ and $\mathbb{F}_{H_0^*}$ and the metrics are K-invariant, it suffices to verify the isometry at the origin. Equivariance also implies that $(R_{w_0})_* \tilde{A} = \tilde{A}$. Thus, for $x \in \mathbb{F}_{H_0}$,

$$((\mathrm{d}R_{w_0})_x \tilde{A}(x), (\mathrm{d}R_{w_0})_x \tilde{B}(x))^{\mathrm{B}}_{R_{w_0}(x)} = (\tilde{A}(R_{w_0}(x)), \tilde{B}(R_{w_0}(x)))^{\mathrm{B}}_{R_{w_0}(x)}.$$

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At $x = H_0 \in \mathbb{F}_{H_0}$ we have $R_{w_0}(H_0) = w_0 H^* = -H_0$. Now, if $\alpha(H_0) > 0$, then

$$(\tilde{A}_\alpha(H_0), \tilde{A}_\alpha(H_0))^{\mathrm{B}}_{H_0} = \alpha(H_0)$$

and the second term of the previous equality for $A = B = A_\alpha$ is

$$(\tilde{A}_\alpha(-H_0), \tilde{A}_\alpha(-H_0))^{\mathrm{B}}_{H_0} = -\alpha(-H_0) = \alpha(H_0).$$

The same holds true for Z_α corresponding to any root α with $\alpha(H_0) > 0$, so

$$((\mathrm{d}R_{w_0})_{H_0}\tilde{A}(H_0), (\mathrm{d}R_{w_0})_{H_0}\tilde{B}(H_0))^{\mathrm{B}}_{-H_0} = (\tilde{A}(H_0), \tilde{B}(H_0))^{\mathrm{B}}_{H_0}$$

for arbitrary A and B. This shows that R_{w_0} is an isometry. □

Having obtained the isometry R_{w_0}, its holomorphicity provides us with the symplectic isomorphism.

Proposition 5.7. *The map $R_{w_0} : \mathbb{F}_{H_0} \to \mathbb{F}_{H_0^*}$ is* anti-*holomorphic with respect to the canonical complex structures on $\mathbb{F}_{H_0}$ and $\mathbb{F}_{H_0^*}$.*

Proof. Let $\tilde{w}_0$ be a representative of w_0 such that $\mathrm{Ad}(\tilde{w}_0)H_0^* = -H_0$, and denote by J_0 and J_{w_0} the complex structures on $T_{H_0}\mathbb{F}_{H_0}$ and $T_{-H_0}\mathbb{F}_{H_0^*}$, respectively. Take a root α with $\alpha(H_0) > 0$, that is, $(-\alpha)(-H_0) > 0$. We have

$$J_0(\tilde{A}_\alpha(H_0)) = \tilde{Z}_\alpha(H_0), \qquad J_0(\tilde{Z}_\alpha(H_0)) = -\tilde{A}_\alpha(H_0),$$

since $\alpha(H_0) > 0$, and

$$J_{w_0}(\tilde{A}_\alpha(-H_0)) = -J_{w_0}(\tilde{A}_{-\alpha}(-H_0)) = -\tilde{Z}_\alpha(-H_0),$$
$$J_{w_0}(\tilde{Z}_\alpha(-H_0)) = -\tilde{A}_{-\alpha}(-H_0) = \tilde{A}_\alpha(-H_0),$$

since $(-\alpha)(-H_0) > 0$.

On the other hand, $(R_{w_0})_*\tilde{A}_\alpha = \tilde{A}_\alpha$ and $(R_{w_0})_*\tilde{Z}_\alpha = \tilde{Z}_\alpha$. Therefore,

$$J_{w_0}((\mathrm{d}R_{w_0})_{H_0}\tilde{A}_\alpha(H_0)) = J_{w_0}(\tilde{A}_\alpha(-H_0)) = -\tilde{Z}_\alpha(-H_0),$$
$$J_{w_0}((\mathrm{d}R_{w_0})_{H_0}\tilde{Z}_\alpha(H_0)) = J_{w_0}(\tilde{Z}_\alpha(-H_0)) = \tilde{A}_\alpha(-H_0),$$

whereas

$$(\mathrm{d}R_{w_0})_{H_0}J_0(\tilde{A}_\alpha(H_0)) = \tilde{Z}_\alpha(-H_0),$$
$$(\mathrm{d}R_{w_0})_{H_0}J_0(\tilde{Z}_\alpha(H_0)) = -\tilde{A}_\alpha(-H_0),$$

which shows that R_{w_0} is anti-holomorphic at the origin and, consequently, on the whole flag by the invariance of the complex structures. □

