Problem Corner

Solutions are invited to the following problems. They should be addressed to **Nick Lord** at **Tonbridge School, Tonbridge, Kent TN9 1JP** (e-mail: **njl@tonbridge-school.org**) and should arrive not later than 10 August 2020.

Proposals for problems are equally welcome. They should also be sent to Nick Lord at the above address and should be accompanied by solutions and any relevant background information.

104.A (Michael Fox)

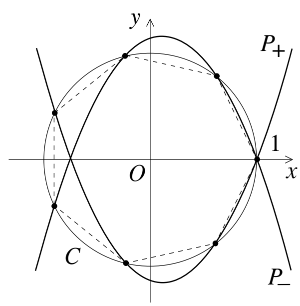
How many points (x, y) with integer coordinates are on the curve

$$9x^2 - 12xy + 4y^2 + 3x + 2y - 12 = 0$$

and inside the square with sides $x = \pm 10^6$ and $y = \pm 10^6$?

104.B (Yasuo Matsuda)

The diagram shows a regular heptagon inscribed in the unit circle C. One vertex of the heptagon is $(1, 0)$. Find the equations of the two parabolas P_+ and P_- which are symmetrical about the x-axis and pass through the vertices of the heptagon as shown.



104.C (K. S. Bhanu and M. N. Deshpande)

The probability of getting a head with a certain coin is p. One player tosses the coin n times and the set A consists of the toss numbers where a head occurred. A second player tosses the coin n times to produce a set B in the same way. What is the expectation of X, the number of elements of $A \cup B$?

[For example, if $n = 3$ and the first player obtains THH and the second player obtains HTH, then $A = \{2, 3\}$, $B = \{1, 3\}$ and the value of X is 3.]

104.D (Stan Dolan)

For given positive integers a and b let $f(x) = \left\lfloor \frac{x + a}{b} \right\rfloor$ and $g(x) = \left\lfloor \frac{x + b}{a} \right\rfloor$.

(a) Show that $fg(x) - gf(x)$ takes at most two values for real x. Under what condition on a and b does $fg(x) - gf(x)$ take only one value?

(b) As $n \to \infty$, find the limit of $\frac{1}{n} \int_0^n fg(x) - gf(x)\, dx$.

N.B. $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

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Solutions and comments on **103.E**, **103.F**, **103.G**, **103.H** (July 2019).

103.E (M. G. Elliott)

Let N be a positive integer and let S_N be the set of points in the plane with coordinates $(m^2 - n^2, 2mn)$ for positive integers m, n with $n < m \leqslant N$. Find, in terms of N, the radius of the circle with centre $(0, 0)$ which minimises the sum of the squares of the perpendicular distances from the points of S_N to the circle.

Answer: The minimising radius is $\frac{1}{3}(N + 1)(2N + 1)$.

Solutions to this popular problem generally went along the following lines.

There are $\binom{N}{2} = \frac{1}{2}N(N-1)$ choices of (n, m) with $1 \leqslant n < m \leqslant N$ and (as Gregory Dresden observed) each creates a distinct point in S_N. For if $(m^2 - n^2, 2mn) = (s^2 - t^2, 2st)$ then $m^2 - n^2 = s^2 - t^2$ and $m^2 + n^2 = s^2 + t^2$ quickly leads to $(m, n) = (s, t)$. The perpendicular distance of $(m^2 - n^2, 2mn)$ to a circle centre $(0, 0)$, radius R is $|R - (m^2 + n^2)|$, so we must minimise $T = \sum_S [R - (m^2 + n^2)]^2$ where $S = \{(m, n) : 1 \leqslant n < m \leqslant N\}$.

To do this, some solvers used calculus or completing the square or quoted the minimising property of the arithmetic mean. The upshot is that T is minimised when

$$R = R_{\min} = \frac{\sum_S (m^2 + n^2)}{\frac{1}{2}N(N-1)}.$$

But
$$\sum_S (m^2 + n^2) = \frac{1}{2}\left[\sum_{1 \leqslant n, m \leqslant N} (m^2 + n^2) - 2\sum_{i=1}^N i^2\right]$$
$$= \frac{1}{2}\left[N\sum_{m=1}^N m^2 + N\sum_{n=1}^N n^2 - 2\sum_{i=1}^N i^2\right]$$
$$= (N-1)\sum_{i=1}^N i^2 = \frac{1}{6}(N-1)N(N+1)(2N+1)$$

from which $R_{\min} = \frac{1}{3}(N + 1)(2N + 1)$.

Peter Johnson showed that this gives the minimum value of T as

$$\tfrac{1}{180}(N-2)(N-1)N(N+1)(2N+1)(8N+11).$$

Correct solutions were received from: N. Curwen, S. Dolan, G. Dresden, A. P. Harrison, G. Howlett, P. F. Johnson, G. Jolly, A. Mahajan, J. A. Mundie, J. Siehler, C. Starr, G. Strickland, E. Swylan, A. Tee and the proposer M. G. Elliott.