Corollary 5.8. *If $k \in K$, then the composites $R_{w_0} \circ k$ and $k \circ R_{w_0}$ are anti-holomorphic.*

Corollary 5.9. *Let Ω_{H_0} and $\Omega_{H_0^*}$ be the Kähler forms of the Hermitian structures on $\mathbb{F}_{H_0}$ and $\mathbb{F}_{H_0^*}$ given by the Borel metric and the canonical complex structures. Then R_{w_0} is anti-symplectic, that is, $R_{w_0}^*\Omega_{H_0^*} = -\Omega_{H_0}$.*

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5.6. Hermitian structures on products

The product $\mathbb{F}_{H_0} \times \mathbb{F}_{H_0^*}$ is a flag of the product $G \times G$ associated with (H_0, H_0^*), that is, $\mathbb{F}_{H_0} \times \mathbb{F}_{H_0^*} = \mathbb{F}_{(H_0,H_0^*)}$. This flag has Borel metric and invariant complex structures.

The adjoint orbit $\mathcal{O}(H_0)$ is identified to the orbit $G \cdot (H_0, -H_0)$ by the diagonal representation (recall that $-H_0 \in \mathbb{F}_{H_0^*}$ since $\mathrm{Ad}(\tilde{w}_0)H_0^* = -H_0$ if $\tilde{w}_0$ is a representative of w_0). The adjoint orbit $\mathcal{O}(H_0)$ has a complex structure inherited from the inclusion into $\mathfrak{g}$. On the other hand, the graphs considered above are Lagrangian with respect to a symplectic form defined from the complex structures of the flags. Hence, to continue our analysis we must compare these different complex structures.

We take $\mathfrak{h} \times \mathfrak{h}$ as a Cartan subalgebra in $\mathfrak{g} \times \mathfrak{g}$. The roots of $\mathfrak{h} \times \mathfrak{h}$ are those of $\mathfrak{h}$ in each component, and the root spaces are of the form $\mathfrak{g}_\alpha \times \{0\}$ or $\{0\} \times \mathfrak{g}_\alpha$.

The tangent space $T_{(H_0,-H_0)}\mathbb{F}_{(H_0,H_0^*)}$ is generated by $(A_\alpha, 0)$, $(Z_\alpha, 0)$, $(0, A_\alpha)$ and $(0, Z_\alpha)$. To obtain these generators, we can take the positive roots $\alpha > 0$. Here, if α is a positive root, then $\alpha(H_0) > 0$ but $\alpha(-H_0) < 0$, determining a difference between the complex structures of the first and second components.

In fact, if $\alpha > 0$ and $(\mathfrak{u} \times \mathfrak{u})_\alpha$ denotes the space generated by the four vectors above, then the canonical complex structure on $(\mathfrak{u} \times \mathfrak{u})_\alpha \subset T_{(H_0,-H_0)}\mathbb{F}_{(H_0,H_0^*)}$ is given by

$$\left.\begin{aligned} J(A_\alpha, 0) &= (Z_\alpha, 0), & J(Z_\alpha, 0) &= -(A_\alpha, 0), \\ J(0, A_\alpha) &= -(0, Z_\alpha), & J(0, Z_\alpha) &= (0, A_\alpha). \end{aligned}\right\} \tag{5.1}$$

Remark 5.10. These expressions show that the canonical complex structure on $\mathbb{F}_{(H_0,H_0^*)} = \mathbb{F}_{H_0} \times \mathbb{F}_{H_0^*}$ is the product of the canonical complex structures on $\mathbb{F}_{H_0}$ and $\mathbb{F}_{H_0^*}$.