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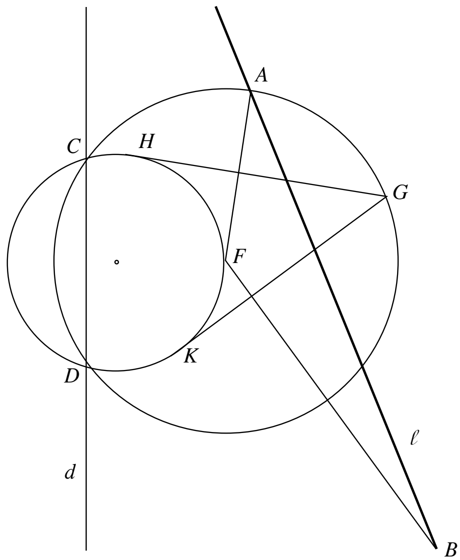
103.F (Geoff Strickland)

In the diagram d is a fixed straight line and F a fixed point. The point G is the reflection of F in a variable straight line ℓ which is drawn in such a way that the circle centre F and radius FG intersects d at two points C and D. The points H and K are the points of contact of the tangents from G to the circle CFD. The points A and B are points on ℓ such that FA and FB are perpendicular to GH and GK respectively.

Prove that A and B are the points of intersection of ℓ with the parabola having directrix d and focus F.



This problem attracted a range of approaches illustrated by the two solutions below.

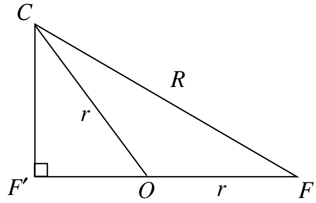
First we give Stan Dolan's neat direct proof.

Let AF and GH intersect at X. Let $FG = R$ and let the circle CFD have centre O and radius r. Let the perpendiculars from A and F to line d intersect the line at A' and F', respectively.

Let $F'F = L$. Then, by Pythagoras,

$$R^2 - L^2 = r^2 - (L - r)^2$$

$$\Rightarrow R^2 = 2Lr.$$



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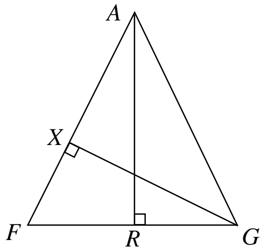
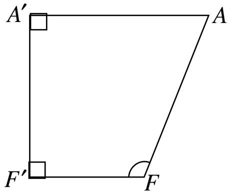
By similar triangles

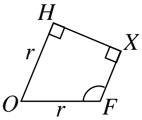
$$FX : R = \tfrac{1}{2}R : FA$$
$$\Rightarrow R^2 = 2FA.FX.$$



Now consider the two quadrilaterals $FAA'F'$ and $FOHX$. From the above results, we note that

$$\frac{L}{FX} = \frac{FA}{r} \Rightarrow \frac{FF'}{FX} = \frac{FA}{FO}.$$





The two quadrilaterals also have the same angles and are therefore similar. In particular, $FA = AA'$. The point A (and similarly B) therefore lies on the parabola.

The proposer, Geoff Strickland, gave the following succinct proof which uses inversion.

Invert with respect to F, radius of inversion FG. Circle FCD inverts to line d and vice versa. Straight line FA inverts into itself. Straight line GH inverts into a circle through G and F and tangential to d (at P say) and since GH is perpendicular to FA it follows that this circle must have its centre on line FA. But its centre must also lie on ℓ, because ℓ is the perpendicular bisector of FG. Thus A is its centre and $AP = AF$ which is sufficient to show that A lies on the given parabola. The proof for B is of course similar.

Finally, it was not the intention of the Figure to suggest that A lies on the large circle: my apologies if any solvers were misled by this.

Correct solutions were received from: S. Dolan, G. Howlett, E. Swylan, L. Wimmer (2 solutions) and the proposer Geoff Strickland.

103.G (Stan Dolan)

The quadratic $f(x)$ has integer coefficients and discriminant D. Given that $\frac{2f(-1)}{f'(0)}$ is an integer greater than or equal to $|D|$, prove that $f(x)$ has rational roots.

Most unusually, only one solution other than the proposer's was received for this problem. Stan Dolan's ingenious solution went as follows.

Let $f(x) = ax^2 + bx + c$ with a, b, c integers. By hypothesis

$$\frac{2f(-1)}{f'(0)} = \frac{2a + 2c}{b} - 2$$

is an integer so that $2a + 2c = bd$ with d an integer satisfying $d \geqslant |D| + 2$.

If $b = 0$, then $a + c = 0$ and $D = 4a^2$ is a square. By replacing f with $-f$, if necessary, we can suppose $b > 0$ and therefore $a + c > 0$.

Consider the integer sequence $\dots, 2a, b, 2c, \dots$ extended in both directions by $x_n + x_{n+2} = dx_{n+1}$. Then

$$x_n^2 + x_{n+1}^2 - dx_nx_{n+1} = x_n^2 + (dx_n - x_{n-1})^2 - dx_n\left(dx_n - x_{n-1}\right)$$
$$= x_{n-1}^2 + x_n^2 - dx_{n-1}x_n$$

is constant on successive terms of the sequence. But

$$x_n^2 + x_{n+1}^2 - dx_nx_{n+1} = x_{n+1}^2 - x_nx_{n+2} = \dots = b^2 - 4ac = D.$$

Note that $x_{n+2} \equiv x_n \pmod 2$ for all n and so at least one of every two successive terms is even.

If any $x_n = 0$, then $D = x_{n+1}^2$ is a square.

If any $x_nx_{n+1} < 0$, then $D = x_n^2 + x_{n+1}^2 + d|x_nx_{n+1}| > d$, a contradiction. Therefore all the x_n have the same sign as b, i.e. positive.

Let u, v, w be three successive terms, where v is the least term of the entire sequence. Then $u + w = dv$ and $uw - v^2 = |D| \leqslant d - 2$. Let $u = v + x$, $w = v + y$ with $x, y \geqslant 0$. Then $x + y = u + w - 2v = (d - 2)v$ and

$$uw - v^2 = (v + x)(v + y) - v^2 = (x + y)v + xy \leqslant d - 2$$

so that $(d - 2)\left(v^2 - 1\right) + xy \leqslant 0$.

If $d = 2$, then $D = 0$. Otherwise, $v = 1$ and either x or $y = 0$. Then two adjacent terms are both 1, contradicting the fact that at least one of them must be even,

In all cases D is a square and so the roots are rational.

Correct solutions were received from: E. Swylan and the proposer Stan Dolan.

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103.H (Peter Shiu)

Find positive integers a, b, c and A, B, C such that

$$\int_0^1 \frac{a(x(1-x))^4 + b(x(1-x))^{12}}{c(x^2+1)}\,dx = \frac{355}{113} - \pi$$

and

$$\int_0^1 \frac{A(x(1-x))^{12} + B(x(1-x))^{20}}{C(x^2+1)}\,dx = \frac{104348}{33215} - \pi.$$

Answer: The smallest triples of integer values are

$$a = 4127,\ b = 1\,225\,785,\ c = 19\,616\,687$$

and

$$A = 206\,953\,445,\ B = 15\,866\,451,\ C = 7\,373\,066\,576.$$

This problem attracted a large number of correct, tightly argued solutions. Several respondents commented that the integrals in **103.H** are extensions of D. P. Dalzell's famous integral from [1]

$$\int_0^1 \frac{(x(1-x))^4}{x^2+1}\,dx = \frac{22}{7} - \pi$$

to the third and fifth convergents $\frac{355}{113}, \frac{104348}{33215}$ in the continued fraction expansion for π.

Most solutions followed the following pattern.

Let $I_n = \int_0^1 \frac{(x(1-x))^n}{x^2+1}\,dx$ and evaluate $I_4 = \frac{22}{7} - \pi$, $I_{12} = \frac{431302721}{8580495} - 16\pi$. Thus the condition $\frac{a}{c}I_4 + \frac{b}{c}I_{12} = \frac{355}{113} - \pi$ leads to the simultaneous equations

$$\frac{22}{7}\frac{a}{c} + \frac{431302721}{8580495}\frac{b}{c} = \frac{355}{113} \text{ and } \frac{a}{c} + 16.\frac{b}{c} = 1$$

which solve to give $\frac{a}{c} = \frac{4127}{19616687}$ and $\frac{b}{c} = \frac{1225785}{19616687}$ and thus the answer given above for a, b, c.

The calculation for A, B, C follows the same lines using

$$I_{20} = \frac{26856502742629699}{33393321606645} - 256\pi.$$

The variations in the solutions occurred in the evaluation of I_n. Many were content to use computer algebra systems, but others gave schemes that could be implemented by hand. The shortest route, given by Seán Stewart and, in an essentially equivalent form, by Stan Dolan is to use the identity

$$\frac{x^{4n}(1-x)^{4n}}{x^2+1} = \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4\right)\sum_{k=0}^{n-1}(-4)^{n-1-k}x^{4k}(1-x)^{4k} + \frac{(-4)^n}{x^2+1}.$$

This occurs in [2] and may be established by induction.