Another basis of the tangent space $T_{(H_0,-H_0)}\mathbb{F}_{(H_0,H_0^*)}$ is given by

$$(\tilde{X}_{-\alpha}(H_0), 0), (\widetilde{\mathrm{i}X}_{-\alpha}(H_0), 0), (0, \tilde{X}_\alpha(-H_0)), (0, \widetilde{\mathrm{i}X}_\alpha(-H_0)),$$

with α running over the *positive* roots. These satisfy

$$\left.\begin{aligned} J(\tilde{X}_{-\alpha}(H_0), 0) = -(\widetilde{\mathrm{i}X}_{-\alpha}(H_0), 0) J(\widetilde{\mathrm{i}X}_{-\alpha}(H_0), 0) = (\tilde{X}_{-\alpha}(H_0), 0), \\ J(0, \tilde{X}_\alpha(-H_0)) = -(0, \widetilde{\mathrm{i}X}_\alpha(-H_0)) J(0, \widetilde{\mathrm{i}X}_\alpha(-H_0)) = (0, \tilde{X}_\alpha(-H_0)). \end{aligned}\right\} \tag{5.2}$$

Using these calculations we obtain the following statement.

Proposition 5.11. *Let J^{in} and J be the following complex structures on $\mathcal{O}(H_0) \approx G \cdot (H_0, -H_0)$.*

(1) *J^{in} is the complex structure on $\mathcal{O}(H_0) \subset \mathfrak{g}$ inherited from $\mathfrak{g}$.*

(2) *J is the complex structure on $G \cdot (H_0, -H_0)$ obtained by restriction of the complex structure on $\mathbb{F}_{H_0} \times \mathbb{F}_{H_0^*}$, defined at the origin by (5.2).*

Then $J^{\mathrm{in}} = -J$.

Proof. It suffices to verify that equality holds at the origin, since both complex structures are G-invariant. The tangent space to $\mathcal{O}(H_0)$ at the origin is generated by $\tilde{W}(H_0) = [W, H_0]$, with W in $\mathfrak{g}_{\pm\alpha}$ and α running over all positive roots. If $W \in \mathfrak{g}_{-\alpha}$, $\alpha > 0$, then $\tilde{W}(H_0)$ is 'horizontal' in the identification with $G \cdot (H_0, -H_0)$, whereas $\tilde{W}(H_0)$ is 'vertical' if $W \in \mathfrak{g}_{\alpha}$, $\alpha > 0$. For the complex structure on $\mathfrak{g}$ we have $X_\alpha \mapsto \mathrm{i}X_\alpha$ and $\mathrm{i}X_\alpha \mapsto -X_\alpha$. Thus the complex structure J^{in} is given in the product by

$$J^{\mathrm{in}}(\tilde{X}_{-\alpha}(H_0), 0) = (\widetilde{\mathrm{i}X}_{-\alpha}(H_0), 0) J^{\mathrm{in}}(\widetilde{\mathrm{i}X}_{-\alpha}(H_0), 0) = -(\tilde{X}_{-\alpha}(H_0), 0),$$
$$J^{\mathrm{in}}(0, \tilde{X}_{\alpha}(-H_0)) = (0, \widetilde{\mathrm{i}X}_{\alpha}(-H_0)) J^{\mathrm{in}}(0, \widetilde{\mathrm{i}X}_{\alpha}(-H_0)) = -(0, \tilde{X}_{\alpha}(-H_0)),$$

which is precisely the negative of (5.2). □

Let $(\cdot,\cdot)^{\mathrm{B}}$ be the Borel metric on $\mathbb{F}_{H_0} \times \mathbb{F}_{H_0^*} = \mathbb{F}_{(H_0, H_0^*)}$. If follows immediately from the definition that $(\cdot,\cdot)^{\mathrm{B}}$ is the product of the Borel metrics on $\mathbb{F}_{H_0}$ and $\mathbb{F}_{H_0^*}$.

This metric together with the canonical complex structure J define a Hermitian structure on $\mathbb{F}_{H_0} \times \mathbb{F}_{H_0^*}$, which is invariant by $K \times K$ (compact group) but is not invariant by $G \times G$, because the metric itself is only invariant by $K \times K$. This Hermitian structure restricts to a Hermitian structure in the open orbit $G \cdot (H_0, -H_0) \approx \mathcal{O}(H_0)$, which is invariant by the action of K (but not by that of G). Denote by $\Omega(\cdot,\cdot) = (\cdot, J(\cdot))^{\mathrm{B}}$ the corresponding Kähler form, which is a symplectic form. Since $(\cdot,\cdot)^{\mathrm{B}}$ is the product metric and J is the product complex structure, it follows that Ω is the product of the Kähler forms in $\mathbb{F}_{H_0}$ and $\mathbb{F}_{H_0^*}$.