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Integrating, using $\int_0^1 x^m (1-x)^n\,dx = \frac{m!\,n!}{(m+n+1)!}$ in the terms of the summation, gives I_{4n} =

$$(-1)^n 4^{n-1}\pi + (-1)^{n+1}4^{n-1}\sum_{k=0}^{n-1}(-1)^k 2^{4-2k}\frac{(4k)!\,(4k+3)!}{(8k+7)!}\left(820k^3+1533k^2+902k+165\right)$$

and substituting $n = 1, 3, 5$ gives I_4, I_{12}, I_{20} needed above.

The proposer, Peter Shiu, proved the more general result below, which makes it clear why *positive* integers occur in **103.H**.

Let $0 \leqslant x \leqslant 1$ and, for $h = 0, 1, 2, \ldots$, write

$$F_h(x) = (x(1-x))^{4h} \text{ and } J_h = \int_0^1 \frac{F_h(x)}{x^2+1}\,dx, \text{ so that } \pi = 4J_0.$$

The equations in the problem are particular cases taken from the following:

Proposition: Let P, Q be coprime positive integers satisfying $\pi < \frac{P}{Q} < \frac{22}{7}$. Then there exists an odd h, and coprime positive integers r, s such that

$$\frac{rJ_h + sJ_{h+2}}{4^{h-1}(r+16s)} = \frac{P}{Q} - \pi.$$

Proof: From $F_h(x) \equiv (-4)^h \pmod{x^2+1}$ we deduce that $\frac{F_h(x) - (-4)^h}{x^2+1}$ is a polynomial in x with integer coefficients, and with degree $8h-2$. On integration we see that, if h is odd, then there are positive integers u_h, v_h, with v_h being a divisor of LCM$(1, 2, \ldots, 8h-1)$, such that

$$J_h = \frac{u_h}{v_h} - 4^{h-1}\pi, \Rightarrow \mu_h = \frac{u_h}{4^{h-1}v_h}, \qquad h = 1, 3, 5, \ldots,$$

are rational approximations to π, because $J_h < F_h\left(\frac{1}{2}\right) = 1/4^{4h}$. The fractions μ_h are decreasing with respect to odd h, and the intervals $I_h = \left\{\mu_{h+2} \leqslant \mu < \mu_h\right\}$ $(h = 1, 3, 5, \ldots)$ form a partition of the interval $\pi < \mu < 22/7 = \mu_1$, so that each such $\mu \in I_h$ for a unique h.

For $h = 1, 3, 5, \ldots$, let $(J, J') = (J_h, J_{h+2})$, $(u, v) = (u_h, v_h)$, $(u', v') = (u_{h+2}, v_{h+2})$. Then, for a real $\lambda \geqslant 0$, we now have

$$\lambda J + J' = \frac{\lambda u}{v} + \frac{u'}{v'} - 4^{h-1}(\lambda + 16)\pi.$$

As λ increases in $0 \leqslant \lambda < \infty$, the bilinear expression (in λ)

$$f(\lambda) = f_h(\lambda) = \frac{\frac{\lambda u}{v} + \frac{u'}{v'}}{4^{h-1}(\lambda + 16)}$$

increases strictly in I_h, and $\mu = f(\lambda)$ is rational if, and only if, λ is rational. The inverse $g : I_h \to [0, \infty)$ of f is given by the bilinear expression (in μ)

$$\lambda = g(\mu) = \frac{4^{h+1}\mu - \frac{u'}{v'}}{4^{h-1}\mu - \frac{u}{v}}. \qquad (*)$$

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Now, let $\pi < \frac{P}{Q} < \frac{22}{7}$. Then $\mu = \frac{P}{Q} \in I_h$ for a unique h, and u, v, u', v' can be computed accordingly. The reduced fraction $\lambda = g(\mu) = \frac{r}{s}$ can then be evaluated from $(*)$, together with the computation of GCD (r, s). The proposition is proved.

James Mundie and Seán Stewart raised the question of whether there are analogous results for the 2nd and 4th convergents, $\frac{333}{106}$ and $\frac{103993}{33102}$, which approximate π from below.

References

1. D. P. Dalzell, On $\frac{22}{7}$, *Journal London Maths. Soc* **19** (1944) pp. 133-134.
2. H. A. Medina, A sequence of polynomials for approximating inverse tangent, *Amer. Math. Monthly* **113** (February 2006) pp. 156-161.

Correct solutions were received from: S. Dolan, R. Gordon, A. P. Harrison, G. Howlett, M. Lukarevski, A. Mahajan, J. A. Mundie, S. M. Stewart, E. Swylan, L. Wimmer and the proposer Peter Shiu.

Readers will be saddened by the news of the death of Brian Trustrum in December 2019. He was one of the most loyal respondents to Problem Corner over the years. His solutions were characterised by a meticulous attention to detail and a ready willingness to tackle problems from all areas of pure and applied mathematics. Brian and Kathleen Trustrum were regular attendees at the annual MA Conferences and Brian's acuity, thoughtfulness and enthusiastic participation will be greatly missed.

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