6. Lagrangian graphs in products of flags

By Corollary 5.9 the map $R_{w_0} \colon \mathbb{F}_{H_0} \to \mathbb{F}_{H_0^*}$ is anti-symplectic with respect to the Kähler forms on $\mathbb{F}_{H_0}$ and $\mathbb{F}_{H_0^*}$ given by the Borel metric and canonical complex structures. Therefore, $\mathrm{graph}(R_{w_0})$ is a Lagrangian submanifold of the product symplectic structure. We now obtain further examples of Lagrangian graphs by composites (either on the left or on the right) of R_{w_0} with symplectic maps.

Example 6.1. If $k_1, k_2 \in K$, then the induced maps $k_1 \colon \mathbb{F}_{H_0} \to \mathbb{F}_{H_0}$ and $k_2 \colon \mathbb{F}_{H_0^*} \to \mathbb{F}_{H_0^*}$ are symplectic. Therefore, $k_1 \circ R_{w_0} \circ k_2$ is anti-symplectic, hence its graph is a Lagrangian submanifold of $\mathbb{F}_{H_0} \times \mathbb{F}_{H_0^*} = \mathbb{F}_{(H_0, H_0^*)}$. Such a graph is not contained in $G \cdot (H_0, -H_0)$; nevertheless, its intersection with the orbit is still a Lagrangian submanifold (non-compact if the graph is not contained in the orbit).

The tangent space $T_{(x,\phi(x))}\,\mathrm{graph}(\phi)$ is given by the vectors $(u, \mathrm{d}\phi_x(u))$. For maps $k \circ R_{w_0}$, with $k \in K$, the tangent spaces admit the following description in terms of the adjoint representation.

Proposition 6.2. *Let $k \in K$. The tangent space to* $\mathrm{graph}(k \circ R_{w_0})$ *at $(x, y) = (x, k \circ R_{w_0}(x))$ is given by*

$$\{(A, \mathrm{Ad}(k)A)^\sim(x, k \circ R_{w_0}(x)) \colon A \in \mathfrak{u}\},$$

where $(A, \mathrm{Ad}(k)A)^\sim$ is the vector field on $\mathbb{F}_{H_0} \times \mathbb{F}_{H_0^*} = \mathbb{F}_{(H_0, H_0^*)}$ that is induced by $(A, \mathrm{Ad}(k)A) \in \mathfrak{u} \times \mathfrak{u}$ ($\mathfrak{u}$ is the Lie algebra of K).

Proof. If $A \in \mathfrak{u}$, then $(R_{w_0})_* \tilde{A} = \tilde{A}$, thus $(\mathrm{d}R_{w_0})_x(\tilde{A}(x)) = \tilde{A}(R_w(x))$. Applying $\mathrm{d}k_{R_w(x)}$ to this equality we get

$$\begin{aligned}(\mathrm{d}k \circ R_{w_0})_x(\tilde{A}(x)) &= \mathrm{d}k_{R_w(x)}(\tilde{A}(R_w(x))) \\ &= \widetilde{\mathrm{Ad}(k)}A(k \circ R_{w_0}(x)).\end{aligned}$$

It follows that the tangent space to the graph is $(\tilde{A}(x), \widetilde{\mathrm{Ad}(k)}A(k \circ R_{w_0}(x)))$. But the action of $K \times U$ on $\mathbb{F}_{H_0} \times \mathbb{F}_{H_0^*}$ works coordinatewise. Hence

$$(\tilde{A}(x), \widetilde{\mathrm{Ad}(k)}A(k \circ R_{w_0}(x))) = (A, \mathrm{Ad}(k)A)^{\sim}(x, k \circ R_{w_0}(x)),$$

which completes the proof, because the vectors $\tilde{A}(x)$, $A \in \mathfrak{u}$, exhaust the tangent space at x. □

In conclusion, we have described the following families of Lagrangian submanifolds of the adjoint orbit $\mathcal{O}(H_\Theta) = \mathrm{Ad}(G) \cdot H_\Theta \approx G/Z_\Theta$.

Theorem 6.3. *For $k_1, k_2 \in K$ and for $m \in T$,*

- $\mathrm{graph}(k_1 \circ R_{w_0} \circ k_2)$ *corresponds to a Lagrangian submanifold of $\mathcal{O}(H_\Theta)$ and*
- $\mathrm{graph}(m \circ R_{w_0})$ *corresponds to a Lagrangian submanifold of $\mathcal{O}(H_\Theta)$.*

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